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Finite element computation for scattering problems of micro-hologram using DtN map

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Abstract

Computational results are presented on micro-hologram diffraction for optical data storage using a finite element method. Retrieval of object light from a micro-hologram is formulated as an optical scattering problem in an infinite region. In order to overcome the difficulty of dealing with the infinite region a Dirichlet to Neumann (DtN) map is employed on an artificial boundary. By virtue of the DtN map reflection from the artificial boundary is effectively alleviated and non-reflecting boundary is obtained. Retrieval of the object light is computed for two different models.

Keywords optical scattering, DtN map, finite element method, micro-hologram

Research Activity Group Scientific Computation and Numerical Analysis

1. Introduction

Holographic data storage has been studied as a next generation method for optical data storage with terabyte capacity. A method using a micro-hologram is one of such technologies [1, 2], where a micro-hologram is generated as a set of interference fringes when two counter-propagating focused laser beams intersect at the focus. The data that one of the two lights, namely object light, carries are reconstructed as diffraction from the hologram when the other light called the reference light is illuminated on the hologram. This process is called retrieval of object light; see [3, p. 308] in detail. One of the main interests of the study is to estimate the diffraction efficiency in the retrieval process that determines the signal to noise ratio of the data storage system.

For free space propagation of light where neither free charge nor current exists, the electric field and the magnetic field are decoupled. This reduces the Maxwell equations to a set of vector-valued Helmholtz equations for the electric and magnetic fields assuming that the fields are time harmonic. Furthermore, in the retrieval process, a laser beam is in general polarized so it suffices to analyze one component of the electric field of light instead of all components of the Helmholtz equations [4].

As a result, the retrieval process can be described as an optical scattering problem, which is stated by the scalar Helmholtz equation in an infinite region. In order to avoid computational difficulty in an infinite region, several techniques have been developed to transform the original problem into one in a bounded domain. In the field of optics, there are researches using Boundary Element Method (BEM) [5], hybrid finite element method with BEM coupling [6], Perfectly Matched Layer (PML) [7], Transparent Boundary Condition (TBC) [4].

There is another method called Dirichlet to Neumann (DtN) map [8, 9] that has been used mainly in scattering problems in acoustics and has not been yet used for optical scattering problems. The reason may be because of the large wave number of light making computation more difficult. To the best of our knowledge, the DtN map has not yet been used for optical scattering problems. In this paper, we apply a DtN map to our optical scattering problem and simulate retrieval of object light from a micro-hologram.

2. Formulation

Let Ω_B be a 2-dimensional transmissive scatterer with a smooth boundary Γ and an outward unit normal n; see Fig. 1. We assume the time harmonic field. Let u be the complex amplitude of a scalar component of the electric field of scattered light. The scattering problem is formulated by the following Helmholtz equations in $\mathbb{R}^2$ according to [4, 10]; for a given domain Ω_B, wave numbers k_1 and k_2 in medium 1 and 2, and an incident light u^{inc}, find $u : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\begin{cases} -\Delta u - k_2^2 u = (\Delta + k_2^2) u^{\mathrm{inc}} & \text{in } \Omega_B, \\ -\Delta u - k_1^2 u = 0 & \text{in } \Omega_B^c, \\ [\, u \,] = 0 & \text{on } \Gamma, \\ \left[\dfrac{\partial u}{\partial n} \right] = 0 & \text{on } \Gamma, \\ \lim_{r \to +\infty} \sqrt{r} \left(\dfrac{\partial u}{\partial r} - i k_1 u \right) = 0, & \end{cases} \tag{1}$$

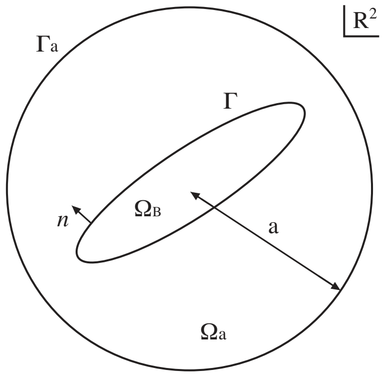

Fig. 1. A scatterer Ω_B and an artificial boundary Γ_{a}.

where i denotes the imaginary unit and $[\,.\,]$ represents a gap across Γ and $r := |x|$ with the orthogonal coordinate system $x = (x_1, x_2)$ in $\mathbb{R}^2$.

The scatterer, incident light and scattered light in the formulation correspond to the micro-hologram, reference light and object light, respectively in the retrieval process.

Let Ω_{a} be a circle with the radius a (> 0), and let Γ_{a} be the boundary of Ω_{a}; see Fig. 1. Suppose that the circle Ω_{a} includes Ω_B strictly. By introducing DtN map [8], the problem (1) becomes equivalent to the following equations in Ω_{a}; find $u : \Omega_{\mathsf{a}} \to \mathbb{C}$ such that

$$\begin{cases} -\Delta u - k_2^2 u = (\Delta + k_2^2) u^{\mathrm{inc}} & \text{in } \Omega_B, \\ -\Delta u - k_1^2 u = 0 & \text{in } \Omega_{\mathsf{a}} \backslash \Omega_B, \\ [\, u \,] = 0 & \text{on } \Gamma, \\ \left[\dfrac{\partial u}{\partial n} \right] = 0 & \text{on } \Gamma, \\ \dfrac{\partial u}{\partial r} = -Su & \text{on } \Gamma_{\mathsf{a}}. \end{cases} \tag{2}$$

Here S is the Steklov-Poincaré operator defined by

$$Su := -k_1 \sum_{n=-\infty}^{\infty} \frac{H_n^{(1)\prime}(k_1 \mathsf{a})}{H_n^{(1)}(k_1 \mathsf{a})}\, u_n(\mathsf{a})\, \phi_n(\theta),$$

where (r, θ) is the polar coordinate system in $\mathbb{R}^2$, $H_n^{(1)}$ Hankel function of the first kind of order n, $\phi_n(\theta)$ the spherical harmonics defined by

$$\phi_n(\theta) := \frac{1}{\sqrt{2\pi}}\, e^{in\theta}$$

and u_n a Fourier coefficient defined by

$$u_n(\mathsf{a}) := \int_0^{2\pi} u(\mathsf{a}, \theta)\, \overline{\phi_n(\theta)}\, d\theta.$$

The Hankel function and its derivative are defined as follows:

$$H_n^{(1)}(x) := J_n(x) + iY_n(x),$$

$$H_n^{(1)}(x)' := \frac{1}{2}\left(H_{n-1}^{(1)}(x) - H_{n+1}^{(1)}(x) \right),$$

where $J_n(x)$ and $Y_n(x)$ are the Bessel function of the first and second kind of order n, respectively.

Let $L^2(\Omega_{\mathsf{a}})$ be the space of complex-valued square-integrable functions defined in Ω_{a}, and let $\|\,.\,\|_{0,\Omega_{\mathsf{a}}}$ be its norm. For $m \in \mathbb{N}$, let $H^m(\Omega)$ be the space of functions in $L^2(\Omega_{\mathsf{a}})$ with derivatives up to the mth order, and let $\|\,.\,\|_{m,\Omega_{\mathsf{a}}}$ be its norm. Set $V := H^1(\Omega_{\mathsf{a}})$. Moreover, bilinear forms a and s are defined by

$$a(u, v) := \int_{\Omega_{\mathsf{a}}} (\nabla u \cdot \nabla \overline{v} - k^2 u\, \overline{v}\,)\, dx, \quad \forall u, v \in V,$$

$$s(u, v) := \int_{\Gamma_{\mathsf{a}}} (Su)\, \overline{v}\, ds, \quad \forall u, v \in V,$$

and a linear functional f is defined by

$$\langle f, v \rangle := \int_{\Omega_{\mathsf{a}}} f\, \overline{v}\, dx, \quad \forall v \in V.$$

Here, k is a piecewise constant function defined by

$$k(x) := \begin{cases} k_1 & \text{in } \Omega_{\mathsf{a}} \backslash \Omega_B, \\ k_2 & \text{in } \Omega_B, \end{cases}$$

and f is a scattering potential defined by

$$f(x) := \begin{cases} 0 & \text{in } \Omega_{\mathsf{a}} \backslash \Omega_B, \\ (\Delta + k_2^2) u^{\mathrm{inc}} & \text{in } \Omega_B. \end{cases}$$

Note that simple calculations make the bilinear form s become

$$s(u, v) = -k_1\, \mathsf{a} \sum_{n=-\infty}^{+\infty} \frac{H_n^{(1)\prime}(k_1 \mathsf{a})}{H_n^{(1)}(k_1 \mathsf{a})}\, u_n\, \overline{v}_n\,.$$

Now, (2) can be written in a weak form as follows: find $u \in V$ such that

$$a(u, v) + s(u, v) = \langle f, v \rangle, \quad \forall v \in V. \tag{3}$$

3. Finite element approximation

Let $\{\mathscr{T}_h\}$ be a uniformly regular family of triangulation of $\overline{\Omega}_{\mathsf{a}}$, where h stands for the maximum diameter of the triangles in $\mathscr{T}_h$. We set $\Omega_{\mathsf{a}h} := \mathrm{int}(\cup\{T; T \in \mathscr{T}_h\})$. By definition, let $V_h \subset V$ be the $P1$ finite element space. Moreover, the bilinear forms a and s, and the linear functional f are approximated by bilinear forms a_h and s_h^N, and a linear functional f_h defined by, for u_h, $v_h \in V_h$,

$$a_h(u_h, v_h) := \int_{\Omega_{\mathsf{a}h}} (\nabla u_h \cdot \nabla \overline{v}_h - k^2 u_h\, \overline{v}_h)\, dx,$$

$$s_h^N(u_h, v_h) := -k_1\, \mathsf{a} \sum_{n=-N}^{N} \frac{H_n^{(1)\prime}(k_1 \mathsf{a})}{H_n^{(1)}(k_1 \mathsf{a})}\, u_{hn}\, \overline{v}_{hn},$$

$$\langle f_h, v_h \rangle := \int_{\Omega_{\mathsf{a}h}} (\Pi_h f)\, \overline{v}_h\, dx,$$

where N is a truncation number and $\Pi_h f$ denotes the $P1$ interpolant of f.

Then, a finite element problem corresponding to (3) is obtained as follows: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) + s_h^N(u_h, v_h) = \langle f_h, v_h \rangle, \quad \forall v_h \in V_h. \tag{4}$$

Remark 1 *Let D be a reflective scatterer with smooth boundary Γ. Instead of the problem in the whole 2-dimensional Euclidean space $\mathbb{R}^2$, we consider the Helmholtz equation in the exterior region $\mathbb{R}^2 \backslash D$. Suppose that the circle Ω_{a} includes D strictly. Then, we can* *obtain the equivalent formula as follows:*

$$\begin{cases} -\Delta u - k_1^2 u = f & in \Omega_{\mathsf{a}} \backslash D, \\ u = g & on\ \Gamma, \\ \dfrac{\partial u}{\partial r} = -Su & on\ \Gamma_{\mathsf{a}}. \end{cases} \tag{5}$$

Under appropriate assumptions, there exists a convergence result for a finite element scheme (5) *corresponding to this problem; see* [9].

4. Numerical examples

Let u_1 and u_2 be the complex amplitude of a reference light and an object light, respectively. An interference pattern is calculated by the intensity field of the sum of u_1 and u_2. The domain for the corresponding scatterer, Ω_B, is found as the region where the interference intensity exceeds a given value. We assume a model that the region suffers index change to n_2 from surrounding index n_1 of a holographic material. We thus prepare a scatterer Ω_B prior to the simulation of a scattering problem. Then simulation is made to compute the scattering distribution and intensity so as to analyze the retrieval process.

In scattering problem, we approximate the retrieval reference light by an incident light with a plane wave, i.e., $u^{\mathrm{inc}} \sim e^{ik_1 x}$, whereas u_1 and u_2 are Gaussian beams. Then the scattering potential f can be simplified to $(-k_1^2 + k_2^2)\, e^{ik_1 x}$ in Ω_B. The complex amplitude of scattered field u is approximated by the conventional conforming $P1$ elements.

Considering that micro-holograms are literally of the size of microns, which is the same order as the wavelength of light, we have nondimensionalized the equations with respect to λ. Throughout these examples, the refractive indices are $n_1 = 1.5$ and $n_2 = 1.51$. The wave numbers then become

$$k_1 := 2\pi n_1 \approx 9.425, \quad k_2 := 2\pi n_2 \approx 9.488.$$

The Hankel function appearing in the Steklov-Poincaré operator is calculated by using the built-in function for the Bessel functions in the compiler. In order to solve the resultant linear systems, Conjugate Residual (CR) method was used. The computations were done by Core 2 Duo 3GHz CPU with 8GB memories.

4.1 Model A

A scatterer Ω_B is given by

$$\Omega_B = \{x \in \mathbb{R}^2;\ |u_1 + u_2|^2 \geq 0.5\},$$

which is a result of interference of two Gaussian beams that intersect at 90 degrees:

$$u_1(x_1, x_2) = \frac{1}{2}\sqrt{\frac{x_R}{q(x_1)}}\exp\left(-\frac{ik_1 x_2^2}{2q(x_1)}\right)\exp(ik_1 x_1),$$

$$u_2(x_1, x_2) = \frac{1}{2}\sqrt{\frac{x_R}{q(x_2)}}\exp\left(-\frac{ik_1 x_1^2}{2q(x_2)}\right)\exp(-ik_1 x_2),$$

where x_R is a given nondimensionalized Rayleigh range ≈ 4.676. The size of the micro-hologram created by these beams corresponds to about $0.7\mu m$.

In this example, u_1 and u_2 represent the reference light propagating along x_1 direction and the object light along $-x_2$ axis, respectively to form a micro-hologram, which is a scatterer in our optical scattering problem. Since we used Gaussian beams, only the vicinity of the Rayleigh range has strong field. This is the reason why the scatterer consists of only three micro-ellipses around the focus. The two beams intersecting at 90 degrees made the micro-hologram orientated at 45 degrees as Fig. 2 depicts. Fig. 2 also shows triangulation, in which the number of triangles is 105,578, and the number of nodal points is 53,046. The truncation number N of DtN map is 115. The CPU time is about 1 hour. Fig. 3 shows the absolute value of scattered light. It is very interesting to see that the retrieved light propagating along $-x_2$ axis, which is the direction of object light, can be clearly seen with relatively stronger intensity as well as transmitted light along x_1 direction as the reference light is incident. It is also interesting to note that some scattering pattern that appears in regular scattering from a cylinder or a sphere can be seen with much weaker intensity. It is worthwhile pointing out that no reflection from the artificial boundary can be observed at all, which means the DtN map is very effectively working as a transparent boundary.

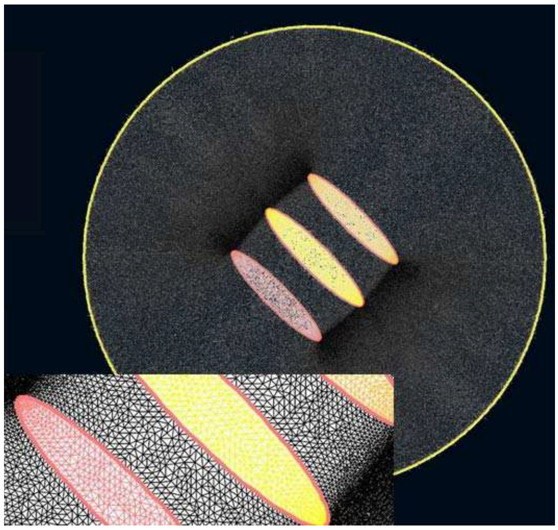

Fig. 2. Model A and its triangulation.

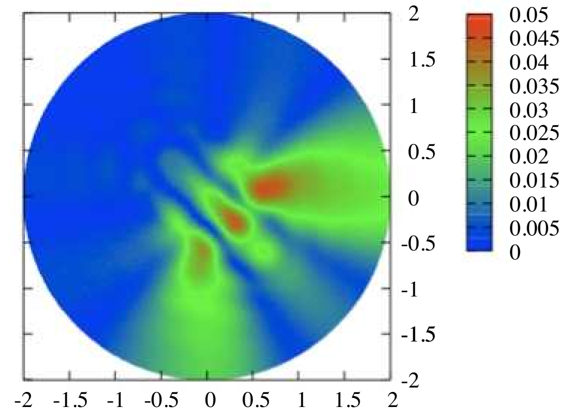

Fig. 3. The absolute value of scattered waves in Model A.

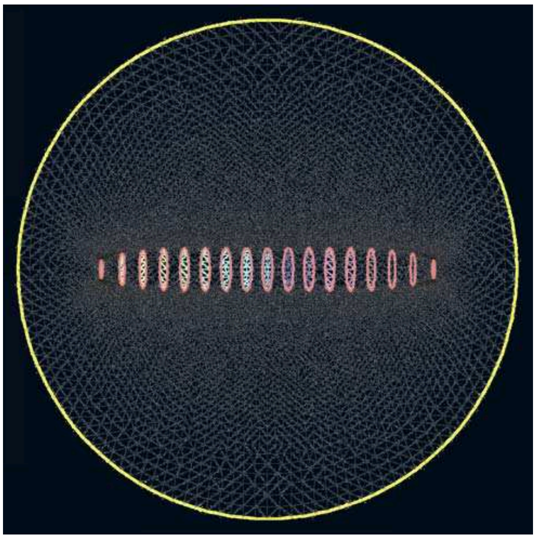

Fig. 4. Model B and its triangulation.

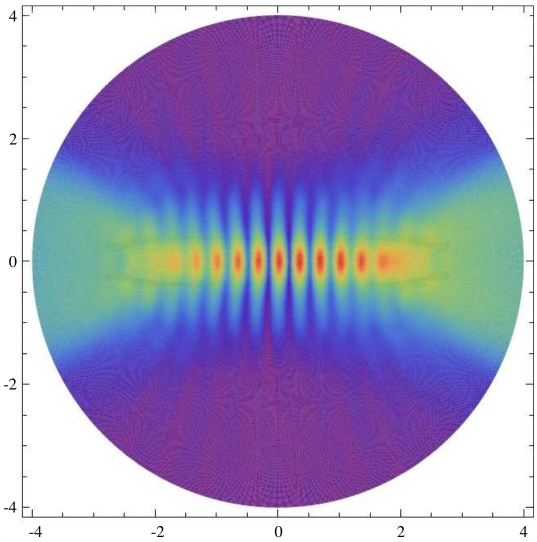

Fig. 5. The absolute value of scattered waves in Model B.

4.2 Model B

In the next model, the scatterer Ω_B is represented by

$$\Omega_B = \{x \in \mathbb{R}^2;\ |u_1 + u_2|^2 \geq 0.5\},$$

which is created by two counter-propagating Gaussian beams at 180 degrees along x_1 axis:

$$u_1(x_1, x_2) = \frac{1}{2}\sqrt{\frac{x_R}{q(x_1)}}\exp\left(-\frac{ik_1 x_2^2}{2q(x_1)}\right)\exp(ik_1 x_1),$$

$$u_2(x_1, x_2) = \frac{1}{2}\sqrt{\frac{x_R}{q(x_1)}}\exp\left(-\frac{ik_1 x_2^2}{2q(x_1)}\right)\exp(-ik_1 x_1),$$

where $x_R \approx 1.618$. The micro-hologram size corresponds to about 2.2 μm in this case.

As shown in Fig. 4 the scatterer Ω_B consists of seventeen micro-ellipses. The number of triangles is 585,019, and the number of nodal points is 1,169,012. The truncation number N of DtN map is 191. The CPU time is about 3 hours.

Fig. 5 shows the absolute value of the scattered field. Retrieval of the object light was successfully simulated with relatively stronger intensity on the left side of the micro-hologram whereas the reference light is transmitting toward the right side. It is very clear that there is no reflection from the artificial boundary.

5. Conclusion

A finite element method with a DtN map was successfully applied to an optical scattering problem. In computational results, no reflection from the artificial boundary was observed, which proved that the DtN map effectively reduced an infinite domain problem to a bounded domain problem even for the case of optical scattering problem.

Retrieval of the object light from a micro-hologram was qualitatively simulated as scattering of an incident reference light in two different configurations. It was confirmed that this method can be effectively used for analyses of holographic data storage based on the micro-hologram.

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