JSIAM Letters Vol.2 (2010) pp.73–76 ©2010 Japan Society for Industrial and Applied Mathematics

Solutions of Sakaki-Kakei equations of type 3, 5 and 6

Koichi Kondo[1]

[1] Graduate School of Engineering, Doshisha University, 1-3 Tatara-Miyakodani, Kyotanabe, Kyoto 610-0394, Japan

E-mail *kokondo@mail.doshisha.ac.jp*

Received March 31, 2010, Accepted April 30, 2010

Abstract

The purpose of this paper is to obtain general solutions of Sakaki-Kakei equations of type 3, 5 and 6. We first obtain general solution of two dimensional discrete dynamical system associated with arithmetic and harmonic mean through a conjugacy of the iteration map. We next show that the arithmetic and harmonic mean system is semiconjugate to Sakaki-Kakei equations of type 3, 5 and 6 under some conditions. From those results, we obtain their general solutions. We finally clarify behaviors of the solutions.

Keywords Sakaki-Kakei equation, arithmetic and harmonic mean, conjugacy of map

Research Activity Group Applied Integrable Systems

1. Introduction

In [1], two dimensional discrete dynamical system associated with arithmetic and harmonic mean (AHM) is considered. It is shown that some particular solutions of AHM are obtained by hyperbolic and trigonometric functions, and that AHM is solvable chaotic system under some conditions with respect to initial values. In [2], the higher order discrete systems of AHM are presented. The general solutions of the systems are obtained by tridiagonal determinants, and the Lyapunov exponents of the systems are obtained through determinant solutions. In [3], Sakaki and Kakei focused on the fact that the conserved quantity of AHM can be obtained by an identity of hypergeometric function. Then, they derived twelve types of two dimensional discrete systems from other identities of hypergeometric function. The derived systems have the conserved quantities in terms of hypergeometric function, however, their solutions are not discussed. The purpose of this paper is to obtain general solutions of Sakaki-Kakei equations of type 3, 5 and 6. Here, the types of Sakaki-Kakei equations are numbered in order of appearance in their paper. For simplicity, the Sakaki-Kakei equations of type 3, 5 and 6 are named as SK3, SK5 and SK6, respectively.

This paper is organized as follows. In Section 2, we first derive a conjugacy of iteration map of AHM from one dimension reduction, and obtain general solution of AHM through conjugacy of map. In Section 3, we next show that AHM is semiconjugate to SK3, SK5 and SK6 under some conditions. From the solution of AHM, we obtain general solutions of SK3, SK5 and SK6. In Section 4, we clarify behaviors of their solutions. In Section 5, some conclusion are mentioned.

2. Conjugacy of AHM

In [1], the equation of AHM is given by

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \frac{2a_n b_n}{a_n + b_n} \tag{1}$$

for $n = 0, 1, 2, \ldots$ and $a_0, b_0 \in \mathbb{R}$.

In [3], the conserved quantity of (1) is derived from an identity of hypergeometric function,

$$ {}_2F_1(\alpha, \beta, \gamma; x) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} x^n, \quad |x| < 1, \tag{2}$$

where $(\alpha)_n = \prod_{j=0}^{n-1} (\alpha + j)$ for $n = 1, 2, 3, \ldots$. Let

$$I_n = \frac{1}{a_n} {}_2F_1\left(\frac{1}{2}, 1, 1; 1 - \frac{b_n}{a_n}\right). \tag{3}$$

Then, it follows that $I_n = I_{n+1}$ for $n = 0, 1, 2, \ldots$. Hence, I_n is the conserved quantity of (1). Moreover, (3) can be rewritten by virtue of the integral expression,

$$ {}_2F_1\left(\frac{1}{2}, 1, 1; x\right) = \frac{\Gamma(1)}{\Gamma(1/2)^2} \int_0^{\infty} \frac{dt}{(t + 1 - x)\sqrt{t}}. \tag{4}$$

Since the integral in (4) is integrable, it holds that

$$ {}_2F_1\left(\frac{1}{2}, 1, 1; x\right) = \frac{1}{\sqrt{1 - x}}. \tag{5}$$

From (3) and (5), it follows that $I_n = 1/\sqrt{a_n b_n}$.

In [1], AHM is reduced to one dimensional system by using conserved quantity $\tilde{I}_n = 1/(I_n)^2 = a_n b_n$. Let $c = a_0 b_0$. Then, $\tilde{I}_n = \tilde{I}_0$ yields $a_n b_n = c$ for $n = 0, 1, 2, \ldots$. By eliminating b_n in (1) with $b_n = c/a_n$, AHM is reduced to

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right), \quad c \in \mathbb{R}\backslash\{0\}, \quad n = 0, 1, \ldots. \tag{6}$$

Let us denote the iteration function of (6) as

$$\Phi(c; x) = \frac{1}{2}\left(x + \frac{c}{x}\right), \quad c \in \mathbb{R}\backslash\{0\}. \tag{7}$$

Then, (6) is expressed as $a_{n+1} = \Phi(a_n)$. Note here that Φ is Newton iteration function $\Phi(x) = x - f(x)/f'(x)$ for $f(x) = x^2 - c$ (cf. [2]).

If maps $\Psi : X \to X$, $\psi : X \to Y$, $\sigma : Y \to Y$ for sets X, Y satisfy $\Psi = \psi^{-1} \circ \sigma \circ \psi$, and ψ is homeomorphic, namely, one-to-one, onto, and continuous function with continuous inverse, then Ψ is dynamically equivalent to σ. We say that $\Psi : X \to X$ is conjugate to $\sigma : Y \to Y$, and ψ is a conjugacy of Ψ (cf. [4, pp. 108–109]).

In [1], [4, p. 172], the map $\Phi(c; x)$ for $c < 0$ is conjugate to the Bernoulli shift, which is well known as chaotic dynamical system. Thus, it turns out that Φ is chaotic system if $c < 0$.

In this paper, we derive another conjugacy of Φ in order to obtain general solution for any $c \in \mathbb{R}\backslash\{0\}$. Let us introduce a function ϕ defined by

$$\phi(c; x) = \frac{x - \sqrt{c}}{x + \sqrt{c}}, \quad c \in \mathbb{R}\backslash\{0\}. \tag{8}$$

Note here that square root in (8) is not single-valued function if its argument value is negative. For simplicity of discussion, all of square roots in this paper are treated as single-valued function such that $\sqrt{c} = i\sqrt{-c}$ if $c < 0$. Here, i is imaginary unit. From (8), we have

$$\phi^{-1}(c; x) = \sqrt{c}\frac{1 + x}{1 - x}. \tag{9}$$

Let $Q(x) = x^2$. From (7), (8) and (9), it formally holds that

$$\Phi = \phi^{-1} \circ Q \circ \phi. \tag{10}$$

Suppose that $c > 0$. Let $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$. From (8), (9), the map $\phi : \mathbb{R}^* \to \mathbb{R}^*$ with satisfying $\phi(\infty) = 1$, $\phi(-\sqrt{c}) = \infty$ is obviously homeomorphic. It holds that $Q : \mathbb{R}^* \to \mathbb{R}^*$. Hence, the map $\Phi : \mathbb{R}^* \to \mathbb{R}^*$ is conjugate to $Q : \mathbb{R}^* \to \mathbb{R}^*$. Suppose that $c < 0$. Let us denote the unit circle in $\mathbb{C}$ as $S = \{z \in \mathbb{C} \mid |z| = 1\}$. From (8) and $\sqrt{c} = i\sqrt{-c}$, it follows that $x \in \mathbb{R}^*$, $\phi(x) = e^{i\theta} \in S$, $\theta = -2\tan^{-1}(\sqrt{-c}/x)$. The map $\phi : \mathbb{R}^* \to S$ with satisfying $\phi(\infty) = 1$ is continuous bijection. From (9), it follows that $\phi^{-1}(e^{i\theta}) = -\sqrt{-c}\sin(\theta)/(1 - \cos(\theta))$. The map ϕ^{-1} is continuous. Hence, the map $\phi : \mathbb{R}^* \to S$ is homeomorphic. It holds that $Q : S \to S$. Thus, the map $\Phi : \mathbb{R}^* \to \mathbb{R}^*$ is conjugate to $Q : S \to S$.

Let us denote the imaginary axis in $\mathbb{C}$ as $T = \{iy \in \mathbb{C} \mid y \in \mathbb{R}\} \cup \{\infty\}$. In order to obtain solutions of SK3, SK5 and SK6, we show a conjugacy of map $\Phi : T \to T$. Similar to above discussion, it follows that $\phi : T \to S$ is homeomorphic if $c > 0$, and that $\phi : T \to \mathbb{R}^*$ is homeomorphic if $c < 0$. Recall that $Q : S \to S$ and $Q : \mathbb{R}^* \to \mathbb{R}^*$. Hence, the map $\Phi : T \to T$ is conjugate to $Q : S \to S$ if $c > 0$, or $Q : \mathbb{R}^* \to \mathbb{R}^*$ if $c < 0$. Thus, we have the following theorem.

Theorem 1 *The map $\Phi : \mathbb{R}^* \to \mathbb{R}^*$ is conjugate to $Q : \mathbb{R}^* \to \mathbb{R}^*$ if $c > 0$, or $Q : S \to S$ if $c < 0$. The map $\Phi : T \to T$ is conjugate to $Q : S \to S$ if $c > 0$, or $Q : \mathbb{R}^* \to \mathbb{R}^*$ if $c < 0$.*

In the case where $c = -1$, this fact was first proved by Cayley in 1879. In the case where $c = \pm 1$, it is shown in [4, pp. 274–275].

It follows from $a_{n+1} = \Phi(a_n)$ and (10) that $\phi(a_{n+1}) = Q(\phi(a_n))$. Let $z_n = \phi(a_n)$. Recall that $Q(x) = x^2$. From Theorem 1, it turns out that the system $a_{n+1} = \Phi(a_n)$ is equivalent to $z_{n+1} = (z_n)^2$ if $a_0 \in \mathbb{R}^*$ or $a_0 \in T$. From $z_0 = \phi(a_0)$, $a_n = \phi^{-1}(z_n)$, and the solution $z_n = (z_0)^{2^n}$ of $z_{n+1} = (z_n)^2$, we have the following theorem.

Theorem 2 *Suppose that $c \in \mathbb{R}\backslash\{0\}$. If $a_0 \in \mathbb{R}^*$ or $a_0 \in T$, then the general solution of $a_{n+1} = \Phi(a_n)$ is*

$$a_n = (\phi^{-1} \circ x^{2^n} \circ \phi)(a_0), \quad n = 0, 1, 2, \ldots. \tag{11}$$

Recall that $b_n = c/a_n$, $c = a_0 b_0$. From (8), (9) and Theorem 2, we have the following theorem.

Theorem 3 *Let $c = a_0 b_0$, $\lambda_1 = a_0 + \sqrt{c}$, and $\lambda_2 = a_0 - \sqrt{c}$. The general solution of AHM (1) is*

$$a_n = \sqrt{c}\frac{\lambda_1^{2^n} + \lambda_2^{2^n}}{\lambda_1^{2^n} - \lambda_2^{2^n}}, \quad b_n = \sqrt{c}\frac{\lambda_1^{2^n} - \lambda_2^{2^n}}{\lambda_1^{2^n} + \lambda_2^{2^n}} \tag{12}$$

for $n = 0, 1, 2, \ldots$. Here, a_0, b_0 are both real numbers, or both pure imaginary numbers, which satisfy $a_0 b_0 \neq 0$ and $a_0 + b_0 \neq 0$. If $a_0 = 0$, $b_0 \neq 0$, AHM has singular solution $a_n = b_0/2^n$, $b_n = 0$ for $n = 1, 2, 3, \ldots$. If $a_0 \neq 0$, $b_0 = 0$, AHM has singular solution $a_n = a_0/2^n$, $b_n = 0$ for $n = 1, 2, 3, \ldots$. If $a_0 + b_0 = 0$, AHM does not have solution.

The solution (12) can be also obtained through the solution in [2], which is expressed by tridiagonal determinant. The tridiagonal determinant satisfies linear difference equation of the second order. Solving the equation and rewriting its solution, we can obtain (12).

3. Solutions of SK3, SK5 and SK6

In [3], the equation of SK3 is given by

$$a_{n+1} = \frac{(a_n + b_n)^2}{a_n - b_n}, \quad b_{n+1} = \frac{4a_n b_n}{a_n - b_n}, \tag{13}$$

which has the conserved quantity,

$$I_n^{(3)} = \frac{1}{a_n}{}_2F_1\left(\frac{1}{2}, \frac{3}{4}, \frac{3}{4}; \frac{b_n}{a_n}\right). \tag{14}$$

Note here that $I_n^{(3)}$ in [3] is erratum. The equation of SK5 is given by

$$a_{n+1} = \frac{(2a_n - b_n)^2}{4a_n}, \quad b_{n+1} = \frac{{b_n}^2}{4a_n}, \tag{15}$$

which has the conserved quantity,

$$I_n^{(5)} = \frac{1}{\sqrt{a_n}}{}_2F_1\left(\frac{1}{2}, 1, 1; \frac{b_n}{a_n}\right). \tag{16}$$

The equation of SK6 is given by

$$a_{n+1} = \frac{4a_n(a_n - b_n)^2}{(2a_n - b_n)^2}, \quad b_{n+1} = \frac{-{b_n}^2(a_n - b_n)}{(2a_n - b_n)^2}, \tag{17}$$

which has same conserved quantity $I_n^{(6)}$ as (16).

Along the line similar to AHM, we first rewrite the conserved quantities of SK3, SK5 and SK6. It follows from (2) that

$${}_2F_1\left(\frac{1}{2}, \frac{3}{4}, \frac{3}{4}; x\right) = 1 + \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{n!}x^n = \frac{1}{\sqrt{1 - x}}. \tag{18}$$

Substituting (18) into (14) and substituting (5) into (16), we have the following theorem.

Theorem 4 *All of the conserved quantities of SK3, SK5 and SK6 are $I_n^{(3)} = I_n^{(5)} = I_n^{(6)} = 1/\sqrt{a_n - b_n}$.*

From Theorem 4, we next reduce SK3, SK5 and SK6 to one dimensional discrete systems by using $\hat{I}_n =$

$(I_n^{(3)})^{-2} = (I_n^{(5)})^{-2} = (I_n^{(6)})^{-2} = a_n - b_n$. Let $\hat{c} = a_0 - b_0$. Then, $\hat{I}_n = \hat{I}_0$ yields $a_n - b_n = \hat{c}$ for $n = 0, 1, 2, \ldots$. By eliminating b_n in (13), (15) and (17) with $b_n = a_n - \hat{c}$, we obtain one dimensional systems of SK3, SK5 and SK6, respectively, as following theorem.

Theorem 5 *If $\hat{c} = a_0 - b_0 \neq 0$, SK3 (13) is reduced to*

$$a_{n+1} = \frac{1}{\hat{c}}(2a_n - \hat{c})^2, \quad n = 0, 1, 2, \ldots. \tag{19}$$

If $\hat{c} = a_0 - b_0 \neq 0$, SK5 (15) is reduced to

$$a_{n+1} = \frac{(a_n + \hat{c})^2}{4a_n}, \quad n = 0, 1, 2, \ldots. \tag{20}$$

If $\hat{c} = a_0 - b_0 \neq 0$, SK6 (17) is reduced to

$$a_{n+1} = \frac{4\hat{c}^2 a_n}{(a_n + \hat{c})^2}, \quad n = 0, 1, 2, \ldots. \tag{21}$$

Let us denote the iteration functions $\Phi_3(\hat{c}; x)$, $\Phi_5(\hat{c}; x)$ and $\Phi_6(\hat{c}; x)$ of (19), (20) and (21) as

$$\Phi_3 = \frac{(2x - \hat{c})^2}{\hat{c}}, \quad \Phi_5 = \frac{(x + \hat{c})^2}{4x}, \quad \Phi_6 = \frac{4\hat{c}^2 x}{(x + \hat{c})^2} \tag{22}$$

for $\hat{c} \in \mathbb{R}\backslash\{0\}$, respectively. Then, (19), (20) and (21) are expressed as $a_{n+1} = \Phi_3(a_n)$, $a_{n+1} = \Phi_5(a_n)$ and $a_{n+1} = \Phi_6(a_n)$, respectively.

If maps $\Psi : X \to X$, $\psi : X \to Y$, $\sigma : Y \to Y$ for sets X, Y satisfy $\psi \circ \Psi = \sigma \circ \psi$, and ψ is continuous, onto, and at most m-to-one, then we say that $\Psi : X \to X$ is semiconjugate to $\sigma : Y \to Y$, and ψ is a semiconjugacy of Ψ (cf. [4, p. 125]).

Let us define the functions η_3, η_5 and η_6 by

$$\eta_3(\hat{c}; x) = \frac{\hat{c}x^2}{x^2 - 1}, \quad \eta_5(x) = x^2, \quad \eta_6(x) = \frac{1}{x^2}. \tag{23}$$

From (7), (22) and (23), it formally holds that

$$\eta_3 \circ \Phi = \Phi_3 \circ \eta_3 \quad \text{if } c = 1, \tag{24}$$

$$\eta_5 \circ \Phi = \Phi_5 \circ \eta_5 \quad \text{if } c = \hat{c}, \tag{25}$$

$$\eta_6 \circ \Phi = \Phi_6 \circ \eta_6 \quad \text{if } c = \frac{1}{\hat{c}}. \tag{26}$$

Let us denote that $\mathbb{R}^+ = \{x \in \mathbb{R} \,|\, x \geq 0\} \cup \{\infty\}$, $\mathbb{R}^- = \{x \in \mathbb{R} \,|\, x \leq 0\} \cup \{\infty\}$, $U_1 = \{x \in \mathbb{R} \,|\, |x| \geq 1\} \cup \{\infty\}$, $D_1 = \{x \in \mathbb{R} \,|\, 1 \leq x < +\infty\} \cup \{\infty\}$, and $D_2 = \{x \in \mathbb{R} \,|\, 0 \leq x \leq 1\}$. Let us define $\hat{c}D = \{\hat{c}x \in \mathbb{R}^* \,|\, x \in D\}$ for $\hat{c} \in \mathbb{R}\backslash\{0\}$ and a set D. From (23), it follows that $\eta_3 : U_1 \to \hat{c}D_1$, $\eta_3 : T \to \hat{c}D_2$, $\eta_5 : \mathbb{R}^* \to \mathbb{R}^+$, $\eta_5 : T \to \mathbb{R}^-$, $\eta_6 : \mathbb{R}^* \to \mathbb{R}^+$, and $\eta_6 : T \to \mathbb{R}^-$ are continuous, onto, and at almost two-to-one maps. From (7), (22), it holds that $\Phi : \mathbb{R}^* \to \mathbb{R}^*$, $\Phi : T \to T$, $\Phi : U_1 \to U_1$ if $c = 1$, $\Phi_3 : \hat{c}D_1 \to \hat{c}D_1$, $\Phi_3 : \hat{c}D_2 \to \hat{c}D_2$, $\Phi_5 : \mathbb{R}^+ \to \mathbb{R}^+$, $\Phi_5 : \mathbb{R}^- \to \mathbb{R}^-$, $\Phi_6 : \mathbb{R}^+ \to \mathbb{R}^+$, and $\Phi_6 : \mathbb{R}^- \to \mathbb{R}^-$. Thus, we have the following theorems.

Theorem 6 *If $c = 1$, $\hat{c} \in \mathbb{R}\backslash\{0\}$, the maps $\Phi : U_1 \to U_1$ and $\Phi : T \to T$ are semiconjugate to $\Phi_3 : \hat{c}D_1 \to \hat{c}D_1$ and $\Phi_3 : \hat{c}D_2 \to \hat{c}D_2$, respectively.*

Theorem 7 *If $c = \hat{c} \in \mathbb{R}\backslash\{0\}$, the maps $\Phi : \mathbb{R}^* \to \mathbb{R}^*$ and $\Phi : T \to T$ are semiconjugate to $\Phi_5 : \mathbb{R}^+ \to \mathbb{R}^+$ and $\Phi_5 : \mathbb{R}^- \to \mathbb{R}^-$, respectively.*

Theorem 8 *If $c = 1/\hat{c} \in \mathbb{R}\backslash\{0\}$, the maps $\Phi : \mathbb{R}^* \to \mathbb{R}^*$ and $\Phi : T \to T$ are semiconjugate to $\Phi_6 : \mathbb{R}^+ \to \mathbb{R}^+$ and $\Phi_6 : \mathbb{R}^- \to \mathbb{R}^-$, respectively.*

Let $\tilde{a}_n$ be a solution of AHM for $\tilde{a}_0 \in \mathbb{R}^*$ or $\tilde{a}_0 \in T$. Namely, $\tilde{a}_n$ satisfies $\tilde{a}_{n+1} = \Phi(\tilde{a}_n)$ for $n = 0, 1, 2, \ldots$. From the map η_j for $j = 3, 5, 6$, we derive $\eta_j(\tilde{a}_{n+1}) = \eta_j(\Phi(\tilde{a}_n))$ for $j = 3, 5, 6$, respectively. Suppose that Φ satisfies one of the conditions of Theorems 6, 7 and 8. From (24), (25) and (26), it follows that

$$\eta_j(\tilde{a}_{n+1}) = \Phi_j(\eta_j(\tilde{a}_n)). \tag{27}$$

Let $a_n = \eta_j(\tilde{a}_n)$. Then, we have $a_{n+1} = \Phi_j(a_n)$. Namely, $a_n = \eta_3(\tilde{a}_n)$, $a_n = \eta_5(\tilde{a}_n)$ and $a_n = \eta_6(\tilde{a}_n)$ for $n = 0, 1, 2, \ldots$ are solutions of SK3, SK5 and SK6, respectively. The solution $\tilde{a}_n$ of AHM is given by (11), where a_n, a_0 are replaced with $\tilde{a}_n$, $\tilde{a}_0$, respectively, and c in ϕ takes a value $c = 1$ for SK3, $c = \hat{c}$ for SK5, and $c = 1/\hat{c}$ for SK6. Recall that $b_n = a_n - \hat{c}$, $\hat{c} = a_0 - b_0$. All of the solutions b_n for SK3, SK5 and SK6 are given by $b_n = a_n - a_0 + b_0$ for $n = 0, 1, 2, \ldots$.

The initial value $\tilde{a}_0$ of AHM is determined from a_0 such that $\eta_j(\tilde{a}_0) = a_0$. Though $\tilde{a}_0$ can take two real or pure imaginary values, each of $a_n = \eta_j(\tilde{a}_n)$ given by a chosen $\tilde{a}_0$ becomes a solution of $a_{n+1} = \Phi_j(a_n)$. Obviously, the systems (13), (15) and (17) generate unique solutions, so that both of solutions are equivalent. For simplicity, we choose $\tilde{a}_0$ as $\tilde{a}_0 = \sqrt{a_0/(a_0 - \hat{c})}$ for SK3, $\tilde{a}_0 = \sqrt{a_0}$ for SK5, and $\tilde{a}_0 = 1/\sqrt{a_0}$ for SK6.

From Theorems 7 and 8, there exist solutions $a_n = \eta_5(\tilde{a}_n)$, $a_n = \eta_6(\tilde{a}_n)$ for $a_0 \in \mathbb{R}^* = \mathbb{R}^+ \cup \mathbb{R}^-$. From Theorem 6, there exists solution $a_n = \eta_3(\tilde{a}_n)$ for $a_0 \in \hat{c}D_1 \cup \hat{c}D_2$. Moreover, we can obtain solution of SK3 for $a_0 \in \hat{c}\mathbb{R}^-$ as follows. Let $U_2 = \{x \in \mathbb{R} \,|\, |x| \leq 1\}$. From (7), (22) and (23), it holds that $\eta_3 : U_2 \to \hat{c}\mathbb{R}^-$, $\Phi_3 : \hat{c}\mathbb{R}^- \to \hat{c}D_1$, and $\Phi : U_2 \to U_1$ if $c = 1$. Note here that the map $\Phi : U_2 \to U_1$ is not semiconjugate to $\Phi_3 : \hat{c}\mathbb{R}^- \to \hat{c}D_1$. If $\tilde{a}_0 \in U_2$, $\eta_3(\tilde{a}_0) = a_0 \in \hat{c}\mathbb{R}^-$, then it follows that $\Phi(\tilde{a}_0) = \tilde{a}_1 \in U_1$, $\eta_3(\tilde{a}_1) \in \hat{c}D_1$, and $\Phi_3(a_0) = \Phi_3(\eta_3(\tilde{a}_0)) \in \hat{c}D_1$. Hence, it holds that (27) for $n = 0$. We have $a_1 = \eta_3(\tilde{a}_1)$. Since $a_1 \in \hat{c}D_1$, it holds that (27) for $n = 1, 2, 3, \ldots$ by Theorem 6. There exists solution $a_n = \eta_3(\tilde{a}_n)$ for $a_0 \in \hat{c}\mathbb{R}^-$. The domain of initial value of SK3 is $\mathbb{R}^* = \hat{c}D_1 \cup \hat{c}D_2 \cup \hat{c}\mathbb{R}^-$.

Thus, we have the following theorems.

Theorem 9 *Let $\hat{c} = a_0 - b_0$, $\lambda_3 = (\sqrt{a_0} + \sqrt{b_0})/\sqrt{\hat{c}}$, and $\lambda_4 = (\sqrt{a_0} - \sqrt{b_0})/\sqrt{\hat{c}}$. The general solution of SK3 (13) is*

$$a_n = \frac{\hat{c}}{4}\left(\lambda_3^{2^n} + \lambda_4^{2^n}\right)^2, \quad b_n = \frac{\hat{c}}{4}\left(\lambda_3^{2^n} - \lambda_4^{2^n}\right)^2 \tag{28}$$

for $n = 0, 1, 2, \ldots$ and real initial values a_0, b_0 such that $a_0 \neq b_0$. If $a_0 = b_0$, SK3 does not have solution.

Theorem 10 *Let $\hat{c} = a_0 - b_0$, $\lambda_5 = \sqrt{a_0} + \sqrt{\hat{c}}$, and $\lambda_6 = \sqrt{a_0} - \sqrt{\hat{c}}$. The general solution of SK5 (15) is*

$$a_n = \hat{c}\left(\frac{\lambda_5^{2^n} + \lambda_6^{2^n}}{\lambda_5^{2^n} - \lambda_6^{2^n}}\right)^2, \quad b_n = a_n - a_0 + b_0 \tag{29}$$

for $n = 0, 1, 2, \ldots$ and real initial values a_0, b_0 such that $a_0 \neq 0$, $a_0 \neq b_0$ and $2a_0 \neq b_0$. If $a_0 = b_0$, SK5 has singular solution $a_n = a_0/4^n$, $b_n = b_0/4^n$ for $n = 0, 1, 2, \ldots$. If $a_0 = 0$ or $2a_0 - b_0 = 0$, SK5 does not have *solution.*

Theorem 11 *Let $\hat{c} = a_0 - b_0$, $\lambda_7 = \sqrt{\hat{c}} + \sqrt{a_0}$, and $\lambda_8 = \sqrt{\hat{c}} - \sqrt{a_0}$. The general solution of SK6* (17) *is*

$$a_n = \hat{c}\left(\frac{\lambda_7^{2^n} - \lambda_8^{2^n}}{\lambda_7^{2^n} + \lambda_8^{2^n}}\right)^2, \quad b_n = a_n - a_0 + b_0, \qquad (30)$$

for $n = 0, 1, 2, \ldots$ and real initial values a_0, b_0 such that $a_0 \neq b_0$, $2a_0 \neq b_0$. If $a_0 - b_0 = 0$ or $2a_0 - b_0 = 0$, SK6 does not have solution.

4. Behaviors of solutions

In this section, we clarify behaviors of the general solutions (28), (29) and (30). Let us introduce the functions ϕ_3, ϕ_5, ϕ_6 and their inverse by

$$\phi_3(x) = \frac{\sqrt{x} - \sqrt{x - \hat{c}}}{\sqrt{\hat{c}}}, \quad \phi_3^{-1}(x) = \frac{\hat{c}}{4}\left(x + \frac{1}{x}\right)^2, \quad (31)$$

$$\phi_5(x) = \frac{\sqrt{x} - \sqrt{\hat{c}}}{\sqrt{x} + \sqrt{\hat{c}}}, \quad \phi_5^{-1}(x) = \hat{c}\left(\frac{1+x}{1-x}\right)^2, \qquad (32)$$

$$\phi_6(x) = \frac{\sqrt{\hat{c}} - \sqrt{x}}{\sqrt{\hat{c}} + \sqrt{x}}, \quad \phi_6^{-1}(x) = \hat{c}\left(\frac{1-x}{1+x}\right)^2. \qquad (33)$$

Thus, we have the following theorem.

Theorem 12 *Solutions a_n of* (28), (29) *and* (30) *are expressed as*

$$a_n = (\phi_j^{-1} \circ x^{2^n} \circ \phi_j)(a_0), \quad n = 0, 1, 2, \ldots \qquad (34)$$

for $j = 3, 5, 6$, respectively.

It may seem that ϕ_3, ϕ_5 and ϕ_6 are conjugacies of Φ_3, Φ_5 and Φ_6, respectively, because (34) is same as (11) in Theorem 2. However, the maps $\phi_j : \mathbb{R}^* \to S$ for $\hat{c} < 0$, $j = 3, 5, 6$ are not homeomorphic, so that they cannot be conjugacies of Φ_j, respectively.

Let $z_0 = \phi_j(a_0) \in \mathbb{C}$, $z_n = (z_0)^{2^n} \in \mathbb{C}$. From (34), it holds that $a_n = \phi_j^{-1}(z_n)$. Let us denote z_n in polar form $z_n = r_n e^{i\theta_n}$, $0 \leq r_n$, $0 \leq \theta_n < 2\pi$ for $n = 0, 1, 2, \ldots$. Then, we have $z_n = (r_0)^{2^n} e^{i2^n\theta_0}$. The behavior of the solution $a_n = \phi_j^{-1}(z_n)$ depends on r_0, θ_0. From (31), (32) and (33), it turns out that there exist the following six cases. (i) If $0 < r_0 < 1$, $\theta_0 = 0$, then z_n monotonically converges to 0. (ii) If $0 < r_0 < 1$, $\theta_0 \neq 0$, then z_n oscillatory converges to 0. (iii) If $r_0 > 1$, $\theta_0 = 0$, then z_n monotonically diverges. (iv) If $r_0 > 1$, $\theta_0 \neq 0$, then z_n oscillatory diverges. (v) If $r_0 = 1$, $\theta_0 \neq 0$, then it follows that $|z_n| = 1$ and the map $\theta_n/(2\pi) \mapsto \theta_{n+1}/(2\pi)$ is conjugate to the Bernoulli shift (cf. [4, p. 125]). Hence, z_n is chaotic. (vi) If $z_0 = 1$ or $z_0 = 0$, then z_n is fixed point. Thus, we have the following theorems.

Theorem 13 *The solution* (28) *of SK3* (13) *behaves as follows. If $a_0(a_0 - b_0) > 0$, then a_n and b_n monotonically diverge. If $a_0(a_0 - b_0) < 0$, then a_n and b_n oscillatory diverge. If $a_0 b_0 < 0$, then a_n and b_n are chaotic. If $a_0 \neq 0$, $b_0 = 0$, then $a_n = a_0$, $b_n = 0$ is fixed point.*

Theorem 14 *The solution* (29) *of SK5* (15) *behaves as follows. If $a_0(a_0 - b_0) > 0$, then a_n and b_n monotonically converge to $a_0 - b_0$, 0, respectively. If $a_0 b_0 < 0$, then a_n and b_n oscillatory converge to $a_0 - b_0$, 0, respectively. If $a_0(a_0 - b_0) < 0$, then a_n and b_n are chaotic. If $a_0 \neq 0$, $b_0 = 0$, then $a_n = a_0$, $b_n = 0$ is fixed point.*

Theorem 15 *The solution* (30) *of SK6* (17) *behaves as follows. If $a_0 b_0 < 0$, then a_n and b_n monotonically converge to $a_0 - b_0$, 0, respectively. If $a_0(a_0 - b_0) > 0$, then a_n and b_n oscillatory converge to $a_0 - b_0$, 0, respectively. If $a_0(a_0 - b_0) < 0$, then a_n and b_n are chaotic. If $a_0 = 0$, $b_0 \neq 0$, then $a_n = 0$, $b_n = b_0$ is fixed point. If $a_0 \neq 0$, $b_0 = 0$, then $a_n = a_0$, $b_n = 0$ is fixed point.*

In special cases for initial values a_0, b_0, the solutions are expressed by hyperbolic and trigonometric functions. Thus, we have the following theorems.

Theorem 16 *Let $\hat{c} = a_0 - b_0$. If $a_0 > b_0 > 0$, then the solution of SK3* (13) *is given by $a_n = \hat{c}\cosh^2(2^n\mu)$, $b_n = \hat{c}\sinh^2(2^n\mu)$ where $\mu = \tanh^{-1}\sqrt{b_0/a_0}$. If $a_0 > 0$, $b_0 < 0$, then the solution of SK3 is given by $a_n = \hat{c}\cos^2(2^n\mu)$, $b_n = -\hat{c}\sin^2(2^n\mu)$ where $\mu = \tan^{-1}\sqrt{-b_0/a_0}$.*

Theorem 17 *Let $\hat{c} = a_0 - b_0$. If $a_0 > b_0 > 0$, then the solution of SK5* (15) *is given by $a_n = \hat{c}\coth^2(2^n\mu)$, $b_n = \hat{c}(\coth^2(2^n\mu) - 1)$ where $\mu = \tanh^{-1}\sqrt{\hat{c}/a_0}$. If $b_0 > a_0 > 0$, then the solution of SK5 is given by $a_n = -\hat{c}\cot^2(2^n\mu)$, $b_n = -\hat{c}(\cot^2(2^n\mu) + 1)$ where $\mu = \tan^{-1}\sqrt{-\hat{c}/a_0}$.*

Theorem 18 *Let $\hat{c} = a_0 - b_0$. If $a_0 > 0$, $b_0 < 0$, then the solution of SK6* (17) *is given by $a_n = \hat{c}\tanh^2(2^n\mu)$, $b_n = \hat{c}(\tanh^2(2^n\mu) - 1)$ where $\mu = \tanh^{-1}\sqrt{a_0/\hat{c}}$. If $b_0 > a_0 > 0$, then the solution of SK5 is given by $a_n = -\hat{c}\tan^2(2^n\mu)$, $b_n = -\hat{c}(\tan^2(2^n\mu) + 1)$ where $\mu = \tan^{-1}\sqrt{-a_0/\hat{c}}$.*

Theorems 16, 17 and 18 can be proved by using double angle formulae of hyperbolic and trigonometric functions, similar to proof about particular solutions of AHM in [1].

5. Conclusion

In this paper, we first obtain the general solution of AHM for real and pure imaginary initial values through a conjugacy of the iteration map. We next show AHM is semiconjugate to SK3, SK5 and SK6 under some conditions. We obtain the general solutions of SK3, SK5 and SK6 from the general solution of AHM. We finally show behaviors of their solutions. Moreover, we obtain particular solutions by hyperbolic and trigonometric functions for special cases of initial values. Further problems are to obtain solutions of the other types of Sakaki-Kakei equations, and to derive an identity of hypergeometric function associated with the higher order systems of AHM.

References

[1] Y. Nakamura, Algorithms associated with arithmetic, geometric and harmonic means and integrable systems, J. Comput. Appl. Math., **131** (2001), 161–174.

[2] K. Kondo and Y. Nakamura, Determinantal solutions of solvable chaotic systems, J. Comput. Appl. Math., **145** (2002), 361–372.

[3] T. Sakaki and S. Kakei, Difference equations with an invariant expressed in terms of the hypergeometric function (in Japanese), Trans. JSIAM, **17** (2007), 455–462.

[4] R. L. Devaney, A first course in chaotic dynamical systems: theory and experiment, Addison-Wesley, Reading, Massachusetts, 1992.