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On boundedness of the condition number of the coefficient matrices appearing in Sinc-Nyström methods for Fredholm integral equations of the second kind

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Abstract

Sinc-Nyström methods for Fredholm integral equations of the second kind have been independently proposed by Muhammad et al. and Rashidinia-Zarebnia. They also gave error analyses, but the results did not claim the convergence of their schemes in a precise sense. This is because in their error estimates there remained an unestimated term: the norm of the inverse of the coefficient matrix of the resulting linear system. In this paper, we estimate the term theoretically to complete the convergence estimate of their methods. Furthermore, we also prove the boundedness of the condition number of each coefficient matrix.

Keywords Sinc method, Fredholm integral equation, condition number, Nyström method

Research Activity Group Scientific Computation and Numerical Analysis

1. Introduction

We are concerned with Fredholm integral equations of the second kind of the form

$$\lambda u(t) - \int_a^b k(t,s)u(s)\,\mathrm{d}s = g(t), \quad a \le t \le b, \tag{1}$$

where λ is a given constant, $g(t)$ and $k(t,s)$ are given continuous functions, and $u(t)$ is the solution to be determined. Various numerical methods have been proposed to solve (1), and the convergence rate of most existing methods has been polynomial with respect to the number of discretization points N [1].

One of the exceptions is the Sinc-Nyström method, which has firstly been developed by Muhammad et al. [2]. According to their error analysis, the method can converge exponentially if the coefficient matrix of the resulting linear equations, say A_N, does not behave badly. To be more precise, the error of the numerical solution $u_N(t)$ has been estimated as

$$\max_{t\in[a,b]} |u(t) - u_N(t)| \le C\|A_N^{-1}\|_2 \exp\left(\frac{-cN}{\log N}\right), \tag{2}$$

where C and c are positive constants independent of N. In their numerical experiments the term $\|A_N^{-1}\|_2$ remained low for all N, which suggested that the method can converge exponentially. Afterwards Rashidinia-Zarebnia [3] have proposed another type of Sinc-Nyström methods, and claimed that the error can be estimated as

$$\max_{t\in[a,b]} |u(t) - \tilde{u}_N(t)| \le \tilde{C}\|\tilde{A}_N^{-1}\|_2 \exp(-\tilde{c}\sqrt{N}), \tag{3}$$

which also suggested the exponential convergence of their method. Strictly speaking, however, the exponential convergence of those two methods still has not been established at this point since the dependence of the terms $\|A_N^{-1}\|_2$ and $\|\tilde{A}_N^{-1}\|_2$ on N has not been clarified. It seems direct estimates of them are difficult, and that was the reason why they have remained open.

In this paper, we take a different approach: we give estimates in ∞-norm as

$$\|A_N^{-1}\|_\infty \le K, \quad \|\tilde{A}_N^{-1}\|_\infty \le \tilde{K},$$

for some constants K and $\tilde{K}$. Through $\|X\|_2 \le \sqrt{n}\|X\|_\infty$ for any $n \times n$ matrix X, the estimates imply the desired exponential convergence estimates. The key here is the analysis of Sinc-collocation methods previously given by the present authors [4].

The above approach has another virtue that we can show a stronger result; we also show

$$\|A_N\|_\infty \le K', \quad \|\tilde{A}_N\|_\infty \le \tilde{K}',$$

from which the condition numbers of the matrices are bounded (in the sense of ∞-norm). This result guarantees not only that the two methods converge exponentially, but also that the resulting linear equations do not become ill-conditioned as N increases.

This paper is organized as follows. In Section 2, we explain the concrete procedure of the Sinc-Nyström methods. New theoretical results are described in Section 3 with their proofs. In Section 4 a numerical example is shown. Section 5 is devoted to conclusions.

2. Sinc-Nyström methods

2.1 Sinc quadrature

In the Sinc-Nyström methods, the Sinc quadrature:

$$\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x \approx h \sum_{j=-N}^{N} F(jh) \tag{4}$$

is employed to approximate the integral. Although the interval of the integral in (1) is finite, we can apply the Sinc quadrature by combining it with a variable transformation. Rashidinia-Zarebnia [3] utilized the Single-Exponential (SE) transformation defined by

$$t = \psi^{\mathrm{SE}}(x) = \frac{b-a}{2}\tanh\left(\frac{x}{2}\right) + \frac{b+a}{2},$$

which enables us to apply the Sinc quadrature as follows:

$$\int_a^b f(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(\psi^{\mathrm{SE}}(x))\psi^{\mathrm{SE}\prime}(x)\,\mathrm{d}x \approx h \sum_{j=-N}^{N} f(\psi^{\mathrm{SE}}(jh))\psi^{\mathrm{SE}\prime}(jh). \tag{5}$$

Muhammad et al. [2] utilized another one:

$$t = \psi^{\mathrm{DE}}(x) = \frac{b-a}{2}\tanh\left(\frac{\pi}{2}\sinh x\right) + \frac{b+a}{2},$$

which is called the Double-Exponential (DE) transformation. By using the DE transformation we have:

$$\int_a^b f(t)\,\mathrm{d}t \approx h \sum_{j=-N}^{N} f(\psi^{\mathrm{DE}}(jh))\psi^{\mathrm{DE}\prime}(jh). \tag{6}$$

In order to achieve quick convergence with the Sinc quadrature (4), it is necessary that the integrand F is analytic and bounded in the strip domain: $\mathscr{D}_d = \{z \in \mathbb{C} : |\operatorname{Im} z| < d\}$ for a positive constant d. Accordingly, as for the approximations (5) and (6), it is appropriate to introduce the following function space.

Definition 1 *Let $\mathscr{D}$ be a bounded and simply-connected domain (or Riemann surface). Then we denote by $\mathbf{H}^\infty(\mathscr{D})$ the family of all functions that are analytic and bounded in $\mathscr{D}$.*

The domain $\mathscr{D}$ should be either $\psi^{\mathrm{SE}}(\mathscr{D}_d)$ or $\psi^{\mathrm{DE}}(\mathscr{D}_d)$, i.e., we may assume $f \in \mathbf{H}^\infty(\psi^{\mathrm{SE}}(\mathscr{D}_d))$ for the approximation (5), and $f \in \mathbf{H}^\infty(\psi^{\mathrm{DE}}(\mathscr{D}_d))$ for the approximation (6).

2.2 SE-Sinc-Nyström method

Firstly we explain the method derived by Rashidinia-Zarebnia [3]. Assume the following two conditions:

(SE1) $u \in \mathbf{H}^\infty(\psi^{\mathrm{SE}}(\mathscr{D}_d))$,
(SE2) $k(t,\cdot) \in \mathbf{H}^\infty(\psi^{\mathrm{SE}}(\mathscr{D}_d))$ for all $t \in [a,b]$.

Then the integral $\mathcal{K}[u](t) := \int_a^b k(t,s)u(s)\,\mathrm{d}s$ in (1) can be approximated by

$$\mathcal{K}_N^{\mathrm{SE}}[u](t) := h \sum_{j=-N}^{N} k(t,\psi^{\mathrm{SE}}(jh))u(\psi^{\mathrm{SE}}(jh))\psi^{\mathrm{SE}\prime}(jh).$$

The mesh size h here is chosen as $h = \sqrt{2\pi d/N}$. Then, corresponding to the original equation $u = (g + \mathcal{K}u)/\lambda$, we consider the new equation:

$$u_N^{\mathrm{SE}}(t) = \frac{g(t) + \mathcal{K}_N^{\mathrm{SE}}[u_N^{\mathrm{SE}}](t)}{\lambda}. \tag{7}$$

The approximated solution u_N^{SE} is obtained by determining the unknown coefficients in $\mathcal{K}_N^{\mathrm{SE}}u_N^{\mathrm{SE}}$, i.e.,

$$\boldsymbol{u}_n^{\mathrm{SE}} = [u_N^{\mathrm{SE}}(\psi^{\mathrm{SE}}(-Nh)), \ldots, u_N^{\mathrm{SE}}(\psi^{\mathrm{SE}}(Nh))]^{\mathrm{T}},$$

where $n = 2N+1$. To this end, let us discretize (7) at $t = \psi^{\mathrm{SE}}(ih)$ $(i = -N, \ldots, N)$, and consider the resulting system of linear equations

$$(\lambda I_n - K_n^{\mathrm{SE}})\boldsymbol{u}_n^{\mathrm{SE}} = \boldsymbol{g}_n^{\mathrm{SE}}, \tag{8}$$

where K_n^{SE} is an $n \times n$ matrix whose (i,j) element is

$$(K_n^{\mathrm{SE}})_{ij} = k(\psi^{\mathrm{SE}}(ih), \psi^{\mathrm{SE}}(jh)), \quad i,j = -N, \ldots, N,$$

and $\boldsymbol{g}_n^{\mathrm{SE}}$ is an n-dimensional vector defined by

$$\boldsymbol{g}_n^{\mathrm{SE}} = [g(\psi^{\mathrm{SE}}(-Nh)), \ldots, g(\psi^{\mathrm{SE}}(Nh))]^{\mathrm{T}}.$$

By solving the system (8), the desired solution u_N^{SE} is obtained. This is called the SE-Sinc-Nyström method.

2.3 DE-Sinc-Nyström method

Next we explain the method derived by Muhammad et al. [2]. Assume the following two conditions:

(DE1) $u \in \mathbf{H}^\infty(\psi^{\mathrm{DE}}(\mathscr{D}_d))$,
(DE2) $k(t,\cdot) \in \mathbf{H}^\infty(\psi^{\mathrm{DE}}(\mathscr{D}_d))$ for all $t \in [a,b]$.

Then the integral $\mathcal{K}u$ in (1) can be approximated by

$$\mathcal{K}_N^{\mathrm{DE}}[u](t) := h \sum_{j=-N}^{N} k(t,\psi^{\mathrm{DE}}(jh))u(\psi^{\mathrm{DE}}(jh))\psi^{\mathrm{DE}\prime}(jh).$$

The mesh size h here is chosen as $h = \log(4dN)/N$. Then, instead of the original equation $u = (g + \mathcal{K}u)/\lambda$, we consider the new equation:

$$u_N^{\mathrm{DE}}(t) = \frac{g(t) + \mathcal{K}_N^{\mathrm{DE}}[u_N^{\mathrm{DE}}](t)}{\lambda}. \tag{9}$$

To obtain the approximated solution u_N^{DE}, we have to determine the unknown coefficients in $\mathcal{K}_N^{\mathrm{DE}}u_N^{\mathrm{DE}}$, i.e.,

$$\boldsymbol{u}_n^{\mathrm{DE}} = [u_N^{\mathrm{DE}}(\psi^{\mathrm{DE}}(-Nh)), \ldots, u_N^{\mathrm{DE}}(\psi^{\mathrm{DE}}(Nh))]^{\mathrm{T}}.$$

By discretizing (9) at $t = \psi^{\mathrm{DE}}(ih)$ $(i = -N, \ldots, N)$, we have the linear system:

$$(\lambda I_n - K_n^{\mathrm{DE}})\boldsymbol{u}_n^{\mathrm{DE}} = \boldsymbol{g}_n^{\mathrm{DE}}, \tag{10}$$

where K_n^{DE} is an $n \times n$ matrix whose (i,j) element is

$$(K_n^{\mathrm{DE}})_{ij} = k(\psi^{\mathrm{DE}}(ih), \psi^{\mathrm{DE}}(jh)), \quad i,j = -N, \ldots, N,$$

and $\boldsymbol{g}_n^{\mathrm{DE}}$ is an n-dimensional vector defined by

$$\boldsymbol{g}_n^{\mathrm{DE}} = [g(\psi^{\mathrm{DE}}(-Nh)), \ldots, g(\psi^{\mathrm{DE}}(Nh))]^{\mathrm{T}}.$$

By solving the system (10), the desired solution u_N^{DE} is obtained. This is called the DE-Sinc-Nyström method.

3. Boundedness of the condition numbers

3.1 Main result

The main contribution of this paper is the following theorem.

Theorem 2 *Let the function k be continuous on $[a,b]\times[a,b]$. Furthermore, suppose that the homogeneous equation $(\lambda\mathcal{I}-\mathcal{K})f=0$ has only the trivial solution $f\equiv 0$. Then there exists a positive integer N_0 such that for all $N\geq N_0$ the matrices $(\lambda I_n-K_n^{\mathrm{SE}})$ and $(\lambda I_n-K_n^{\mathrm{DE}})$ have bounded inverses. Furthermore, there exist constants C^{SE} and C^{DE} independent of N such that for all $N\geq N_0$*

$$\|(\lambda I_n-K_n^{\mathrm{SE}})\|_\infty\|(\lambda I_n-K_n^{\mathrm{SE}})^{-1}\|_\infty\leq C^{\mathrm{SE}}, \tag{11}$$

$$\|(\lambda I_n-K_n^{\mathrm{DE}})\|_\infty\|(\lambda I_n-K_n^{\mathrm{DE}})^{-1}\|_\infty\leq C^{\mathrm{DE}}. \tag{12}$$

3.2 Sketch of the proof

In what follows we write $\mathbf{C}=C([a,b])$ for short. The next result plays an important role to prove Theorem 2.

Lemma 3 (Okayama et al. [4, in the proofs of Theorems 6.3 and 8.2]) *Suppose that the assumptions in Theorem 2 are fulfilled. Then there exist constants C_1 and C_2 independent of N such that for all N*

$$\|\mathcal{K}_N^{\mathrm{SE}}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}\leq C_1,$$
$$\|\mathcal{K}_N^{\mathrm{DE}}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}\leq C_2.$$

Furthermore, there exists a positive integer N_0 such that for all $N\geq N_0$ the operators $(\lambda\mathcal{I}-\mathcal{K}_N^{\mathrm{SE}})$ and $(\lambda\mathcal{I}-\mathcal{K}_N^{\mathrm{DE}})$ have bounded inverses, and

$$\|(\lambda\mathcal{I}-\mathcal{K}_N^{\mathrm{SE}})^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}\leq C_3,$$
$$\|(\lambda\mathcal{I}-\mathcal{K}_N^{\mathrm{DE}})^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}\leq C_4,$$

hold, where C_3 and C_4 are constants independent of N.

In view of this, we see that Theorem 2 is established if the following lemma is shown.

Lemma 4 *Suppose that the assumptions in Theorem 2 are fulfilled. Then we have*

$$\|(\lambda I_n-K_n^{\mathrm{SE}})\|_\infty\leq\|(\lambda\mathcal{I}-\mathcal{K}_N^{\mathrm{SE}})\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}, \tag{13}$$

$$\|(\lambda I_n-K_n^{\mathrm{DE}})\|_\infty\leq\|(\lambda\mathcal{I}-\mathcal{K}_N^{\mathrm{DE}})\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}. \tag{14}$$

Furthermore, if the inverse operators $(\lambda\mathcal{I}-\mathcal{K}_N^{\mathrm{SE}})^{-1}$ and $(\lambda\mathcal{I}-\mathcal{K}_N^{\mathrm{DE}})^{-1}$ exist, then the matrices $(\lambda I_n-K_n^{\mathrm{SE}})^{-1}$ and $(\lambda I_n-K_n^{\mathrm{DE}})^{-1}$ also exist, and we have

$$\|(\lambda I_n-K_n^{\mathrm{SE}})^{-1}\|_\infty\leq\|(\lambda\mathcal{I}-\mathcal{K}_N^{\mathrm{SE}})^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}, \tag{15}$$

$$\|(\lambda I_n-K_n^{\mathrm{DE}})^{-1}\|_\infty\leq\|(\lambda\mathcal{I}-\mathcal{K}_N^{\mathrm{DE}})^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}. \tag{16}$$

We prove this lemma below.

3.3 Proofs

The existence of the inverse matrix: $(\lambda I_n-K_n^{\mathrm{SE}})^{-1}$ is shown by the following lemma.

Lemma 5 *Suppose that the assumptions in Theorem 2 are fulfilled, and let $g\in C([a,b])$. Then the following two statements are equivalent:*

(A) *The equation $(\lambda\mathcal{I}-\mathcal{K}_N^{\mathrm{SE}})v=g$ has a unique solution $v\in\mathbf{C}$.*

(B) *The system of linear equations $(\lambda I_n-K_n^{\mathrm{SE}})\boldsymbol{c}_n=\boldsymbol{g}_n^{\mathrm{SE}}$ has a unique solution $\boldsymbol{c}_n\in\mathbb{R}^n$.*

Proof We show (A) $\Rightarrow$ (B) first. Using the unique solution $v\in\mathbf{C}$, define the vector $\boldsymbol{c}_n\in\mathbb{R}^n$ as $\boldsymbol{c}_n=[v(\psi^{\mathrm{SE}}(-Nh)),\ldots,v(\psi^{\mathrm{SE}}(Nh))]^{\mathrm{T}}$. Clearly this $\boldsymbol{c}_n$ is a solution of the linear system in (B), which shows the existence of a solution. The uniqueness is shown as follows. Suppose that there exists another solution $\tilde{\boldsymbol{c}}_n=[\tilde{c}_{-N},\ldots,\tilde{c}_N]^{\mathrm{T}}$. Define a function $\tilde{v}\in\mathbb{R}$ as

$$\tilde{v}(t)=\frac{1}{\lambda}\left(g(t)+h\sum_{j=-N}^{N}k(t,\psi^{\mathrm{SE}}(jh))\tilde{c}_j\psi^{\mathrm{SE}\prime}(jh)\right). \tag{17}$$

At the points $t_i=\psi^{\mathrm{SE}}(ih)$ $(i=-N,\ldots,N)$, clearly

$$\lambda\tilde{v}(t_i)=g(t_i)+h\sum_{j=-N}^{N}k(t_i,t_j)\tilde{c}_j\psi^{\mathrm{SE}\prime}(jh) \tag{18}$$

holds. On the other hand,

$$\lambda\tilde{c}_i=g(t_i)+h\sum_{j=-N}^{N}k(t_i,t_j)\tilde{c}_j\psi^{\mathrm{SE}\prime}(jh) \tag{19}$$

holds since $\tilde{\boldsymbol{c}}_n$ is a solution of the linear system. And since the right-hand side of (18) is equal to that of (19), we conclude $\tilde{v}(t_i)=\tilde{c}_i$. Therefore (18) can be rewritten as $(\lambda\mathcal{I}-\mathcal{K}_N^{\mathrm{SE}})\tilde{v}=g$, which means $\tilde{v}$ is a solution of the equation in (A). From the uniqueness of the equation, $v\equiv\tilde{v}$ holds, which implies $\boldsymbol{c}_n=\tilde{\boldsymbol{c}}_n$. This shows the desired uniqueness.

Next we show (B) $\Rightarrow$ (A). Let $\tilde{\boldsymbol{c}}_n=[\tilde{c}_{-N},\ldots,\tilde{c}_N]^{\mathrm{T}}$ be a unique solution in (B), and define a function $\tilde{v}\in\mathbf{C}$ by (17). Then by the same argument as above, we can conclude $\tilde{v}$ is a solution of the equation in (A), which shows the existence. The uniqueness is shown as follows. Suppose that there exists another solution $v\in\mathbf{C}$. Define the vector $\boldsymbol{c}_n\in\mathbb{R}^n$ as $\boldsymbol{c}_n=[v(\psi^{\mathrm{SE}}(-Nh)),\ldots,v(\psi^{\mathrm{SE}}(Nh))]^{\mathrm{T}}$. Then clearly $\boldsymbol{c}_n$ is a solution of the linear system in (B). From the uniqueness of the linear system, we have $\boldsymbol{c}_n=\tilde{\boldsymbol{c}}_n$. Therefore $v(\psi^{\mathrm{SE}}(jh))=\tilde{c}_j$, and v can be rewritten as

$$v(t)=\frac{1}{\lambda}\left(g(t)+h\sum_{j=-N}^{N}k(t,\psi^{\mathrm{SE}}(jh))\tilde{c}_j\psi^{\mathrm{SE}\prime}(jh)\right). \tag{20}$$

In view of (17) and (20), we have $v\equiv\tilde{v}$, which shows the desired uniqueness.

(QED)

In the same manner we can prove the following lemma for the DE-Sinc-Nyström method. The proof is omitted.

Lemma 6 *Suppose that the assumptions in Theorem 2 are fulfilled, and let $g\in C([a,b])$. Then the following two statements are equivalent:*

(A) *The equation $(\lambda\mathcal{I}-\mathcal{K}_N^{\mathrm{DE}})v=g$ has a unique solution $v\in\mathbf{C}$.*

(B) *The system of linear equations $(\lambda I_n-K_n^{\mathrm{DE}})\boldsymbol{c}_n=\boldsymbol{g}_n^{\mathrm{DE}}$ has a unique solution $\boldsymbol{c}_n\in\mathbb{R}^n$.*

Thus the existence of the inverse matrix is guaranteed in both cases (SE and DE). The remaining task is to show (13)–(16). We show only (13) and (15) since (14) and (16) are shown in the same manner.

Proof of Lemma 4 We show (13) first. Let $\boldsymbol{c}_n=[c_{-N},\ldots,c_N]^{\mathrm{T}}$ be an arbitrary n-dimensional vector. Pick a function $\gamma\in\mathbf{C}$ that satisfies $\gamma(\psi^{\mathrm{SE}}(ih))=c_i$ $(i=-N,\ldots,N)$ and $\|\gamma\|_{\mathbf{C}}=\|\boldsymbol{c}_n\|_\infty$. Using this function γ, define a function $f\in\mathbf{C}$ as $f=(\lambda\mathcal{I}-\mathcal{K}_N^{\mathrm{SE}})\gamma$, and

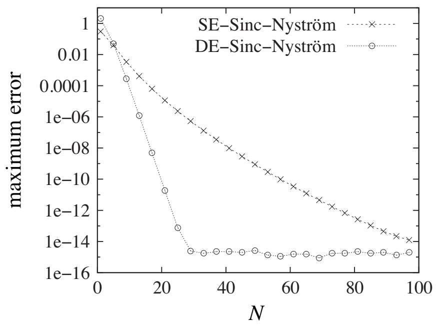

Fig. 1. Error of the Sinc-Nyström methods for (21).

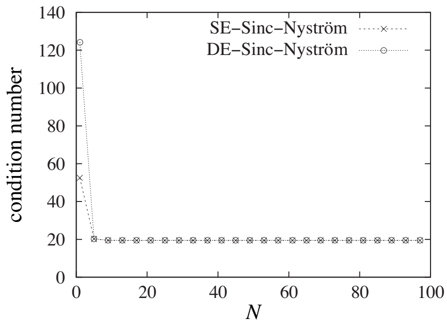

Fig. 2. Condition number of the coefficient matrix appearing in the Sinc-Nyström methods for (21).

a vector $\boldsymbol{f}_n$ as $\boldsymbol{f}_n = [f(\psi^{\mathrm{SE}}(-Nh)), \ldots, f(\psi^{\mathrm{SE}}(Nh))]^{\mathrm{T}}$. Then we have

$$\begin{aligned}\|(\lambda I_n - K_n^{\mathrm{SE}})\boldsymbol{c}_n\|_\infty &= \|\boldsymbol{f}_n\|_\infty \\ &\leq \|f\|_{\mathbf{C}} \\ &= \|(\lambda \mathcal{I} - \mathcal{K}_N^{\mathrm{SE}})\gamma\|_{\mathbf{C}} \\ &\leq \|(\lambda \mathcal{I} - \mathcal{K}_N^{\mathrm{SE}})\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}\|\gamma\|_{\mathbf{C}} \\ &= \|(\lambda \mathcal{I} - \mathcal{K}_N^{\mathrm{SE}})\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}\|\boldsymbol{c}_n\|_\infty,\end{aligned}$$

from which (13) follows.

Next we show (15). Notice that the inverse matrix $(\lambda I_n - K_n^{\mathrm{SE}})^{-1}$ exists from Lemma 5. Let $\boldsymbol{c}_n$ be an arbitrary n-dimensional vector. In the same manner as the above, pick a function $\gamma \in \mathbf{C}$. Define a function $f \in \mathbf{C}$ as $f = (\lambda \mathcal{I} - \mathcal{K}_N^{\mathrm{SE}})^{-1}\gamma$, and a vector $\boldsymbol{f}_n$ in the same way as the above. The difference from the above is in f; $(\lambda \mathcal{I} - \mathcal{K}_N^{\mathrm{SE}})$ is replaced with $(\lambda \mathcal{I} - \mathcal{K}_N^{\mathrm{SE}})^{-1}$. Then we have

$$\begin{aligned}\|(\lambda I_n - K_n^{\mathrm{SE}})^{-1}\boldsymbol{c}_n\|_\infty &= \|\boldsymbol{f}_n\|_\infty \\ &\leq \|f\|_{\mathbf{C}} \\ &= \|(\lambda \mathcal{I} - \mathcal{K}_N^{\mathrm{SE}})^{-1}\gamma\|_{\mathbf{C}} \\ &\leq \|(\lambda \mathcal{I} - \mathcal{K}_N^{\mathrm{SE}})^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}\|\gamma\|_{\mathbf{C}} \\ &= \|(\lambda \mathcal{I} - \mathcal{K}_N^{\mathrm{SE}})^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}\|\boldsymbol{c}_n\|_\infty,\end{aligned}$$

from which (15) follows. This completes the proof.

(QED)

4. Numerical example

In this section we show numerical results for

$$u(t) - \int_0^{\pi/2} (ts)^{3/2} u(s)\,\mathrm{d}s = \sqrt{t}\left(1 - \frac{\pi^3}{24}t\right), \quad 0 \leq t \leq \frac{\pi}{2}, \tag{21}$$

which has also been conducted by Muhammad et al. [2, Example 4.3]. The exact solution is $u(t) = \sqrt{t}$. Let us first check the conditions described in Sections 2.2 and 2.3. The conditions (SE1) and (SE2) are satisfied with $d = \pi - \epsilon$, and (DE1) and (DE2) are satisfied with $d = (\pi - \epsilon)/2$, where ϵ is an arbitrary small positive number (we set $\epsilon = \pi - 3.14$ in our computation).

Based on the information, we implemented the SE-Sinc-Nyström method and DE-Sinc-Nyström method in C++ with double-precision floating-point arithmetic. The errors $|u(t) - u_N^{\mathrm{SE}}(t)|$ and $|u(t) - u_N^{\mathrm{DE}}(t)|$ were investigated on equally-spaced 1000 points on $[0, \pi/2]$, and the maximum of them is shown in Fig. 1. We can observe the rate $\mathrm{O}(\exp(-c_1\sqrt{N}))$ in the SE-Sinc-Nyström method, and $\mathrm{O}(\exp(-c_2 N/\log N))$ in the DE-Sinc-Nyström method. These results can be explained by combining the existing estimates (2) and (3) with the new result (Theorem 2). Furthermore from Fig. 2, we can also confirm boundedness of the condition numbers, i.e., the estimates (11) and (12).

5. Concluding remarks

The Sinc-Nyström methods for (1) have been known as efficient methods in the sense that exponential convergence can be attained. However, the convergence has not been guaranteed theoretically, since in the existing estimates (2) and (3), there remained unestimated terms: $\|A_N^{-1}\|_2$ and $\|\tilde{A}_N^{-1}\|_2$ ($A_N = I_n - K_n^{\mathrm{DE}}$ and $\tilde{A}_N = I_n - K_n^{\mathrm{SE}}$). In this paper we showed theoretically that $\|A_N^{-1}\|_\infty$ and $\|\tilde{A}_N^{-1}\|_\infty$ are bounded, from which exponential convergence of the methods is guaranteed. Furthermore we showed that $\|A_N\|_\infty$ and $\|\tilde{A}_N\|_\infty$ are also bounded, and consequently the condition number of them is bounded, as stated in Theorem 2.

Muhammad et al. [2] have also developed the Sinc-Nyström methods for Volterra integral equations, and the similar result to this paper can be shown for them. We are now working on this issue, and the result will be reported somewhere else soon.

Acknowledgments

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