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The existence of solutions to topology optimization problems

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Abstract

Topology optimization is to determine a shape or topology, having minimum cost. We are devoted entirely to minimum compliance (maximum stiffness) as minimum cost. An optimal shape Ω is realized as a distribution of material on a reference domain D, strictly larger than Ω in general. The optimal shape Ω and an equilibrium $u(\Omega)$ on Ω are approximated by material distributions on the domain D and equilibriums also on D, respectively. This note gives a sufficient setting to the existence of an optimal material distribution.

Keywords topology optimization, the Hausdorff metric, minimum compliance, maximum stiffness, material distribution problem

Research Activity Group Mathematical Design

1. Introduction

Let D be a bounded domain of $\mathbf{R}^d, d = 2, 3$, and ω be an open subset included in D together with its closure $\overline{\omega}$, i.e., $\omega \subset \overline{\omega} \subset D$. The set ω may have several connected components. Let $\Omega = D \setminus \overline{\omega}$. The domain Ω could be multiply connected in general. Some material with density value one is filled into Ω and D, so the weights of Ω and D are given by $|\Omega| = \int_\Omega dx$ and $|D|$, respectively. We assume that Ω has the weight $|\Omega|$ $(\leq \overline{c}_\mathrm{v})$, where $\overline{c}_\mathrm{v}$ is strictly smaller than $|D|$. Let $\Gamma = \partial D$ be the boundary of D such that $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D^o \cap \Gamma_N^o = \emptyset$, where Γ_D^o and Γ_N^o are the interiors of Γ_D and Γ_N, respectively. Let $H^1(\Omega)$ and $H^{1/2}(\Gamma_N)$ be the usual Sobolev spaces and let $H^{-1/2}(\Gamma_N)$ be the dual space of $H^{1/2}(\Gamma_N)$.

For $f \in L^2(D)$, $f \neq 0$ and $g \in H^{-1/2}(\Gamma_N)$ we consider the problem $\mathbf{BVP}(\Omega)$: Find $u^\Omega \in H^1(\Omega)$ such that

$$\begin{cases} -\Delta u^\Omega + u^\Omega = f \quad \text{in } \Omega, \\ u^\Omega = 0 \quad \text{on } \Gamma_D, \\ \partial_\nu u^\Omega = g \quad \text{on } \Gamma_N, \\ \partial_\nu u^\Omega = 0 \quad \text{on } \Gamma_\omega (= \partial\omega \cap \partial\Omega). \end{cases} \tag{1}$$

Let $J(\Omega)$ be cost of $\Omega (\subset D)$ given by

$$J(\Omega) = \textstyle\int_\Omega f u^\Omega dx + \int_{\Gamma_N} g u^\Omega d\Gamma. \tag{2}$$

After giving an *admissible family* $\mathcal{U}$ of domains Ω adequately, we consider a minimizing problem, called the *topology optimization problem* (confer [1,2] and refereces therein), $\mathbf{TOP}(D)$: Find $\Omega^* \in \mathcal{U}$ such that

$$J(\Omega^*) = \inf\nolimits_{\Omega \in \mathcal{U}} J(\Omega). \tag{3}$$

Our aim is to assure the existence of a solution $\Omega^* \in \mathcal{U}$ of $\mathbf{TOP}(D)$. For this aim a choice of admissible sets is significant (confer Theorem 10). Another aim is to approximate such a solution Ω^* by density functions $\phi \in L^\infty(D)$. Before precise description of $\mathcal{U}$ we approximate the problem (1) by boundary value problems (5).

2. Approximation

We owe the idea of the approximation (5) to the spirit in the SIMP model by Bendsøe and Sigmund [2]. Let $\mathcal{O}_D$ be a family of open, connected sets Ω $(\subset D, \Gamma \subset \overline{\Omega}, |\Omega| \leq \overline{c}_\mathrm{v})$ and let $\Phi = \{\chi^\Omega \mid \Omega \in \mathcal{O}_D\}$, where χ^Ω denotes the characteristic function of Ω. For Ω of $\mathcal{O}_D$, χ^Ω $(\in \Phi)$ can be obviously approximated by simple functions $\chi_\kappa^{\Omega,\omega} = \chi^\Omega + \kappa\chi^\omega$ using small κ (> 0). Let $\Phi_\kappa = \{\chi_\kappa^{\Omega,\omega} \mid \Omega \in \mathcal{O}_D\}$ and $V(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D\}$. The problem $\mathbf{BVP}(\Omega)$ (1) is equivalent to the problem: Find $u^\Omega \in V(D)$ such that

$$\begin{cases} a^\Omega(u^\Omega, v) = (f, v)_\Omega + \langle g, v\rangle_{\Gamma_N}, \quad v \in V(\Omega), \\ a^\Omega(v, w) = \int_D \chi^\Omega(\nabla v \cdot \nabla w + vw)dx, \\ (f, v)_\Omega = \int_D \chi^\Omega f v dx, \quad \langle g, v\rangle_{\Gamma_N} = \int_{\Gamma_N} g v d\Gamma. \end{cases} \tag{4}$$

Let $V = \{v \in H^1(D) \mid v = 0 \text{ on } \Gamma_D\}$. Replacing the function χ^Ω by a function $\chi_\kappa^{\Omega,\omega}$ in the equality (4) implies $\mathbf{BVP}_\kappa^{\Omega,\omega}(D)$: Find $u_\kappa^{\Omega,\omega} \in V$ such that

$$\begin{cases} a_\kappa^{\Omega,\omega}(u_\kappa^{\Omega,\omega}, v) = (f, v)_\Omega + \kappa(f, v)_\omega \\ \qquad\qquad + \langle g, v\rangle_{\Gamma_N}, \quad v \in V, \\ a_\kappa^{\Omega,\omega}(v, w) = \int_\Omega(\nabla v \cdot \nabla w + vw)dx \\ \qquad\qquad + \kappa\int_\omega(\nabla v \cdot \nabla w + vw)dx. \end{cases} \tag{5}$$

For simplicity let $u_\kappa^\Omega = u_\kappa^{\Omega,\omega}|_\Omega$, $u_\kappa^\omega = u_\kappa^{\Omega,\omega}|_\omega$ and $\partial_\nu^G v$ be the outer normal derivative of v on a smooth open domain G. Further if $u_\kappa^{\Omega,\omega}$ is smooth enough, then $u_\kappa^{\Omega,\omega}$ is a unique solution of the following problem:

$$\begin{cases} -\Delta u_\kappa^{\Omega,\omega} + u_\kappa^{\Omega,\omega} = f \quad \text{in } \Omega \cup \omega, \\ u_\kappa^\Omega = u_\kappa^\omega \quad \text{on } \partial\Omega \cap \partial\omega, \\ \partial_\nu^\Omega u_\kappa^\Omega + \kappa\partial_\nu^\omega u_\kappa^\omega = 0 \quad \text{on } \partial\Omega \cap \partial\omega, \\ \partial_\nu^\Omega u_\kappa^\Omega = g \quad \text{on } \Gamma_N, \\ u_\kappa^\Omega = 0 \quad \text{on } \Gamma_D. \end{cases}$$

We shall approximate the cost function $J(\Omega)$ by

$$\mathcal{J}(\chi_\kappa^{\Omega,\omega}) = (f, u_\kappa^{\Omega,\omega})_\Omega + \kappa(f, u_\kappa^{\Omega,\omega})_\omega + \langle g, u_\kappa^{\Omega,\omega}\rangle_{\Gamma_N}. \quad (6)$$

3. Admissible family

Our one aim is to prescribe $\mathcal{U}$ precisely and another aim is to assure approximation of u^Ω and $J(\Omega)$ by $u_\kappa^{\Omega,\omega}$ and $\mathcal{J}(\chi_\kappa^{\Omega,\omega})$ as $\kappa \to +0$, respectively. A family $\mathcal{U} = \mathrm{Lip}(k, r)$, constructed by uniformly Lipschitz continuous domains, due to Chenais [3], is a good admissible family. Here $k > 0$ and $r > 0$. For simplicity we define $\mathrm{Lip}(k, r)$ with a slight modification from the original.

Definition 1 *A domain Ω $(\in \mathcal{O}_D)$ is admissible and belongs to* $\mathrm{Lip}(k, r)$*, if and only if, for any $x \in \partial\Omega$, there exists a local coordinate system with a real valued function φ of $d-1$ variables with Lipschitz constant k such that $B(x, r) \cap \Omega = \{(\hat{x}, x_d) \in \Omega \mid x_d < \varphi(\hat{x})\}$, where $\hat{x} \in \mathbf{R}^{d-1}$, $B(x, r) = \{y \in \mathbf{R}^d \mid |y - x| \le r\}$ and $|\cdot|$ denotes the Euclid norm in $\mathbf{R}^d$.*

The advantage of using $\mathcal{U} = \mathrm{Lip}(k, r)$ is that Theorems 2, 3 and 10, a target in this paper, are available. We write $\Phi(k, r) = \{\chi^\Omega \mid \Omega \in \mathrm{Lip}(k, r)\}$ and $\Phi_\kappa(k, r) = \{\chi_\kappa^{\Omega,\omega} \mid \Omega \in \mathrm{Lip}(k, r)\}$. For $m \in \mathbf{N}$, the usual Sobolev norm of $v \in H^m(G)$ is denoted by $\|v\|_{m,G}$.

Theorem 2 (Chenais [3]) *For $\Omega \in \mathrm{Lip}(k, r)$ there exists a linear continuous extension operator p^Ω : $H^m(\Omega) \ni v \mapsto \tilde{v} = p^\Omega(v) \in H^m(\mathbf{R}^d)$ with operator norm $\|p^\Omega\| \le \bar{c}(k, r)$, where $\bar{c}(k, r)$ depends only on m, k, r and d.*

Recall that we have set $u_\kappa^\Omega = u_\kappa^{\Omega,\omega}|_\Omega$ and $u_\kappa^\omega = u_\kappa^{\Omega,\omega}|_\omega$.

Theorem 3 *Let $\Omega \in \mathcal{O}_D$ and $0 < \kappa \le 1$. Then we have*

$$\|u_\kappa^\Omega\|_{1,\Omega}^2 + \kappa\|u_\kappa^\omega\|_{1,\omega}^2 \le 2(\|f\|_{0,D}^2 + \|g\|_{-1/2,\Gamma_N}^2). \quad (7)$$

Let $\bar{c}(k, r)$ be a constant in Theorem 2 *with $m = 1$ and let $\bar{c}_1(k, r) = 1 + 2\bar{c}(k, r)^2$. Further we assume $\Omega \in \mathrm{Lip}(k, r)$. Then we have*

$$\|u_\kappa^\Omega - u^\Omega\|_{1,\Omega}^2 + \kappa\|u_\kappa^\omega\|_{1,\omega}^2 \le 3\kappa(\bar{c}_1(k, r)\|f\|_{0,D}^2 + 2\bar{c}(k, r)^2\|g\|_{-1/2,\Gamma_N}^2). \quad (8)$$

Proof Let $\|v\|_{1,\Omega,\omega,\kappa} = (\|v\|_{1,\Omega}^2 + \kappa\|v\|_{1,\omega}^2)^{1/2}$. Then $\max\{|a_\kappa^{\Omega,\omega}(u_\kappa^{\Omega,\omega}, v)| \mid \|v\|_{1,\Omega,\omega,\kappa} = 1 \text{ and } v \in V\} \le 2^{1/2}(\|f\|_{0,D}^2 + \|g\|_{-1/2,\Gamma_N}^2)^{1/2}$. Moreover the maximum of the left-hand side attains at $v = u_\kappa^{\Omega,\omega}$. This shows (7). Putting $v = u^\Omega$ into (4) implies

$$\|u^\Omega\|_{1,\Omega}^2 \le 2(\|f\|_{0,D}^2 + \|g\|_{-1/2,\Gamma_N}^2). \quad (9)$$

We show (8). Connecting (9) with Theorem 2 with $m = 1$ we have an extension $\tilde{u}^\Omega (\in V)$ of u^Ω such that

$$\|\tilde{u}^\Omega\|_{1,\omega}^2 \le 2\bar{c}(k, r)^2(\|f\|_{0,D}^2 + \|g\|_{-1/2,\Gamma_N}^2). \quad (10)$$

Recall that notation $a^G(\cdot,\cdot)$, where $G = \Omega, \omega$. Putting the right-hand side of (4) into the sum of the first and the last terms in the right-hand side of (5) implies

$$a^\Omega(u_\kappa^\Omega - u^\Omega, v) + \kappa a^\omega(u_\kappa^\omega, v) = \kappa(f, v)_\omega, \quad v \in V. \quad (11)$$

Putting $v = u_\kappa^{\Omega,\omega} - \tilde{u}^\Omega$ into (11) gives

$$\|u_\kappa^\Omega - u^\Omega\|_{1,\Omega}^2 + \kappa\|u_\kappa^\omega\|_{1,\omega}^2 \le 3\kappa(\|f\|_{0,\omega}^2 + \|\tilde{u}^\Omega\|_{1,\omega}^2). \quad (12)$$

The inequalities (9), (10) and (12) imply (8).

(QED)

Definition 4 *Let E be a bounded closed subset of $\mathbf{R}^d$ and $\mathcal{F}$ be the totality of compact subsets of E. For $F \in \mathcal{F}$ we set $[F]_c = \{x \in \mathbf{R}^d \mid \exists y \in F \text{ such that } |y - x| \le c\}$. For $F_i \in \mathcal{F}, i = 1, 2$, the Hausdorff metric $d(F_1, F_2)$ between F_1 and F_2 is defined by*

$$d(F_1, F_2) = \inf\{c > 0 \mid F_1 \subset [F_2]_c, F_2 \subset [F_1]_c\}.$$

We define an equivalent relation $\Omega_1 \sim \Omega_2$ for $\Omega_i \in \mathcal{O}_D, i = 1, 2$, defined by $d(\overline{\Omega}_1, \overline{\Omega}_2) = 0$. The equivalent relation $\sim$ determines a metric $\tilde{d}(\cdot,\cdot)$ by $\tilde{d}(\Omega_1, \Omega_2) = d(\overline{\Omega}_1, \overline{\Omega}_2)$ on $\mathcal{O}_D/\sim$. Hereafter we use a notation $d(\Omega_1, \Omega_2)$ instead of $\tilde{d}(\Omega_1, \Omega_2)$, if no confusion occurs.

Theorem 5 (Blachke selection theorem [4]) *The family $\mathcal{F}$ with topology $\mathcal{T}_H$ induced by metric d is compact.*

Due to Chenais [3, Theorems III.1 and III.2], the set $\Phi(k, r) = \{\chi^\Omega \in L^2(D) \mid \Omega \in \mathrm{Lip}(k, r)\}$ with the topology $\mathcal{T}_C$ induced from $L^2(D)$ is compact. Theorem 5 is applied to $E = \overline{D}$. We see that the identity map $\mathcal{I}$: $(\Phi(k, r), \mathcal{T}_H) \overset{\mathcal{I}}{\mapsto} (\Phi(k, r), \mathcal{T}_C)$, is continuous by the definition of the Hausdorff metric. Further, $\mathcal{I}^{-1}$ is also continuous, because any closed set F in $(\Phi(k, r), \mathcal{T}_H)$ is also closed in $(\Phi(k, r), \mathcal{T}_C)$. In fact, the set F of $(\Phi(k, r), \mathcal{T}_H)$ is compact in $(\Phi(k, r), \mathcal{T}_H)$ clearly. So it is compact in $(\Phi(k, r), \mathcal{T}_C)$, because $\mathcal{I}$ is continuous. Thus the set F is closed in $(\Phi(k, r), \mathcal{T}_C)$. This means that the inverse $\mathcal{I}^{-1}$ is continuous. Therefore the map $\mathcal{I}$ is homeomorphism between $(\Phi(k, r), \mathcal{T}_H)$ and $(\Phi(k, r), \mathcal{T}_C)$.

Theorem 6 *Topologies $\mathcal{T}_\mathrm{C}$ and $\mathcal{T}_\mathrm{H}$ are compact and equivalent to each other.*

4. The continuity of the cost function and solvability of TOP(D)

We consider the convergence of $\mathcal{J}_n = \mathcal{J}(\chi_{\kappa_n}^{\Omega,\omega})$ to $J(\Omega)$ as $\kappa_n \to +0$. The estimate (8) shows $u_{\kappa_n}^{\Omega,\omega} \to u^\Omega$ strongly in $V(\Omega)$ as $\kappa_n \to +0$. Actually, by (8) we have

$$|\mathcal{J}_n - J(\Omega)| \le (\|f\|_{0,\Omega} + \|g\|_{-1/2,\Gamma_N})\|u_{\kappa_n}^{\Omega,\omega} - u^\Omega\|_{1,\Omega} + \kappa_n^{1/2}\|f\|_{0,D} \cdot \kappa_n^{1/2}\|u_{\kappa_n}^{\Omega,\omega}\|_{1,\omega} \to 0 \quad \text{as } \kappa_n \to +0.$$

Lemma 7 *Let $\Omega \in \mathrm{Lip}(k, r)$. And let $\{\kappa_n\}_n$ be a sequence tends to 0 as $n \to \infty$. Then two sequences $\{u_{\kappa_n}^{\Omega,\omega}\}_n$ and $\{\mathcal{J}(\chi_{\kappa_n}^{\Omega,\omega})\}_n$ converge to u^Ω strongly in $V(\Omega)$ and to $J(\Omega)$, respectively, as $n \to \infty$. Both the sequences converge to their limits uniformly over* $\mathrm{Lip}(k, r)$ *or $\Phi(k, r)$.*

Lemma 8 *Let κ be fixed and let $\Omega_n, \Omega \in \mathcal{O}_D$ such that $d(\Omega_n, \Omega) \to 0$ as $n \to \infty$. Then we have $u_\kappa^{\Omega_n,\omega_n} \to u_\kappa^{\Omega,\omega}$ strongly in V as $n \to \infty$.*

Proof Let $u_n = u_\kappa^{\Omega_n,\omega_n}$ and $u = u_\kappa^{\Omega,\omega}$. Since $\{u_n\}_n$ is bounded in V by Theorem 3, we have a subsequence of $\{u_n\}_n$, still denoted by $\{u_n\}_n$, having a weak limit $\hat{u} \in V$. To see $u = \hat{u}$ it suffices to show $R_n \to 0$, where R_n is given by $a_\kappa^{\Omega,\omega}(u_n, \zeta) = (f, \zeta)_\Omega + \kappa(f, \zeta)_\omega + \langle g, \zeta\rangle_{\Gamma_N} + R_n$. Here ζ denotes any smooth function of $V \cap C^1(\overline{D})$ with notation: $\|\zeta\|_{C^0(\overline{D})} = \max_{x\in\overline{D}} |\zeta(x)|$ and $\|\zeta\|_{C^1(\overline{D})}$

$= \|\zeta\|_{C^0(\overline{D})} + \max_{x\in\overline{D},1\le i\le d}|\partial\zeta(x)/\partial x_i|$. A tedious calculation gives

$$\begin{aligned}
R_n &= R_n^1 + R_n^2,\\
R_n^1 &= (a^{\Omega\setminus\Omega_n}(u_n,\zeta) - a^{\Omega_n\setminus\Omega}(u_n,\zeta))\\
&\quad + \kappa(a^{\omega\setminus\omega_n}(u_n,\zeta) - a^{\omega_n\setminus\omega}(u_n,\zeta))\\
&= a_\kappa^{\Omega\setminus\Omega_n,\omega\setminus\omega_n}(u_n,\zeta) - a_\kappa^{\Omega_n\setminus\Omega,\omega_n\setminus\omega}(u_n,\zeta),\\
R_n^2 &= l_\kappa^{\Omega_n\setminus\Omega,\omega_n\setminus\omega}(\zeta) - l_\kappa^{\Omega\setminus\Omega_n,\omega\setminus\omega_n}(\zeta),\\
l_\kappa^{\Omega_n\setminus\Omega,\omega_n\setminus\omega}(\zeta) &= \textstyle\int_{\Omega_n\setminus\Omega} f\zeta dx + \kappa\int_{\omega_n\setminus\omega} f\zeta dx,\\
l_\kappa^{\Omega\setminus\Omega_n,\omega\setminus\omega_n}(\zeta) &= \textstyle\int_{\Omega\setminus\Omega_n} f\zeta dx + \kappa\int_{\omega\setminus\omega_n} f\zeta dx.
\end{aligned}$$

Since $|F| = 0$ for a measurable set $F \subset \mathbf{R}^d$ such that $(\overline{F})^o = \emptyset$, we see that $|\partial G| = 0$, where $\partial G = \overline{G}\setminus G$ for an open subset G in $\mathbf{R}^d$. We have

$$\begin{cases} \omega_n\setminus\omega = \Omega\setminus\Omega_n & \text{a.e. in } D,\\ \omega\setminus\omega_n = \Omega_n\setminus\Omega & \text{a.e. in } D,\\ \omega\ominus\omega_n = \Omega\ominus\Omega_n & \text{a.e. in } D. \end{cases} \tag{13}$$

Let $\|v\|_{1,\Omega\setminus\Omega_n,\omega\setminus\omega_n,\kappa} = (\|v\|^2_{1,\Omega\setminus\Omega_n} + \kappa\|v\|^2_{1,\omega\setminus\omega_n})^{1/2}$ for $v \in V$. Let $\overline{c}_2(f,g)^2$ be a constant described as the right-hand side of the inequality (7). Applying the Schwarz inequality to $a_\kappa^{\Omega\setminus\Omega_n,\omega\setminus\omega_n}(u_n,\zeta)$, we have

$$\begin{aligned}
&|a_\kappa^{\Omega\setminus\Omega_n,\omega\setminus\omega_n}(u_n,\zeta)(u_n,\zeta)|\\
&\quad \le \|u_n\|_{1,\Omega\setminus\Omega_n,\omega\setminus\omega_n,\kappa}\|\zeta\|_{1,\Omega\setminus\Omega_n,\omega\setminus\omega_n,\kappa}\\
&\quad \le \overline{c}_2(f,g)\|\zeta\|_{C^1(\overline{D})}\sqrt{|\Omega\setminus\Omega_n|}.
\end{aligned}$$

All the estimates of remaining terms in R_n^1 and R_n^2 have the same upper bounds as above deriving by similar considering. Thus we have

$$|R_n| \le 4\overline{c}_2(f,g)\|\zeta\|_{C^1(\overline{D})}\sqrt{|\Omega\ominus\Omega_n|}.$$

Let $\delta_n = d(\Omega_n,\Omega)$, then the definition of the Hausdorff metric implies $\Omega_n\setminus\Omega \subset [\Omega]_{\delta_n}\setminus\Omega$ and $\Omega\setminus\Omega_n \subset [\Omega_n]_{\delta_n}\setminus\Omega_n$. The assumption says $\delta_n \to 0$. Thus we have $|\Omega_n\ominus\Omega| \to 0$ as $n\to\infty$. So we see $R_n \to 0$ as $n\to\infty$. Thus the full sequence $\{u_\kappa^{\Omega_n,\omega_n}\}_n$ converges to u weakly in V.

Next we show $u_n \to u$ strongly in V. For the aim we notice that the bilinear form $a_\kappa^{\Omega,\omega}(\cdot,\cdot)$ is an inner product equivalent to the usual one, because κ is a positive constant. So the value $(a_\kappa^{\Omega,\omega}(v,v))^{1/2}$ could play as a norm on V. Since V is a Hilbert space, so if we can show that

$$\lim_{n\to\infty} a_\kappa^{\Omega,\omega}(u_n,u_n) = a_\kappa^{\Omega,\omega}(u,u), \tag{14}$$

then u_n converges to u strongly in V. Actually it is true. In fact, let $\overline{l}_\kappa^{\Omega,\omega}(v) = l_\kappa^{\Omega,\omega}(v) + \langle g,v\rangle_{\Gamma_N}$. Then $\overline{l}_\kappa^{\Omega,\omega}(v)$ belongs to the dual space V' of V and we notice that u_n converges to u weakly in V, thus we have

$$\lim_{n\to\infty} \overline{l}_\kappa^{\Omega,\omega}(u_n) = \overline{l}_\kappa^{\Omega,\omega}(u). \tag{15}$$

Beside we see $\lim_{n\to\infty} a_\kappa^{\Omega,\omega}(u_n,u_n) = \lim_{n\to\infty}\overline{l}_\kappa^{\Omega,\omega}(u_n)$ and $\overline{l}_\kappa^{\Omega,\omega}(u) = a_\kappa^{\Omega,\omega}(u,u)$. The equality (14) holds true.

(QED)

Lemma 9 *Let $\Omega_n,\Omega\in \mathrm{Lip}(k,r)$ such that $d(\Omega_n,\Omega)\to 0$ as $n\to\infty$. Then we have $J(\Omega_n)\to J(\Omega)$ as $n\to\infty$.*

Proof Since we have $J(\Omega_n) - J(\Omega) = (J(\Omega_n) - \mathcal{J}(\chi_\kappa^{\Omega_n,\omega_n})) + (\mathcal{J}(\chi_\kappa^{\Omega_n,\omega_n}) - \mathcal{J}(\chi_\kappa^{\Omega,\omega})) + (\mathcal{J}(\chi_\kappa^{\Omega,\omega}) - J(\Omega)) = j_1(n,\kappa) + j_2(n,\kappa) + j_3(\kappa)$, for any $\epsilon > 0$ it suffices to show the existence of n_ϵ and κ_ϵ such that, for all $n(\ge n_\epsilon)$, we have

$$\max\{|j_1(n,\kappa_\epsilon)|,|j_3(n,\kappa_\epsilon)|\} \le \frac{\epsilon}{3}, \tag{16}$$

$$|j_2(n,\kappa_\epsilon)| \le \frac{\epsilon}{3}. \tag{17}$$

Let $3\kappa\overline{c}_3(f,g)^2$ be a constant described at the right-hand side of (8). Then, applying the Schwarz inequality and the definition of $\|\cdot\|_{-1/2,\Gamma_N}$ to $j_1(n,\kappa)$ and $j_3(\kappa)$, we have

$$\begin{aligned}
&\max\{|j_1(n,\kappa)|,|j_3(\kappa)|\}\\
&\quad \le \sqrt{3\kappa}\overline{c}_3(f,g)[(1+\sqrt{\kappa})\|f\|_{0,D} + \|g\|_{-1/2,\Gamma_N}].
\end{aligned}$$

Thus there exists $\kappa_\epsilon(>0)$ satisfying (16), independent of $n\in\mathbf{N}$. Finally we have $n_\epsilon\in\mathbf{N}$ such that (17) holds by Lemma 8 with $\kappa=\kappa_\epsilon$. The proof is completed.

(QED)

Although the problem $\mathbf{TOP}(D)$ is known well, we don't assure the existence of solutions to the problem together with its uniqueness generally.

Theorem 10 *For $\mathcal{U} = \mathrm{Lip}(k,r)$ there exists a solution $\Omega^*\in\mathrm{Lip}(k,r)$ of $\mathbf{TOP}(D)$ (3).*

Proof First, we see $\inf_{\Omega\in\mathrm{Lip}(k,r)} J(\Omega) \ge 0$, because we see $J(\Omega) = \|u^\Omega\|^2_{1,\Omega}$. So there exists a minimizing sequence of $\{J(\Omega_n)\}_n, \Omega_n \in \mathrm{Lip}(k,r)$. Applying Theorem 6 together with Lemma 9 to $\{J(\Omega_n)\}_n$, we have the assertion. Here we notice $J(\Omega^*) > 0$ provided $\|f\|_{0,D} + \|g\|_{-1/2,\Gamma_N} > 0$.

(QED)

5. Approximation of TOP(D) by simple functions

We consider the minimizing problem $\mathbf{P}_\kappa(D)$: Find $\phi^*\in\Phi_\kappa$ such that

$$\mathcal{J}(\phi_\kappa^*) = \inf_{\phi\in\Phi_\kappa}\mathcal{J}(\phi). \tag{18}$$

We show that the existence of a solution of $\mathbf{P}_\kappa(D)$ and that the problem $\mathbf{TOP}(D)$ is approximated by the problems $\mathbf{P}_{\kappa_n}(D)$ as $\kappa_n\to 0$ provided that Φ_κ is replaced by $\Phi_\kappa(k,r)$. Now, we denote a topology on the set Φ_κ induced through $L^2(D)$ by $\mathcal{T}_{L^2}^{\Phi_\kappa}$.

Lemma 11 *Let $\phi_n = \chi_\kappa^{\Omega_n,\omega_n}$, $\phi = \chi_\kappa^{\Omega,\omega}\in\Phi_\kappa$ for $n\in\mathbf{N}$. Then $\phi_n\to\phi$ as $n\to\infty$ with respect to $\mathcal{T}_{L^2}^{\Phi_\kappa}$, if and only if $\Omega_n\to\Omega$ as $n\to\infty$ with respect to $\mathcal{T}_H$. Thus Φ_κ is compact with respect to $\mathcal{T}_{L^2}^{\Phi_\kappa}$.*

Proof The relation (13) implies $|\phi_n - \phi| = (1-\kappa)(|\chi^{\Omega_n\setminus\Omega}| + |\chi^{\Omega\setminus\Omega_n}|)$ a.e. in D. So $\|\phi_n-\phi\|^2_{L^2(D)} = (1-\kappa)^2|\Omega_n\ominus\Omega|$. Since $d(\Omega_n,\Omega)\to 0$ implies $|\Omega_n\ominus\Omega|\to 0$, then $\|\phi_n-\phi\|^2_{L^2(D)}\to 0$.

Next we show the inverse assertion. Assume $\|\phi_n - \phi\|^2_{L^2(D)}\to 0$ doesn't imply $d(\Omega_n,\Omega)\to 0$. Then, by Theorem 5 with $E=\overline{D}$ there exists a subsequence $\{\Omega_{n_m}\}_m$ of $\{\Omega_n\}_n$, where $d(\Omega_{n_m},\hat\Omega)\to 0$ for some $\hat\Omega\in\mathcal{O}_D$, $\hat\Omega\ne\Omega$. Let $\hat\phi = \chi_\kappa^{\hat\Omega,\hat\omega}$. So $\|\phi_{n_m} - \hat\phi\|_{L^2(D)}\to 0$. It contradicts to $\hat\phi\ne\phi$ and $\lim_{m\to\infty}\phi_{n_m} = \phi$.

(QED)

It is to be noticed that the equivalence between $\mathcal{T}_H$ and $\mathcal{T}_{L^2}^{\Phi_\kappa}$ restricted to $\Phi(k,r)$ is shown already by Theorem 6.

Theorem 12 *Let κ be fixed. Then the minimization problem* $\mathrm{P}_\kappa(D)$ (18) *admits a solution* $\phi^* \in \Phi_\kappa$.

Proof Lemmas 8 and 11 imply the existence of a solution $\phi^* \in \Phi_\kappa$ of the problem $\mathbf{P}_\kappa(D)$.

(QED)

Lemma 13 *Let* $\phi_n = \chi_{\kappa_n}^{\Omega_n,\omega_n} \in \Phi_{\kappa_n}(k,r)$, $\phi = \chi^\Omega \in \Phi(k,r)$ *and let* $u_n = u_{\kappa_n}^{\Omega_n,\omega_n} \in V$. *We assume that* $\kappa_n \to 0$ *and* $d(\Omega_n,\Omega) \to 0$. *Then* $\{u_n\}_n$ *weakly converges to* $u \in V$, *where* $u|_\Omega$ *and* $u|_\omega$, $\omega = D \setminus \overline{\Omega}$ *are given by* (4) *and the equalities below, respectively.*

$$u|_\omega = U|_\omega, \tag{19}$$

where $U \in V$ *is determined by* $a^D(U,v) = (f,v)_\omega, v \in V$. *Further we have*

$$\lim_{n\to\infty} \mathcal{J}(\phi_n) = J(\Omega). \tag{20}$$

Proof Recall that $\bar{c}_2(f,g)^2$ and $3\kappa\bar{c}_3(f,g)^2$ be constants used in the right-hand sides of (7) and (8) (cf. the proofs of Lemmas 8 and 9), respectively. The estimate (7) shows that $\bar{c}_2(f,g)$ denotes an upper bound of $\{\|u_n\|_{1,\Omega_n}\}_n$. After applying (8) to $\{\Omega_n,\omega_n\}$, deviding the both hands side of (8) by κ_n, we see

$$\frac{\|u_n - u^{\Omega_n}\|_{1,\Omega_n}^2}{\kappa_n} + \|u_n\|_{1,\omega_n}^2 \le 3\bar{c}_3(f,g)^2. \tag{21}$$

So $3^{1/2}\bar{c}_3(f,g)$ denotes an upper bound of $\{\|u_n\|_{1,\omega_n}\}_n$. Thus $\{u_n\}_n$ is bounded in V. We have a weakly convergent subsequence, still denoted by $\{u_n\}_n$, with its weak limit u. We show $u|_\Omega = u^\Omega$. For this aim it suffices to show $R_n \to 0$ as $n \to \infty$, where R_n denotes a constant given by below.

$$\begin{aligned}
&a^\Omega(u_n,\zeta) = (f,\zeta)_\Omega + \langle g,\zeta\rangle_{\Gamma_N} + R_n,\\
&R_n = R_n^1 + R_n^2,\\
&R_n^1 = a^{\Omega\setminus\Omega_n}(u_n,\zeta) - a^{\Omega_n\setminus\Omega}(u_n,\zeta) - \kappa_n a^{\omega_n}(u_n,\zeta),\\
&R_n^2 = (f,\zeta)_{\Omega_n\setminus\Omega} - (f,\zeta)_{\Omega\setminus\Omega_n} + \kappa_n(f,\zeta)_{\omega_n},
\end{aligned}$$

where ζ denotes a function belonging to $V \cap C^1(\overline{D})$. It is shown similarly as in the proof of Lemma 8. Thus, $u|_\Omega$ satisfies (4). Now we show (19). We consider spaces $V_n = \{v \in L^2(D) \mid v|_{\Omega_n} \in V(\Omega_n), v|_{\omega_n} \in H^1(\omega_n)\}, n \in \mathbf{N}$ and orthogonal projections $p_n v = \bar{v}$ from V_n onto the space V defined by $a^D(\bar{v},w) = a^{\Omega_n}(v,w) + a^{\omega_n}(v,w)$ for all $w \in V$. Applying (11) to $v = \zeta \in V \cap C^1(\overline{D})$, we have

$$\begin{aligned}
&a^{\Omega_n}\left(\tfrac{u_n - u^{\Omega_n}}{\kappa_n},\zeta\right) + a^{\omega_n}(u_n,\zeta)\\
&\quad = a^{\Omega_n}\left(p_n\left(\tfrac{u_n-u^{\Omega_n}}{\kappa_n}\right),\zeta\right) + a^{\omega_n}(p_n(u_n),\zeta)\\
&\quad = (f,\zeta)_{\omega_n}.
\end{aligned} \tag{22}$$

Because of (22) it suffices to $R_n \to 0$, where R_n is given by

$$\begin{aligned}
&a^\omega(u_n,\zeta) + a^\Omega(U,\zeta) - (f,\zeta)_\omega = R_n,\\
&R_n = R_n^1 + R_n^2 + R_n^3 + R_n^4,\\
&R_n^1 = a^{\omega\setminus\omega_n}(u_n,\zeta) - a^{\omega_n\setminus\omega}(u_n,\zeta),\\
&R_n^2 = -a^{\Omega\setminus\Omega_n}(U,\zeta) - a^{\Omega_n\setminus\Omega}(U,\zeta),\\
&R_n^3 = -a^{\Omega_n}\left(U - p_n\left(\tfrac{u_n-u^{\Omega_n}}{\kappa_n}\right),\zeta\right),\\
&R_n^4 = -(f,\zeta)_{\omega_n\setminus\omega} + (f,\zeta)_{\omega\setminus\omega_n}.
\end{aligned}$$

It is shown also similarly as in the proof of Lemma 8 together with a fact such that $p_n(z_n)$ weakly converges to U, where $z_n(x) = (u_n - u^{\Omega_n})/\kappa_n$ for $x \in \Omega_n$ and $z_n(x) = u_n(x)$ for $x \in \omega_n$. Finally we shall show (20). First we see

$$\begin{aligned}
&\mathcal{J}(\phi_n) - J(\Omega)\\
&\quad = (f,u_n)_{\Omega_n\setminus\Omega} - (f,u)_{\Omega\setminus\Omega_n} + (f,u_n-u)_{\Omega_n\cap\Omega}\\
&\qquad + \kappa_n(f,u_n)_{\omega_n} + \langle g,u_n-u\rangle_{\Gamma_N}.
\end{aligned} \tag{23}$$

The last two terms of the right hand side of (23) vanish respectively, because of the weak convergence of u_n to u in V, $g \in V'$ and (8). For the third term we have $|(f,u_n-u)_{\Omega_n\cap\Omega}| \le \|f\|_{0,D}\|u_n-u\|_{0,D} \to 0$, because of the Rellich theorem. The Sobolev imbedding theorem implies that $\|v\|_{L^2(G)} \le \bar{c}\|v\|_V|G|^{1/3}$, where G and $\bar{c}$ denote an open subset of $\mathbf{R}^d$ and a constant independent of $v \in V$, respectively. Thus, we have

$$\begin{aligned}
&|(f,u_n)_{\Omega_n\setminus\Omega} - (f,u)_{\Omega\setminus\Omega_n}|\\
&\quad \le \|f\|_{0,D}(\|u_n\|_{0,\Omega_n\setminus\Omega} + \|u\|_{0,\Omega\setminus\Omega_n})\\
&\quad \le \bar{c}\|f\|_{0,D}\max\{\|u_n\|_V,\|u\|_V\}|\Omega_n \ominus \Omega|^{1/3}.
\end{aligned}$$

The estimate (7) and $d(\Omega_n,\Omega) \to 0$ imply that the sum of the first two terms goes to zero.

(QED)

The uniqueness of solutions of $\mathbf{TOP}(D)$ is not known in general, we have to rely on subsequences of a minimizing sequence for cost as follows.

Theorem 14 *Let* $\{\kappa_n\}_n$ *be a sequence decreasing to zero. We assume that* $\phi_n = \chi_{\kappa_n}^{\Omega_n,\omega_n} \in \Phi_{\kappa_n}(k,r)$ *satisfies* $\mathcal{J}(\phi_n) = \inf_{\phi\in\Phi_{\kappa_n}(k,r)}\mathcal{J}(\phi)$, *where* u_n *and* $\mathcal{J}(\phi_n)$ *are given by* (5) *and* (6) *with* $\phi = \phi_n$, *respectively. Then we have* $\liminf_{n\to\infty}\mathcal{J}(\phi_n) = \inf_{\Omega\in\mathrm{Lip}(k,r)} J(\Omega)$. *Moreover we have* $\Omega^* \in \mathrm{Lip}(k,r)$ *and a subsequence* $\{\phi_{n_m}\}_m$ *of* $\{\phi_n\}_n$ *such that* $\liminf_{n\to\infty}\mathcal{J}(\phi_n) = \lim_{m\to\infty}\mathcal{J}(\phi_{n_m})$ *and* $d(\Omega_{n_m},\Omega^*) \to 0$, *where* Ω^* *is a solution of the problem* $\mathbf{TOP}(D)$.

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