JSIAM Letters Vol.5 (2013) pp.33–36 ©2013 Japan Society for Industrial and Applied Mathematics JSIAM Letters

Complete low-cut filter and the best constant of Sobolev inequality

Hiroyuki Yamagishi[1], Yoshinori Kametaka[2], Atsushi Nagai[3], Kohtaro Watanabe[4] and Kazuo Takemura[3]

[1] Tokyo Metropolitan College of Industrial Technology, 1-10-40 Higashi-ooi, Shinagawa, Tokyo 140-0011, Japan
[2] Osaka University, 1-3 Machikaneyama-cho, Toyonaka 560-8531, Japan
[3] Nihon University, 2-11-1 Shinei, Narashino 275-8576, Japan
[4] National Defense Academy, 1-10-20 Yokosuka 239-8686, Japan

E-mail *yamagisi@s.metro-cit.ac.jp*

Received September 5, 2012, Accepted November 28, 2012

Abstract
We obtained the best constants of Sobolev inequalities corresponding to complete low-cut filter. In the background, we have an n-dimensional boundary value problem and a one-dimensional periodic boundary value problem. The best constants of the corresponding Sobolev inequalities are equal to diagonal values of Green's functions for these boundary value problems.

Keywords Sobolev inequality, best constant, Green's function, Bessel function

Research Activity Group Applied Integrable Systems

1. The Sobolev inequality for a boundary value problem in n-dimensional Euclidean space

We consider the problem on the basis of a complete low-cut filter, which is a device that passes only high frequency.

We assume $M = 1, 2, \dots,\ n = 1, 2, \dots, 2M-1,\ 0 < A < \infty$, $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$, $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbf{R}^n$. We set the unitary inner product

$$\langle \xi, x\rangle = \sum_{j=1}^{n} \xi_j \overline{x}_j, \quad |\xi|^2 = \langle \xi, \xi\rangle.$$

We define Fourier transform as

$$u(x) \ \overset{\widehat{}}{\longrightarrow}\ \widehat{u}(\xi) = \int_{\mathbf{R}^n} e^{-\sqrt{-1}\,\langle \xi, x\rangle} u(x)\,dx,$$

where $dx = dx_1 dx_2 \cdots dx_n$. We introduce Sobolev space with low-cut frequency

$$H = \left\{ u \in W^{M,2} \mid \widehat{u}(\xi) = 0 \quad (|\xi| < A) \right\},$$

Sobolev inner product

$$(u, v)_H = \left(\frac{1}{2\pi}\right)^n \int_{|\xi| \ge A} |\xi|^{2M} \widehat{u}(\xi)\overline{\widehat{v}}(\xi)\,d\xi,$$

and Sobolev energy

$$\|u\|_H^2 = \left(\frac{1}{2\pi}\right)^n \int_{|\xi| \ge A} |\xi|^{2M} \left|\widehat{u}(\xi)\right|^2 d\xi.$$

$(\cdot, \cdot)_H$ is proved to be an inner product of H in Proof of Theorem 1. H is Hilbert space with inner product $(\cdot, \cdot)_H$. Our conclusion is as follows.

Theorem 1 *For any $u \in H$, there exists a positive constant C which is independent of u, such that a Sobolev inequality*

$$\left(\sup_{y \in \mathbf{R}^n} |u(y)|\right)^2 \le C\, \|u\|_H^2 \tag{1}$$

holds. Among such C, the best constant is

$$C_0 = G(0) = \frac{2}{(4\pi)^{\frac{n}{2}} \Gamma(\frac{n}{2})(2M-n)A^{2M-n}}. \tag{2}$$

If one relpaces C by C_0 in the above inequality (1), the equality holds for $u(x) = cG(x-y_0)$ with arbitrary $c \in \mathbf{C}$ and $y_0 \in \mathbf{R}^n$. Green's function $G(x, y) = G(x-y)$ is explained later in Lemma 1.

In the background of this theorem, we have the following n-dimensional boundary value problem. Concerning the uniqueness and existence of the solution to the boundary value problem, we have the following lemma.

Lemma 1 *For an arbitrary bounded continuous function $f(x)$ satisfying the solvability condition $\widehat{f}(\xi) = 0$ $(|\xi| < A)$, the boundary value problem*

$$\text{BVP} \quad \begin{cases} (-\Delta)^M u = f(x) & (x \in \mathbf{R}^n), \\ \widehat{u}(\xi) = 0 & (|\xi| < A) \end{cases}$$

has a unique solution

$$u(x) = \int_{\mathbf{R}^n} G(x, y) f(y) dy \quad (x \in \mathbf{R}^n). \tag{3}$$

$G(x, y) = G(x-y)$ $(x, y \in \mathbf{R}^n)$ is Green's function given

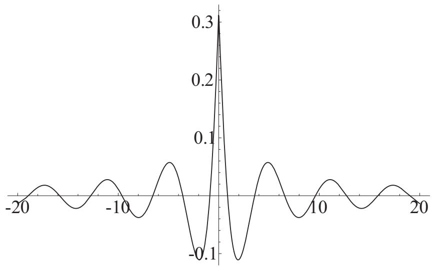

Fig. 1. $G(x)$ $(M = 1)$.

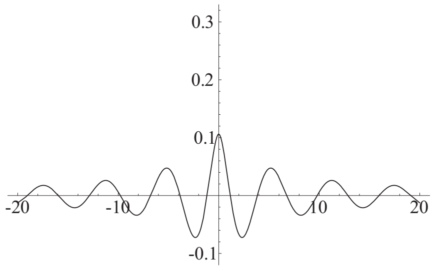

Fig. 2. $G(x)$ $(M = 2)$.

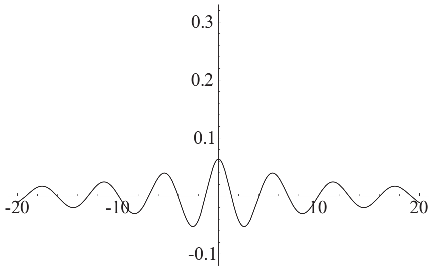

Fig. 3. $G(x)$ $(M = 3)$.

by

$$G(x) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_A^\infty r^{-(2M-n)-1} \left(|x|r\right)^{-\frac{n-2}{2}} \times J_{\frac{n-2}{2}}(|x|r)\, dr, \tag{4}$$

where $J_\nu(z)$ $(z \geq 0)$ *is the Bessel function. From the expansion of* $J_\nu(z)$ [1, p.145]*, we have*

$$G(x) = \frac{2}{(4\pi)^{\frac{n}{2}}} \sum_{j=0}^{\infty} \left[\frac{(-1)^j}{j!\Gamma(\frac{n}{2}+j)(2M-n-2j)A^{2M-n-2j}} \times \left(\frac{|x|}{2}\right)^{2j} \right]. \tag{5}$$

Figs. 1–3 illustrate graphs of $G(x)$ in $M = 1, 2, 3$, $n = 1$ and $A = 1$.

$$G(x) = \frac{1}{\pi} \int_1^\infty r^{-2M} \cos(|x|r)\, dr.$$

Proof of Lemma 1 Through Fourier transform, BVP is transformed into $|\xi|^{2M}\widehat{u}(\xi) = \widehat{f}(\xi)$ $(\xi \in \mathbf{R}^n)$. From $\widehat{f}(\xi) = 0$ and $\widehat{u}(\xi) = 0$ $(|\xi| < A)$, we have

$$\widehat{u}(\xi) = \widehat{G}(\xi)\widehat{f}(\xi) \quad (\xi \in \mathbf{R}^n),$$

$$\widehat{G}(\xi) = \begin{cases} |\xi|^{-2M} & (|\xi| \geq A), \\ 0 & (|\xi| < A). \end{cases}$$

Through inverse Fourier transform, we have (3) and

$$G(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^n} e^{\sqrt{-1}\langle x,\xi\rangle} \widehat{G}(\xi)\, d\xi \quad (x \in \mathbf{R}^n). \tag{6}$$

Let $T = (t_{ij})$ be an orthogonal matrix. We introduce a new variable $y \in \mathbf{R}^n$ by the relation $\xi = Ty$, or equivalently

$$\xi_i = \sum_{j=1}^n t_{ij} y_j \quad (1 \leq i \leq n).$$

It is easy to see that the corresponding Jacobian J is

$$J = \det\left(\frac{\partial \xi}{\partial y}\right) = \det T = \pm 1$$

and therefore $|J| = 1$. Here, we consider a special case ${}^tTx = |x|^t(1, 0, \ldots, 0)$. From (6), we have

$$\begin{aligned}
(2\pi)^n G(x) &= \int_{|\xi|\geq A} e^{\sqrt{-1}\,\langle x,\xi\rangle} |\xi|^{-2M}\, d\xi \\
&= \int_{|Ty|\geq A} e^{\sqrt{-1}\,\langle x,Ty\rangle} |Ty|^{-2M} |J|\, dy \\
&= \int_{|y|\geq A} e^{\sqrt{-1}\,\langle {}^tTx,y\rangle} |y|^{-2M}\, dy \\
&= \int_{|y|\geq A} e^{\sqrt{-1}\,|x|y_1} |y|^{-2M}\, dy.
\end{aligned}$$

For $|y| = r$, y is expressed as the following polar coordinates.

$$\begin{aligned}
y_1 &= r\cos(\theta_1), \\
y_2 &= r\sin(\theta_1)\cos(\theta_2), \\
y_3 &= r\sin(\theta_1)\sin(\theta_2)\cos(\theta_3), \\
&\vdots \\
y_{n-2} &= r\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{n-3})\cos(\theta_{n-2}), \\
y_{n-1} &= r\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{n-2})\cos(\varphi), \\
y_n &= r\sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{n-2})\sin(\varphi),
\end{aligned}$$

where $A < r < \infty$, $0 < \theta_1, \theta_2, \ldots, \theta_{n-2} < \pi$, $0 < \varphi < 2\pi$. Its Jacobian is

$$\frac{\partial(y_1, \ldots, y_n)}{\partial(r, \theta_1, \ldots, \theta_{n-2}, \varphi)} = r^{n-1}(\sin(\theta_1))^{n-2}(\sin(\theta_2))^{n-3}\cdots\sin(\theta_{n-2}).$$

Here, ω_n is surface area of n dimensional unit sphere as

$$\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

Green's function (6) is rewritten as follows:

$$\begin{aligned}
&(2\pi)^n G(x) \\
&= \int_A^\infty \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} e^{\sqrt{-1}\,|x|r\cos(\theta_1)} r^{-(2M-n)-1} (\sin(\theta_1))^{n-2} \\
&\qquad \times (\sin(\theta_2))^{n-3} \cdots \sin(\theta_{n-2})\, d\varphi d\theta_{n-2} \cdots d\theta_1 dr \\
&= \omega_{n-1} \int_A^\infty \int_0^\pi \cos\left(|x|r\cos(\theta_1)\right)
\end{aligned}$$

$$\times (\sin(\theta_1))^{n-2}\, d\theta_1 r^{-(2M-n)-1}\, dr,$$

where we use

$$\int_0^{\pi} \sin(|x|r\cos(\theta_1))\,(\sin(\theta_1))^{n-2} d\theta_1 = 0.$$

Using Lommel's formula [1, p.179], we have

$$(2\pi)^n G(x) = 2^{\frac{n-2}{2}} \pi^{\frac{1}{2}} \Gamma\left(\frac{n-1}{2}\right) \omega_{n-1} \times \int_A^{\infty} r^{-(2M-n)-1} (|x|r)^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(|x|r)\, dr.$$

From

$$\omega_{n-1} = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right)}\omega_n = \frac{2\,\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)},$$

so we have (4). Moreover, using expansion of Bessel function [1, p145], we have

$$G(x) = \frac{2}{2^n \pi^{\frac{n}{2}}} \times \int_A^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma\left(\frac{n}{2}+j\right)} \left(\frac{|x|r}{2}\right)^{2j} r^{-(2M-n)-1} dr.$$

From the assumption $2M > n$, we have (5). Taking the limit as $x \to 0$ of (5), we have (2). Thus we proved Lemma 1.

(QED)

We next show that Green's function $G(x,y)$ is simultaeously a reproducing kernel for a set of Hilbert space H and its inner product $(\cdot,\cdot)_H$.

Lemma 2 *For any $u \in H$ and fixed $y \in \mathbf{R}^n$, we have the following reproducing relations:*

$$u(y) = (u(x), G(x,y))_H, \tag{7}$$

$$G(0) = \|G(x,y)\|_H^2. \tag{8}$$

Proof of Lemma 2 Since Fourier transform of $G(x,y) = G(x-y)$ with respect to x is $e^{-\sqrt{-1}\,\langle \xi, y\rangle}\widehat{G}(\xi)$, the relation (7) is rewritten as

$$\begin{aligned}
&(u(x), G(x,y))_H \\
&= \left(\frac{1}{2\pi}\right)^n \int_{|\xi|\ge A} |\xi|^{2M} \widehat{u}(\xi) \overline{e^{-\sqrt{-1}\,\langle \xi, y\rangle}\widehat{G}(\xi)}\, d\xi \\
&= \left(\frac{1}{2\pi}\right)^n \int_{|\xi|\ge A} e^{\sqrt{-1}\,\langle y, \xi\rangle}\widehat{u}(\xi)\, d\xi = u(y).
\end{aligned}$$

(8) is shown by putting $u(x) = G(x,y)$ in (7). This completes the proof of Lemma 2.

(QED)

Finally, we prove Theorem 1.

Proof of Theorem 1 Applying Schwarz inequality to (7) and using (8), we have

$$|u(y)|^2 \le \|G(x,y)\|_H^2\, \|u\|_H^2 = G(0)\, \|u\|_H^2\,.$$

Taking the supremum with respect to $y \in \mathbf{R}^n$, we have

$$\left(\sup_{y\in\mathbf{R}^n} |u(y)|\right)^2 \le G(0)\, \|ux\|_H^2\,. \tag{9}$$

This inequality shows that the inner product $\|u\|_H^2 = (u,u)_H$ is positive definite. For any fixed $y_0 \in \mathbf{R}^n$, if we take $u(x) = G(x-y_0) \in H$ in the above inequality, then we have

$$\left(\sup_{y\in\mathbf{R}^n} |G(y-y_0)|\right)^2 \le G(0)\, \|G(x-y_0)\|_H^2 = G(0)^2.$$

Together with a trivial inequality

$$G(0)^2 \le \left(\sup_{y\in\mathbf{R}^n} |G(y-y_0)|\right)^2,$$

we have

$$\left(\sup_{y\in\mathbf{R}^n} |G(y-y_0)|\right)^2 = G(0)\, \|G(x-y_0)\|_H^2\,.$$

This shows that $G(0)$ is the best constant of (9) and the equality holds for $G(x-y_0)$. This completes the proof of Theorem 1.

(QED)

For (6), we have

$$|G(x)| \le \left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^n} \left|\widehat{G}(\xi)\right| d\xi = G(0) \quad (x \in \mathbf{R}^n).$$

So we see that the maximum of $G(x)$ is $G(0)$.

2. The Sobolev inequality under a periodic boundary condition

We here consider a one-dimensional case. For $M, N = 1, 2, \dots$ and $x \in \mathbf{R}$, we introduce the function

$$\varphi(j,x) = e^{\sqrt{-1}\, a_j x},$$
$$a_j = 2\pi j \quad (j = 0, \pm 1, \pm 2, \dots).$$

We define Fourier transform as

$$u(x) \ \stackrel{\widehat{}}{\longrightarrow}\ \widehat{u}(j) = \int_0^1 u(x)\overline{\varphi}(j,x)\, dx.$$

We introduce Sobolev space with periodic boundary condition and low-cut frequency

$$\begin{aligned}
H = \Big\{ u \,\Big|\, & u^{(M)} \in L^2(0,1), \\
& u^{(i)}(1) - u^{(i)}(0) = 0 \quad (0 \le i \le M-1), \\
& \widehat{u}(j) = 0 \quad (|j| < N) \Big\},
\end{aligned}$$

Sobolev inner product

$$(u,v)_H = \sum_{|j|\ge N} a_j^{2M} \widehat{u}(j)\overline{\widehat{v}}(j),$$

and Sobolev energy

$$\|u\|_H^2 = \sum_{|j|\ge N} a_j^{2M}\, |\,\widehat{u}(j)|^2\,.$$

$(\cdot,\cdot)_H$ is proved to be an inner product of H in Proof of Theorem 2. H is Hilbert space with inner product $(\cdot,\cdot)_H$. Our conclusion is as follows.

Theorem 2 *For any $u \in H$, there exists a positive constant C which is independent of u, such that a Sobolev inequality*

$$\left(\sup_{0\le y\le 1} |u(y)|\right)^2 \le C\|u\|_H^2 \tag{10}$$

holds. Among such C, the best constant is

$$C_0 = G(0) = \begin{cases} \dfrac{2}{(2\pi)^{2M}}\zeta(2M) & (N=1), \\ \dfrac{2}{(2\pi)^{2M}}\left(\zeta(2M) - \displaystyle\sum_{j=1}^{N-1}\frac{1}{j^{2M}}\right) & (N=2,3,\dots), \end{cases}$$

where $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ $(\mathrm{Re}z > 1)$ *is the well-known Riemann-zeta function. If one relpaces C by C_0 in the above inequality* (10)*, the equality holds for* $u(x) = c\,G(x-y_0)$ *with arbitrary* $c \in \mathbf{C}$ *and* $y_0 \in \mathbf{R}$*. Green's function* $G(x)$ *is explained in Lemma 3.*

It should be noted that if we put $N = 1$ in the above theorem, we have Theorem 2 in our previous work [2].

In the background of this theorem, we have the following one-dimensional periodic boundary value problem. Concerning the uniqueness and existence of the solution to the boundary value problem, we have the following lemma.

Lemma 3 *For an arbitrary bounded continuous function* $f(x)$ *satisfying the solvability condition* $\widehat{f}(j) = 0$ $(|j| < N)$*, the boundary value problem*

BVP
$$\begin{cases} (-1)^M u^{(2M)} = f(x) & (0 < x < 1), \\ u^{(i)}(1) - u^{(i)}(0) = 0 & (0 \le i \le 2M-1), \\ \widehat{u}(j) = 0 & (|j| < N) \end{cases}$$

has a unique solution

$$u(x) = \int_0^1 G(x,y) f(y)\,dy \quad (0 < x < 1). \tag{11}$$

$G(x,y) = G(x-y)$ $(0 < x,y < 1)$ *is Green's function given by*

$$G(x) = 2\sum_{j=N}^{\infty} a_j^{-2M}\cos(a_j x). \tag{12}$$

Proof of Lemma 3 Through Fourier transform as

$$\sum_{j\in\mathbf{Z}} \widehat{f}(j)\varphi(j,x) = f(x) = (-1)^M u^{(2M)} = (-1)^M \sum_{j\in\mathbf{Z}} \widehat{u}(j)\varphi^{(2M)}(j,x) = \sum_{j\in\mathbf{Z}} a_j^{2M}\widehat{u}(j)\varphi(j,x),$$

BVP is transformed into

$$a_j^{2M}\widehat{u}(j) = \widehat{f}(j) \quad (j \in \mathbf{Z}).$$

From $\widehat{f}(j) = 0$ and $\widehat{u}(j) = 0$ $(|j| < N)$, we have

$$\widehat{u}(j) = \widehat{G}(j)\widehat{f}(j) \quad (j \in \mathbf{Z}),$$

$$\widehat{G}(j) = \begin{cases} a_j^{-2M} & (|j| \ge N), \\ 0 & (|j| < N). \end{cases}$$

Through inverse Fourier transform, we have (11) and (12) as

$$G(x) = \sum_{j=-\infty}^{\infty} \widehat{G}(j)\varphi(j,x) = \sum_{|j|\ge N} a_j^{-2M}\varphi(j,x) = 2\sum_{j=N}^{\infty} a_j^{-2M}\cos(a_j x).$$

We see that the maximum of $G(x)$ is $G(0)$. This completes the proof.

(QED)

We next show that Green's function $G(x,y)$ is simultaeously a reproducing kernel for a set of Hilbert space H and its inner product $(\cdot,\cdot)_H$.

Lemma 4 *For any* $u \in H$ *and fixed* y $(0 \le y \le 1)$*, we have the following reproducing relations.*

$$u(y) = (u(x), G(x,y))_H, \tag{13}$$

$$G(0) = \|G(x,y)\|_H^2. \tag{14}$$

Proof of Lemma 4 Fourier transform of $G(x,y) = G(x-y)$ wiht respect to x is $\overline{\varphi}(j,y)\widehat{G}(j)$. Hence, for any $u \in H$, we have (13) as

$$(u(x), G(x,y))_H = \sum_{|j|\ge N} a_j^{2M}\widehat{u}(j)\,\overline{\overline{\varphi}(j,y)\widehat{G}(j)} = \sum_{|j|\ge N} a_j^{2M}\widehat{G}(j)\widehat{u}(j)\varphi(j,y) = \sum_{|j|\ge N}\widehat{u}(j)\varphi(j,y) = u(y).$$

(14) is shown by putting $u(x) = G(x,y)$ in (13). This completes the proof of Lemma 4.

(QED)

Finally, we prove Theorem 2.

Proof of Theorem 2 Applying Schwarz inequality to (13) and using (14), we have

$$|u(y)|^2 \le \|G(x,y)\|_H^2\,\|u\|_H^2 = G(0)\,\|u\|_H^2.$$

Taking the supremum with respect to y $(0 \le y \le 1)$, we have

$$\left(\sup_{0\le y\le 1}|u(y)|\right)^2 \le G(0)\,\|u\|_H^2. \tag{15}$$

This inequality shows that the inner product $\|u\|_H^2 = (u,\ u)_H$ is positive definite. For any fixed y_0 $(0 \le y_0 \le 1)$, if we take $u(x) = G(x-y_0) \in H$ in the above inequality, then we have

$$\left(\sup_{0\le y\le 1}|G(y-y_0)|\right)^2 \le G(0)\,\|G(x-y_0)\|_H^2 = G(0)^2.$$

Combining this and a trivial inequality

$$G(0)^2 \le \left(\sup_{0\le y\le 1}|G(y-y_0)|\right)^2,$$

we have

$$\left(\sup_{0\le y\le 1}|G(y-y_0)|\right)^2 = G(0)\,\|G(x-y_0)\|_H^2.$$

This shows that $G(0)$ is the best constant of (15) and the equality holds for $G(x-y_0)$. This completes the proof of Theorem 2.

(QED)

References

[1] S. Moriguchi, K. Udagawa and S. Hitotsumatsu, Iwanami Sugaku Koshiki III (in Japanese), Iwanami, Tokyo, 1960.

[2] Y. Kametaka et al., Riemann zeta function, Bernoulli polynomials and the best constant of Sobolev inequality, Sci. Math. Jpn., **e-2007** (2007), 63–89.