JSIAM Letters Vol.6 (2014) pp.41–44 ©2014 Japan Society for Industrial and Applied Mathematics

Some examples of multidimensional Shintani zeta distributions

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Received January 22, 2014, Accepted January 29, 2014

Abstract

In the studies of mathematical statistics, we often consider discrete distributions and their corresponding stochastic processes. Especially, probabilistic limit theorems of them may give us some progress in mathematical finance. There exist not so many properties of discrete distributions on $\mathbb{R}^d$. In this paper, we treat multiple zeta functions as to define several forms of discrete distributions on $\mathbb{R}^d$ including those with infinitely many mass points. Our purpose is to obtain new methods in the relations between multiple infinite series and high dimensional integral calculus, which can provide us more opportunities to handle high dimensional phenomenon.

Keywords Lévy measure, multinomial distribution, zeta function

Research Activity Group Mathematical Finance

1. Infinitely divisible distributions

Infinitely divisible distributions are known as one of the most important class of distributions in probability theory. They are the marginal distributions of stochastic processes having independent and stationary increments such as Brownian motion and Poisson processes. In 1930's, such stochastic processes were well-studied by P. Lévy and now we usually call them Lévy processes. They often appear in mathematical finance as standard stochastic processes. We can find the detail of Lévy processes in Sato [1].

In this section, we mention some known properties of infinitely divisible distributions.

Definition 1 (Infinitely divisible distribution) *A probability measure μ on $\mathbb{R}^d$ is infinitely divisible if, for any positive integer n, there is a probability measure μ_n on $\mathbb{R}^d$ such that*

$$\mu = \mu_n^{n*},$$

where μ_n^{n*} is the n-fold convolution of μ_n.

Example 2 *Normal, degenerate and Poisson distributions are infinitely divisible.*

Denote by $I(\mathbb{R}^d)$ the class of all infinitely divisible distributions on $\mathbb{R}^d$. Let $\widehat{\mu}(\vec{t}) := \int_{\mathbb{R}^d} e^{\mathrm{i}\langle \vec{t},x\rangle}\mu(dx)$, $\vec{t} \in \mathbb{R}^d$, be the characteristic function of a distribution μ, where $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^d$. We also write $a \wedge b = \min\{a,b\}$.

The following is well-known.

Proposition 3 (Lévy–Khintchine representation (see, e.g. Sato [1])) *(i) If $\mu \in I(\mathbb{R}^d)$, then*

$$\widehat{\mu}(\vec{t}) = \exp\biggl(-\frac{1}{2}\langle \vec{t}, A\vec{t}\rangle + \mathrm{i}\langle \gamma, \vec{t}\rangle + \int_{\mathbb{R}^d}\left(e^{\mathrm{i}\langle \vec{t},x\rangle} - 1 - \frac{\mathrm{i}\langle \vec{t},x\rangle}{1+|x|^2}\right)\nu(dx)\biggr), \quad \vec{t} \in \mathbb{R}^d, \tag{1.1}$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, ν is a measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d}\left(|x|^2 \wedge 1\right)\nu(dx) < \infty, \tag{1.2}$$

and $\gamma \in \mathbb{R}^d$.

(ii) The representation of $\widehat{\mu}$ in (i) by A, ν, and γ is unique.

(iii) Conversely, if A is a symmetric nonnegative-definite $d \times d$ matrix, ν is a measure satisfying (1.2)*, and $\gamma \in \mathbb{R}^d$, then there exists an infinitely divisible distribution μ whose characteristic function is given by* (1.1).

The measure ν is called the Lévy measure and it generates a jump type Lévy process. The following is also known as one of the most important classes of infinitely divisible distributions.

Definition 4 (Compound Poisson distribution) *A distribution μ on $\mathbb{R}^d$ is called compound Poisson if, for some $c > 0$ and some probability measure ρ on $\mathbb{R}^d$ with $\rho(\{0\}) = 0$,*

$$\widehat{\mu}(\vec{t}) = \exp\left(c[\widehat{\rho}(\vec{t}) - 1]\right), \quad \vec{t} \in \mathbb{R}^d.$$

Here the measure ρ is the Lévy measure of the compound Poisson distribution μ and is finite. The Poisson distribution is a special case where $d = 1$ and $\rho = \delta_1$, where δ_x is a delta measure at x.

Remark 5 *We have to note that any infinitely divisible distribution can be expressed as the weak limit of a certain sequence of compound Poisson distributions.*

2. Zeta distributions

In one dimensional case, there exists a class of discrete distribution generated by the Riemann zeta function. Our research is focused on this class and expanded to obtain several exact expressions of discrete multidimensional distributions with Lévy measures if they have. We rarely see that both of the discrete distributions and Lévy measures on $\mathbb{R}^d$ are computable in mathematical statistics as well as such relations between multiple series and high dimensional measure theories even in pure mathematics.

First, we introduce the Riemann zeta function and distribution. We can find the basic properties of zeta functions in Apostol [2].

Definition 6 (Riemann zeta function) *The Riemann zeta function is a function of a complex variable $s = \sigma + \mathrm{i}t \in \mathbb{C}$, for $\sigma > 1$, $t \in \mathbb{R}$ given by*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The Riemann zeta function converges absolutely in the region $\sigma > 1$. In this region of absolutely convergence, we have the following well-known distribution on $\mathbb{R}$.

Definition 7 (Riemann zeta distribution) *For each $\sigma > 1$, a probability measure μ_σ on $\mathbb{R}$ is called a Riemann zeta distribution, if*

$$\mu_\sigma(\{-\log n\}) = \frac{n^{-\sigma}}{\zeta(\sigma)}, \quad n \in \mathbb{N}.$$

Then its characteristic function f_σ can be written as follows:

$$f_\sigma(t) = \int_{\mathbb{R}} e^{\mathrm{i}tx} \mu_\sigma(dx) = \frac{\zeta(\sigma + \mathrm{i}t)}{\zeta(\sigma)}, \quad t \in \mathbb{R}.$$

This class of distribution is first introduced in Jessen and Wintner [3] without normalization for an example in the studies of infinitely many times convolutions. As a probability distribution, it is first appeared in Khintchine [4].

Proposition 8 (See, e.g. Gnedenko and Kolmogorov [5]) *The characteristic function $f_\sigma(t)$ is a compound Poisson with a finite Lévy measure N_σ on $\mathbb{R}$:*

$$\log f_\sigma(t) = \int_0^\infty (\exp(-\mathrm{i}tx) - 1) N_\sigma(dx),$$

where

$$N_\sigma(dx) = \sum_{p \in \mathbb{P}} \sum_{r=1}^{\infty} \frac{p^{-r\sigma}}{r} \delta_{r \log p}(dx).$$

This proposition shows that the Riemann zeta function is treatable in the theory of Lévy processes as some other well-known functions. Further properties of this class is also studied in Hu and Lin [6].

The Riemann zeta function is variously extended such as Hurwitz or Barnes types. (See, e.g. Apostol [2] in detail.) Also, several generalized zeta distributions are introduced but most of them are not infinitely divisible. The cases having the infinite divisibility are the following.

A special case of Hurwitz zeta function generates a compound Poisson distribution on $\mathbb{R}$ which is given in Hu and Lin [7].

Other cases are given by using multivariable zeta functions and their corresponding distributions are on $\mathbb{R}^d$. Aoyama and Nakamura [8] introduced multidimensional Shintani zeta functions which are generalized to be multivariable and multiple infinite series as in the following.

Definition 9 (Multidimensional Shintani zeta function (Aoyama and Nakamura [8])) *Let $d, m, r \in \mathbb{N}$, $\vec{s} \in \mathbb{C}^d$ and $(n_1, \dots, n_r) \in \mathbb{Z}^r_{\geq 0}$. For λ_{lj}, $u_j > 0$, $\vec{c}_l \in \mathbb{R}^d$, where $1 \leq j \leq r$ and $1 \leq l \leq m$, and a function $\theta(n_1, \dots, n_r) \in \mathbb{C}$ satisfying $|\theta(n_1, \dots, n_r)| = O((n_1 + \cdots + n_r)^\varepsilon)$, for any $\varepsilon > 0$, we define a multidimensional Shintani zeta function $Z_S(\vec{s})$ given by*

$$\sum_{n_1, \dots, n_r = 0}^{\infty} \frac{\theta(n_1, \dots, n_r)}{\prod_{l=1}^{m} (\sum_{j=1}^{r} \lambda_{lj}(n_j + u_j))^{\langle \vec{c}_l, \vec{s} \rangle}}.$$

We call the function $\theta(n_1, \dots, n_r)$ a generalized Dirichlet character of the multidimensional Shintani zeta function and write $\langle \vec{c}, \vec{s} \rangle := \langle \vec{c}, \vec{\sigma} \rangle + \mathrm{i}\langle \vec{c}, \vec{t} \rangle$ for $\vec{c} \in \mathbb{R}^d$ and $\vec{s} \in \mathbb{C}^d$, where $\vec{\sigma}, \vec{t} \in \mathbb{R}^d$ and $\vec{s} = \vec{\sigma} + \mathrm{i}\vec{t}$. The series $Z_S(\vec{s})$ converges absolutely in the region $\min_{1 \leq l \leq m} \langle \vec{c}_l, \vec{\sigma} \rangle > r/m$ (see, Aoyama and Nakamura [8]), which we denote by $\mathrm{D_S}$. Suppose that $\theta(n_1, \dots, n_r)$ is non-negative or non-positive definite, then we can define the following class of distribution on $\mathbb{R}^d$.

Definition 10 (Multidimensional Shintani zeta distribution (Aoyama and Nakamura [8])) *For each $\vec{\sigma} \in \mathrm{D_S}$, a probability measure $\mu_{\vec{\sigma}}$ on $\mathbb{R}^d$ is called a multidimensional Shintani zeta distribution, if for all $(n_1, \dots, n_r) \in \mathbb{Z}^r_{\geq 0}$,*

$$\mu_{\vec{\sigma}}\left(\left\{-\sum_{l=1}^{m} c_{l1} \log\left(\sum_{k=1}^{r} \lambda_{lk}(n_k + u_k)\right), \dots, -\sum_{l=1}^{m} c_{ld} \log\left(\sum_{k=1}^{r} \lambda_{lk}(n_k + u_k)\right)\right\}\right) = \frac{\theta(n_1, \dots, n_r)}{Z_S(\vec{\sigma})} \prod_{l=1}^{m} \left(\sum_{k=1}^{r} \lambda_{lk}(n_k + u_k)\right)^{-\langle \vec{c}_l, \vec{\sigma} \rangle}.$$

Then its characteristic function $f_{\vec{\sigma}}$ is given by

$$f_{\vec{\sigma}}(\vec{t}) := \int_{\mathbb{R}^d} e^{\mathrm{i}\langle \vec{t}, x \rangle} \mu_{\vec{\sigma}}(dx) = \frac{Z_S(\vec{\sigma} + \mathrm{i}\vec{t})}{Z_S(\vec{\sigma})}, \quad \vec{t} \in \mathbb{R}^d,$$

which can be regarded as a generalization of the Riemann zeta distribution.

Remark 11 *This class contains both infinitely divisible and non infinitely divisible distributions on $\mathbb{R}^d$. By applying Euler products, some simple examples of compound Poisson case on $\mathbb{R}^2$ and generalized cases on $\mathbb{R}^d$ are given in [9] and [10], respectively.*

3. Relation between distributions and characters

Many kinds of discrete distributions can be represented in the sense of multidimensional Shintani zeta functions by choosing suitable characters, and so their characteristic functions can be written by multiple infinite series. In this section, we pick up the multinomial distribution which is well-known as a multidimensional discrete one, and show the relation with the character.

3.1 Main result 1 (A character corresponding to a multinomial distribution)

Fix $N \in \mathbb{N}$. For each $1 \le l, k \le m$, let $u_l = 1$, $\lambda_{ll} = 1$, $\lambda_{lk} = 0$, $(l \ne k)$. We also take $\vec{\sigma} \in \mathrm{D_S}, \vec{c}_l = (c_{lj})_{j=1}^d \in \mathbb{R}^d$, $\phi(l) \in \mathbb{R}$ and $j(1), \ldots, j(m) \in \mathbb{N} \setminus \{1\}$ with relatively prime each other. Define a character by

$$\theta_N(n_1, \ldots, n_m) := \begin{cases} N\,! \prod_{l=1}^m \dfrac{(\phi(l))^{k_l}}{k_l\,!} \\ \left(n_l + 1 = (j(l))^{k_l}, \quad \sum_{l=1}^m k_l = N \right), \\ 0 \quad \text{(otherwise)}. \end{cases}$$

Then, for each $\vec{s} = \vec{\sigma} + \mathrm{i}\vec{t} \in \mathbb{C}^d$, $\vec{t} \in \mathbb{R}^d$,

$$Z_S(\vec{s}) = \sum_{k_1 + \cdots + k_m = N} N\,! \prod_{l=1}^m \frac{(\phi(l))^{k_l} (j(l)^{-\langle \vec{c}_l, \vec{s} \rangle})^{k_l}}{k_l\,!} = \left(\sum_{l=1}^m \phi(l)(j(l))^{-\langle \vec{c}_l, \vec{s} \rangle} \right)^N,$$

$$f_{\vec{\sigma}}(\vec{t}) := \frac{Z_S(\vec{\sigma} + \mathrm{i}\vec{t})}{Z_S(\vec{\sigma})} = \left(\sum_{l=1}^m q(l) e^{\mathrm{i}\langle \vec{x}_l, \vec{t} \rangle} \right)^N,$$

where

$$q(l) := \frac{\phi(l)(j(l))^{-\langle \vec{c}_l, \vec{\sigma} \rangle}}{\sum_{l=1}^m \phi(l)(j(l))^{-\langle \vec{c}_l, \vec{\sigma} \rangle}},$$

$$\vec{x}_l := (x_{lk})_{k=1}^d, \quad x_{lk} := -c_{lk} \log j(l).$$

If $\phi(1), \ldots, \phi(m)$ have the same sign, then the character θ_N is non-negative or non-positive definite, and so that $f_{\vec{\sigma}}$ is a characteristic function of a Shintani zeta distribution. That is, it is the characteristic function of a random variable $X_{\vec{\sigma}}$ defined by

$$\Pr\left(X_{\vec{\sigma}} = \left(\sum_{l=1}^m x_{l1} n_l, \ldots, \sum_{l=1}^m x_{ld} n_l \right) \right) = N\,! \prod_{l=1}^m \frac{(q(l))^{n_l}}{n_l!}, \quad \text{when} \sum_{l=1}^m n_l = N.$$

Especially, if $m = d$ and $\vec{x}_1, \ldots, \vec{x}_d$ are the standard basis of $\mathbb{R}^d$, then $X_{\vec{\sigma}}$ belongs to a multinomial distribution.

Since multinomial distributions are the distributions which have densities at most finitely many points, their characteristic functions are also multiple finite sum. However, multidimensional Shintani zeta distributions whose characteristic functions defined by multiple infinite series may have densities at countably many points. In the following, we give an example of them and mention whether it is infinitely divisible or not.

3.2 Main result 2 (A character corresponding to a compound Poisson distribution on $\mathbb{R}^d$)

We use the settings in Main result 1. For any non-negative integer valued random variable T, define a character

$$\theta_T(n_1, \ldots, n_m) := \sum_{N=0}^\infty Pr(T = N) \frac{\theta_N(n_1, \ldots, n_m)}{(\sum_{l=1}^m \phi(l)(j(l))^{-\langle \vec{c}_l, \vec{\sigma} \rangle})^N}.$$

Then the characteristic function $F_{\vec{\sigma},T}$ of a multidimensional Shintani zeta distribution with a character θ_T has the form of

$$\begin{aligned} F_{\vec{\sigma},T}(\vec{t}) &= \sum_{N=0}^\infty \frac{Pr(T = N)}{(\sum_{l=1}^m \phi(l)(j(l))^{-\langle \vec{c}_l, \vec{\sigma} \rangle})^N} \\ &\quad \times \sum_{n_1, \ldots, n_m = 0}^\infty \frac{\theta_N(n_1, \ldots, n_m)}{\prod_{l=1}^m (\sum_{k=1}^m (\lambda_{lk}(n_k + u_k))^{\langle \vec{c}_l, \vec{\sigma} + i\vec{t} \rangle}} \\ &= \sum_{N=0}^\infty \frac{Pr(T = N)}{(\sum_{l=1}^m \phi(l)(j(l))^{-\langle \vec{c}_l, \vec{\sigma} \rangle})^N} \\ &\quad \times \left(\sum_{l=1}^m \phi(l)(j(l))^{-\langle \vec{c}_l, \vec{\sigma} + i\vec{t} \rangle} \right)^N \\ &= \sum_{N=0}^\infty Pr(T = N) \left(\sum_{l=1}^m q(l)\, e^{\mathrm{i}\langle \vec{x}_l, \vec{t} \rangle} \right)^N, \quad \vec{t} \in \mathbb{R}^d. \end{aligned}$$

Especially, if T belongs to a Poisson distribution with mean λ, then

$$\begin{aligned} F_{\vec{\sigma}}(\vec{t}) &= \sum_{N=0}^\infty \frac{\lambda^N}{N\,!} e^{-\lambda} \left(\sum_{l=1}^m q(l)\, e^{\mathrm{i}\langle \vec{x}_l, \vec{t} \rangle} \right)^N \\ &= \exp\left(\lambda \left(\sum_{l=1}^m q(l)\, e^{\mathrm{i}\langle \vec{x}_l, \vec{t} \rangle} - 1 \right) \right), \quad \vec{t} \in \mathbb{R}^d. \end{aligned}$$

This is the characteristic function of a compound Poisson distribution with a finite Lévy measure $N_{\vec{\sigma}}$ on $\mathbb{R}^d$ given by

$$N_{\vec{\sigma}}(dx) = \lambda \sum_{l=1}^m q(l) \delta_{\vec{x}_l}(dx).$$

Remark 12 *In mathematical finance, models caused by some one-dimensional Lévy process are studied. Now we have discrete infinitely divisible distributions on $\mathbb{R}^d$ with finite Lévy measures, which make us possible to simulate some models associating with $\mathbb{R}^d$-valued Lévy processes.*

4. Conditions to be characteristic functions

Non-negative or non-positive definiteness of characters are not necessary conditions for distributions to be defined by multidimensional Shintani zeta functions. Therefore, now we consider the case when they are not non-negative nor non-positive definite. We have the following lemma which holds under the settings in Main result 1 and 2.

Lemma 13 *Suppose that $\mathbb{R}^d$-valued vectors $\vec{c}_1, \ldots, \vec{c}_m$ are linearly independent over $\mathbb{R}$ or $\vec{c}_1 = \cdots = \vec{c}_m$ $(\ne 0)$. If $\phi(1), \ldots, \phi(m)$ do not have the same sign, then there* *exist $\mathbb{R}^d$-valued vectors $\vec{t}_1$, $\vec{t}_2$ such that*

$$|f_{\vec{\sigma}}(\vec{t}_1)| > 1, \quad |F_{\vec{\sigma}}(\vec{t}_2)| > 1.$$

It is known that characteristic functions $\widehat{\mu}$ of any probability measures μ on $\mathbb{R}^d$ satisfies $|\widehat{\mu}(\vec{t})| \le 1$, $\vec{t} \in \mathbb{R}^d$. Hence, if normalized functions $f_{\vec{\sigma}}(\vec{t})$ and $F_{\vec{\sigma}}(\vec{t})$ have vectors $\vec{t}_1$, $\vec{t}_2$ such that $|f_{\vec{\sigma}}(\vec{t}_1)| > 1$, $|F_{\vec{\sigma}}(\vec{t}_2)| > 1$ holds, then they can not be characteristic functions. Now we have the following result.

Theorem 14 (A necessary and sufficient condition to be characteristic functions) *Suppose that $\mathbb{R}^d$-valued vectors $\vec{c}_1, \dots, \vec{c}_m$ are linearly independent over $\mathbb{R}$ or $\vec{c}_1 = \dots = \vec{c}_m$ $(\neq 0)$. Then, $f_{\vec{\sigma}}$, $F_{\vec{\sigma}}$ are characteristic functions if and only if $\phi(1), \dots, \phi(m)$ have the same sign.*

We give the proof of Lemma 13 of the case of $f_{\vec{\sigma}}$ when $\vec{c}_1 = \dots = \vec{c}_m$ $(\neq 0)$ as in the same way as in Aoyama and Nakamura [9, 10]. The following proposition plays a key role in its proof.

Proposition 15 (Kronecker's approximation theorem (see, e.g. [11])) *If $r_1, \dots, r_n$ are arbitrary real numbers, if real numbers $\theta_1, \dots, \theta_n$ are linearly independent over the rationals, and if $\epsilon > 0$ is arbitrary, then there exist a real number t and integers $h_1, \dots, h_n$ such that*

$$|t\theta_k - h_k - r_k| < \epsilon, \quad 1 \le k \le n.$$

Proof of Lemma 13 Since $\phi(1), \dots, \phi(m)$ do not have the same sign, there exists l_0 such that $q(l_0) < 0$. Put

$$L := \sum_{l \neq l_0} q(l) - q(l_0) > \sum_{l=1}^{m} q(l) = 1,$$

and take $n_0 \in \mathbb{N}$ and $\epsilon > 0$ such that $L - \epsilon > 1$.

Define

$$\theta_l = \frac{\log j(l)}{2\pi} \quad (1 \le l \le m).$$

Then, $\theta_1, \dots, \theta_m$ are linearly independent over the rationals. Therefore, the Kronecker's approximation theorem shows that there exists $T_0 \in \mathbb{R}$ such that

$$|e^{\mathrm{i}2\pi T_0 \theta_{l_0}} + 1| < \frac{\epsilon}{(\sum_{l=1}^{m} |q(l)|)},$$
$$|e^{\mathrm{i}2\pi T_0 \theta_{l}} - 1| < \frac{\epsilon}{(\sum_{l=1}^{m} |q(l)|)} \quad (l \neq l_0).$$

Thus, we have

$$\left| \sum_{l=1}^{m} e^{\mathrm{i}2\pi T_0 \theta_l} - L \right| \le \sum_{l \neq l_0} |q(l)||e^{\mathrm{i}2\pi T_0 \theta_l} - 1| + |q(l_0)||e^{\mathrm{i}2\pi T_0 \theta_{l_0}} + 1| < \epsilon$$

and that is

$$\left| Re \left(\sum_{l=1}^{m} q(l) e^{\mathrm{i}2\pi T_0 \theta_l} - L \right) \right| \le \left| \left(\sum_{l=1}^{m} q(l) e^{\mathrm{i}2\pi T_0 \theta_l} - L \right) \right| < \epsilon.$$

Take $\vec{t}_1 \in \mathbb{R}^d$ such that $T_0 = \langle \vec{c}_1, \vec{t}_1 \rangle$. Then

$$Re \left(\sum_{l=1}^{m} q(l) e^{\mathrm{i}\langle \vec{x}_l, \vec{t}_1 \rangle} \right) = Re \left(\sum_{l=1}^{m} q(l) e^{\mathrm{i}2\pi T_0 \theta_l} \right) > L - \epsilon > 1.$$

Hence

$$|f_{\vec{\sigma}}(\vec{t}_1)| = \left| \sum_{l=1}^{m} q(l) e^{\mathrm{i}\langle \vec{x}_l, \vec{t}_1 \rangle} \right|^N > 1.$$

(QED)

The rest of the proofs and further results are given in Aoyama and Yoshikawa [12].

By following our story, we can see that the characters seem to be an important key of multidimensional Shintani zeta functions in view of defining distributions. We still need new facts and methods of them as to make things in stochastic models clearer and more useful.

Acknowledgments

The authors would like to express sincere appreciations to Professor Jirô Akahori for his valuable comments. This work was partially supported by JSPS KAKENHI Grant Numbers 23330190, 24340022, 23654056 and 25285102.

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