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Some results on Parisian walks

Jirô Akahori[1] and Yuuki Ida[1]

[1] Department of Mathematical Sciences, Ritsumeikan University, 1-1-1 Nojihigashi, Kusatsu, Shiga 525-8577, Japan

E-mail *akahori@se.ritsumei.ac.jp*

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Abstract

In the present paper, we introduce a framework of a discrete stochastic calculus based on *Parisian walk*, a special kind of symmetric random walk in the complex plane, listing some results analogue to those for complex Brownian motion. We also discuss, as an application to mathematical finance, a Parisian-walk analogue of Heston's stochastic volatility model.

Keywords Parisian walk, discrete stochastic calculus, conformal martingale, Heston model

Research Activity Group Mathematical Finance

1. Introduction

Let $\tau_1, \ldots, \tau_n, \ldots$ be an i.i.d. sequence with $\mathsf{P}(\tau = 1) = \mathsf{P}(\tau = \zeta) = \mathsf{P}(\tau = \zeta^2) = 1/3$, where we denote $\zeta = (-1+\sqrt{-3})/2$. The filtration generated by $\{\tau\}$ will be denoted by $\mathbf{F} \equiv \{\mathcal{F}_t\}$.

Definition 1 *An* $\mathbf{F}$*-adapted complex valued process* $\{Z_t\}$ *is called* **Parisian** (*walk*) *if* (i) *it is a martingale starting from a point in* $\mathbb{Z}[\zeta] \equiv \{a + b\zeta : a, b \in \mathbb{Z}\}$, *and* (ii) $Z_{t+1} - Z_t =: \Delta Z_t \in \{1, \zeta, \zeta^2\}$ *for all* t.

Thus, a Parisian walk (PW for short) is a random walk on $\mathbb{Z}[\zeta]$. Note that there are a lot of Parisian walks as functions of $\{\tau\}$, but the law is unique up to the initial point. The main aim of the present paper is to claim that PW is a discrete analogue of the complex Brownian motion (cBM for short), just as the simple symmetric random walk on $\mathbb{Z}$ is to the real one dimensional Brownian motion.

The latter analogy, the real case, is already established since many similar properties are known (see e.g. [1,2]). For our complex case, we have the following "evidences" so far;

(i) The scaling limit of PW converges to cBM.

(ii) The Itô's formula for PW (Proposition 4 below) looks very much like (symbolically the same as) the one for cBM.

(iii) We can define a discrete analogue of the conformal martingale, which is shown to be a time changed Parisian walk (Theorem 10). The result is analogue to the well-known fact with the conformal martingales.

The first fact (i) is due to the orthogonality between $\mathrm{Re}(\Delta Z)$ and $\mathrm{Im}(\Delta Z)$. The second one is coming from the fact that the martingale dimension of $\mathbf{F}$ is two. We give a proof for (ii) in Section 2, and to discuss (iii) we first argue the conformalilty in $\mathbb{Z}[\zeta]$ in Section 3.2, then give a proof of (ii) in Section 4.@

In Section 5, we present a potential application of our Parisian discrete stochastic calculus to mathematical finance. We first remark that a parametric restriction of

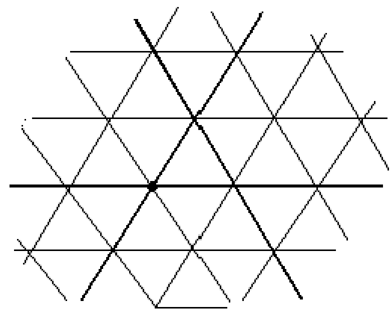

Fig. 1. Parisian walkways $\mathbb{Z}[\zeta]$.

the Heston model, one of the most popular stochastic volatility model, has a representation in terms of the squared norm and the area of a two-dimensional Ornstein-Uhlenbeck process (Proposition 11). Then relying on the fact that the pair of the squared norm and the area of a two-dimensional process has a representation in terms of a complex martingale (Lemma 12), we construct a Parisian discrete analogue of the Heston model.

Remark 2 *A fully general discrete Itô formula is given in* [3], *where the convergence rate of a scaling limit is also discussed, but analogies with complex stochastic calculus was out of the scope. Discrete analogues of Malliavin calculus are also studied in* [4–6], *etc, but, again, none of them is interested in analogies with complex stochastic calculus.*

2. An Itô formula for Parisian walks

We begin with a lemma.

Lemma 3 *Let* Z *be a Parisian walk. Then the two dimensional process* $(Z, \bar{Z})$ *enjoys martingale representation property; every complex valued* $\mathbf{F}$*-martingale is represented as a stochastic integral with respect to* $(Z, \bar{Z})$.

Proof Denote $\Delta Z_t := Z_t - Z_{t-1}$ for $t \in \mathbb{Z}_{>0}$. Fix t and set

$$\Delta Z_S := \prod_{s_i=\zeta} \Delta Z_i \prod_{s_i=\zeta^2} \overline{\Delta Z_i}$$

for $S = (s_1, \dots, s_t) \in \{1, \zeta, \zeta^2\}^t$. Then we have $\mathbf{E}[\Delta Z_S \overline{\Delta Z_{S'}}] = 1$ if $S = S'$ and $= 0$ otherwise because of the martingale property and of the fact that $(\Delta Z_t)^2 = \overline{\Delta Z_t}$. Therefore $\{\Delta Z_S | S \in \{1, \zeta, \zeta^2\}^t\}$ forms an orthonormal basis (ONB) of $L^2(\mathcal{F}_t)$ since $\sharp\{1, \zeta, \zeta^2\}^t = \dim L^2(\mathcal{F}_t) = 3^t$.

For an adapted $\{X_t\}$, expanding $X_t - X_{t-1}$ with respect to this ONB and denoting

$$\mathbf{E}[(X_t - X_{t-1})\overline{\Delta Z_S}] = x_S,$$

we have

$$\begin{aligned} X_t - X_{t-1} &= \sum_{s_t=\zeta} x_S \Delta Z_S + \sum_{s_t=\zeta^2} x_S \Delta Z_S + \sum_{s_t=1} x_S \Delta Z_S \\ &= \left(\sum_{s_t=\zeta} x_S \Delta Z_{(s_1,\dots,s_{t-1})} \right) \Delta Z_t \\ &\quad + \left(\sum_{s_t=\zeta^2} x_S \Delta Z_{(s_1,\dots,s_{t-1})} \right) \Delta \bar{Z}_t \\ &\quad + \sum_{s_t=1} x_S \Delta Z_{(s_1,\dots,s_{t-1})}. \qquad (1) \end{aligned}$$

By summing up the above equations, we obtain the Doob decomposition of X, and this completes the proof. **(QED)**

The above lemma can be easily extended to general unit root cases. The point here is that Parisian walk is the right discrete analogue of planar Brownian motion as can be seen by the following proposition.

Proposition 4 *Let $\{Z_t\}$ be a Parisian walk, and let f be a complex valued function on $\mathbb{Z}[\zeta]$. Then we have the following formula, which would correspond to an Itô's formula in $\mathbf{F}$. For $t = 0, 1, 2, \dots,$ we have*

$$\begin{aligned} & f(Z_{t+1}) - f(Z_t) \\ &= \frac{1}{3}(Z_{t+1} - Z_t) \\ &\qquad \left(f(Z_t+1) + \zeta^2 f(Z_t+\zeta) + \zeta f(Z_t+\zeta^2) \right) \\ &\quad + \frac{1}{3}(\bar{Z}_{t+1} - \bar{Z}_t) \\ &\qquad \left(f(Z_t+1) + \zeta f(Z_t+\zeta) + \zeta^2 f(Z_t+\zeta^2) \right) \\ &\quad + \frac{1}{3}\left(f(Z_t+1) + f(Z_t+\zeta) \right. \\ &\qquad \left. + f(Z_t+\zeta^2) - 3f(Z_t) \right). \qquad (2) \end{aligned}$$

Proof As in the expression (1),

$$f(Z_{t+1}) - f(Z_t) = \alpha \Delta Z_{t+1} + \beta \Delta \bar{Z}_{t+1} + \gamma$$

for some $\mathcal{F}_t$-measurable α, β and γ. On the set of $\Delta Z_{t+1} = 1$, $\Delta Z_{t+1} = \zeta$, and $\Delta Z_{t+1} = \zeta^2$ respectively, we have

$$\begin{aligned} & f(Z_t+1) - f(Z_t) = \alpha + \beta + \gamma, \\ & f(Z_t+\zeta) - f(Z_t) = \alpha\zeta + \beta\zeta^2 + \gamma, \qquad (3) \\ & \text{and } f(Z_t+\zeta^2) - f(Z_t) = \alpha\zeta^2 + \beta\zeta + \gamma. \end{aligned}$$

Solving (3) in terms of (α, β, γ), we obtain (2). **(QED)**

3. A discrete analogue of conformality

3.1 Analogy in Itô formulas

If we put

$$Df(z) = \frac{1}{3} \sum_{j=0,1,2} \zeta^{-j} f(z+\zeta^j),$$

$$\bar{D}f(z) = \frac{1}{3} \sum_{j=0,1,2} \zeta^{j} f(z+\zeta^j),$$

and

$$Lf(z) = \frac{1}{3} \sum_{j=0,1,2} \left[f(z+\zeta^j) - f(z) \right],$$

the formula (2) becomes

$$\begin{aligned} \Delta f(Z_t) &:= f(Z_t) - f(Z_{t-1}) \\ &= Df(Z_{t-1})\Delta Z_t + \bar{D}f(Z_{t-1})\overline{\Delta Z_t} + Lf(Z_{t-1}). \end{aligned} \qquad (4)$$

The discrete Itô's formula symbolically coincides with the one of cBM $\mathbf{Z}_t$; for $f(x+iy) = f_1(x,y) + if_2(x,y)$ with $f_1, f_2 \in C^2(\mathbb{R})$, we have that

$$df(\mathbf{Z}_t) = \partial_z f(\mathbf{Z}_t) d\mathbf{Z} + \partial_{\bar{z}} f(\mathbf{Z}_t) d\overline{\mathbf{Z}} + \mathcal{L}f(\mathbf{Z}_t) dt,$$

where

$$\partial_z = \frac{1}{2}(\partial_x + i\partial_y), \qquad (5)$$

and

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x - i\partial_y), \qquad (6)$$

the right-hand-side of (5) and (6) acting on f_1 and f_2,

$$\mathcal{L} = \frac{1}{2}\partial_z \partial_{\bar{z}},$$

is the Laplacian (see e.g. [7]).

3.2 Conformality in $\mathbb{Z}[\zeta]$

If f is analytic, or equivalently $\partial_{\bar{z}} = 0$, we have that

$$df(\mathbf{Z}_t) = f'(\mathbf{Z}_t) d\mathbf{Z}.$$

With this in mind, we define the conformality in $\mathbb{Z}[\zeta]$ as follows:

Definition 5 *We say a map $f : \mathbb{Z}[\zeta] \to \mathbb{Z}[\zeta]$ is* **Parisian conformal** *or* **p-conformal** *if $\bar{D}f(z) = 0$ and $Lf(z) = 0$ for all $z \in K$.*

We give the following basic result, which insists that our definition of conformality is proper in a geometric sense.

Proposition 6 *A map $f : \mathbb{Z}[\zeta] \to \mathbb{Z}[\zeta]$ is p-conformal if and only if it has the following property; any triangle of the form $\{z+1, z+\zeta, z+\zeta^2\}$ is mapped to a triangle $\{f(z)+c, f(z)+c\zeta, f(z)+c\zeta^2\}$ for some $c \in \mathbb{Z}[\zeta]$ with $f(z+\zeta^j) = f(z) + c\zeta^j$, $j = 0, 1, 2$.*

Proof The "if" part is straightforward since we can calculate directly $\bar{D}f$ and Lf by the assumed property.

The converse is also easy to see; since $\bar{D}f(z) = Lf(z) = 0$, we obtain by the Itô's formula (4),

$$f(z+\zeta^j) = f(z) + Df(z)\zeta^j, \quad j = 0, 1, 2.$$

Here we notice in particular that $Df(z) \in \mathbb{Z}[\zeta]$, which we state separately as a corollary.

(QED)

Corollary 7 *If $f : \mathbb{Z}[\zeta] \to \mathbb{Z}[\zeta]$ is p-conformal, then the image of the map Df is in $\mathbb{Z}[\zeta]$.*

Example 8 *Except the trivial ones like constant or linear functions in z, the simplest example of p-conformal map would be $z^2 - |z|$. Note that z^2 is not p-conformal. In general, among monic polynomials of a fixed degree, there is only one p-conformal map. A proof will be given in* [8].

4. Discrete analogue of conformal martingales

In this section, we give a probabilistic "credit" that ours is a discrete analogue of the conformality. First we give a definition of a "Parisian" conformal martingale.

Definition 9 *We say an* **F** *martingale M is p-conformal if it is $\mathbb{Z}[\zeta]$-valued and is represented by a martingale transform with respect to Z.*

From the definitions and the Itô's formula (4), it is straightforward that $f(Z)$ is a p-conformal martingale if and only if $f : \mathbb{Z}[\zeta] \to \mathbb{Z}[\zeta]$ is p-conformal.

We yet have the following theorem, which also implies that the definition is proper since it is also a discrete analogue of the fact that "*a conformal martingale is a time changed cBM*" (see e.g. [7, Proposition 3.6.2]).

Theorem 10 *For a p-conformal martingale M with respect to a Parisian walk Z, there exists another Parisian Walk $\tilde{Z}$ and a sequence of stopping times $T_1 < T_2 < \cdots < \infty$ such that $\{M\}$ is identically distributed as $\{\tilde{Z}_{T_t}\}$ as a stochastic process.*

Proof We first note that a Parisian walk is recurrent, which can be proven in a similar way as the case with the simple random walk on $\mathbb{Z}^2$. We choose a sequence of stopping times $T_1 < T_2 < \cdots < \infty$ recursively as $T_0 = 0$,

$$\begin{aligned} T_k &:= \\ &\inf\{t > T_{k-1} : \tilde{Z}_t \in \{\tilde{Z}_{T_{k-1}} + M_{k-1}\zeta^j, k = 0, 1, 2\}\}, \\ &k = 1, 2, \ldots \end{aligned}$$

Then it is easy to see that the sequence satisfies the desired property.

(QED)

5. Parisian analogue of Heston's stochastic volatility model

In this section, we discuss a potential application of our Parisian stochastic calculus to mathematical finance.

5.1 Heston's stochastic volatility model

In Heston's model [9], the stock price at time t is given by

$$S_t = S_0 \exp\left(\int_0^t \sqrt{\nu_s}\, dB_s + rt - \frac{1}{2}\int_0^t \nu_s ds\right),$$

where $r > 0$ stands for the risk-free rate, B is a 1-dimensional standard Brownian motion (under an equivalent martingale measure), and (the square root) of the volatility ν_t is a Cox-Ingersol-Ross process;

$$d\nu_t = \xi\sqrt{\nu_t}\, dB'_t + \kappa(\theta - \nu_t)dt, \tag{7}$$

where ξ is the *volatility of volatility* assumed to be constant, κ is the rate at which ν_t reverts to θ, the *long variance*. Here B' is another standard Brownian motion such that

$$d\langle B, B'\rangle = \rho dt$$

for some $\rho \in [-1, 1]$.

It is known that when $\xi^2 \leq 2\kappa\theta$, the process stays strictly positive (see e.g. [10, Chapter 6, Section 3.1]).

5.2 A decomposition of a Heston process

We work on the special case $\xi^2 = 2\kappa\theta$. It is also well known that in this case the unique strong solution to (7) is given by

$$\nu_t = |O_t|^2,$$

where $O_t = (O_t^1, O_t^2)$ is a two-dimensional Ornstein-Uhlenbeck process; $O^j, j = 1, 2$ solve

$$dO_t^j = \frac{\xi}{2}dW_t^j - \frac{\kappa}{2}O_t^j dt. \tag{8}$$

Here (W^1, W^2) is a two-dimensional Brownian motion.

In fact, since

$$\begin{aligned} d(|O_t|^2) &= 2O^1 dO_t^1 + 2O^2 dO_t^2 + \frac{\xi^2}{2}dt \\ &= \xi(O^1 dW^1 + O^2 dW^2) \\ &\qquad - \kappa\{(O^1)^2 + (O^2)^2\}dt + \frac{\xi^2}{2}dt \\ &= \xi|O_t|\frac{O^1 dW^1 + O^2 dW^2}{|O_t|} + \kappa(\theta - |O_t|^2)dt, \end{aligned}$$

and since

$$\left\langle \frac{O^1 dW^1 + O^2 dW^2}{|O_t|} \right\rangle = dt,$$

it is a Brownian motion, we see that $|O_t|^2$ solves (7).

Further, we have the following fact, which may not be new.

Proposition 11 *When $\xi^2 = 2\kappa\theta$, $X := \log S_t$ has the following identity in law as a stochastic process:*

$$\begin{aligned} &X_t - X_0 \\ &\quad = \sqrt{1-\rho^2}\left(\int_0^t O_s^1 dO_s^2 - \int_0^t O_s^2 dO_s^1\right) \\ &\qquad + \frac{\rho}{\xi}\left(|O_t|^2 - |O_0|^2\right) + \left(\frac{\rho\xi}{\theta} - \frac{1}{2}\right)\int_0^t |O_t|^2 ds \end{aligned}$$

$$+\left(r-\frac{\rho\xi}{2}\right)t. \tag{9}$$

Proof Observe that

$$\begin{aligned} O_t^1 dO_t^2 - O_t^2 dO_t^1 &= O_t^1 dW_t^2 - O_t^2 dW_t^1 \\ &= |O_t| \frac{O_t^1 dW_t^2 - O_t^2 dW_t^1}{|O_t|} \\ &=: |O_t| dB_t''. \end{aligned}$$

Here B'' is a Brownian motion independent of B' since it is a martingale, $d\langle B''\rangle = dt$, and $d\langle B', B''\rangle = 0$.

On the other hand, since $d\langle B, B'\rangle = \rho dt$, we have

$$\begin{aligned} &X_t - X_0 \\ &\overset{d}{=} \int_0^t \sqrt{\nu_s}\left(\rho dB_s' + \sqrt{1-\rho^2}\, dB_s''\right) + rt - \frac{1}{2}\int_0^t \nu_s ds \\ &= \frac{\rho}{\xi}(\nu_t - \nu_0) + \sqrt{1-\rho^2}\int_0^t \sqrt{\nu_s}\, dB_s'' \\ &\quad + \left(r - \frac{\rho\xi}{2}\right)t + \left(\frac{\rho\xi}{\theta} - \frac{1}{2}\right)\int_0^t \nu_s ds, \end{aligned}$$

which leads to (9).

(QED)

5.3 Discrete analogue of the stochastic area and the squared Bessel processes

For a discrete deterministic process $X : \mathbb{Z}_0 \to \mathbb{C}$, define $H(X) : \mathbb{Z}_0 \to \mathbb{C}$ by

$$H_0(X) = |X_0|^2, \quad \Delta H_t(X) = \bar{X}_{t-1}\Delta X_t, \quad t = 1, 2, \ldots$$

Then, we have the following.

Lemma 12 (i) *The squared distance from* 0 *of* X_t *is represented by the real part of* $H(X)_t$ *for each* $t > 0$*;*

$$|X_t|^2 = 2\mathrm{Re}H_t(X) + \sum_{j=1}^t |\Delta X_j|^2,$$

and (ii) *the aggregation of the* (*oriented*) *areas of the triangle drawn by* X_{s-1}, X_s *and* 0*,* $s = 1, \ldots, t$ *is represented by the imaginary part of* $H(X)_t$ *for each* $t > 0$*;*

$$\begin{aligned} A(X)_t &:= \frac{1}{2}\sum_{s=1}^t \{(\mathrm{Re}X_{s-1})(\mathrm{Im}X_s) - (\mathrm{Re}X_s)(\mathrm{Im}X_{s-1})\} \\ &= \frac{1}{2}\mathrm{Im}H_t(X). \end{aligned}$$

Proof For (i),

$$\begin{aligned} &(X_0 + \Delta X_1 + \cdots + \Delta X_t)\overline{(X_0 + \Delta X_1 + \cdots + \Delta X_t)} \\ &= |X_0|^2 + \sum_{j=1}^t |\Delta X_j|^2 + 2\mathrm{Re}(X_0 \sum_{j=1}^t \Delta X_j) \\ &\quad + 2\mathrm{Re}\sum_{j=1}^t \sum_{i=1}^{j-1} \overline{\Delta X_i}\Delta X_j \\ &= 2\mathrm{Re}H_t(X) + \sum_{j=1}^t |\Delta X_j|^2. \end{aligned}$$

The relation (ii) is obvious.

(QED)

Since the Ornstein-Uhlenbeck process (8) can be approximate by our Parisian walk by taking a scaling limit with a Girsanov-Maruyama type measure-change, we may claim that

$$\begin{aligned} S_t^P &:= \sqrt{1-\rho^2}\,\mathrm{Im}H_t(Z) + \frac{2\rho}{\xi}\left(2\mathrm{Re}H_t(Z) - |Z_0|^2\right) \\ &\quad + \left(\frac{2\rho\xi}{\theta} - 1\right)\sum \mathrm{Re}H_t(Z)\Delta t + \left(r - \frac{\rho\xi}{2}\right)t, \end{aligned} \tag{10}$$

where Z is a Parisian walk, is a discrete analogue of Heston's model, with a proper change of measures.

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