JSIAM Letters Vol.7 (2015) pp.61–64 ©2015 Japan Society for Industrial and Applied Mathematics

Conditional stability of the Lagrange–Galerkin scheme with numerical quadrature

Shinya Uchiumi[1,2]

[1] Graduate School of Fundamental Science and Engineering, Waseda University, 3-4-1 Ohkubo, Shinjuku, Tokyo 169-8555, Japan
[2] Research Fellow of Japan Society for the Promotion of Science, Tokyo, Japan

E-mail *su48@fuji.waseda.jp*

Received August 6, 2015, Accepted September 1, 2015

Abstract
We show conditional stability of the Lagrange–Galerkin scheme with numerical quadrature for the convection-diffusion equation. We consider the scheme under the assumption that quadrature points are inside the element and that the time increments are sufficiently small. Our analysis covers general triangular or tetrahedral meshes and arbitrary smooth velocities. We present some numerical examples that reflect the theoretical result.

Keywords convection diffusion equation, Lagrange–Galerkin scheme, characteristics finite element scheme, numerical quadrature

Research Activity Group Scientific Computation and Numerical Analysis

1. Introduction

The Lagrange–Galerkin scheme, also called the characteristics finite element scheme or Galerkin-characteristics scheme, is a powerful numerical scheme for flow problems such as the convection-diffusion equations and the Navier–Stokes equations.

Although stability and convergence analysis of the scheme has been performed successfully, the results are proved under the condition that the integration of the composite function term, which characterizes the method, is computed exactly. Since in real problems it is difficult to get the exact integration value, numerical quadrature is usually used. However, it has been reported that instability can occur as a result of the numerical quadrature error [1–5].

Several methods have been investigated to avoid instability. Morton–Priestley–Suli [1] proposed a method called area weighting. Tabata–Fujima [2] and Tabata [3] used a scheme of second order in time increments and take large time increments to suppress instability. Tanaka–Suzuki–Tabata [4] used a locally linearized velocity and calculated the integral exactly for the P_1-element. Recently, the method has been extended to the P_2-element and provided with error estimates [5]. On the other hand, the stability condition of the scheme with quadrature is analyzed in [1], where it is assumed that the velocity is constant and the mesh is uniform in each coordinate direction.

In this paper, we show conditional stability of the Lagrange–Galerkin scheme with numerical quadrature for the convection-diffusion equation. We only analyze the numerical quadrature whose quadrature points are inside the element. Our analysis is performed on general triangular or tetrahedral meshes and arbitrary smooth velocities. We also provide numerical examples using numerical quadrature with and without quadrature points on the edges.

2. Preliminaries

In this section, we state the problem and prepare notation used throughout this paper.

For a set ω we use the Sobolev spaces $L^p(\omega)$ with the norm $\|\cdot\|_{0,p,\omega}$, $W^{s,p}(\omega)$ and $W_0^{s,p}(\omega)$ with the norm $\|\cdot\|_{s,p,\omega}$ for $1 \le p \le \infty$ and a positive integer s. We write $H^s(\omega) = W^{s,2}(\omega)$ and drop the subscript $p = 2$ in the corresponding norms.

We consider the convection-diffusion problem: find $\phi : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\frac{\partial \phi}{\partial t} + u \cdot \nabla\phi - \nu\Delta\phi = f \quad \text{in } \Omega \times (0,T), \tag{1a}$$

$$\phi = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1b}$$

$$\phi = \phi^0 \quad \text{in } \Omega \text{ at } t = 0, \tag{1c}$$

where Ω is a polygonal or polyhedral domain of $\mathbb{R}^d$ ($d = 2,3$), $\partial\Omega$ is its boundary, $T > 0$ is a time and $\nu > 0$ is a diffusion constant. Functions $u : \Omega \times (0,T) \to \mathbb{R}^d$, $f \in C([0,T]; L^2(\Omega))$ and $\phi^0 : \Omega \to \mathbb{R}$ are given.

Let $\Delta t > 0$ be a time increment, $N_T \equiv \lfloor T/\Delta t \rfloor$, $t^n \equiv n\Delta t$ and $\psi^n \equiv \psi(\cdot, t^n)$ for a function ψ defined in $\Omega \times (0,T)$. For a set of functions $\psi = \{\psi^n\}_{n=0}^{N_T}$, two norms $\|\cdot\|_{\ell^\infty(L^2)}$ and $\|\cdot\|_{\ell^2(L^2)}$ are defined by

$$\|\psi\|_{\ell^\infty(L^2)} \equiv \max\{\|\psi^n\|_{0,\Omega}\,; n = 0,\ldots,N_T\},$$

$$\|\psi\|_{\ell^2(L^2)} \equiv \left(\Delta t \sum_{n=1}^{N_T} \|\psi^n\|_{0,\Omega}^2\right)^{1/2}.$$

For $w : \Omega \to \mathbb{R}^d$ we define the mapping $X_1(w) : \Omega \to$

$\mathbb{R}^d$ by

$$(X_1(w))(x) \equiv x - w(x)\Delta t. \tag{2}$$

Then, it holds that

$$\frac{\partial \phi^n}{\partial t} + u^n \cdot \nabla \phi^n - \frac{\phi^n - \phi^{n-1} \circ X_1(u^n)}{\Delta t} = O(\Delta t),$$

where the symbol $\circ$ stands for the composition of functions, e.g., $(g \circ f)(x) \equiv g(f(x))$.

Let $\mathcal{T}_h \equiv \{K\}$ be a triangulation of $\bar{\Omega}$, $h_K \equiv \mathrm{diam}(K)$ and $h \equiv \max_{K \in \mathcal{T}_h} h_K$. Throughout this paper we consider a regular family of triangulations $\{\mathcal{T}_h\}_{h \downarrow 0}$ satisfying the inverse assumption [6], that is, there exists a positive constant σ such that $\sigma h \leq h_K$, $\forall K \in \mathcal{T}_h$ and $\forall h$. Let k be a fixed positive integer and V_h be the P_k-finite element space,

$$V_h \equiv \{v_h \in H_0^1(\Omega); v_{h|K} \in P_k(K), \forall K \in \mathcal{T}_h\},$$

where $P_k(K)$ is the set of polynomials on K whose degrees are less than or equal to k. The parentheses $(\cdot,\cdot)$ indicate the $L^2(\Omega)$-inner product $(f,g) \equiv \int_\Omega fg dx$.

3. The Lagrange–Galerkin schemes

In this section we introduce the Lagrange–Galerkin schemes. Let $\phi_h^0 \in V_h$ be an approximation of ϕ^0 and set $X_1^n = X_1(u^n)$. The Lagrange–Galerkin scheme, which we call Scheme LG, is described as follows.

Scheme LG Find $\{\phi_h^n\}_{n=1}^{N_T} \subset V_h$ such that for $n = 1, \ldots, N_T$

$$\left(\frac{\phi_h^n - \phi_h^{n-1} \circ X_1^n}{\Delta t}, \psi_h\right) + \nu(\nabla \phi_h^n, \nabla \psi_h) = (f^n, \psi_h),$$

$\forall \psi_h \in V_h$.

For a general velocity u it is difficult to calculate the composite function term $(\phi_h^{n-1} \circ X_1^n, \psi_h)$ exactly. In practice, the following numerical quadrature has been used. We set a pair of a weight and a quadrature point on the reference element $\hat{K}$ by $(w_i, \hat{a}_i) \in \mathbb{R} \times \hat{K}$ for $i = 1, \ldots, N_q$, where N_q is the number of the points. For a general element K we define the quadrature point by $a_i \equiv F_K(\hat{a}_i)$, where F_K is the invertible affine mapping from $\hat{K}$ onto K. For a continuous function $v: K \to \mathbb{R}$ a numerical quadrature $I_h[v;K]$ of $\int_K v\,dx$ is defined by

$$I_h[v;K] \equiv \mathrm{meas}(K) \sum_{i=1}^{N_q} w_i v(a_i). \tag{3}$$

We call the practical scheme using numerical quadrature Scheme LG′.

Scheme LG′ Find $\{\phi_h^n\}_{n=1}^{N_T} \subset V_h$ such that for $n = 1, \ldots, N_T$

$$\begin{aligned} &\frac{1}{\Delta t}(\phi_h^n, \psi_h) - \frac{1}{\Delta t} \sum_{K \in \mathcal{T}_h} I_h[(\phi_h^{n-1} \circ X_1^n)\psi_h; K] \\ &\quad + \nu(\nabla \phi_h^n, \nabla \psi_h) = (f^n, \psi_h), \quad \forall \psi_h \in V_h. \end{aligned} \tag{4}$$

It has been reported that stability does not hold in general [1–5].

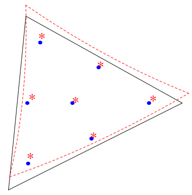

Fig. 1. An element K, the image $X_1^n(K)$, quadrature points of seven points formula a_i (•) and $X_1^n(a_i)$ for $i = 1, \ldots, 7$ ($*$).

4. Conditional stability

In this section, we consider the stability of Scheme LG′ for sufficiently small time increments Δt. We consider the quadrature formula (3) whose quadrature points $\{a_i\}$ are inside the element. For example, the seven-point formula of degree five [7] satisfies the condition. We restrict the time increment so small that the point $X_1^n(a_i)$ is also in the element (Fig. 1).

Hypothesis 1 $u \in C([0,T]; W^{k,\infty}(\Omega)^d)$.

Hypothesis 2 *Numerical quadrature I_h satisfies that*

(1) *each quadrature point satisfies $a_i \in \mathrm{int}(K)$,*
(2) *each weight w_i is positive,*
(3) *the quadrature formula is of degree r,*

where r is a positive integer.

Theorem 3 *Suppose that Hypotheses 1 and 2 hold with $r \geq 2k$. Let ϕ_h be the solution of (4). Then, there exist positive constants c_1 and c_2 independent of $h, \Delta t, \phi_h, f$ and T, and a positive constant $\Delta t_0 = O(h)$ such that if $\Delta t \leq \Delta t_0$, it holds that*

$$\begin{aligned} &\|\phi_h\|_{\ell^\infty(L^2)}, \sqrt{2\nu}\|\nabla \phi_h\|_{\ell^2(L^2)} \\ &\quad \leq \exp\left(\left(c_1 + c_2\frac{\Delta t}{h^2}\right)T\right)^{1/2} (\|\phi_h^0\|_{0,\Omega} + \|f\|_{\ell^2(L^2)}). \end{aligned} \tag{5}$$

Remark 4 (i) *If we take $\Delta t \leq ch^2$, the scheme is stable.* (ii) *One may consider that this result is not consistent with that of [1], wherein it was concluded that the Lagrange–Galerkin methods with quadrature whose quadrature points are inside the element are unstable. However, their criterion for stability is $|\lambda| \leq 1$, where λ is the amplification factor in the Fourier analysis. The von Neumann condition $|\lambda| \leq 1 + c\Delta t$ is sufficient for stability on the time interval $(0,T)$.*

We prepare lemmas before the proof of Theorem 3.

Lemma 5 *Suppose that the family of triangulations $\{\mathcal{T}_h\}_{h\downarrow 0}$ is regular. Let $p = 2$ or ∞ and a non-negative integer s be given. Then, there exists a positive constant γ_1 independent of $K \in \mathcal{T}_h$ and h such that*

$$\|\psi_h\|_{s,p,K} \leq \gamma_1 h_K^{-s-d/2+d/p} \|\psi_h\|_{0,K}, \psi_h \in P_k(K).$$

Lemma 5 is the inverse inequality on an element K [6].

Lemma 6 *Assume that the quadrature formula is of degree $2k-1$. Then, there exists a positive constant γ_2 independent of $K \in \mathcal{T}_h$ such that*

$$\begin{aligned} &\left|\int_K v\frac{\partial \psi_{1h}}{\partial x_i}\psi_{2h}dx - I_h\left[v\frac{\partial \psi_{1h}}{\partial x_i}\psi_{2h}; K\right]\right| \\ &\quad \leq \gamma_2 h_K^k \|v\|_{k,\infty,K} \|\psi_{1h}\|_{k,K} \|\psi_{2h}\|_{0,K}, \\ &\forall v \in W^{k,\infty}(K), \quad \forall \psi_{1h}, \psi_{2h} \in P_k(K), \quad i = 1, \ldots, d. \end{aligned}$$

The proof of Lemma 6 is same as that of [6, Theorem 4.1.4].

Lemma 7 *Let $v \in W^{k,\infty}(\Omega)^d$. Suppose that Hypothesis 2 holds with $r \geq 2k$, $\Delta t \leq h$ and $(X_1(v))(a_i) \in K, i = 1, \ldots, N_q$. Then, there exist positive constants $c_3(v)$ and $c_4(v)$ such that*

$$\sum_{K \in \mathcal{T}_h} I_h[(\psi_h \circ X_1(v))^2; K] \leq \left(1 + c_3 \Delta t + c_4 \frac{\Delta t^2}{h^2}\right) \|\psi_h\|_{0,\Omega}^2, \quad \forall \psi_h \in V_h. \quad (6)$$

Proof In this proof, we write $X_1(v)$ as X_1. Since quadrature point a_i and $X_1(a_i) = a_i - v(a_i)\Delta t$ are in the same element K, we apply the Taylor series expansion to obtain

$$\begin{aligned}
&(\psi_h(a_i - v(a_i)\Delta t))^2 \\
&= \left(\sum_{|\alpha| \leq k} \frac{1}{\alpha!}(-\Delta t)^{|\alpha|} v(a_i)^\alpha D^\alpha \psi_h(a_i)\right)^2 \\
&= \sum_{|\alpha_1| \leq k, |\alpha_2| \leq k} \left\{\frac{1}{\alpha_1! \alpha_2!}(-\Delta t)^{|\alpha_1|+|\alpha_2|} \right. \\
&\qquad \left. \times v(a_i)^{\alpha_1+\alpha_2} D^{\alpha_1}\psi_h(a_i) D^{\alpha_2}\psi_h(a_i)\right\},
\end{aligned}$$

where α, α_1 and α_2 are multi indexes. We divide $I_h[(\psi_h \circ X_1)^2; K]$ as

$$\begin{aligned}
&I_h[(\psi_h \circ X_1)^2; K] \\
&= I_h[\psi_h^2; K] - 2\Delta t I_h[(v \cdot \nabla \psi_h)\psi_h; K] \\
&\quad + \sum_{2 \leq |\alpha_1|+|\alpha_2| \leq 2k} \left\{\frac{1}{\alpha_1!\alpha_2!}(-\Delta t)^{|\alpha_1|+|\alpha_2|} \right. \\
&\qquad \left. \times I_h[v^{\alpha_1+\alpha_2} D^{\alpha_1}\psi_h D^{\alpha_2}\psi_h; K]\right\} \\
&\equiv J_0^K + J_1^K + J_2^K. \quad (7)
\end{aligned}$$

Since the quadrature formula is of degree r $(\geq 2k)$,

$$J_0^K = \|\psi_h\|_{0,K}^2. \quad (8)$$

Using Lemmas 6 and 5, we have

$$\begin{aligned}
J_1^K &\leq -2\Delta t \int_K (v \cdot \nabla \psi_h)\psi_h dx \\
&\quad + 2\Delta t \left| \int_K (v \cdot \nabla\psi_h)\psi_h dx - I_h[(v \cdot \nabla \psi_h)\psi_h; K] \right| \\
&\leq -2\Delta t \int_K (v \cdot \nabla \psi_h)\psi_h dx \\
&\quad + c(\gamma_2)\Delta t h_K^k \|v\|_{k,\infty,K} \|\psi_h\|_{k,K} \|\psi_h\|_{0,K} \\
&\leq -2\Delta t \int_K (v \cdot \nabla \psi_h)\psi_h dx \\
&\quad + c(\gamma_1, \gamma_2, v)\Delta t \|\psi_h\|_{0,K}^2. \quad (9)
\end{aligned}$$

Since it holds that $|D^{\alpha_l}\psi_h(a_i)| \leq \gamma_1 h_K^{-|\alpha_l|-d/2} \|\psi_h\|_{0,K}^2$ from Lemma 5, we have

$$\begin{aligned}
J_2^K &= \mathrm{meas}(K) \sum_{2 \leq |\alpha_1|+|\alpha_2| \leq 2k} \left(\frac{1}{\alpha_1!\alpha_2!}(-\Delta t)^{|\alpha_1|+|\alpha_2|} \right. \\
&\qquad \left. \times \sum_{i=1}^{N_q} w_i v(a_i)^{\alpha_1+\alpha_2} D^{\alpha_1}\psi_h(a_i) D^{\alpha_2}\psi_h(a_i)\right) \\
&\leq c(\gamma_1) \|\psi_h\|_{0,K}^2 \sum_{j=2}^{2k} \Delta t^j \|v\|_{0,\infty,K}^j h_K^{-j}, \quad (10)
\end{aligned}$$

where we have used the assumption $w_i > 0$, the identity $\sum_{i=1}^{N_q} w_i = 1$ and the inequality $\mathrm{meas}(K) \leq h_K^d$. Summing over all $K \in \mathcal{T}_h$ in (7), combining with (8)–(10) and using $\Delta t \leq h$, we have

$$\begin{aligned}
&\sum_{K \in \mathcal{T}_h} I_h[(\psi_h \circ X_1)^2; K] \\
&\leq \|\psi_h\|_{0,\Omega}^2 - 2\Delta t \int_\Omega (v \cdot \nabla \psi_h)\psi_h dx \\
&\quad + c(\gamma_1, \gamma_2, v)\Delta t \|\psi_h\|_{0,\Omega}^2 + c_4(\gamma_1, v)\frac{\Delta t^2}{h^2} \|\psi_h\|_{0,\Omega}^2.
\end{aligned}$$

Noting that $\psi_h = 0$ on $\partial\Omega$ and

$$\int_\Omega (v \cdot \nabla \psi_h)\psi_h dx = -\frac{1}{2}\int_\Omega (\nabla \cdot v)\psi_h^2 dx,$$

we obtain (6).

(QED)

Lemma 8 (discrete Gronwall inequality) *Let a_0 and a_1 be non-negative numbers, $\Delta t \in (0, 1/(2a_0)]$ a real number, and $\{x^n\}_{n \geq 0}, \{y^n\}_{n \geq 1}$ and $\{b^n\}_{n \geq 1}$ non-negative sequences. Suppose that*

$$\tfrac{1}{\Delta t}(x^n - x^{n-1}) + y^n \leq a_0 x^n + a_1 x^{n-1} + b^n, \ \forall n \geq 1.$$

Then, it holds that, for $n \geq 1$,

$$x^n + \Delta t \sum_{i=1}^n y^i \leq \exp\{(2a_0 + a_1)n\Delta t\} \left(x^0 + \Delta t \sum_{i=1}^n b^i\right).$$

Lemma 8 is shown by using the inequalities

$$(1 - a_0\Delta t)^{-1} \leq 1 + 2a_0\Delta t \leq \exp(2a_0\Delta t).$$

Proof of Theorem 3 Substituting $\psi_h = \phi_h^n$ in (4) and noting that the quadrature formula is of degree r $(\geq 2k)$, we have

$$\begin{aligned}
&\frac{1}{2\Delta t}\|\phi_h^n\|_{0,\Omega}^2 - \frac{1}{2\Delta t}\sum_{K \in \mathcal{T}_h} I_h[(\phi_h^{n-1} \circ X_1^n)^2; K] \\
&\quad + \frac{1}{2\Delta t}\sum_{K \in \mathcal{T}_h} I_h[(\phi_h^n - \phi_h^{n-1} \circ X_1^n)^2; K] + \nu \|\nabla \phi_h^n\|_{0,\Omega}^2 \\
&= (f^n, \phi_h^n).
\end{aligned}$$

Let Δt_0 be so small that $X_1^n(a_i) \in K$ for $\Delta t \leq \Delta t_0$ and $i = 1, \ldots, N_q$, and $\Delta t_0 \leq \min\{1/2, h\}$. Applying Lemma 7 with $\psi_h = \phi_h^{n-1}$ and $v = u^n$, we have

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\phi_h^n\|_{0,\Omega}^2 - \|\phi_h^{n-1}\|_{0,\Omega}^2) + \nu \|\nabla \phi_h^n\|_{0,\Omega}^2 \\
&\leq \frac{1}{2}\left(c_3 + c_4 \frac{\Delta t}{h^2}\right)\|\phi_h^{n-1}\|_{0,\Omega}^2 + \frac{1}{2}\|f^n\|_{0,\Omega}^2 + \frac{1}{2}\|\phi_h^n\|_{0,\Omega}^2,
\end{aligned}$$

where we have used the fact that

$$I_h[(\phi_h^n - \phi_h^{n-1} \circ X_1^n)^2; K] \geq 0$$

for each K since all weights are positive. We obtain (5) by Lemma 8.

(QED)

5. Numerical results

In this section, we show numerical results in $d = 2$. We compare Scheme LG′ with two quadrature formulas and Scheme LG. The P_2-element is employed. We use the rotating Gaussian hill problem [3] as a test problem.

Example In (1), Ω is a unit disk, and we set $T = 2\pi, \nu = 10^{-5}$, $u(x,t) \equiv (-x_2, x_1)$, $f \equiv 0$ and $\phi^0 \equiv \phi_e(\cdot, 0)$, where

$$\phi_e(x,t) \equiv \frac{\sigma}{\sigma + 4\nu t} \exp\left(-\frac{|x - \bar{x}(t; x_c)|^2}{\sigma + 4\nu t}\right),$$

$$\bar{x}(t; x_c) \equiv (x_{c1} \cos t - x_{c2} \sin t, x_{c1} \sin t + x_{c2} \cos t),$$

$$x_c \equiv (0.25, 0) \text{ and } \sigma \equiv 0.01.$$

This problem does not satisfy our setting since Ω is not a polygon. The function ϕ_e satisfies (1a) and (1c) but does not satisfy the boundary condition (1b). However, we treat ϕ_e as the exact solution since the value of ϕ_e on $\partial\Omega$ is less than 10^{-15} and thus almost equal to zero.

We use two meshes obtained by FreeFem++ [8] by setting N vertices on the circle for $N = 64$ and 128. We take Δt as 0.01×2^i $(i = -1, \dots, 4)$, 0.001×2^i $(i = -1, \dots, 2)$ and 0.007. In this problem, we can obtain the numerical solution by **Scheme LG** using exact integration since the velocity u is linear [4,5]. We use the seven-point (**S**) and Newton–Cotes (**NC**) formulas for Scheme LG′ (see Figs 1 and 2). Both formulas are of degree five and have positive weights. For formula S we check $X^n(a_i) \in K$ for $\Delta t \leq 0.002$ in the case of $N = 64$ and $\Delta t \leq 0.001$ in the case of $N = 128$. Note that we do not have theoretical results for formula NC since it has quadrature points on the edge. The relative error E is defined by

$$E \equiv \|\Pi_h^{(2)} \phi_e - \phi_h\|_{\ell^\infty(L^2)} / \|\Pi_h^{(2)} \phi_e\|_{\ell^\infty(L^2)},$$

where $\Pi_h^{(2)}$ is the Lagrange interpolation operator to the P_2-finite element space.

Fig. 3 shows the log-log graphs of E versus Δt. In the case of $N = 64$, the error of Scheme LG′ with formula S is large for Δt between 0.01 and 0.005 while it is as small as that of Scheme LG for $\Delta t \leq 0.004$. The result is consistent with Theorem 3. Although the change of error with formula NC is small for $\Delta t \leq 0.02$, the error is larger than that of Scheme LG. In the case of $N = 128$, the error of Scheme LG′ with formula S is large for Δt between 0.01 and 0.002 while it is as small as that of Scheme LG for $\Delta t \leq 0.001$. The error with formula NC is ten times larger than that of Scheme LG for $\Delta t = 0.0005$.

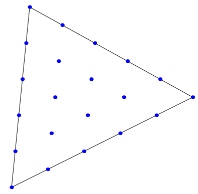

Fig. 2. Quadrature points of NC.

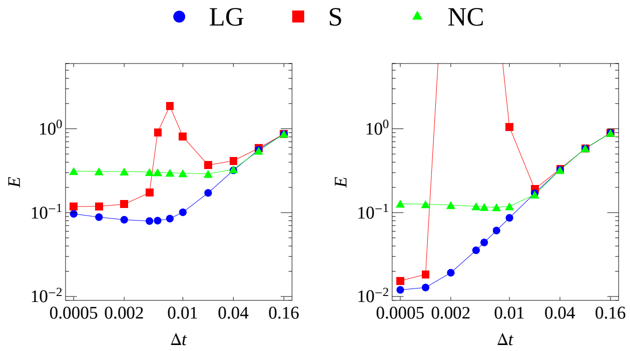

Fig. 3. Graphs of E versus Δt for $N = 64$ (left) and $N = 128$ (right).

6. Conclusions

We proved conditional stability of the Lagrange–Galerkin scheme with numerical quadrature having interior quadrature points for sufficiently small time increments. Numerical examples were consistent with the theoretical result. We did not analyze quadrature with quadrature points on the boundary but numerical results showed that Newton–Cotes quadrature was robust with respect to time increments.

Acknowledgments

The author would like to express his gratitude to Professor Masahisa Tabata for his considerable encouragement and valuable discussions. The author also thanks the anonymous reviewer for the careful reading and helpful comments. This work was supported by Japan Society for the Promotion of Science under Grant-in-Aid for JSPS Fellows No. 26·964.

References

[1] K. W. Morton, A. Priestley and E. Suli, Stability of the Lagrange-Galerkin method with non-exact integration, M2AN, **22** (1988), 625–653.

[2] M. Tabata and S. Fujima, Robustness of a characteristic finite element scheme of second order in time increment, in: Proc. of Computational Fluid Dynamics 2004, pp. 177–182, Springer, Berlin Heidelberg, 2006.

[3] M. Tabata, Discrepancy between theory and real computation on the stability of some finite element schemes, J. Comput. Appl. Math., **199** (2007), 424–431.

[4] K. Tanaka, A. Suzuki and M. Tabata, A characteristic finite element method using the exact integration (in Japanese), Annual Report of Research Institute for Information Technology, Kyushu University, **2** (2002), 11–18.

[5] M. Tabata and S. Uchiumi, A genuinely stable Lagrange–Galerkin scheme for convection-diffusion problems, arXiv:1505.05984 [math.NA].

[6] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, SIAM, Philadelphia, 2002.

[7] P. C. Hammer, O. J. Marlowe and A. H. Stroud, Numerical integration over simplexes and cones, Math. Comp., **10** (1956), 130–137.

[8] F. Hecht, New development in FreeFem++, J. Numer. Math., **20** (2012), 251–265.