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On LP-based approximation for copositive formulation of stable set problem

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Abstract

De Klerk-Pasechnik (2002) showed a way to compute the stability number $\alpha(G)$ via copositive programming and proposed LP- and SDP-based approximation schemes for the copositive program. In this paper, we show that their LP-based approximation for the stable set problem is equivalent to a problem of minimizing a quadratic form over a rational grid on the simplex, which can be viewed as a discretized version of the Motzkin-Straus theorem. Furthermore, we provide an algorithm to recover a maximum stable set from an optimal solution of the LP-based approximation and propose a simple local search heuristics for the stable set problem.

Keywords stable set problem, linear programming, copositive programming, Motzkin-Straus theorem, heuristics

Research Activity Group Discrete Systems

1. Introduction

The stable set problem is a classical problem in combinatorial optimization, which has important applications in various fields. A pioneering work by Lovász [1] introduced an SDP relaxation for the stable set problem to obtain an upper bound $\theta(G)$ (called theta number) of the stability number $\alpha(G)$. De Klerk-Pasechnik [2] refined this approach and provided a way to obtain $\alpha(G)$ via copositive programming. They also provided LP- and SDP-based approximation schemes by replacing the copositive cone $\mathcal{C}_n$ with a sequence of cones that converges to $\mathcal{C}_n$ and proved that both of the schemes yield $\alpha(G)$ after rounding down if the degree r of approximation is sufficiently large.

In this paper, we establish a new explicit formula for the optimal value of their LP-based approximation and reformulate it as a minimization of a quadratic form over a rational grid on the simplex. Our reformulation sheds a new insight on the LP-based approximation and clarifies its power of approximation. Our discrete quadratic program may be viewed as a discretized version of a classical result by Motzkin-Straus [3] on representing the stability number as a quadratic program. We provide an algorithm to recover a stable set from the support of a feasible solution. Our algorithm actually gives a maximum stable set from any optimal solution, provided the degree r of approximation is at least $\alpha(G)-2$. This lower bound sharpens the result of Peña-Vera-Zuluaga [4]. Furthermore, on the basis of these results, we provide a quite simple local search heuristics for the stable set problem. The efficiency of the proposed heuristics is confirmed by computational experiments on DIMACS benchmarks.

2. Preliminaries

2.1 Stable set problem

Throughout the paper $G=(V,E)$ will denote a simple undirected graph with vertex set $V=\{1,\ldots,n\}$ and edge set E. Also, let A be the adjacency matrix of G, I the $n\times n$ identity matrix, and e the n-dimensional all-one vector. A subset $V'\subseteq V$ is *stable* if $\{i,j\}\notin E$ for all $i,j\in V'$. A stable set is *maximum* if there are no larger stable sets in G and the *stability number* $\alpha(G)$ is the cardinality of a maximum stable set in G. The *stable set problem* is to find a maximum stable set and is known to be NP-hard [5].

2.2 Copositive programming

Let $\mathcal{S}_n$ be the set of all $n\times n$ real symmetric matrices. A matrix $X\in\mathcal{S}_n$ is said to be *copositive* if $y^\top Xy$ is nonnegative for all n-dimensional nonnegative vectors $y\in\mathbb{R}^n_+$. The set of all $n\times n$ copositive matrices is denoted by $\mathcal{C}_n$. A *Copositive program* is a convex optimization problem of the following form:

$$\begin{aligned}&\text{Minimize} && \mathrm{Tr}(CX)\\ &\text{subject to} && \mathrm{Tr}(A_iX)=b_i \quad (i=1,\ldots,m), \quad X\in\mathcal{C}_n,\end{aligned}$$

where $A_i,X,C\in\mathbb{R}^{n\times n}$ and $b_i\in\mathbb{R}$. The stability number $\alpha(G)$ can be obtained by solving a copositive program.

Theorem 1 (De Klerk-Pasechnik [2]) *The stability number $\alpha(G)$ equals the optimal value of*

$$\begin{aligned}&\text{Minimize} && \lambda\\ &\text{subject to} && \lambda(I+A)-ee^\top\in\mathcal{C}_n, \quad \lambda\in\mathbb{R}.\end{aligned} \tag{1}$$

Theorem 1 implies that copositive programming is intractable. In fact, determining whether a matrix is copositive is co-NP-complete [6].

2.3 LP-based approximation

De Klerk-Pasechnik [2] introduced an LP-based approximation hierarchy for $\mathcal{C}_n$. We consider the equivalent definition of copositivity to construct the approximate cone. We can see that $M \in \mathcal{S}_n$ is copositive if and only if the fourth order form given by

$$P_M(x) = (x \circ x)^\top M(x \circ x) = \sum_{i,j=1}^{n} M_{ij} x_i^2 x_j^2$$

is nonnegative, where "$\circ$" indicates the componentwise product. Obviously, a sufficient condition for M to be copositive is that all the coefficients of $P_M(x)$ are nonnegative. Then higher-order sufficient conditions can be derived by considering whether the coefficients of the polynomial

$$P_M^{(r)}(x) = \left(\sum_{i,j=1}^{n} M_{ij} x_i^2 x_j^2\right)\left(\sum_{i=1}^{n} x_i^2\right)^r$$

take nonnegative values. For any integer $r \geq 0$, we define $\mathcal{C}_n^r$ as the cone of matrices $M \in \mathcal{S}_n$ such that all the coefficients of $P_M^{(r)}(x)$ are nonnegative. Then the following inclusions hold:

$$\mathcal{C}_n^0 \subseteq \mathcal{C}_n^1 \subseteq \cdots \subseteq \mathcal{C}_n. \tag{2}$$

We define $\zeta^{(r)}(G)$ as the minimum of the LP-based approximation of (1):

$$\begin{array}{ll} \text{Minimize} & \lambda \\ \text{subject to} & \lambda(I + A) - ee^\top \in \mathcal{C}_n^r, \quad \lambda \in \mathbb{R}, \end{array} \tag{3}$$

where we set $\zeta^{(r)}(G) = \infty$ if the problem is infeasible. Then it follows from (2) that

$$\zeta^{(0)}(G) \geq \zeta^{(1)}(G) \geq \cdots \geq \alpha(G).$$

De Klerk-Pasechnik [2] showed that $\lfloor \zeta^{(r)}(G) \rfloor = \alpha(G)$ if $r \geq \alpha(G)^2$. Peña-Vera-Zuluaga [4] strengthened and sharpened their result as follows.

Theorem 2 (Peña-Vera-Zuluaga [4]) *It holds that $\lfloor \zeta^{(r)}(G) \rfloor = \alpha(G)$ if and only if $r \geq \alpha(G)^2 - 1$. Furthermore, $\zeta^{(r)}(G) < \infty$ if and only if $r \geq \alpha(G) - 1$.*

Thus we can regard problem (3) as an LP-based formulation of the stable set problem for sufficiently large r.

3. Results

3.1 Discrete version of Motzkin-Straus theorem

We present a new explicit expression of $\zeta^{(r)}(G)$ as follows.

Theorem 3 *For $r \geq \alpha(G) - 1$, we have*

$$\zeta^{(r)}(G) = \max_{w \in I_n(r+2)} \frac{(r+2)(r+1)}{w^\top (I + A)w - (r+2)}, \tag{4}$$

where

$$I_n(t) = \{w \in \mathbb{Z}_+^n \mid e^\top w = t\}.$$

Considering (3) as an LP with a single variable λ, we can solve it easily by deriving conditions for each coefficient of $P_M^{(r)}(x)$ to be nonnegative. We can calculate them by expanding the polynomial.

Lemma 4 (Bomze-de Klerk [7]) *Let $M \in \mathcal{S}^n$ and introduce the multinomial coefficients*

$$c(m) = \frac{(\sum_{i=1}^n m_i)!}{m_1! \ldots m_n!}$$

for any $m \in \mathbb{Z}_+^n$. Then we have

$$P_M^{(r)}(x) = \sum_{w \in I_n(r+2)} a_w x_1^{w_1} \ldots x_n^{w_n},$$

where

$$a_w = \frac{c(w)}{(r+2)(r+1)}(w^\top M w - w^\top \mathrm{diag} M),$$

$$\mathrm{diag} M = (M_{11}, \ldots, M_{nn})^\top.$$

Now we can obtain (4) immediately from Lemma 4.

Proof of Theorem 3 The constraint in Problem (3), $\lambda(I + A) - ee^\top \in \mathcal{C}_n^r$, means that every coefficient of $P^r_{\lambda(I+A)-ee^\top}(x)$ is nonnegative. By Lemma 4, this is equivalent to

$$\begin{aligned} & w^\top(\lambda(I + A) - ee^\top)w - w^\top \mathrm{diag}(\lambda(I + A) - ee^\top) \\ &= \lambda w^\top (I + A)w - (e^\top w)^2 - (\lambda - 1)w^\top e \\ &= \lambda w^\top (I + A)w - (r + 2)^2 - (\lambda - 1)(r + 2) \\ &= \lambda[w^\top (I + A)w - (r + 2)] - (r + 2)(r + 1) \geq 0 \end{aligned}$$

for every $w \in I_n(r+2)$. If $r \geq \alpha(G) - 1$, we have $w^\top(I + A)w - (r+2) > 0$ for every $w \in I_n(r+2)$ from Theorem 2. Therefore (4) holds.

(QED)

Our formula can be viewed as a discretized version of the Motzkin-Straus formula.

Theorem 5 (Motzkin-Straus [3]) *We have*

$$\alpha(G) = \max_{x \in \Delta} \frac{1}{x^\top (I + A)x}, \tag{5}$$

where Δ denotes the n-dimensional standard simplex. Moreover, let $\{1, \ldots, k\}$ be a maximum stable set of G. Then $x_1 = \cdots = x_k = 1/k$, $x_{k+1} = \cdots = x_n = 0$ is an optimal solution of (5).

The relation between (4) and (5) becomes more explicit if we rewrite $\zeta^{(r'-2)}(G)$ for $r' \geq \alpha(G) + 1$ as

$$\zeta^{(r'-2)}(G) = \max_{x \in \Delta(r')} \frac{r' - 1}{r' x^\top (I + A)x - 1},$$

where $\Delta(r')$ denotes the set of $1/r'$-integral vectors in Δ for $r' \in \mathbb{N}$. Theorem 5 also states that the support of an optimal solution of (5) is a maximum stable set. Correspondingly, we can derive a maximum stable set from the support of an optimal solution of (4).

3.2 Recovery of stable set

We provide an algorithm to obtain a maximum stable set from the support of an arbitrary optimal solution of (4).

Definition 6 *Let e_i be the unit vector of the ith coordinate direction. We denote by $\widehat{x}$ the vector obtained from $w \in I_n(r + 2)$ by applying the following procedure:*

(i) *If there are $\{i, j\} \in E$ such that $w_i > 0$, $w_j > 0$,* *choose $w+w_i(e_j-e_i)$ or $w+w_j(e_i-e_j)$ as w' that makes $w'^\top(I+A)w'$ smaller and replace w with w'.*

(ii) *Repeat* (i) *until the support of w corresponds to a stable set of G.*

We show that this procedure recovers a maximum stable set if w is optimal. Note that it holds for $r \geq \alpha(G)-1$ that

$$\begin{aligned}&\mathop{\arg\min}_{w\in I_n(r+2)} w^\top(I+A)w\\ &\quad = \mathop{\arg\max}_{w\in I_n(r+2)} \frac{(r+2)(r+1)}{w^\top(I+A)w-(r+2)}.\end{aligned}$$

Lemma 7 *It holds for any $w \in I_n(r+2)$ that*

$$w^\top(I+A)w \geq \widehat{w}^\top(I+A)\widehat{w}.$$

Proof At each choice of w' in Definition 6, if $w' = w+w_j(e_i-e_j)$, we have

$$\begin{aligned}&w'^\top(I+A)w' - w^\top(I+A)w\\ &\quad = w_j^2(I_{ii}-A_{ij}-A_{ji}+I_{jj})+2w_jw^\top(I+A)(e_i-e_j)\\ &\quad = 2w_jw^\top(I+A)(e_i-e_j),\end{aligned}$$

since $\{i,j\} \in E$. Similarly, if $w' = w+w_i(e_j-e_i)$, we have

$$w'^\top(I+A)w' - w^\top(I+A)w = 2w_iw^\top(I+A)(e_j-e_i).$$

Since one of these values are nonpositive, $w^\top(I+A)w \geq w'^\top(I+A)w'$. Thus we have $w^\top(I+A)w \geq \widehat{w}^\top(I+A)\widehat{w}$ by repeating the process.

(QED)

Theorem 8 *Let $w^* \in \arg\min_{w\in I_n(r+2)} w^\top(I+A)w$ and $S(w) = \{i \mid w_i \neq 0\}$. Then $S(\widehat{w}^*)$ is a maximum stable set if and only if $r \geq \alpha(G)-2$.*

Proof If $r < \alpha(G)-2$, it follows from the definition of $I_n(r+2)$ that $|S(\widehat{w}^*)| < \alpha(G)$, which implies that $S(\widehat{w}^*)$ is not a maximum stable set.

To show the sufficiency, suppose $|S(\widehat{w}^*)| < \alpha(G)$. Then there exists $k \notin S(\widehat{w}^*)$ such that $S(\widehat{w}^*)\cup\{k\}$ is a stable set and $l \in S(\widehat{w}^*)$ such that $\widehat{w}_l^* \geq 2$. We consider the vector $\tilde{w}^*-e_l+e_k \in I_n(r+2)$. It follows from the stability of $S(\widehat{w}^*)$ and $S(\widehat{w}^*-e_l+e_k)$ that

$$\begin{aligned}&\widehat{w}^{*\top}(I+A)\widehat{w}^* - (\widehat{w}^*-e_l+e_k)^\top(I+A)(\widehat{w}^*-e_l+e_k)\\ &\quad = \widehat{w}^{*\top}\widehat{w}^* - (\widehat{w}^*-e_l+e_k)^\top(\widehat{w}^*-e_l+e_k)\\ &\quad = 2(\widehat{w}_l^*-1) > 0.\end{aligned}$$

Now, from the optimality of w^* and Lemma 7,

$$\begin{aligned}w^*(I+A)w^* &= \widehat{w}^{*\top}(I+A)\widehat{w}^*\\ &> (\widehat{w}^*-e_l+e_k)^\top(I+A)(\widehat{w}^*-e_l+e_k).\end{aligned}$$

This contradicts $w^* \in \arg\min_{w\in I_n(r+2)} w^\top(I+A)w$. By contradiction, $|S(\widehat{w}^*)| = \alpha(G)$.

(QED)

Thus we can solve the stable set problem by minimizing the quadratic form over $I_n(r+2)$ for $r \geq \alpha(G)-2$, although $\lfloor\zeta^{(r)}(G)\rfloor \neq \alpha(G)$ if $\alpha(G) \leq r < \alpha(G)^2-1$. Since we need $r' \geq \alpha(G)^2-1$ to obtain $\alpha(G)$ in (4), Theorem 8 sharpens Theorem 2 with regard to the degree of approximation.

Algorithm 1 Local search for the stable set problem

> $w := e$
> **while** w is not a local optimum **do**
>   choose $w' \in N(w)$
>   **if** $w'^\top(I+A)w' \leq w^\top(I+A)w$ **then**
> $w := w'$
>   **end if**
> **end while**
> compute $\widehat{w}$
> **return** $S(\widehat{w})$

3.3 Local search heuristics

We propose a simple heuristics for the stable set problem using the results in the previous subsections. For each $w \in I_n(r+2)$, we regard

$$N(w) = \{w+e_i-e_j \mid i,j \in \{1,\ldots,r\}, w_j > 0\}$$

as a neighborhood of w. This neighborhood leads to a local search shown in Algorithm 1. The heuristic starts from the initial point $w = e$, which implies that we set $r = n-2$. Then we repeatedly pick $w' \in N(w)$ to get the objective value smaller until w reaches a local optimum. In the algorithm, we take w as a local optimum if the objective value does not change after n updates of w. Finally, we compute $\widehat{w}$ and its support $S(\widehat{w})$.

The performance of this heuristics has been tested on the complement graphs of the DIMACS clique benchmarks. See for details of the graphs at

http://dimacs.rutgers.edu/Challenges/.

We applied the heuristics 10 times for each graph. All computations were executed with 2.4GHz Intel CPU Core i7 and 16GB of memory. The results are given in Table 1. The columns "Name", "$\alpha(\bar{G})$", "Solution", "Average", and "Time" represent the name of the graph, the stability number of the complement graph, the maximum cardinality of the stable sets obtained, the average cardinality of them, and CPU time in seconds.

The proposed heuristics found a maximum stable set in 24 of the 36 instances in the categories of CFAT, Johnson, Hamming, PHAT, and MANN. However, it did not perform well on the graphs in the categories of Keller, SAN, SANR, and BROCK.

4. Conclusion

In this paper, we have reformulated the LP-based approximation for the stable set problem as a discrete version of the Motzkin-Straus theorem. This reformulation leads to a procedure to obtain a maximum stable set from an optimal solution and a local search heuristics for the stable set problem. Furthermore, we showed the strict lower bound for our procedure to yield a maximum stable set. This lower bound is less than the strict bound to compute $\alpha(G)$ as the optimal value of the LP-based approximation.

It remains as a future work to investigate whether we can apply a similar idea to other problems in combinato-

Table 1. Results on the DIMACS benchmarks.

Name	$\alpha(\bar{G})$	Solution	Average	Time (s)
c-fat200-1	12	12	12.0	0.5
c-fat200-2	24	24	23.0	0.2
c-fat200-5	58	58	55.2	0.1
c-fat500-1	14	14	13.4	4.6
c-fat500-2	26	26	26.0	2.0
c-fat500-5	64	64	64.0	0.8
c-fat500-10	$\geq$126	126	125.8	0.5
johnson8-2-4	4	4	2.6	<0.1
johnson8-4-4	14	14	12.0	0.1
johnson16-2-4	8	8	7.7	0.2
johnson32-2-4	16	16	15.7	3.6
keller4	11	9	8.0	0.3
keller5	27	19	17.3	5.5
keller6	$\geq$59	38	35.8	125.4
hamming6-2	32	32	28.1	<0.1
hamming6-4	4	4	2.4	<0.1
hamming8-2	128	128	119.0	0.1
hamming8-4	16	16	14.0	0.8
hamming10-2	512	512	442.0	1.6
hamming10-4	$\geq$40	34	31.2	9.9
san200_0.7_1	30	15	15.0	<0.1
san200_0.7_2	18	12	12.0	<0.1
san200_0.9_1	70	45	45.0	<0.1
san200_0.9_2	60	38	36.1	0.1
san200_0.9_3	44	33	31.4	0.1
san400_0.5_1	13	7	7.0	<0.1
san400_0.7_1	40	20	20.0	0.1
san400_0.7_2	30	15	15.0	0.1
san400_0.7_3	22	12	12.0	0.1
san400_0.9_1	100	52	50.6	0.2
san1000	15	8	8.0	0.5
sanr200_0.7	18	17	15.2	0.3
sanr200_0.9	42	41	37.4	0.1
sanr400_0.5	13	11	9.9	4.2
sanr400_0.7	21	21	16.3	1.6
brock200_1	21	19	17.0	0.3
brock200_2	12	9	7.7	0.6
brock200_3	15	13	11.2	0.5
brock200_4	17	15	13.1	0.3
brock400_1	27	22	20.4	1.4
brock400_2	29	22	20.2	1.3
brock400_3	31	22	19.6	1.5
brock400_4	33	24	20.3	1.3
brock800_1	23	18	16.1	15.3
brock800_2	24	18	16.2	14.9
brock800_3	25	18	16.4	15.6
brock800_4	26	19	16.7	19.3
p_hat300-1	8	7	6.0	3.2
p_hat300-2	25	25	23.5	0.6
p_hat300-3	36	36	31.8	0.4
p_hat500-1	9	8	6.8	10.0
p_hat500-2	36	36	34.4	1.2
p_hat500-3	$\geq$49	49	47.2	0.9
p_hat700-1	11	9	6.7	25.2
p_hat700-2	44	44	41.8	2.8
p_hat700-3	62	60	58.4	1.8
p_hat1000-1	10	10	7.1	56.7
p_hat1000-2	46	45	42.9	6.8
p_hat1000-3	65	64	61.6	4.1
p_hat1500-1	12	11	7.7	186.1
p_hat1500-2	63	63	60.7	13.4
p_hat1500-3	94	90	87.5	8.8
MANN_a9	16	16	14.7	<0.1
MANN_a27	126	118	117.2	0.1
MANN_a45	345	332	330.4	0.5
MANN_a81	$\geq$1100	1081	1080.2	5.8

rial optimization which can be formulated as a copositive program. Also, the performance of our heuristics can be expected to improve by using a more efficient technique, such as tabu search, for the local search.

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