JSIAM Letters Vol.7 (2015) pp.73–76 ©2015 Japan Society for Industrial and Applied Mathematics

Numerical verification of positiveness for solutions to semilinear elliptic problems

Kazuaki Tanaka[1], Kouta Sekine[2], Makoto Mizuguchi[1] and Shin'ichi Oishi[2,3]

[1] Graduate School of Fundamental Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
[2] Faculty of Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
[3] CREST/JST, Japan

E-mail *imahazimari@fuji.waseda.jp*

Received September 10, 2015, Accepted October 13, 2015

Abstract

In this paper, we propose a numerical method for verifying the positiveness of solutions to semilinear elliptic boundary value problems. We provide a sufficient condition for a solution to an elliptic problem to be positive in the domain of the problem, which can be checked numerically without requiring a complicated computation. Although we focus on the homogeneous Dirichlet case in this paper (in fact, it is often possible that solutions are not positive near the boundary in this case), our method can be applied naturally to other boundary conditions. We present some numerical examples.

Keywords elliptic problems, verifying positiveness, verified numerical computation

Research Activity Group Quality of Computations

1. Introduction

We are concerned with the following elliptic problem:

$$\begin{cases} -Lu = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain (i.e., an open connected bounded set) in $\mathbb{R}^n$ $(n = 1, 2, 3, \dots)$, f is a given nonlinear operator from $H_0^1(\Omega)$ to $L^2(\Omega)$, and L is a uniformly elliptic self-adjoint operator from its domain $D(L)$ to $L^2(\Omega)$ (the domain $D(L)$ depends on the smoothness of the boundary $\partial\Omega$). Here, letting $H^1(\Omega)$ denote the first order L^2-Sobolev space on Ω, $H_0^1(\Omega)$ is defined as $H_0^1(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega \text{ in the trace sense}\}$, with inner product $(\cdot,\cdot)_{H_0^1(\Omega)} := (\nabla\cdot, \nabla\cdot)_{L^2(\Omega)}$ and norm $\|\cdot\|_{H_0^1(\Omega)} := \|\nabla\cdot\|_{L^2(\Omega)}$. To be precise, L can be written in the form

$$L = \sum_{i,j=1}^{n} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + c, \tag{2}$$

where the following properties hold:

- $a_{i,j} \in L^\infty(\Omega)\,(i, j = 1, 2, \dots, n)$ and $c \in L^\infty(\Omega)$;
- $a_{i,j} = a_{j,i}$ $(i, j = 1, 2, \dots, n)$;
- There exists a positive number μ_0 such that

$$\sum_{i,j=1}^{n} a_{i,j}(x)\,\xi_i \xi_j \geq \mu_0 \sum_{i=1}^{n} \xi_i^2 \tag{3}$$

for all $x \in \Omega$ and all n-tuples of real numbers $(\xi_1, \xi_2, \dots, \xi_n)$.

Here, $L^\infty(\Omega)$ is the functional space of Lebesgue measurable functions over Ω with the norm $\|u\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}\{|u(x)| \mid x \in \Omega\}$ for $u \in L^\infty(\Omega)$.

Eq. (1) arises from various models, including examples from biology and physics. In addition, it has many mathematical applications, such as in the analysis of solution structures of partial differential equations and optimization problems. In other words, it is often necessary to distinguish positive solutions from others.

There have been a number of numerical methods for verifying solution to elliptic problems (see, e.g., [1–4]) and related works, e.g., [5,6]. Such methods enable us to obtain a concrete ball containing exact solutions to the problem

$$\begin{cases} -Lu = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

typically in the sense of one of the norms $\|\cdot\|_{H_0^1(\Omega)}$ or $\|\cdot\|_{L^\infty(\Omega)}$. No matter how small the radius of the ball is, it is possible for a verified solution u not to be positive near the boundary $\partial\Omega$, because the solution u vanishes exactly on $\partial\Omega$.

In this paper, we will propose a numerical method for verifying the positiveness of solutions to (4), in order to verify solutions of (1). Theorem 2 provides a sufficient condition for positiveness. Moreover, this enables us to numerically verify positiveness in the whole of Ω, even near the boundary, and this only requires a simple numerical computation.

2. Verification for positiveness

Throughout this paper, we omit the expression "almost everywhere" for Lebesgue measurable functions, for simplicity. For example, we employ the notation

$u > 0$ in the place of $u(x) > 0$ a.e. $x \in \Omega$. We introduce the following lemma that is required to prove Theorem 2.

Lemma 1 *Suppose that there exists a weak solution $u \in H_0^1(\Omega)$ to (1), such that $f(u) \geq 0$ ($f(u) \not\equiv 0$) and $(g_u =) f(u) u^{-1} \in L^\infty(\Omega)$. Then,*

$$\operatorname{ess\,sup}\{g_u(x) \,|\, x \in \Omega\} \geq \lambda_1, \tag{5}$$

where λ_1 is the first eigenvalue of the problem

$$(-L\phi, v)_{L^2(\Omega)} = \lambda (\phi, v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega), \tag{6}$$

in the weak sense.

Proof Let $\phi_1 \geq 0$ ($\phi_1 \not\equiv 0$) be the first eigenfunction corresponding to λ_1 (see, e.g., [7, Theorems 1.2.5 and 1.3.2] for ensuring the nonnegativeness of the first eigenfunction). Since the weak solution u satisfies

$$(-Lu, v)_{L^2(\Omega)} = (f(u), v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega)$$

and L is self-adjoint, it follows that

$$(f(u), \phi_1)_{L^2(\Omega)} = \lambda_1 (u, \phi_1)_{L^2(\Omega)}.$$

Therefore,

$$\begin{aligned}
&(f(u), \phi_1)_{L^2(\Omega)} \\
&= \int_\Omega f(u(x)) u(x)^{-1} \{u(x) \phi_1(x)\} dx \\
&\leq \operatorname{ess\,sup}\{g_u(x) \,|\, x \in \Omega\} (u, \phi_1)_{L^2(\Omega)} \\
&= \lambda_1^{-1} \operatorname{ess\,sup}\{g_u(x) \,|\, x \in \Omega\} (f(u), \phi_1)_{L^2(\Omega)}.
\end{aligned}$$

The positiveness of $\int_\Omega f(u(x)) \phi(x) dx$ implies (5).

(QED)

Using Lemma 1, we are able to prove the following theorem, which provides a sufficient condition for the positiveness of solutions to (4).

Theorem 2 *Suppose that a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to (4) satisfies the following properties:*

i) *u is positive in a nonempty subdomain $\Omega' \subset \Omega$;*
ii) *$|g_u| < \infty$ and $f(|u|) \geq 0$;*
iii) *$\sup\{g_{u_-}(x) \,|\, x \in \Omega\} < \lambda_1(\Omega)$.*

Then, $u > 0$ in the original domain Ω; that is, u is also a solution to (1). Here, $g_u := f(u) u^{-1}$, $\lambda_1(\Omega)$ is the first eigenvalue of the problem (6), and u_- is defined by

$$u_-(x) := \begin{cases} -u(x), & u(x) < 0, \\ 0, & u(x) \geq 0. \end{cases}$$

Proof Assume that u is not always positive in Ω. The strong maximum principle ensures that u is also not always nonnegative in Ω (the case that $u \equiv 0$ in Ω is generally allowed, but this case is also ruled out owing to assumption i) in the statement). In other words, there exists a nonempty subdomain $\Omega'' \subset \Omega \backslash \Omega'$ such that $u < 0$ in Ω'' and $u = 0$ on $\partial\Omega''$. Therefore, the restricted function $v := -u|_{\Omega''}$ can be regarded as a solution to

$$\begin{cases} -Lv = f(v) & \text{in } \Omega'', \\ v > 0 & \text{in } \Omega'', \\ v = 0 & \text{on } \partial\Omega''. \end{cases}$$

From Lemma 1, we have that

$$\begin{aligned}
\sup_{x\in\Omega} g_{u_-}(x) &\geq \sup_{x\in\Omega''} g_v(x) \\
&\geq \lambda_1(\Omega''),
\end{aligned}$$

where $\lambda_1(\Omega'')$ is the first eigenvalue of (6), with the notational replacement $\Omega = \Omega''$. Since the inclusion $\Omega'' \subset \Omega$ ensures that all functions in $H_0^1(\Omega'')$ can be regarded as functions in $H_0^1(\Omega)$ by considering the zero extension outside Ω'', the inequality $\lambda_1(\Omega'') \geq \lambda_1(\Omega)$ follows. This contradicts the property iii).

(QED)

Remark 3 *Since the strong maximum principle requires the regularity $u \in C^2(\Omega) \cap C(\overline{\Omega})$, we require this regularity to obtain the result in Theorem 2. For each $h \in L^2(\Omega)$, the problem*

$$\begin{cases} -\Delta u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in H^2(\Omega)$, such as when Ω is a bounded convex domain with a piecewise C^2 boundary (see, e.g., [8, Section 3.3]*). Therefore, the so called bootstrap argument ensures that a weak solution $u \in H_0^1(\Omega)$ to (4) on such a domain Ω, is in $C^\infty(\Omega) (\subset C^2(\Omega))$, such as when L is the Laplace operator and f is given by $f(u) = |u|^{p-1} u$ (many other choices exist). Moreover, the strong maximum principle can be applied to other boundary conditions as well. To be precise, this principle claims that if $-Lu > 0$ in Ω, then u cannot have a minimum in Ω independently of its boundary condition. Therefore, this theorem can be naturally applied to other boundary value problems. Details for the strong maximum principle can be found in, e.g.,* [9, 10].

The following corollary immediately follows from Theorem 2 and will be convenient for presenting our numerical examples in the next section.

Corollary 4 *Let $f(u) = |u|^{p-1} u$ with $p > 1$. If a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to (4) is positive in a nonempty subdomain $\Omega' \subset \Omega$ and $\sup\{(u_-(x))^{p-1} | x \in \Omega\} < \lambda_1(\Omega)$, then $u > 0$ in the original domain Ω. Here, $\lambda_1(\Omega)$ and u_- are defined as in Theorem 2.*

Proof Since property ii) in Theorem 2 clearly holds when $f(u) = |u|^{p-1} u$ ($p > 1$), this corollary holds.

(QED)

3. Numerical example

In this section, we will present two numerical examples where the positiveness of solutions to (4) is verified. In both examples, we set $L = \Delta$ and $f(u) = u^3 (= |u|^2 u)$, where Δ is the Laplace operator. All computations were carried out on a computer with Intel Xeon E7-4830 at 2.20 GHz×40, 2 TB RAM, CentOS 6.6, and MATLAB 2012b. All rounding errors were strictly estimated using toolboxes the INTLAB version 9 [11] and KV library version 0.4.16 [12] for verified numerical computations. Therefore, the accuracy of all results was mathematically guaranteed. In this section, $\overline{B}(x, r; \|\cdot\|)$ denotes the closed ball whose center is x, and whose radius is $r \geq 0$ in the sense of the norm $\|\cdot\|$.

For the first example, we selected the case in which $\Omega = (0,1)^2 \subset \mathbb{R}^2$. We computed an approximate solution $\hat{u}$ to (4), which is displayed in Fig. 1, using the Fourier-Galerkin method (the number of basis elements was $N \times N$). We then proved the existence of a solution u to (4) in an H_0^1-ball $\overline{B}(\hat{u}, r_1; \|\cdot\|_{H_0^1(\Omega)})$ and an L^∞-ball $\overline{B}(\hat{u}, r_2; \|\cdot\|_{L^\infty(\Omega)})$, both centered around the approximation $\hat{u}$, using the method in [2] combined with the method in [6]. Note that the verified solution has the regularity to be in $C^2(\Omega) \cap C(\overline{\Omega})$ regardless of the regularity of the approximation $\hat{u}$, owing to the argument given in Remark 3. Table 1 presents the verification result, which ensures the positiveness of the verified solution to (4) centered around $\hat{u}$, owing to the condition that $\sup\{(u_-(x))^2 | x \in \Omega\} < \lambda_1$. Here, the upper bounds of $\sup\{(u_-(x))^2 | x \in \Omega\}$ were calculated by $(|\min\{\hat{u}(x) | x \in \Omega\}| + r_2)^2$ with verification.

Remark 5 *The verified solution centered around $\hat{u}$ corresponds to the unique solution of* (1)*, since the problem*

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits only one solution for $p > 1$ when $\Omega \subset \mathbb{R}^2$ is bounded and convex (*a proof can be found in, e.g.,* [13])*. On the other hand, the problem without the property of positivity, given by*

$$\begin{cases} -\Delta u = u^3 & \text{in } (0,1)^2, \\ u = 0 & \text{on } \partial(0,1)^2, \end{cases} \tag{7}$$

admits an infinite number of solutions. Indeed, if u is a solution to (7)*, then $v(x,y) := a^{-1}u(a^{-1}x, a^{-1}y)$ and $-v$ are solutions to*

$$\begin{cases} -\Delta v = v^3 & \text{in } (0,a)^2, \\ v = 0 & \text{on } \partial(0,a)^2. \end{cases}$$

Therefore, by setting $a = 2^{-m}$ $(m = 1,2,3,\dots)$ as one example, one can construct an infinite number of solutions to (7) *through suitable symmetrical reflections. In Fig. 2, we present some approximations of solutions to* (7)*. Solutions to* (7) *might exist in neighborhoods of these approximations; however, they are not allowed to be positive. Just for reference, we indicate here that the minimum value of an approximate solution to* (7) *should be greater than $-\sqrt{\lambda_1} (= -\sqrt{2}\pi \le -4.44)$ to ensure that $\sup\{(u_-(x))^2 | x \in \Omega\} < \lambda_1$, and the approximations in Fig. 2 do not satisfy this condition.*

For our second example, we selected the case in which Ω is the pentagon shaped domain Ω displayed in Fig. 3. The vertexes of Ω are $(1, -0.1)$, $(0.5, 1)$, $(-0.5, 0.5)$, $(-0.75, -0.875)$, and $(0.5, -0.75)$. In this case, the problem (1) again admits only one solution. We computed the approximate solution $\hat{u}$ to (4), as displayed in Fig. 3, using a piecewise quadratic finite element basis (the mesh size was 0.02525). Using the same method as in the first example, as detailed in [2, 6], we verified the existence of a solution to (4) in the balls $\overline{B}(\hat{u}, r_1; \|\cdot\|_{H_0^1(\Omega)})$ and $\overline{B}(\hat{u}, r_2; \|\cdot\|_{L^\infty(\Omega)})$, which also has C^2 regularity. It can be seen from Table 2 that the positiveness of the verified solution is again ensured. The lower bound of λ_1 (denoted by $\underline{\lambda_1}$ in Table 2) was verified using the method from [14]. The upper bound of $\sup\{(u_-(x))^2 | x \in \Omega\}$ in Table 2 was calculated in the same way as in the first example.

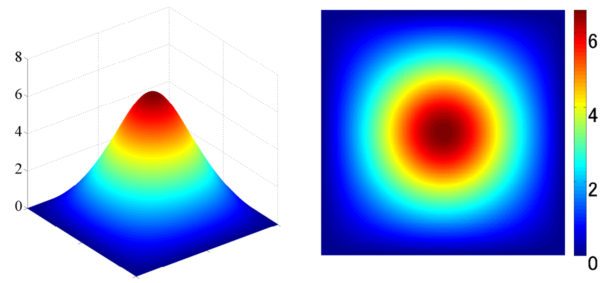

Fig. 1. An approximation of the unique solution to (1) on the square $\Omega = (0,1)^2$.

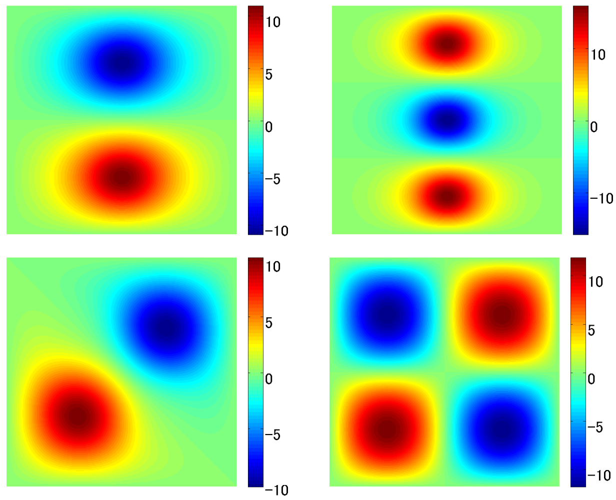

Fig. 2. Approximations of nonpositive solutions to (4) on the square $\Omega = (0,1)^2$.

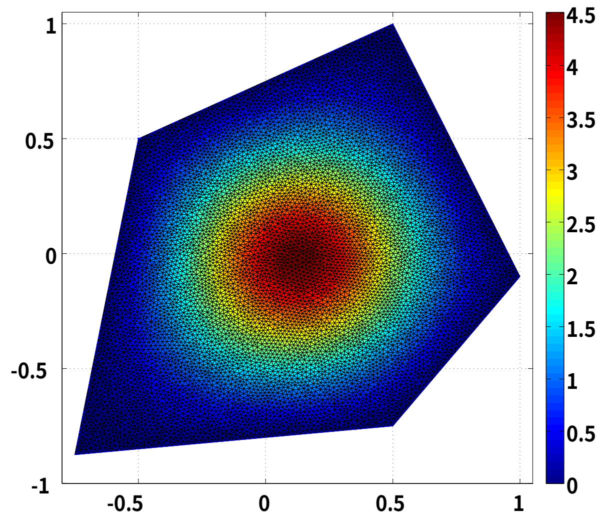

Fig. 3. An approximate solution to (4) on the pentagon shaped domain Ω.

4. Conclusion

We have proposed a numerical method for verifying the positiveness of solutions to (4). We have demonstrated that the positiveness of solutions to (4) can be

Table 1. Verification result for the approximation displayed in Fig. 2 on the square $\Omega = (0,1)^2$.

N	r_1	r_2	$\sup(u_-(x))^2$	λ_1
10	3.03867E-02	9.88804E-02	9.77733E-03	$(19 \leq) 2\pi^2$
20	1.59823E-06	5.21958E-06	2.72440E-11	″
30	6.34399E-11	2.07371E-10	4.30024E-20	″

Table 2. Verification result on the pentagon shaped domain Ω.

r_1	r_2	$\sup(u_-(x))^2$	$\underline{\lambda_1}$
2.55806E-02	1.26535E-01	4.00060E-02	9.12780

verified using a simple calculation on the basis of Theorem 2, which provides a sufficient condition for the positiveness of solutions to elliptic problems. We have also noted that this method can be naturally applied to other boundary value problems, although we focused on homogeneous Dirichlet boundary value problems in this paper.

Acknowledgments

The authors express their sincere thanks to Professor M. Plum in Karlsruhe Institute of Technology, Germany for his valuable comments and kind remarks. They also express their profound gratitude to an anonymous referee for attentive reviews and helpful comments.

References

[1] M. T. Nakao, Numerical verification methods for solutions of ordinary and partial differential equations, Numer. Func. Anal. Opt., **22** (2001), 321–356.
[2] M. Plum, Computer-assisted enclosure methods for elliptic differential equations, Linear Algebra Appl., **324** (2001), 147–187.
[3] M. Plum, Existence and multiplicity proofs for semilinear elliptic boundary value problems by computer assistance, Jahresber. Dtsch. Math.-Ver., **110** (2008), 19–54.
[4] A. Takayasu, X. Liu, and S. Oishi, Remarks on computable a priori error estimates for finite element solutions of elliptic problems, NOLTA, **5** (2014), 53–63.
[5] M. T. Nakao, K. Hashimoto and Y. Watanabe, A numerical method to verify the invertibility of linear elliptic operators with applications to nonlinear problems, Computing, **75** (2005), 1–14.
[6] K. Tanaka, A. Takayasu, X. Liu and S. Oishi, Verified norm estimation for the inverse of linear elliptic operators using eigenvalue evaluation, Jpn. J. Ind. Appl. Math., **31** (2014), 665–679.
[7] A. Henrot, Extremum Problems for Eigenvalues of Elliptic Operators, Birkhäuser-Verlag, Basel, 2006.
[8] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Classics Appl. Math., SIAM, Philadelphia, 2011.
[9] M. H. Protter and H. F. Weinberger, Maximum Principles in Differential Equations, Springer, New York, 2012.
[10] M. Renardy and R. C. Rogers, An Introduction to Partial Differential Equations, Springer, New York, 2006.
[11] S. M. Rump, INTLAB - INTerval LABoratory, Developments in Reliable Computing, Tibor Csendes ed., pp. 77–104, Kluwer Academic Publishers, Dordrecht, 1999, `http://www.ti3.tuhh.de/rump/`.
[12] M. Kashiwagi, KV library, 2014, `http://verifiedby.me/kv/`.
[13] C. S. Lin, Uniqueness of least energy solutions to a semilinear elliptic equation in $\mathbb{R}^2$, Manuscripta Math., **84** (1994), 13–19.
[14] X. Liu and S. Oishi, Verified eigenvalue evaluation for the laplacian over polygonal domains of arbitrary shape, SIAM J. Numer. Anal., **51** (2013), 1634–1654.