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Comparison of local risk minimization and delta hedging strategy for exponential Lévy models

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Abstract

We discuss the differences of local risk minimization (LRM) and delta hedging strategies, in exponential Lévy models, where delta hedging strategies in this paper are defined under the minimal martingale measures (MMM). First of all we give inequality estimations for the differences of LRM and delta hedging strategies, and then show numerical examples for the two typical exponential Lévy models, Merton models and variance Gamma (VG) models.

Keywords local risk minimization, delta hedging strategy, fast Fourier transform, exponential Lévy models

Research Activity Group Mathematical Finance

1. Introduction

The concept of local risk minimization (LRM) is widely used for contingent situations in an incomplete market framework. LRM is closely related to the equivalent martingale measure which is well known as the minimal martingale measure (MMM). For more details on LRM, see [1, 2]. Delta hedging strategies, which are also well-known and often used by practitioners, are given by differentiating the option price under a certain martingale measure with respect to the underlying asset price. Due to the relationship between LRM and the MMM, we consider delta hedging strategies under the MMM. Its precise definition will be introduced in Section 2.

The paper [2] showed explicit representations of LRM for call options by using Malliavin calculus for Lévy processes based on the canonical Lévy space. Carr and Madan introduced a numerical method for valuing options based on the fast Fourier transform (FFT) in [3]. In [1], the authors adopted Carr and Madan's method to compute LRM of call options for exponential Lévy models. In particular, the authors discussed Merton models and variance Gamma (VG) models as typical examples of exponential Lévy models.

This paper aims to illustrate, based on [2], how different is LRM from delta hedging strategies for call options in exponential Lévy models. Furthermore, we show that delta hedging strategies are easily calculated by using the numerical scheme developed in [1]. We give inequality estimations of the differences of LRM and delta hedging strategies for the typical exponential Lévy models, known as Merton models and VG models. Merton models are composed of a Brownian motion and compound Poisson jumps with normally distributed jump sizes. VG models, which are exponential Lévy processes with infinitely many jumps in any finite time interval and no Brownian component, are the second example. We show that the difference of LRM and delta hedging strategies converges to zero when moneyness tends to zero or infinity. In addition to this, we give numerical results of the difference of LRM and delta hedging strategies since there are mathematical difficulties to follow the behaviours of the option prices around at the money.

2. Notations and preliminaries

We consider a financial market composed of one risk-free asset and one risky asset with finite maturity $T > 0$. For simplicity, we assume that market's interest rate is zero, that is, the price of the risk-free asset is 1 at all times. The fluctuation of the risky asset is assumed to be described by an exponential Lévy process S on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, described by

$$S_t := S_0 \exp\left(\mu t + \sigma W_t + \int_{\mathbb{R}_0} x\widetilde{N}([0,t], dx)\right)$$

for any $t \in [0,T]$, where $S_0 > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, and $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. Here W is a one-dimensional Brownian motion and $\widetilde{N}$ is the compensated version of a Poisson random measure N. Denoting the Lévy measure of N by ν, we have $\widetilde{N}([0,t], A) = N([0,t], A) - t\nu(A)$ for any $t \in [0,T]$ and $A \in \mathcal{B}(\mathbb{R}_0)$. Moreover, S is also a solution of the stochastic differential equation

$$dS_t = S_{t-}\left[\mu^S dt + \sigma dW_t + \int_{\mathbb{R}_0} (e^x - 1)\widetilde{N}(dt, dx)\right],$$

where $\mu^S := \mu + (1/2)\sigma^2 + \int_{\mathbb{R}_0}(e^x - 1 - x)\nu(dx)$. Without loss of generality, we may assume that $S_0 = 1$ for simplicity. Now, defining $L_t := \log S_t$ for all $t \in [0,T]$, we obtain a Lévy process L. Moreover, $dM_t := S_{t-}\big[\sigma dW_t + \int_{\mathbb{R}_0}(e^x - 1)\widetilde{N}(dt, dx)\big]$ is the martingale part of S.

Our focus is to compare LRM to delta hedging strate-

gies with respect to a call option $(S_T - K)^+$ with strike price $K > 0$. We first give some preparations and assumptions to introduce an explicit LRM representation of such options in exponential Lévy models. Define the MMM $\mathbb{P}^*$ as an equivalent martingale measure under which any square-integrable $\mathbb{P}$-martingale orthogonal to M remains a martingale. Its density is given by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(-\xi W_T - \frac{\xi^2}{2}T + \int_{\mathbb{R}_0} \log(1-\theta_x) N([0,T], dx) + T\int_{\mathbb{R}_0} \theta_x \nu(dx)\right),$$

where

$$\xi := \frac{\mu^S \sigma}{\sigma^2 + \int_{\mathbb{R}_0}(e^y - 1)^2 \nu(dy)},$$

$$\theta_x := \frac{\mu^S (e^x - 1)}{\sigma^2 + \int_{\mathbb{R}_0}(e^y - 1)^2 \nu(dy)}$$

for $x \in \mathbb{R}_0$. In the development of our approach, we rely on the following assumption.

Assumption 1

(1) $\int_{\mathbb{R}_0}(|x| \vee x^2)\nu(dx) < \infty$, *and* $\int_{\mathbb{R}_0}(e^x - 1)^n \nu(dx) < \infty$ *for* $n = 2, 4$.

(2) $0 \geq \mu^S > -\sigma^2 - \int_{\mathbb{R}_0}(e^x - 1)^2 \nu(dx)$.

The first condition ensures that μ^S, ξ, and θ_x are well defined, the square integrability of L, and the finiteness of $\int_{\mathbb{R}_0}(e^x - 1)^n \nu(dx)$ for $n = 1, 3$. The second condition guarantees that $\theta_x < 1$ for any $x \in \mathbb{R}_0$. Moreover, by the Girsanov theorem, $W_t^{\mathbb{P}^*} := W_t + \xi t$ and $\widetilde{N}^{\mathbb{P}^*}([0,t], dx) := \theta_x \nu(dx) t + \widetilde{N}([0,t], dx)$ are a $\mathbb{P}^*$-Brownian motion and the compensated Poisson random measure of N under $\mathbb{P}^*$, respectively. We can then rewrite L_t as $L_t = \mu^* t + \sigma W_t^{\mathbb{P}^*} + \int_{\mathbb{R}_0} x \widetilde{N}^{\mathbb{P}^*}([0,t], dx)$, where $\mu^* := -(1/2)\sigma^2 + \int_{\mathbb{R}_0}(x - e^x + 1)(1 - \theta_x)\nu(dx)$. Note that L is a Lévy process even under $\mathbb{P}^*$, with Lévy measure given by $\nu^{\mathbb{P}^*}(dx) := (1 - \theta_x)\nu(dx)$. LRM will be given as a predictable process LRM_t, which represents the number of units of the risky asset the investor holds at time t. We introduce a representation of LRM for call option. We define

$$I_1 := \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{S_T > K\}} S_T \mid \mathcal{F}_{t-}], \tag{1}$$

$$I_2 := \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[(S_T e^x - K)^+ - (S_T - K)^+ \mid \mathcal{F}_{t-}] \times (e^x - 1)\nu(dx), \tag{2}$$

where $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ is the $\mathbb{P}$-completed filtration generated by W and N. By using these symbols, we can write an explicit representation of LRM for call option $(S_T - K)^+$ as follows:

Proposition 2 ([2, Proposition 4.6]) *For any* $K > 0$ *and* $t \in [0, T]$,

$$LRM_t = \frac{\sigma^2 I_1 + I_2}{S_{t-}\left[\sigma^2 + \int_{\mathbb{R}_0}(e^x - 1)^2 \nu(dx)\right]}. \tag{3}$$

Next, we introduce integral representations of I_1 and I_2 given in [2] in order to show we can adopt Carr and Madan's method. The characteristic function of L_{T-t} under $\mathbb{P}^*$ is denoted by $\phi_{T-t}(z) := \mathbb{E}_{\mathbb{P}^*}[e^{izL_{T-t}}]$ for $z \in \mathbb{C}$. We induce an integral representation for I_1 with ϕ_{T-t} firstly.

$$\begin{aligned} I_1 &= \mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{S_T > K\}} \cdot S_T \mid \mathcal{F}_{t-}] \\ &= \frac{1}{\pi}\int_0^\infty \frac{K^{-iv-\alpha+1}}{\alpha - 1 + iv} \phi_{T-t}(v - i\alpha) S_{t-}^{\alpha+iv} dv \\ &= \frac{e^k}{\pi}\int_0^\infty e^{-i(v-i\alpha)k} \psi_1(v - i\alpha) dv \end{aligned} \tag{4}$$

where $k := \log K$ and $\psi_1(z) := \left(\phi_{T-t}(z) S_{t-}^{iz}\right)/(iz - 1)$ and $\alpha \in (1, 2]$. Note that the right-hand side is independent of the choice of α. We turn next to I_2. Denoting $\psi_2(z) := \left(\phi_{T-t}(z) S_{t-}^{iz}\right)/[(iz-1)iz]$ and $\zeta := v - i\alpha$, we have

$$\begin{aligned} I_2 &= \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*}[(S_T e^x - K)^+ - (S_T - K)^+ \mid \mathcal{F}_{t-}] \times (e^x - 1)\nu(dx) \\ &= \frac{1}{\pi}\int_0^\infty K^{-i\zeta+1} \int_{\mathbb{R}_0} (e^{i\zeta x} - 1)(e^x - 1)\nu(dx)\psi_2(\zeta) dv. \end{aligned} \tag{5}$$

Note that we can not calculate (5) numerically as it stands, because it is not possible to compute the integral $\int_{\mathbb{R}_0}(e^{i\zeta x} - 1)(e^x - 1)\nu(dx)$ directly. Thus, we introduce model-dependent calculations for Merton models in Section 3 and for VG models in Section 4, respectively. Regarding LRM_t, I_1, and I_2 as functions of S_{t-} and K, we have $I_i(S_{t-}, K)/S_{t-} = I_i(1, K/S_{t-})$ for $i = 1, 2$. We obtain

$$LRM_t(S_{t-}, K) = \frac{\sigma^2 I_1(1, K/S_{t-}) + I_2(1, K/S_{t-})}{\sigma^2 + \int_{\mathbb{R}_0}(e^x - 1)^2 \nu(dx)}$$

from (3). As a result, LRM_t is given as a function of $K/S_{t-} =: \chi_{t-}$, where χ_{t-} is called *moneyness*. Thus, we denote LRM_t by $LRM_t(\chi_{t-})$. Hereinafter we fix $\alpha \in (1, 2]$ arbitrarily. Moreover, we denote $\zeta := v - i\alpha$ for $v \in \mathbb{R}$, so we may regard ζ as a function of v.

Next, we define delta hedging strategies.

Definition 3 *For any* $K > 0$ *and* $s > 0$, *a delta hedging strategy under the minimal martingale measure is defined as*

$$\Delta_t^{\mathbb{P}^*}(\chi_{t-}) := \frac{\partial \mathbb{E}_{\mathbb{P}^*}[(S_T - K)^+ \mid S_{t-} = s]}{\partial s}.$$

Remark that the above definition of delta hedging strategies coincide with the usual delta hedging strategies in the case of Black–Scholes. The next theorem follows from the direct calculation.

Theorem 4

$$\Delta_t^{\mathbb{P}^*}(\chi_{t-}) = \frac{I_1}{S_{t-}}.$$

Remark 5 *Using the numerical scheme developed in* [1], *we can calculate* $\Delta_t^{\mathbb{P}^*}(\chi_{t-})$ *easily from Theorem* 4.

Remark 6 *The paper* [4] *introduced the definition of*

Δ-strategies which are generalized delta hedging strategies. The authors derived semi-explicit formulas for the mean-squared hedging error of a European-style contingent claim in terms of Δ-strategies. This has been done for delta hedging strategies including Black-Scholes hedging strategies. They also showed two numerical examples. First, they compared the performance of Black-Scholes strategies and variance-optimal strategies in the normal Gaussian Lévy model. Second, they assessed the hedging errors of Black-Scholes strategies, the delta hedge and the variance-optimal strategy in a diffusion-extended CGMY Lévy model. As in Example 3.2, they discussed the delta hedge by computing the derivatives of a price process with respect to the underlying exponential Lévy models. This delta hedge is equivalent to our $\Delta_t^{\mathbb{P}^*}$.

3. Merton jump-diffusion models

We consider the case where L is given as a Merton jump-diffusion process, which consists of a diffusion component with volatility $\sigma > 0$ and compound Poisson jumps with three parameters, $m \in \mathbb{R}$, $\delta > 0$, and $\gamma > 0$. Note that γ represents the jump intensity, and that the sizes of the jumps are distributed normally with mean m and variance δ^2. Thus, its Lévy measure ν is given by

$$\nu(dx) = \frac{\gamma}{\sqrt{2\pi}\,\delta} \exp\left(-\frac{(x-m)^2}{2\delta^2}\right) dx.$$

Note that the first condition of Assumption 1 is satisfied for any $m \in \mathbb{R}$, $\delta > 0$, and $\gamma > 0$. We consider only parameter sets satisfying the second condition of Assumption 1.

3.1 Mathematical preliminaries

Our aim here is to give an inequality estimation of $|LRM_t - \Delta_t^{\mathbb{P}^*}|$. An analytic form of ϕ_{T-t} was given in [1, Proposition 3.1] and of $\nu^{\mathbb{P}^*}$ can be seen in [1, Proposition 3.2] also.

Theorem 7 *There exists a positive constant $\mathcal{C}$ such that*

$$|LRM_t(\chi_{t-}) - \Delta_t^{\mathbb{P}^*}(\chi_{t-})| \le \mathcal{C}\chi_{t-}^{1-\alpha}. \tag{6}$$

We obtain, furthermore,

$$\lim_{\chi_{t-}\to 0} |LRM_t(\chi_{t-}) - \Delta_t^{\mathbb{P}^*}(\chi_{t-})| = 0. \tag{7}$$

This constant $\mathcal{C}$ can be written explicitly, and depends on the model parameters. We only give a rough proof for Theorem 7 here.

Step 1. Eq. (7) is shown by Lebesgue's dominated convergence theorem.

Step 2. Eq. (6) is implied by the following lemma:

Lemma 8 ([1, Proposition 3.4]) *We have*

$$|\phi_{T-t}(v - i\alpha)| \le \mathcal{C}_1 \exp\left(-\frac{\sigma^2 v^2 (T-t)}{2}\right)$$

for any $v \in \mathbb{R}$, where

$$\mathcal{C}_1 := \exp\left((T-t)\left[\alpha\mu^* + \frac{\sigma^2\alpha^2}{2} + \int_{\mathbb{R}_0} (e^{\alpha x} - 1 - \alpha x)\nu^{\mathbb{P}^*}(dx)\right]\right).$$

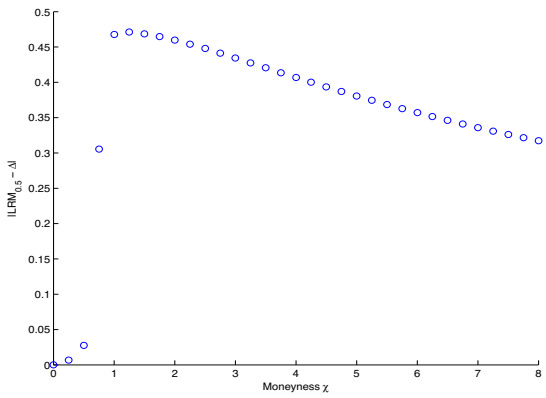

Fig. 1. Merton models, plotted $|LRM_{0.5} - \Delta_{0.5}^{\mathbb{P}^*}|$.

This lemma implies the next estimation:

$$|LRM_t - \Delta_t^{\mathbb{P}^*}| \le \frac{\mathcal{C}_1\mathcal{C}_2}{\sigma\sqrt{2\pi(T-t)}} \frac{\chi_{t-}^{1-\alpha}}{\sigma^2 + \int_{\mathbb{R}_0}(e^x - 1)^2\nu(dx)} = \mathcal{C}\chi_{t-}^{1-\alpha}$$

for some $\mathcal{C}_2$ depending on the parameters m, δ and α.

3.2 Numerical results

We compute $|LRM_t - \Delta_t^{\mathbb{P}^*}|$ with the FFT. In this subsection, we provide a numerical result for a Merton jump-diffusion model with parameters $T = 0.5$, $L_t = 0$, $\mu = -0.7$, $\sigma = 0.2$, $\gamma = 1$, $m = 0$, and $\delta = 1$. Note that μ^S is given by -0.03, which satisfies the second condition of Assumption 1. We compute and plot the data of $|LRM_{0.5} - \Delta_t^{\mathbb{P}^*}|$ shown as Fig. 1. FFT parameters are chosen as $N = 2^{14}$, $\eta = 0.025$ and $\alpha = 1.75$.

4. Variance Gamma models

We now consider the case where L is given as a variance Gamma process, which has three parameters $\kappa > 0$, $m \in \mathbb{R}$, and $\delta > 0$. This is defined as a time-changed Brownian motion with volatility δ, drift m, and subordinator G_t, where G_t is a Gamma process with parameters $(1/\kappa, 1/\kappa)$. In summary, L is represented as

$$L_t = mG_t + \delta B_{G_t} \quad \text{for } t \in [0, T],$$

where B is a one-dimensional standard Brownian motion. Moreover, the Lévy measure of L is given by

$$\nu(dx) = C\left(\mathbf{1}_{\{x<0\}}e^{-G|x|} + \mathbf{1}_{\{x>0\}}e^{-M|x|}\right)\frac{dx}{|x|}$$

where

$$C := \frac{1}{\kappa} > 0,$$

$$G := \frac{1}{\delta^2}\sqrt{m^2 + \frac{2\delta^2}{\kappa}} + \frac{m}{\delta^2} > 0,$$

$$M := \frac{1}{\sqrt{m^2 + \frac{2\delta^2}{\kappa}}}\delta^2 - \frac{m}{\delta^2} > 0.$$

In addition, we assume $M > 4$, which ensures that the first condition of Assumption 1 holds. An analytic form of ϕ_{T-t} was given in [1, Proposition 4.5], and that of $\nu^{\mathbb{P}^*}$

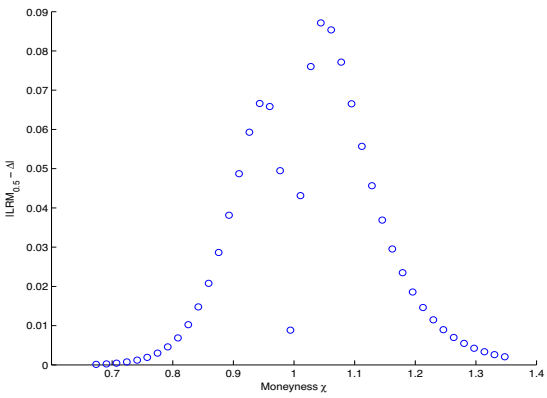

Fig. 2. Variance Gamma models, plotted $|LRM_{0.5} - \Delta_{0.5}^{\mathbb{P}^*}|$.

can be seen in [1, Proposition 4.3] also.

The following theorem is the same estimation as Theorem 7.

Theorem 9 *There exists a positive constant $\mathcal{C}$ such that*

$$|LRM_t(\chi_{t-}) - \Delta_t^{\mathbb{P}^*}(\chi_{t-})| \leq \mathcal{C}\chi_{t-}^{1-\alpha}.$$

We obtain, furthermore,

$$\lim_{\chi_{t-}\to 0} |LRM_t(\chi_{t-}) - \Delta_t^{\mathbb{P}^*}(\chi_{t-})| = 0.$$

We can prove this theorem in the same manner as Theorem 7, but **Step 2** is slightly different.

In **Step 2**, we use the next lemma instead of Lemma 8.

Lemma 10 ([1, Proposition 4.7]) *For any $v \in \mathbb{R}$,*

$$|\phi_{T-t}(v - i\alpha)| \leq \mathcal{C}_2 |v|^{-2C(T-t)},$$

where

$$\mathcal{C}_2 = (GM)^{(1+h)(T-t)C}[(G+1)(M-1)]^{-h(T-t)C} \times \exp\left((T-t)\alpha\left[\mu^* + (1+h)C\frac{M-G}{GM} - hC\frac{M-G-2}{(G+1)(M-1)}\right]\right).$$

4.1 Numerical results

In this subsection, we compute $|LRM_t - \Delta_t^{\mathbb{P}^*}|$ for a VG model with a parameter set based on market data. We use the Nikkei 225 index for March 2014, as in [1]. We need to set the log price $L_t := \log(S_t/S_0)$, where S_0 is the price on 28 February 2014, which is 14841.07. The parameters C, G, and M are estimated from the mean, variance, and skewness of the log price by using the generalized method of moments and the Levenberg–Marquardt method. The values of C, G and M are $C = 2.469395026815120$, $G = 23.743109051760964$ and $M = 24.903251787154687$. For $G - M \approx -1.16$, this parameter set satisfies Assumption 1. We take $T = 1$ and $S_{t-} = 14841.07$, that is, $L_{t-} = 0$. We fix t to 0.5, the values of $LRM_{0.5}$ and $\Delta_{0.5}^{\mathbb{P}^*}$ are computed for $K = 10000, 10250, \ldots, 20000$. The computational results are given as Fig. 2.

5. Conclusion

For Merton models and VG models, we have derived inequality estimations for the differences of LRM_t and $\Delta_t^{\mathbb{P}^*}$. Moreover the difference converges to zero when moneyness tends to zero or infinity. We have computed the behaviours of $|LRM_t - \Delta_t^{\mathbb{P}^*}|$ for two cases. The first case is a Merton model with an artificial parameter set. The other is a VG model with a parameter set based on market data. Numerical examples have shown that the behaviours of $|LRM_t - \Delta_t^{\mathbb{P}^*}|$ are different between the two cases. We have deduced four points from the numerical experiments: (i) the differences in VG models have converged faster than the Merton models when moneyness tends to zero or infinity. (ii) Under the given conditions, the values of $|LRM_t - \Delta_t^{\mathbb{P}^*}|$ for the Merton models are larger than that for the VG models. (iii) For the Merton model, $|LRM_t - \Delta_t^{\mathbb{P}^*}|$ has the maximum value around at the money. (iv) For the VG model, the behaviours of $|LRM_t - \Delta_t^{\mathbb{P}^*}|$ are unstable around at the money.

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References

[1] T. Arai, Y. Imai and R. Suzuki, Numerical local risk minimization for exponential Lévy models, arXiv:1506.03898 [q-fin.CP], 2015

[2] T. Arai and R. Suzuki, Local risk-minimization for Lévy markets, J. Finan. Eng., 2015, in press.

[3] P. Carr and D. Madan, Option valuation using the fast Fourier transform, J. Comput. Finan., **2** (1999), 61–73.

[4] S. Denkl, M. Goy, J. Kallsen, J. Muhle-Karbe and A. Pauwels, On the performance of delta hedging strategies in exponential Lévy models, Quant. Finance, **13** (2013), 1173–1184.