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Generalization of log-aesthetic curves by Hamiltonian formalism

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Abstract

In the field of industrial shape design, the plane curves which have radii of curvature proportional to the power of linear functions of their arc-length parameters are called the log-aesthetic curves (LAC) and have been investigated. However, the well-used curves, for example, the parabolic arcs and the typical curves of Mineur et al. are not contained in the family of LACs. In this letter we generalize LAC by the Hamiltonian formalism. This extended family of curves contains some well-known plane curves in classical differential geometry.

Keywords log-aesthetic curve, similarity curvature, Riccati equation, Hamiltonian system, quasi aesthetic curve

Research Activity Group Applied Integrable Systems

1. Introduction

This letter deals with the mathematical formulation of certain family of plane curves with monotonous curvature, which are used in the digital style design of industrial products.

From the designers' viewpoints Harada et al. [1–3] analyzed quantitatively characteristics of the plane curves with monotonous curvature radii utilized in the style design of industrial products such as automobiles. Their important observation is that "logarithmic distribution diagrams of curvature" (LDDC) of automobiles' keylines are approximately linear.

In [1–3] LDDC is defined by using the histogram, but in this letter we use the analytic definition formulated by Nakano et al. [4] and Miura et al. [5]. Let $\boldsymbol{r}(\mathrm{s}) = (\mathrm{x}(\mathrm{s}), \mathrm{y}(\mathrm{s}))$ be a smooth plane curve with monotonous curvature parametrized by the arc-length parameter s and $\rho = \rho(s)$ the radius of curvature. Then LDDC of the curve $\boldsymbol{r}(\mathrm{s})$ is a curve on the XY plane defined by $(X(s), Y(s)) = (\log\rho, \log(|ds/d(\log\rho)|))$.

Miura et al. [5] derived the general formula of the radius of curvature of aesthetic curves whose LDDC are given by a straight line with the slope α as follows:

$$\rho(s) = \begin{cases} c_0 e^{c_1 s} & (\alpha = 0), \\ (c_0 s + c_1)^{1/\alpha} & (\alpha \neq 0). \end{cases} \tag{1}$$

Yoshida et al. [6] clarified the overall shapes of the above aesthetic curves. Now the above family of curves is called the log-aesthetic curve (LAC) and has been studied actively in the field of digital style design. The family of LACs contains well-known curves with monotonous curvature. For example, $\alpha = -1$ corresponds to the clothoid curves and $\alpha = 1$ corresponds to the logarithmic spirals.

The authors [7] noticed that the slopes of LDDC of LAC can be formulated from the similarity geometry. In the similarity geometry the angular parameter θ is invariant in stead of the arc-length and the invariant $S(\theta) := -\rho_\theta/\rho$ is called the similarity curvature of the curve $\boldsymbol{r}(\theta) = (\mathrm{x}(\theta), \mathrm{y}(\theta))$. Then the slope α of LDDC of LAC can be expressed by

$$\alpha = \frac{S_\theta}{S^2} + 1. \tag{2}$$

For the rest of this letter we refer the slope of LDDC to $\gamma + 1$, because we want to use α for the other purpose. Rewriting (2), we show that the similarity curvature of LAC satisfies the Riccati equation of the constant coefficient γ of the following form:

$$S_\theta = \gamma S^2. \tag{3}$$

Solving (3) by quadrature, we obtain the similarity curvature of LAC as follows:

$$S(\theta) = \frac{-1}{\gamma\theta + \delta}, \tag{4}$$

where δ is certain integral constant.

We will take certain family of plane curves including parabolas as research targets.

2. Parabolic arcs and typical curves of Mineur et al.

Vertex-free parabolic arcs are often-used plane curves with monotonous curvature in the industrial style design. Harada et al. [1–3] also regarded parabolas as important examples of aesthetic curves in their early researches. However, since the LDDC of parabolic arcs are only approximately linear, the general expressions of LAC, which were developed by Miura et al. [5], can not be applied to the parabolic arcs.

From the viewpoint of the similarity geometry, the similarity curvature of the parabolic arc $y = x^2 (x \geq 0)$

parametrized by the angular parameter $\theta (0 \leq \theta < \pi/2)$ is expressed by

$$S(\theta) = -3\tan(\theta) \tag{5}$$

and satisfies the Riccati equation

$$S_\theta = \frac{-1}{3}S^2 - 3. \tag{6}$$

Eq. (5) and (6) show that the parabola differs from LAC.

Other examples of the aesthetic curves excluded from LAC are the typical curves of Mineur et al. [8]. The typical curve of degree $m (m \geq 3)$ has the features generalizing the parabola. The similarity curvature of the typical curve of degree $m (m \geq 3)$ satisfies the following Riccati equation

$$S_\theta = \frac{-1}{(m+1)}S^2 - \frac{(m+1)}{(m-1)^2}, \tag{7}$$

which is regarded as a generalization of the one of the parabola.

Yoshida et al. [9, 10] investigated the quasi aesthetic curves approximating the linearity of LDDC of LAC or approximating the Taylor series expansions of LAC by the polynomial forms.

In the next section we will investigate the quasi aesthetic curve whose similarity curvature satisfies the following Riccati equation

$$S_\theta = \gamma S^2 + \alpha, \tag{8}$$

where $\gamma \neq 0, \alpha$ are constants. Since the above equation (8) is the generalization of (6) and (7), the proposed family of the aesthetic curves includes the parabola and the typical curves of Mineur et al. We will derive (8) by the Hamiltonian formalism.

3. Generalization of LAC by Hamiltonian formalism

Let $\boldsymbol{r}(\theta)$ be a smooth plane curve parametrized by the angular parameter θ varying with the monotonous curvature in the domain $\theta_0 < \theta < \theta_1$. Assume that the similarity curvature $S(\theta)$ of the curve $\boldsymbol{r}(\theta)$ satisfies the Riccati equation

$$S_\theta = \gamma S^2 + \beta S + \alpha, \tag{9}$$

where $\gamma \neq 0$ and β, α are constant coefficients.

By using the curvature radius $\rho(\theta)$ of the curve, we define the generalized coordinate q, the generalized momentum p and the Hamiltonian H as follows:

$$q := \rho^\gamma, \tag{10}$$

$$p := \frac{\exp(-\beta\theta)}{\gamma^2}q_\theta, \tag{11}$$

$$H(p, q, \theta) := \frac{\gamma^2\exp(\beta\theta)}{2}p^2 + \frac{\alpha\exp(-\beta\theta)}{2\gamma}q^2. \tag{12}$$

We have the above Hamiltonian (12) by the suggestion of [11, Exercise 8–35 (p.367)]. Here we remark that Miura et al. [12, 13] investigated a variational formulation of LAC. They used the Lagrangian based on the linearity of LDDC. In contrast, our Hamiltonian is based on harmonic oscillations and its idea is different from [12, 13].

From the canonical equations of Hamilton

$$q_\theta = \frac{\partial H}{\partial p},\ p_\theta = -\frac{\partial H}{\partial q}, \tag{13}$$

we obtain second-order linear differential equation

$$q_{\theta\theta} - \beta q_\theta + \gamma\alpha q = 0. \tag{14}$$

From (14) the Riccati equation (9) can be derived via the Cole-Hopf transformation

$$S = \frac{-1}{\gamma}\frac{q_\theta}{q}. \tag{15}$$

Now assume that the Hamiltonian (12) does not contain the variable θ explicitly. Based on elementary arguments of Hamiltonian system, we can conclude $\beta = 0$ and we have

$$q_{\theta\theta} + \gamma\alpha q = 0, \tag{16}$$

and the Riccati equation (8) via the Cole-Hopf transformation (15). Here we remark that the equation (16) is satisfied by the eigenfunctions of the eigenvalue $(-\gamma\alpha)$ according to the second-order differential operator $d^2/d\theta^2$.

The following proposition is the main result of this letter.

Proposition 1 *Under the above conditions the similarity curvature $S(\theta)$ of the plane curve except circular arcs and logarithmic spirals is classified by the sign of the eigenvalue $(-\gamma\alpha)$ of the operator $d^2/d\theta^2$:*

(a) case of $(-\gamma\alpha) = 0$

$$S(\theta) = \frac{1}{\textit{a linear function of } \theta},$$

(b) case of $(-\gamma\alpha) < 0$

$$S(\theta) = \frac{\sqrt{\gamma\alpha}}{\gamma}\tan((\sqrt{\gamma\alpha})\theta - \delta),$$

where δ is some real constant,

(c) case of $(-\gamma\alpha) > 0$

$$S(\theta) = \frac{\sqrt{-\gamma\alpha}}{(-\gamma)}\tanh((\sqrt{-\gamma\alpha})\theta + \delta),$$

or

$$S(\theta) = \frac{\sqrt{-\gamma\alpha}}{(-\gamma)}\coth((\sqrt{-\gamma\alpha})\theta + \delta),$$

where δ is some real constant.

Proof (a) In this case q is a linear function of θ. Since q is not constant by excluding circular arcs, we have the conclusion.

(b) Since $q \neq 0$, we have the linear combination

$$q = A\cos(\sqrt{\gamma\alpha}\theta) + B\sin(\sqrt{\gamma\alpha}\theta)$$

with some real coefficients A, B satisfying $A^2 + B^2 > 0$. So we can select some real constant δ such that

$$q = \sqrt{A^2 + B^2}\cos(\sqrt{\gamma\alpha}\theta - \delta),$$

and we reach the conclusion.

(c) We have the linear combination

$$q = A\exp(\sqrt{-\gamma\alpha}\theta) + B\exp(-\sqrt{-\gamma\alpha}\theta)$$

Table 1. The family of parabola.

curve list	(γ, α)
catenary	(-1/2, -2)
cycloid	(1, 1)
asteroid	(1, 4)
lemniscate	(-2, -2/9)
rectangular hyperbola	(-2/3, -6)

with some real coefficients A, B. By using hyperbolic functions, q is expressed as

$$q = (A+B)\cosh(\sqrt{-\gamma\alpha}\theta) + (A-B)\sinh(\sqrt{-\gamma\alpha}\theta).$$

Here we note $AB \neq 0$ by excluding logarithmic spirals. There are two cases.

(c-1) When $AB > 0$, we can select some real constant δ such that

$$\cosh(\delta) = \pm\frac{A+B}{\sqrt{4AB}},$$
$$\sinh(\delta) = \frac{A-B}{\sqrt{4AB}}.$$

Then we have

$$q = \pm\sqrt{4AB}\cosh(\sqrt{-\gamma\alpha}\theta \pm \delta)$$

and

$$S(\theta) = \frac{\sqrt{-\gamma\alpha}}{(-\gamma)}\tanh(\sqrt{-\gamma\alpha}\theta \pm \delta),$$

where double-sign corresponds.

(c-2) When $AB < 0$, we can select some real constant δ such that

$$\sinh(\delta) = \frac{A+B}{\sqrt{-4AB}},$$
$$\cosh(\delta) = \pm\frac{A-B}{\sqrt{-4AB}}.$$

Then we have

$$q = \pm\sqrt{-4AB}\sinh(\sqrt{-\gamma\alpha}\theta \pm \delta)$$

and

$$S(\theta) = \frac{\sqrt{-\gamma\alpha}}{(-\gamma)}\coth(\sqrt{-\gamma\alpha}\theta \pm \delta).$$

where double-sign corresponds.

(QED)

In Proposition 1 the case (a) corresponds to the log-aesthetic curves except circular arcs and logarithmic spirals. Here we will call the family of curves corresponding to the case (b) "the family of parabola", because it contains parabolas. The case (c) includes spirals. We will call the family of curves corresponding to the case (c) "the family of quasi aesthetic spirals". Moreover we will collectively call the curves corresponding to the case (b) and (c) "the quasi aesthetic curves".

4. Examples of the quasi aesthetic curves

The parabolas (6) and the typical curves (7) of the degree m are the quasi aesthetic curves in "the family of parabola". It should be noted that "the family of parabola" contains some well-known plane curves in classical differential geometry. The following list (Table 1) shows some of the examples of well-known curves in "the family of parabola".

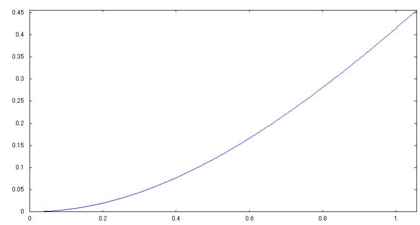

Fig. 1. Rectangular hyperbola.

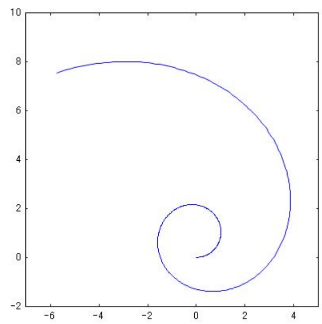

Fig. 2. $\gamma = 0.05, \alpha = -1.0$.

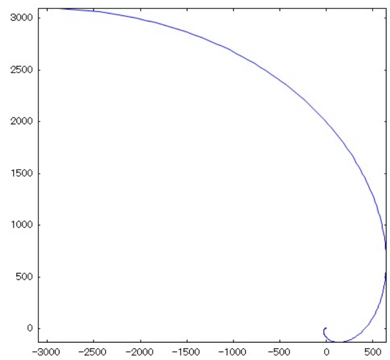

Fig. 3. $\gamma = 1.0, \alpha = -1.0$.

Fig. 1 shows the rectangular hyperbola from "the family of parabola".

The following curves (Figs. 2 and 3) are chosen from "the family of quasi aesthetic spirals". The curve of Fig. 2 corresponds to $\gamma = 0.05$, $\alpha = -1.0$ and $q = \rho^\gamma$ is expressed by cosh.

The curve of Fig. 3 corresponds to $\gamma = 1.0$, $\alpha = -1.0$ and $q = \rho^\gamma$ is expressed by sinh.

5. Concluding remarks and future problems

In this letter we generalized LACs to include the parabolas and obtained the family of plane curves which we will call the quasi aesthetic curves. The guiding principle of our generalization is the fact that the Hamiltonian (12) of the quasi aesthetic curves does not contain the variable θ explicitly. Remarkably, it follows that the family of the quasi aesthetic curves contains some well-known plane curves in classical differential geometry such as catenaries, cycloids, asteroids and lemniscates.

Finally, we will add two short remarks. At first, it turned out after we have completed the body of this letter that the family of parabola contains one of the two cases (i.e. the case where the angle difference is a linear function of the azimuth angle) of the polar-aesthetic curves which Miura et al. [14] presented previously. Here we remark that our approach and the one of [14] are different. Secondly, well-known plane curves in classical differential geometry defined by polar coordinates have been investigated by Sánchez-Reyes [15] from the viewpoint of computer aided geometric design. But our approach is based on the similarity geometry and is different from the approach of [15].

We will investigate the relation of these curves and LACs in the forthcoming publications.

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