

Mehdi Rahimi

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A LOCAL APPROACH TO g -ENTROPY

MEHDI RAHIMI

In this paper, a local approach to the concept of g -entropy is presented. Applying the Choquet's representation Theorem, the introduced concept is stated in terms of g -entropy.

Keywords: fuzzy entropy, g -entropy, local entropy

Classification: 28D20, 28E10

1. INTRODUCTION

The fuzzy entropy of dynamical systems is studied extensively [3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 17, 19, 20]. It is based on the idea of replacing partitions, in the classical ergodic theory, by fuzzy partitions.

The concept of g -entropy, as a generalized form of the fuzzy entropy, is studied by Mesiar and Rybarik [12, 17, 20] and its Hudetz correction is discussed in [16]. It is based on an increasing bijective map $g : [0, \infty] \rightarrow [0, \infty]$ such that $g(0) = 0$ and $g(1) = 1$.

In classical ergodic theory, local studies of entropy of dynamical systems is studied extensively [1, 11, 13, 21, 23]. A local study of the fuzzy entropy of dynamical systems, in the sense of Dumitrescu, was arranged in [14].

The main goal of this paper is to apply the method used in [14] to present a local approach to g -entropy of a dynamical system. This approach is of topological nature, in the sense that, the set of all g -decomposable measures is equipped by a topology which provides the requirements of the Choquet's representation Theorem. It enables us to state the introduced entropy in this paper in terms of the g -entropy [12, 17, 20].

In section 2, we recall some preliminary concepts. In section 3, we define a topology on the set of invariant measures and prove some results which leads to the g -ergodic decomposition of invariant measures. In section 4, a new version of g -entropy, in a local approach, is defined. Finally, the new quantity is stated in terms of g -entropy.

2. PRELIMINARY CONCEPTS

In this section, we provide some known facts which will be used in the remaining of the paper. From now on, $g : [0, \infty] \rightarrow [0, \infty]$ is an increasing bijective function such that $g(0) = 0$ and $g(1) = 1$. The following definitions are mainly from [16].

A fuzzy σ -algebra \mathcal{F} on X is a collection of fuzzy subsets of X , i. e., functions $f : X \rightarrow [0, 1]$, satisfying the following conditions:

- (i) $1_X \in \mathcal{F}$.
- (ii) If $f, g \in \mathcal{F}$ then $f.g \in \mathcal{F}$ and $(f-g)^+ \in \mathcal{F}$ where $(f-g)^+(x) := \max\{(f-g)(x), 0\}$ for all $x \in X$.
- (iii) If $\{f_n\}_{n=1}^\infty \subset \mathcal{F}$ then $\bigvee_{n=1}^\infty f_n \in \mathcal{F}$ where $\bigvee_{n=1}^\infty f_n := \min\{\sum_{n=1}^\infty f_n, 1\}$.

A function $m^* : \mathcal{F} \rightarrow [0, \infty)$ is called a fuzzy measure, if

- (i) $m^*(0_X) = 0$.
- (ii) $m^*(\bigvee_{n=1}^\infty f_n) = \sum_{n=1}^\infty m^*(f_n)$, whenever $\{f_n\}_{n=1}^\infty \subset \mathcal{F}$ and $\sum_{n=1}^\infty f_n \leq 1$.

A g -decomposable measure on \mathcal{F} is a mapping $m : \mathcal{F} \rightarrow [0, 1]$ such that $m(1_X) = 1$, $m(0_X) = 0$ and

$$m\left(g^{-1}\left(\sum_{n=1}^\infty g \circ f_n\right)\right) = g^{-1}\left(\sum_{n=1}^\infty g(m(f_n))\right)$$

whenever $f_n \in \mathcal{F}$ ($n = 1, 2, 3, \dots$) are such that $\sum_{n=1}^\infty g \circ f_n \leq 1$.

If m is a g -decomposable measure on \mathcal{F} then the function

$$m^* := g \circ m \circ g^{-1} \tag{1}$$

is a fuzzy measure on \mathcal{F} .

A family $\xi = \{f_1, f_2, \dots, f_k\}$ of members of \mathcal{F} is a g -fuzzy partition of X , if $\sum_{i=1}^k g \circ f_i = 1$ on X . When $g(x) = x$, a g -fuzzy partition is nothing but a fuzzy partition, i. e., a family $\xi = \{f_1, f_2, \dots, f_k\}$ such that $\sum_{i=1}^k f_i = 1$ on X . Note that, if $\xi = \{f_1, f_2, \dots, f_k\}$ is a g -fuzzy partition then $g(\xi) = \{g \circ f_1, g \circ f_2, \dots, g \circ f_k\}$ is a fuzzy partition.

The g -entropy $H_{m,g}$ of a g -fuzzy partition $\xi = \{f_1, f_2, \dots, f_k\}$ is defined by the formula

$$H_{m,g}(\xi) = g^{-1}\left(\sum_{i=1}^k g(\Phi(m(f_i)))\right)$$

where $\Phi = g^{-1} \circ \phi \circ g$ and $\phi(x) = -x \log x$ for $x \neq 0$, $\phi(0) = 0$. Hence

$$H_{m,g}(\xi) = g^{-1}\left(\sum_{i=1}^k \phi(m^*(g(f_i)))\right).$$

The joint of two g -fuzzy partitions $\xi = \{f_1, f_2, \dots, f_k\}$ and $\eta = \{h_1, h_2, \dots, h_t\}$ is defined by

$$\xi \vee \eta = \{g^{-1}((g \circ f_i)(g \circ h_j)) : i = 1, \dots, k, j = 1, \dots, t\}.$$

Note that, in the case of $g(x) = x$ for all $x \in X$, if ξ and η are two fuzzy partitions then

$$\xi \vee \eta = \{f_i h_j : i = 1, \dots, k, j = 1, \dots, t\}.$$

Suppose that $T : (X, \mathcal{B}) \rightarrow (X, \mathcal{B})$ is a measurable mapping and \mathcal{F} is a fuzzy σ -algebra of \mathcal{B} -measurable fuzzy subsets of X . For a g -decomposable measure m , and a g -fuzzy partition ξ , the g -entropy of T with respect to ξ is defined as:

$$h_{m,g}(T, \xi) := \lim_{n \rightarrow \infty} g^{-1} \left(\frac{1}{n} g \left(H_{m,g} \left(\bigvee_{i=0}^{n-1} T^{-i} \xi \right) \right) \right)$$

where $T^{-i} \xi = \{f_1 \circ T^i, f_2 \circ T^i, \dots, f_k \circ T^i\}$.

Finally, the g -entropy of T is defined by:

$$h_{m,g}(T) := \sup_{\xi} h_{m,g}(T, \xi)$$

where the supremum is taken over all g -fuzzy partitions. Note that, the fuzzy entropy $h_{m^*}(T)$ can be obtained putting $g(u) = u, u \in [0, 1]$. We recall that, the entropy of a fuzzy partition $\xi = \{f_1, f_2, \dots, f_k\}$ is given by $H_{m^*}(\xi) = - \sum_{i=1}^k m^*(f_i) \log m^*(f_i)$. The entropy of a dynamical system T with respect to the fuzzy entropy ξ is given by

$$h_{m^*}(T, \xi) = \lim_{n \rightarrow \infty} \frac{1}{n} H_{m^*} \left(\bigvee_{i=0}^{n-1} T^{-i} \xi \right)$$

and the fuzzy entropy of T is given by

$$h_{m^*}(T) = \sup_{\xi} h_{m^*}(T, \xi)$$

where the supremum is taken over all fuzzy partitions ξ .

The following theorem makes a connection between the concept of g -entropy and fuzzy entropy.

Theorem 2.1. Let T, g and ξ be as above and m^* be a fuzzy measure defined by (1). Then

- (i) $H_{m,g}(\xi) = g^{-1}(H_{m^*}(g(\xi)))$;
- (ii) $h_{m,g}(T, \xi) = g^{-1}(h_{m^*}(T, g(\xi)))$;
- (iii) $h_{m,g}(T) = g^{-1}(h_{m^*}(T))$.

Proof. See [18] Proposition 10.6.6, Proposition 10.6.11 and Theorem 10.6.13. □

Theorem 2.2. (Butnariu and Klement [2]) Suppose that \mathcal{F} is a fuzzy σ -algebra on X and m^* is a fuzzy measure on \mathcal{F} . Let $\mathfrak{B} = \{A \subset X : \chi_A \in \mathcal{F}\}$ and let $\mu_{m^*} : \mathfrak{B} \rightarrow \mathbb{R}$ be defined by $\mu_{m^*}(A) := m^*(\chi_A)$. Then every $f \in \mathcal{F}$ is \mathfrak{B} -measurable and $m^*(f) = \int_X f d\mu_{m^*}$ for all $f \in \mathcal{F}$.

Corollary 2.3. Let \mathcal{F} be a fuzzy σ -algebra on X and m is a g -decomposable measure on X . Then $m^* := g \circ m \circ g^{-1}$ is a fuzzy measure on \mathcal{F} and μ_{m^*} is a measure on the σ -algebra $\mathfrak{B} = \{A \subset X : \chi_A \in \mathcal{F}\}$ such that

$$m(f) = g^{-1} \left(\int_X g \circ f \, d\mu_{m^*} \right)$$

for all $f \in \mathcal{F}$.

3. TOPOLOGY ON INVARIANT MEASURES

In this section, let $T : X \rightarrow X$ be a continuous map on a compact metric space X . Let $\mathcal{F} \subset [0, 1]^X$ be the family of all Borel measurable maps $f : X \rightarrow [0, 1]$. Then the corresponding σ -algebra given by Theorem 2.2 is indeed the σ -algebra of Borel sets of X . The set of all fuzzy measures $m : \mathcal{F} \rightarrow [0, \infty]$, satisfying $m(1_X) = 1$ is denoted by $M^*(X)$. Let $g : [0, \infty] \rightarrow [0, \infty]$ be an increasing bijective function such that $g(0) = 0$ and $g(1) = 1$. The set of g -invariant measures of T is defined by

$$M_g^*(X, T) := \{m \in M^*(X) : m(g^{-1} \circ f \circ T) = m(g^{-1} \circ f) \, \forall f \in \mathcal{F}\}.$$

A g -decomposable measure $m \in M_g^*(X, T)$ is said to be g -ergodic, if the following implication holds for all $f \in \mathcal{F}$:

$$f \circ T = f \Rightarrow m(g^{-1} \circ f) = 0 \text{ or } 1.$$

The set of all g -fuzzy ergodic measures of T is denoted by $E_g^*(X, T)$.

The following lemma connects the g -invariant and g -ergodic measures to the classical invariant and ergodic measures.

Lemma 3.1. Let $T : X \rightarrow X$ be a continuous map on a compact metric space X . Let $M(X, T)$ and $E(X, T)$ be the set of invariant and ergodic measures of T in the classical sense respectively. Let m^* be a fuzzy measure defined by (1). Then

- (i) $m \in M_g^*(X, T)$ if and only if $\mu_{m^*} \in M(X, T)$.
- (ii) If $m \in E_g^*(X, T)$ then $\mu_{m^*} \in E(X, T)$.

Proof. (i) Let $m \in M_g^*(X, T)$. For any Borel set A , if $f = \chi_A$ then $m(g^{-1} \circ \chi_A \circ T) = m(g^{-1} \circ \chi_A)$, therefore

$$\begin{aligned} \mu_{m^*}(T^{-1}(A)) &= \int_X \chi_{T^{-1}(A)} \, d\mu_{m^*} \\ &= \int_X \chi_A \circ T \, d\mu_{m^*} \\ &= m^*(\chi_A \circ T) \\ &= g(m((g^{-1} \circ \chi_A) \circ T)) \\ &= g(m(g^{-1} \circ \chi_A)) \\ &= m^*(\chi_A) \\ &= \mu_{m^*}(A). \end{aligned}$$

This means $\mu_{m^*} \in M(X, T)$.

Conversely, let $\mu_{m^*} \in M(X, T)$. For any characteristic function $f = \chi_A$, where A is a Borel set, we have

$$\begin{aligned}
 m(g^{-1} \circ f \circ T) &= m(g^{-1} \circ \chi_A \circ T) \\
 &= g^{-1}(m^*(\chi_A \circ T)) \\
 &= g^{-1}\left(\int_X \chi_A \circ T \, d\mu_{m^*}\right) \\
 &= g^{-1}\left(\int_X \chi_{T^{-1}(A)} \, d\mu_{m^*}\right) \\
 &= g^{-1}(\mu_{m^*}(T^{-1}(A))) \\
 &= g^{-1}(\mu_{m^*}(A)) \\
 &= g^{-1}(m^*(\chi_A)) \\
 &= m(g^{-1} \circ \chi_A) \\
 &= m(g^{-1} \circ f),
 \end{aligned}$$

which gives the result for characteristic functions. Moreover, if $f = \sum_{i=1}^k c_i \chi_{A_i}$ is a simple function where $c_i, i = 1, 2, \dots, k$, are different real numbers and $A_i, i = 1, 2, \dots, k$, are pairwise disjoint Borel measurable, then

$$\begin{aligned}
 m(g^{-1} \circ f \circ T) &= g^{-1}(m^*(f \circ T)) \\
 &= g^{-1}\left(\int_X f \circ T \, d\mu_{m^*}\right) \\
 &= g^{-1}\left(\int_X \sum_{i=1}^k c_i \chi_{A_i} \, d\mu_{m^*}\right) \\
 &= g^{-1}\left(\sum_{i=1}^k c_i m^*(\chi_{A_i} \circ T)\right) \\
 &= g^{-1}\left(\sum_{i=1}^k c_i m^*(\chi_{A_i})\right) \\
 &= g^{-1}\left(\sum_{i=1}^k c_i \int_X \chi_{A_i} \, d\mu_{m^*}\right) \\
 &= g^{-1}\left(\int_X \sum_{i=1}^k c_i \chi_{A_i} \, d\mu_{m^*}\right) \\
 &= g^{-1}(m^*(f)) \\
 &= m(g^{-1} \circ f),
 \end{aligned}$$

which gives the result for simple functions.

Finally, let $f \in \mathcal{F}$. Let $\{f_n\}_{n=1}^\infty$ be a sequence of simple functions such that $0 \leq f_1 \leq$

$f_2 \leq f_3 \leq \dots$ and $f_n \nearrow f$. Applying Monotone Convergence Theorem we will have

$$\begin{aligned}
 m(g^{-1} \circ f \circ T) &= g^{-1}(m^*(f \circ T)) \\
 &= g^{-1}\left(\int_X f \circ T \, d\mu_{m^*}\right) \\
 &= g^{-1}\left(\lim_{n \rightarrow \infty} \int_X f_n \circ T \, d\mu_{m^*}\right) \\
 &= g^{-1}\left(\lim_{n \rightarrow \infty} m^*(f_n \circ T)\right) \\
 &= g^{-1}\left(\lim_{n \rightarrow \infty} m^*(f_n)\right) \\
 &= g^{-1}\left(\lim_{n \rightarrow \infty} \int_X f_n \, d\mu_{m^*}\right) \\
 &= g^{-1}\left(\int_X f \, d\mu_{m^*}\right) \\
 &= g^{-1}(m^*(f)) \\
 &= m(g^{-1} \circ f).
 \end{aligned}$$

It completes the proof of (i).

(ii) Let $m \in E_g^*(X, T)$. If A is a Borel measurable set such that $T^{-1}(A) = A$ then $\chi_{T^{-1}(A)} = \chi_A$ or equivalently $\chi_A \circ T = \chi_A$, therefore $m(g^{-1} \circ \chi_A) = 0$ or 1 , that is $g^{-1}(m^*(\chi_A)) = 0$ or 1 which means $g^{-1}(\mu_{m^*}(A)) = 0$ or 1 , consequently $\mu_{m^*}(A) = 0$ or 1 , since g is injective. It proves that $\mu_{m^*} \in E(X, T)$. \square

In the following, $M^*(X)$ is equipped by a topology in a natural way.

Definition 3.2. The w^* -topology on $M^*(X)$ is the smallest topology making each of the maps $m^* \mapsto \int_X f \, d\mu_{m^*}$ ($f \in C(X)$) continuous. A basis is given by the collection of all sets of the form

$$V_{m_0^*}(f_1, \dots, f_k; \epsilon) = \left\{ m^* \in M^*(X) : \left| \int_X f_i \, d\mu_{m^*} - \int_X f_i \, d\mu_{m_0^*} \right| < \epsilon, 1 \leq i \leq k \right\}$$

where $m_0^* \in M^*(X)$, $k \geq 1$, $f_i \in C(X)$ and $\epsilon > 0$.

By the correspondence given in Theorem 2.2, the previous topology is indeed the weak* topology defined on $M(X)$ in the classical case. So all of the properties of the weak* topology defined on $M(X)$ is inherited to $M^*(X)$. We summarize the most important properties of $M^*(X)$ in the following theorem.

Theorem 3.3. Let X be a compact metrizable space and let $T : X \rightarrow X$ be continuous. Let $\mathcal{F} \subset [0, 1]^X$ be the σ -algebra of all Borel measurable maps $f : X \rightarrow [0, 1]$. Then

(i) The space $M^*(X)$ is metrizable in the w^* -topology. If $\{f_n\}_{n=1}^\infty$ is a dense subset of $C(X)$ then

$$D(m, m') = \sum_{n=1}^\infty \frac{|\int_X f_n \, d\mu_{m^*} - \int_X f_n \, d\mu_{m'^*}|}{2^n \|f_n\|}$$

is a metric on $M^*(X)$ giving the w^* -topology.

- (ii) For $m_n, m \in M^*(X)$ ($n \geq 1$), $m_n \rightarrow m$ if and only if $\int_X f \, d\mu_{m_n}^* \rightarrow \int_X f \, d\mu_m^*$ for all $f \in C(X)$.
- (iii) $M_g^*(X, T)$ is a compact subset of $M^*(X)$.
- (iv) $M_g^*(X, T)$ is convex.
- (v) $\text{ext}(M_g^*(X, T)) = E_g^*(X, T)$.

Proof. See [24] Theorems 6.4 and 6.10. □

Definition 3.4. Suppose that Y is a non-empty compact subset of a locally convex space E , and let τ be a probability measure on Y . A point x in E is said to be represented by τ if $\Phi(x) = \int_Y \Phi \, d\tau$ for every continuous linear functional Φ on E .

Theorem 3.5. (Choquet) Suppose that Y is a metrizable compact convex subset of a locally convex space E , and that x_0 is an element of Y . Then there exists a probability measure τ on Y which represents x_0 and is supported by the extreme points of Y .

See Phelps [22] for a proof of Choquet’s Theorem.

By Theorem 3.3, $M^*(X)$ is a compact metrizable space and $M_g^*(X, T)$ is a compact metrizable convex set with the extreme points $E_g^*(X, T)$. So applying the Choquet’s Theorem we will have the following corollary.

Corollary 3.6. For any $m \in M_g^*(X, T)$ there exists a unique probability measure τ on the Borel subsets of the compact metrizable space $M_g^*(X, T)$ such that $\tau(E_g^*(X, T)) = 1$ and

$$\int_X f(x) \, d\mu_{m^*}(x) = \int_{E_g^*(X, T)} \left(\int_X f(x) \, d\mu_{\nu^*}(x) \right) \, d\tau(\nu)$$

for every bounded measurable function $f : X \rightarrow \mathbb{R}$.

In particular, if $f \in \mathcal{F}$ then the previous equality is indeed

$$m^*(f) = \int_{E_g^*(X, T)} \nu^*(f) \, d\tau(\nu).$$

Since $m^* = g \circ m \circ g^{-1}$ then

$$g \circ m \circ g^{-1}(f) = \int_{E_g^*(X, T)} g \circ \nu \circ g^{-1}(f) \, d\tau(\nu).$$

Replacing f by $g \circ f$ in the previous relation we will have

$$m(f) = g^{-1} \left(\int_{E_g^*(X, T)} g(\nu(f)) \, d\tau(\nu) \right).$$

Under the assumptions of Corollary 3.6 we write $m = \int_{E_g^*(X, T)} \nu \, d\tau(\nu)$ and it is called the g -ergodic decomposition of m .

4. LOCAL g -ENTROPY

In this section, $T : X \rightarrow X$ is a continuous map on a compact metric space X and \mathcal{F} is the σ -algebra of Borel measurable maps $f : X \rightarrow [0, 1]$. As before, let $g : [0, \infty] \rightarrow [0, \infty]$ be an increasing function such that $g(0) = 0$ and $g(1) = 1$.

Definition 4.1. For $x \in X$ and $f \in \mathcal{F}$, define

$$\omega_g(T, x, f) := g^{-1} \left(\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} ((g \circ f) \circ T^k)(x) \right).$$

We write $\omega(T, x, f)$ for the special case $g(x) = x$, indeed

$$\omega(T, x, f) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} (f \circ T^i)(x).$$

Definition 4.2. Let $\xi = \{f_1, f_2, \dots, f_k\}$ be a g -partition and $x \in X$. Define

$$\begin{aligned} \Omega_g(T, x, \xi) &:= g^{-1} \left(\sum_{i=1}^k g(\Phi(\omega_g(T, x, f_i))) \right) \\ &= g^{-1} \left(\sum_{i=1}^k \phi(g(\omega_g(T, x, f_i))) \right) \\ &= g^{-1} \left(- \sum_{i=1}^k g(\omega_g(T, x, f_i)) \log g(\omega_g(T, x, f_i)) \right). \end{aligned}$$

If $\xi = \{f_1, f_2, \dots, f_k\}$ is a fuzzy partition, the special case $g(x) = x$ of the definition 4.2 is given by

$$\Omega(T, x, \xi) = - \sum_{i=1}^k \omega(T, x, f_i) \log \omega(T, x, f_i) = \sum_{i=1}^k \phi(\omega(T, x, f_i)).$$

Definition 4.3. Let $\xi = \{f_1, f_2, \dots, f_k\}$ be a g -partition and $x \in X$. Define

$$\mathcal{H}_g(T, x, \xi) := \limsup_{n \rightarrow \infty} g^{-1} \left(\frac{1}{n} g \left(\Omega_g(T, x, \bigvee_{i=0}^{n-1} T^{-i} \xi) \right) \right).$$

Setting $g(x) = x$, in Definition 4.3 will result in the following:

$$\mathcal{H}(T, x, \xi) = \limsup_{n \rightarrow \infty} \frac{1}{n} \Omega \left(T, x, \bigvee_{i=0}^{n-1} T^{-i} \xi \right)$$

where $\xi = \{f_1, f_2, \dots, f_k\}$ is a fuzzy partition.

Definition 4.4. Let ξ be a g -partition and $m \in M_g^*(X, T)$. Define

$$h_{m,g}^*(T, \xi) := \int_X \mathcal{H}_g(T, x, \xi) \, d\mu_{m^*}(x)$$

and

$$h_{m,g}^*(T) := \sup_{\xi} h_{m,g}^*(T, \xi)$$

where the supremum is taken over all g -fuzzy partitions.

If m^* is a fuzzy measure and ξ is a fuzzy partition, the special case $g(x) = x$ in Definition 4.4 results in the following:

$$h_{m^*}^*(T, \xi) = \int_X \mathcal{H}(T, x, \xi) \, d\mu_{m^*}(x)$$

and

$$h_{m^*}^*(T) = \sup_{\xi} h_{m^*}^*(T, \xi)$$

where the supremum is taken over all fuzzy partitions.

In [14], the properties of the previous quantities in the case of $g(x) = x$ for all $x \in X$, are discussed.

Theorem 4.5. Let $x \in X$ and ξ be a g -fuzzy partition. Then

- (i) $\Omega_g(T, x, \xi) = g^{-1}(\Omega(T, x, g(\xi)))$;
- (ii) $\mathcal{H}_g(T, x, \xi) = g^{-1}(\mathcal{H}(T, x, g(\xi)))$;
- (iii) If g is convex then $h_{m,g}^*(T) \leq g^{-1}(h_{m^*}^*(T))$.

Proof. (i) First note that, by Definition 4.1, $g(\omega_g(T, x, f)) = \omega(T, x, f)$. Now, the result follows directly from Definition 4.2.

(ii) follows from (i) and the equality $g(\bigvee_{i=0}^{n-1} T^{-i}\xi) = \bigvee_{i=0}^{n-1} T^{-i}g(\xi)$.

(iii) For a g -fuzzy partition ξ , applying part (ii) and Jensen's inequality we will have

$$\begin{aligned} h_{m,g}^*(T, \xi) &= \int_X \mathcal{H}_g(T, x, \xi) \, d\mu_{m^*}(x) \\ &= \int_X g^{-1}(\mathcal{H}(T, x, g(\xi))) \, d\mu_{m^*}(x) \\ &\leq g^{-1}\left(\int_X \mathcal{H}(T, x, g(\xi)) \, d\mu_{m^*}(x)\right) \\ &= g^{-1}(h_{m^*}^*(T, g(\xi))) \\ &\leq g^{-1}(h_{m^*}^*(T)), \end{aligned}$$

where the last inequality holds because g^{-1} is also increasing. Finally, taking supremum over all g -fuzzy partitions we will get the result. □

Theorem 4.6. Suppose that $T : X \rightarrow X$ is a continuous map on a compact metric space X . If ξ, η are g -fuzzy partitions and $x \in X$ then

(i) If $\xi \leq \eta$ then $\Omega_g(T, x, \xi) \leq \Omega_g(T, x, \eta)$.

(ii) If $\xi \leq \eta$ then $\mathcal{H}_g(T, x, \xi) \leq \mathcal{H}_g(T, x, \eta)$.

Proof. Let $\xi = \{f_i\}$ and $\eta = \{h_j\}$ be two g -fuzzy partitions and assume, without loss of generality, that all fuzzy sets are such that $\omega_g(T, x, f) \neq 0$. (Since if $\xi = \{f_1, f_2, \dots, f_k\}$ with $\omega_g(T, x, f) > 0$ for $1 \leq i \leq r$ and $\omega_g(T, x, f) = 0$ for $r < i \leq k$ we can replace ξ by $\{f_1 \vee f_2 \vee \dots \vee f_k\}$).

(i) Since $\xi \leq \eta$ we have $\xi \vee \eta = \eta$. By definition we obtain

$$\begin{aligned}
 g(\Omega_g(T, x, \eta)) &= g(\Omega_g(T, x, \xi \vee \eta)) \\
 &= \Omega(T, x, g(\xi \vee \eta)) \\
 &= \Omega(T, x, g(\xi) \vee g(\eta)) \\
 &= - \sum_{i,j} \omega(T, x, (g \circ f_i)(g \circ h_j)) \log \omega(T, x, (g \circ f_i)(g \circ h_j)) \\
 &= - \sum_{i,j} \omega(T, x, (g \circ f_i)(g \circ h_j)) \log \frac{\omega(T, x, (g \circ f_i)(g \circ h_j))}{\omega(T, x, g \circ f_i)} \omega(T, x, g \circ f_i) \\
 &= - \sum_{i,j} \omega(T, x, (g \circ f_i)(g \circ h_j)) \log \frac{\omega(T, x, (g \circ f_i)(g \circ h_j))}{\omega(T, x, g \circ f_i)} \\
 &\quad - \sum_{i,j} \omega(T, x, (g \circ f_i)(g \circ h_j)) \log \omega(T, x, g \circ f_i) \\
 &\geq - \sum_{i,j} \omega(T, x, (g \circ f_i)(g \circ h_j)) \log \omega(T, x, g \circ f_i) \tag{2}
 \end{aligned}$$

where the last inequality holds since $\omega(T, x, (g \circ f_i)(g \circ h_j)) \leq \omega(T, x, (g \circ f_i))$.

On the other hand, since

$$\omega(T, x, g \circ f_i) \leq \sum_j \omega(T, x, (g \circ f_i)(g \circ h_j))$$

we conclude that

$$- \sum_{i,j} \omega(T, x, (g \circ f_i)(g \circ h_j)) \log \omega(T, x, g \circ f_i) \geq - \sum_i \omega(T, x, (g \circ f_i)) \log \omega(T, x, g \circ f_i). \tag{3}$$

Combining (2) and (3) we will have

$$g(\Omega_g(T, x, \eta)) \geq g(\Omega_g(T, x, \xi)).$$

This gives the result, since g is increasing.

(ii) Replace ξ by $\bigvee_{i=0}^{n-1} T^{-i} \xi$ and η by $\bigvee_{i=0}^{n-1} T^{-i} \eta$ in (i) and apply Definition 4.3 to get the result. \square

The following theorem states the quantities in Definition 4.4 in terms of the g -entropy.

Theorem 4.7. Suppose that $T : X \rightarrow X$ is a continuous map on a compact metric space X and $\mathcal{F} \subset [0, 1]^X$ is the σ -algebra of Borel measurable maps $f : X \rightarrow [0, 1]$. If $m \in M_g^*(X, T)$ and $m = \int_{E_g^*(X, T)} \nu \, d\tau(\nu)$ is the g -ergodic decomposition of m then

(i) If ξ is a g -fuzzy partition then

$$h_{m, g}^*(T, \xi) = \int_{E_g^*(X, T)} h_{\nu, g}(T, \xi) \, d\tau(\nu).$$

(ii) If $\text{card}(E_g^*(X, T)) < \infty$ then

$$h_{m, g}^*(T) = \int_{E_g^*(X, T)} h_{\nu, g}(T) \, d\tau(\nu).$$

Proof. (i) First, let $\nu \in E_g^*(X, T)$. By Lemma 3.1 (ii), $\mu_{\nu^*} \in E(X, T)$. By Birkhoff ergodic Theorem we have

$$\begin{aligned} \omega_g(T, x, f) &= g^{-1} \left(\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} (g \circ f) \circ T^k(x) \right) \\ &= g^{-1} \left(\int_X g \circ f \, d\mu_{\nu^*} \right) \\ &= g^{-1} (\nu^*(g \circ f)) \\ &= g^{-1} \circ \nu^* \circ g(f) \end{aligned}$$

for almost every $x \in X$. Since $\nu^* = g \circ \nu \circ g^{-1}$ we conclude that

$$g(\omega_g(T, x, f)) = g(\nu(f))$$

for almost every $x \in X$.

Therefore, if $\xi = \{f_1, f_2, \dots, f_k\}$ is a g -fuzzy partition then

$$\begin{aligned} \Omega_g(T, x, \xi) &= g^{-1} \left(- \sum_{i=1}^k g(\omega_g(x, f_i)) \log g(\omega_g(x, f_i)) \right) \\ &= g^{-1} \left(- \sum_{i=1}^k g(\nu(f_i)) \log g(\nu(f_i)) \right) \\ &= H_{\nu, g}(\xi) \end{aligned}$$

for almost every $x \in X$.

For every $n \in \mathbb{N}$, replacing ξ by $\bigvee_{i=0}^{n-1} T^{-i}\xi$, and considering the equality $g \left(\bigvee_{i=0}^{n-1} T^{-i}\xi \right) = \bigvee_{i=0}^{n-1} T^{-i}g(\xi)$ we obtain

$$\Omega_g \left(T, x, \bigvee_{i=0}^{n-1} T^{-i}\xi \right) = H_{\nu, g} \left(\bigvee_{i=0}^{n-1} T^{-i}\xi \right)$$

for almost every $x \in X$ which simply results in

$$\mathcal{H}_g(T, x, \xi) = h_{\nu, g}(T, \xi)$$

for almost every $x \in X$. Integrating with respect to μ_{ν^*} we obtain

$$h_{\nu, g}^*(T, \xi) = h_{\nu, g}(T, \xi) \tag{4}$$

for all g -fuzzy partitions ξ . Taking supremum over all g -fuzzy partitions we will have $h_{\nu, g}^*(T) = h_{\nu, g}(T)$.

Now, let $m \in M_g^*(X, T)$. For $n \geq 1$, let $f_n := \min\{\mathcal{H}_g(T, \cdot, \xi), n\}$. Clearly, $\{f_n\}_{n=1}^\infty$ is an increasing sequence of bounded measurable functions such that $f_n \nearrow \mathcal{H}_g(T, \cdot, \xi)$. Applying Corollary 3.6, Monotone Convergence Theorem and (4) we will have

$$\begin{aligned} h_{m, g}^*(T, \xi) &= \int_X \mathcal{H}_g(T, x, \xi) \, d\mu_{m^*}(x) \\ &= \lim_{n \rightarrow \infty} \int_X f_n(x) \, d\mu_{m^*}(x) \\ &= \lim_{n \rightarrow \infty} \int_{E_g^*(X, T)} \left(\int_X f_n(x) \, d\mu_{\nu^*}(x) \right) \, d\tau(\nu) \\ &= \int_{E_g^*(X, T)} \left(\int_X \mathcal{H}_g(T, x, \xi) \, d\mu_{\nu^*}(x) \right) \, d\tau(\nu) \\ &= \int_{E_g^*(X, T)} h_{\nu, g}^*(T, \xi) \, d\tau(\nu) \\ &= \int_{E_g^*(X, T)} h_{\nu, g}(T, \xi) \, d\tau(\nu). \end{aligned}$$

(ii) For $m \in E_g^*(X, T)$, let

$$\mathfrak{D}_m := \{ \{ \xi_n \}_{n \geq 1} : \xi_n \leq \xi_{n+1}, \quad h_{m, g}(T, \xi_n) \rightarrow h_{m, g}(T) \}$$

then, by the supremum property, $\mathfrak{D}_m \neq \emptyset$. Also $\bigcap_{m \in E_g^*(X, T)} \mathfrak{D}_{m_j} \neq \emptyset$. To show this, let $E_g^*(X, T) = \{m_1, m_2, \dots, m_k\}$. For each $j \in \{1, 2, \dots, k\}$ choose $\{ \xi_n^{(j)} \}_{n \geq 1} \in \mathfrak{D}_{m_j}$. For $n \geq 1$, let $\xi_n := \vee_{j=1}^k \xi_n^{(j)}$. For $j \in \{1, 2, \dots, k\}$, applying Theorem 4.6 (ii), we have

$$h_{m_j, g}(T, \xi_n^{(j)}) \leq h_{m_j, g}(T, \xi_n) \leq h_{m_j, g}(T). \tag{5}$$

Since $\{ \xi_n^{(j)} \}_{n \geq 1} \in \mathfrak{D}_{m_j}$, the relation (5) results in

$$\lim_{n \rightarrow \infty} h_{m_j, g}(T, \xi_n) = h_{m_j, g}(T)$$

for all $j \in \{1, 2, \dots, k\}$. It means $\{ \xi_n \}_{n \geq 1} \in \bigcap_{m \in E_g^*(X, T)} \mathfrak{D}_{m_j}$ which proves that $\bigcap_{m \in E_g^*(X, T)} \mathfrak{D}_{m_j} \neq \emptyset$. Now, we can choose a sequence $\{ \xi_n \}_{n=1}^\infty$ of g -fuzzy partitions such

that $\xi_n \leq \xi_{n+1}$ ($n = 1, 2, \dots$) and $\lim_{n \rightarrow \infty} h_{m,g}(T, \xi_n) = h_{m,g}(T)$ for all $m \in E_g^*(X, T)$. Applying part (i) and Monotone Convergence Theorem we will have:

$$\begin{aligned} h_{m,g}^*(T) &\geq \lim_{n \rightarrow \infty} h_{m,g}^*(T, \xi_n) \\ &= \lim_{n \rightarrow \infty} \int_{E_g^*(X, T)} h_{\nu,g}(T, \xi_n) \, d\tau(\nu) \\ &= \int_{E_g^*(X, T)} \lim_{n \rightarrow \infty} h_{\nu,g}(T, \xi_n) \, d\tau(\nu) \\ &= \int_{E^*(X, T)} h_{\nu,g}(T) \, d\tau(\nu). \end{aligned}$$

On the other hand

$$h_{m,g}^*(T, \xi) = \int_{E_g^*(X, T)} h_{\nu,g}(T, \xi) \, d\tau(\nu) \leq \int_{E_g^*(X, T)} h_{\nu,g}(T) \, d\tau(\nu)$$

for any given g -fuzzy partition ξ . This easily results in

$$h_{m,g}^*(T) \leq \int_{E_g^*(X, T)} h_{\nu,g}(T) \, d\tau(\nu)$$

which completes the proof. □

SUMMARY AND CONCLUSIONS

This paper is devoted to a local study of the concept of g -entropy of dynamical systems. The set of g -invariant and g -ergodic fuzzy measures is defined in section 3. It is equipped to a weak* topology such that the set of g -invariant fuzzy measures is the convex hull of the set of g -ergodic fuzzy measures. Then the g -ergodic decomposition is introduced. A new type of g -entropy is defined in section 4. This definition is of local entity. Using the framework constructed in section 4, the new quantity is stated in terms of the known g -entropy.

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Mehdi Rahimi, Department of Mathematics, Faculty of Science, University of Qom, Qom. Iran.

e-mail: m10.rahimi@gmail.com