On the Minkowski-Hölder type inequalities for generalized Sugeno integrals with an application

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Abstract

In this paper, we use a new method to obtain the necessary and sufficient condition guaranteeing the validity of the Minkowski-Hölder type inequality for the generalized upper Sugeno integral in the case of functions belonging to a wider class than the comonotone functions. As a by-product, we show that the Minkowski type inequality for seminormed fuzzy integral presented by Daraby [10] is not true. Next, we study the Minkowski-Hölder inequality for the lower Sugeno integral and the class of μ subadditive functions introduced in [18]. The results are applied to derive new metrics on the space of measurable functions in the setting of nonadditive measure theory. We also give a partial answer to the open problem 2.22 posed in [5].

Keywords: Seminormed fuzzy integral; Semicopula; Monotone measure; Minkowski's inequality; Hölder's inequality; convergence in mean.

1 Introduction

The concepts of fuzzy measures and the Sugeno integral were introduced by Sugeno in [32] as a tool for modeling nondeterministic problems. The study of inequalities for the Sugeno integral was initiated by Román-Flores et al. [28]. Since then, the fuzzy integral counterparts of several classical inequalities, including Chebyshev's, Minkowski's and Hölder's inequalities have been given by Agahi et al. [1], Klement et al. [21], Ouyang et al. [25, 26], Wu et al. [35] and many other researchers. Most of them deal with comonotone functions which highly limit the range of potential applications in probability, statistics, decision theory, risk theory and others.

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Since many classical inequalities are free of the comonotonicity assumption, Agahi and Mesiar [2] asked whether one could omit it. They gave a version of the Cauchy–Schwarz inequality without the comonotonicity condition for two classes of Choquet-like integrals. In [18] the Chebyshev type inequalities were provided for positively dependent functions which form a wider class than the comonotone functions. The aim of this paper is to present another inequalities for nonadditive integrals without the comonotonicity condition.

The paper is organized as follows. In Section 2, we introduce a new concept, called \star -associativity, which extends the notion of comonotonicity. Next, we obtain the necessary and sufficient conditions ensuring that the Minkowski-Hölder type inequality holds for the generalized upper Sugeno integral and \star -associative functions. We give a counterexample showing that the Minkowski type inequality for seminormed fuzzy integral presented in [10], Theorem 3.1, is false. The sufficient conditions for subadditivity of some functionals based on the upper Sugeno integral are also provided. Section 3 presents the Minkowski-Hölder type inequality for the generalized lower Sugeno integral and μ -subadditive functions. The necessary and sufficient condition for subadditivity of the Sugeno integral with respect to a subadditive measure is given. Finally, in Section 4 we propose new metrics on the space of measurable functions when the involved measure is monotone. We also give a partial answer to the open problem posed by Borzová-Molnárová et al. [5].

2 Inequalities for generalized upper Sugeno integral

First, we introduce some basic definitions and properties. Let (X, \mathcal{A}) be a measurable space, where \mathcal{A} is a σ -algebra of subsets of a nonempty set X. A monotone measure on \mathcal{A} is a nondecreasing set function $\mu: \mathcal{A} \to [0, \infty]$ with $\mu(\emptyset) = 0$. We say that μ is finite if $\mu(X) < \infty$. A monotone measure μ is continuous from below if $\lim_{n \to \infty} \mu(A_n) = \mu(\lim_{n \to \infty} A_n)$ for all $A_n \in \mathcal{A}$ such that $A_n \subset A_{n+1}, n \in \mathbb{N}$.

Let Y = [0, m) or Y = [0, m], where $0 < m \leq \infty$; usually Y = [0, 1], $Y = [0, \infty)$ or $Y = [0, \infty]$. The operator $\circ: Y^2 \to Y$ is said to be *nondecreasing* if $a \circ b \leq x \circ y$ for $a \leq x, b \leq y$. We say that $\circ: Y^2 \to Y$ is *right-continuous* if $\lim_{n \to \infty} (a_n \circ b_n) = a \circ b$ for all $a_n, b_n, a, b \in Y$ such that $b_n \searrow b$ and $a_n \searrow a$. Hereafter, $c_n \searrow c$ means that $\lim_{n \to \infty} c_n = c$ and $c_n > c_{n+1}$ for all n.

Recall that $f, g: X \to Y$ are comonotone on D if $(f(x) - f(y))(g(x) - g(y)) \ge 0$ for all $x, y \in D$. If f and g are comonotone on D, then for any $t \in Y$ either $(D \cap \{f \ge t\}) \subset$ $(D \cap \{g \ge t\})$ or $(D \cap \{g \ge t\}) \subset (D \cap \{f \ge t\})$, where $\{f \ge t\} = \{x \in X : f(x) \ge t\}$.

Now we will generalize the concept of comonotonicity.

Definition 1. Given an operator $\star: Y^2 \to Y$, we say that $f, g: X \to Y$ are \star -associated on

D if for any nonempty and measurable subset $A \subset D$,

$$\inf_{x \in A} \left\{ f(x) \star g(x) \right\} = \inf_{x \in A} f(x) \star \inf_{x \in A} g(x).$$
(1)

From now on, $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$ and $a_+ = a \vee 0$.

Example 1. Any functions $f, g: X \to Y$ are \wedge -associated on X.

Example 2. Any comonotone functions $f, g: X \to Y$ are \star -associated on X if the operator \star is nondecreasing and right-continuous. Indeed, $\inf_{x \in A} \{f(x) \star g(x)\} \ge s \star t$ for all $A \subset X$, where $s = \inf_{x \in A} f(x)$ and $t = \inf_{x \in A} g(x)$. Let $\varepsilon > 0$, $A \subset X$ and $B = \{x \in A : f(x) < s + \varepsilon\}$ and $C = \{x \in A : g(x) < t + \varepsilon\}$. From the comonotonicity we obtain that $B \cap C \neq \emptyset$ as $B \subset C$ or $C \subset B$. Thus $\inf_{x \in A} \{f(x) \star g(x)\} \le (s + \varepsilon) \star (t + \varepsilon)$. Because of the right-continuity of \star , we get the assertion.

Example 3. Let $f, g: X \to Y$ be measurable functions and $g = b\mathbb{1}_B$ for $b \in Y$, where $\mathbb{1}_B$ denotes the indicator of $B \subset X$, $B \cap \{f \ge b\} \ne B$ and $B \cap \{f \ge b\} \ne \{f \ge b\}$. Let $\star: Y^2 \to Y$ be a nondecreasing and right-continuous operator. If $x \star 0 = 0$ for all $x \in X$, then f, g are \star -associated on X, but not comonotone. Indeed, if $A \setminus B = \emptyset$, then $A \subset B$ and

$$\inf_{x \in A} \{ f(x) \star g(x) \} = \inf_{x \in A} \{ f(x) \star b \} = \inf_{x \in A} f(x) \star \inf_{x \in A} g(x).$$

If $A \setminus B \neq \emptyset$, then

$$\inf_{x \in A} \left\{ f(x) \star g(x) \right\} = \inf_{x \in A \cap B} \left\{ f(x) \star b \right\} \wedge \inf_{x \in A \setminus B} \left\{ f(x) \star 0 \right\} = 0$$
$$= \inf_{x \in A} f(x) \star \inf_{x \in A} g(x).$$

Example 4. Suppose \star is a nondecreasing operator such that $0 \star y = y \star 0 = 0$ for all $y \in Y$. Let $f = b\mathbb{1}_B + c\mathbb{1}_C$ and $g = b\mathbb{1}_B + c\mathbb{1}_D$, where $b, c \in Y, 0 < b \land c$, and B, C are nonempty sets such that $B \cap C = \emptyset$ and $D = X \setminus (B \cup C) \neq \emptyset$. Clearly, f and g are \star -associated on X, but not comonotone.

Example 5. Functions f, g are +-associated if and only if they are commondene. In fact, the condition (1) for $\star = +$ and $A = \{x, y\}$ is equivalent to $(a + b) \land 0 = (a \land 0) + (b \land 0)$ with a = f(x) - f(y) and b = g(x) - g(y), and this implies that $ab \ge 0$.

Open problem 1. Does there exist an operator $\circ \neq +$ such that the \circ -associativity property is equivalent to the comonotonicity property?

Now we are ready to present the Minkowski-Hölder type inequality for the *generalized* upper Sugeno integral of the form

$$\int_{\circ,D} f \,\mathrm{d}\mu := \sup_{t \in Y} \left\{ t \circ \mu \left(D \cap \{f \ge t\} \right) \right\},\tag{2}$$

where $f: X \to Y$ is a measurable function, μ is a monotone measure on \mathcal{A} and $\circ: Y \times \mu(\mathcal{A}) \to [0, \infty]$ is a nondecreasing operator. The functional in (2) is the universal integral in the sense of Definition 2.5 in [21] if \circ is the pseudomultiplication function (see [21], Definition 2.3).

Put $\mu(\mathcal{A}) = \{\mu(A) : A \in \mathcal{A}\}$ and $\mu(\mathcal{A} \cap D) = \{\mu(A \cap D) : A \in \mathcal{A}\}$. The following theorem gives an answer to open problems from [1], [18] and [26].

Theorem 1. Assume the operators $\star, \Diamond: Y^2 \to Y$ and $\circ_i: Y \times \mu(\mathcal{A}) \to Y$ are such that \Diamond and \circ_i are nondecreasing and $0 \circ_i x = y \circ_i 0 = 0$ for all $y \in Y, x \in \mu(\mathcal{A})$ and i = 1, 2, 3. Suppose $\phi_i: Y \to Y$ are increasing and $\phi_i(Y) = Y$ for all i. Suppose also that f and g are \star -associated on $D \subset X$. Then the Minkowski-Hölder type inequality

$$\phi_1^{-1}\Big(\int_{\circ_1,D}\phi_1(f\star g)\,\mathrm{d}\mu\Big) \leqslant \phi_2^{-1}\Big(\int_{\circ_2,D}\phi_2(f)\,\mathrm{d}\mu\Big) \diamond \phi_3^{-1}\Big(\int_{\circ_3,D}\phi_3(g)\,\mathrm{d}\mu\Big) \tag{3}$$

is satisfied if and only if for all $a, b \in Y$ and $c \in \mu(\mathcal{A} \cap D)$

$$\phi_1^{-1}(\phi_1(a \star b) \circ_1 c) \leqslant \phi_2^{-1}(\phi_2(a) \circ_2 c) \Diamond \phi_3^{-1}(\phi_3(b) \circ_3 c).$$
(4)

Proof. Arguing as in the proof of Lemma 3.8 in [31], we can show that

$$\int_{\circ_i,D} f \,\mathrm{d}\mu = \sup_{A \subset D, A \in \mathcal{A}} \left\{ \inf_{x \in A} f(x) \circ_i \mu(A) \right\}$$
(5)

for all *i* (see also [4], Theorem 2.2). To shorten the notation, we write \sup_{A} instead of $\sup_{A \subset D, A \in \mathcal{A}}$. From the continuity of ϕ_1 and (5) we get

$$L := \phi_1^{-1} \Big(\int_{\circ_1, D} \phi_1(f \star g) \, \mathrm{d}\mu \Big) = \sup_A \phi_1^{-1} \Big(\phi_1 \Big(\inf_{x \in A} \{ f(x) \star g(x) \} \Big) \circ_1 \mu(A) \Big)$$

Since f and g are \star -associated, we have

$$L = \sup_{A} \phi_1^{-1} \Big(\phi_1 \Big(\inf_{x \in A} f(x) \star \inf_{x \in A} g(x) \Big) \circ_1 \mu(A) \Big).$$

Combining (4) with the monotonicity of \Diamond and ϕ_i^{-1} yields

$$L \leqslant \sup_{A} \left\{ \phi_{2}^{-1} \Big(\phi_{2} \Big(\inf_{x \in A} f(x) \Big) \circ_{2} \mu(A) \Big) \Diamond \phi_{3}^{-1} \Big(\phi_{3} \Big(\inf_{x \in A} g(x) \Big) \circ_{3} \mu(A) \Big) \right\}$$

$$\leqslant \left(\sup_{A} \phi_{2}^{-1} \Big(\phi_{2} \Big(\inf_{x \in A} f(x) \Big) \circ_{2} \mu(A) \Big) \Big) \Diamond \left(\sup_{A} \phi_{3}^{-1} \Big(\phi_{3} \Big(\inf_{x \in A} g(x) \Big) \circ_{3} \mu(A) \Big) \Big) \right)$$

$$= \phi_{2}^{-1} \Big(\int_{\circ_{2}, D} \phi_{2}(f) \, \mathrm{d}\mu \Big) \Diamond \phi_{3}^{-1} \Big(\int_{\circ_{3}, D} \phi_{3}(g) \, \mathrm{d}\mu \Big).$$

To obtain the necessary condition (4), put $f = a \mathbb{1}_A$ and $g = b \mathbb{1}_A$ in (3), where $c = \mu(A) \leq \mu(D)$ and $a, b \in Y$.

Observe that the assumption $0 \circ_i x = y \circ_i 0 = 0$ is used only in the proof of the necessity of condition (4). Moreover, the condition (4) is sufficient for inequality (3) to hold if we set $Y = \mathbb{R}$ in (2) and both f and g are bounded from below.

Example 6. Let $a \star b = a \Diamond b = a + b - ab$, where $a, b \in Y = [0, 1]$ and let $\circ_i = \cdot$ for all i. Put $\phi_i(x) = x^{p_i}$ and $c_i = c^{1/p_i}$, where $p_i > 0$ for all i. The condition (4) takes the form

$$0 \leqslant a(c_2 - c_1) + b(c_3 - c_1) + ab(c_1 - c_2c_3) \tag{6}$$

and holds if and only if $p_1 \leq p_j$ for j = 2, 3; in order to see this, put a = 1, b = 0 as well as a = 0, b = 1 in (6) and observe that

$$a(c_2 - c_1) + b(c_3 - c_1) + ab(c_1 - c_2c_3) \ge ab(c_2 - c_1 + c_3(1 - c_2)) \ge 0.$$

We recall that the Sugeno integral and the Shilkret integral are given by

$$(S) \int_{D} f \,\mathrm{d}\mu := \sup_{y \in Y} \left\{ y \wedge \mu \left(D \cap \{f \ge y\} \right) \right\},\tag{7}$$

$$(N) \int_D f \,\mathrm{d}\mu := \sup_{y \in Y} \left\{ y \cdot \mu \left(D \cap \{f \ge y\} \right) \right\},\tag{8}$$

respectively, where Y = [0, m] or Y = [0, m) with $0 < m \leq \infty$, see [30, 32, 33].

Corollary 1. Assume $\star: Y^2 \to Y$ is nondecreasing, $f, g: X \to Y$ are \star -associated on Dand $\phi_i: Y \to Y$ are increasing functions such that $\phi_i(Y) = Y$ for i = 1, 2, 3. The following Minkowski-Hölder type inequality

$$\phi_1^{-1}\Big((S)\int_D\phi_1(f\star g)\,\mathrm{d}\mu\Big)\leqslant\phi_2^{-1}\Big((S)\int_D\phi_2(f)\,\mathrm{d}\mu\Big)\star\phi_3^{-1}\Big((S)\int_D\phi_3(g)\,\mathrm{d}\mu\Big)$$

holds true if and only if for $a, b \in Y$ and $c \in \mu(\mathcal{A} \cap D)$

$$(a \star b) \wedge \phi_1^{-1}(c) \leqslant \left(a \wedge \phi_2^{-1}(c)\right) \star \left(b \wedge \phi_3^{-1}(c)\right).$$

$$\tag{9}$$

The above result generalizes Theorem 3.1 from [1] and Theorem 3.1 from [35]. In fact, since $a \lor b \leqslant a \star b$, we have $c \leqslant a \lor c \leqslant a \star c$, $c \leqslant c \star b$ and $c \leqslant c \star c$, so

$$(a \star b) \land c \leqslant (a \star b) \land (a \star c) \land (c \star b) \land (c \star c) = (a \land c) \star (b \land c).$$

It follows from the assumption $\phi_1 \ge \phi_j$ for j = 2, 3 that

$$(a \star b) \land \phi_1^{-1}(c) \leqslant \left(a \land \phi_1^{-1}(c)\right) \star \left(b \land \phi_1^{-1}(c)\right) \leqslant \left(a \land \phi_2^{-1}(c)\right) \star \left(b \land \phi_3^{-1}(c)\right).$$

Thus, the condition (9) holds.

Suppose that S: $[0,1]^2 \to [0,1]$ is a *semicopula* (also called a *t-seminorm*), i.e., a nondecreasing function with the neutral element equal to 1. It is clear that $S(x,y) \leq x \wedge y$ and S(x,0) = 0 = S(0,x) for all $x, y \in [0,1]$ (see [3, 12, 20]). We denote the class of all semicopulas by \mathfrak{S} . There are three important examples of semicopulas: M, II and S_L, where $M(a,b) = a \wedge b$, $\Pi(a,b) = ab$ and $S_L(a,b) = (a + b - 1)_+$, usually called the *Lukasiewicz t*-norm [20].

Given $S \in \mathfrak{S}$, the *seminormed fuzzy integral* is defined by

$$\int_{\mathcal{S},D} f \, \mathrm{d}\mu := \sup_{t \in [0,1]} \mathcal{S}\big(t, \mu(D \cap \{f \ge t\})\big),\tag{10}$$

see [25, 31]. Replacing semicopula S with M, we get the Sugeno integral (7) for Y = [0, 1]. Moreover, if $S = \Pi$, then we get the *Shilkret integral* (8) for Y = [0, 1].

Corollary 2. Let $S \in \mathfrak{S}$ and $f, g: X \to [0, 1]$ be \star -associated, where $\star: [0, 1]^2 \to [0, 1]$ is a nondecreasing operator. Let $0 and <math>\mu(\mathcal{A}) = 1$. The following inequality holds

$$\left(\int_{\mathcal{S},D} (f \star g)^p \,\mathrm{d}\mu\right)^{1/p} \leqslant \left(\int_{\mathcal{S},D} f^p \,\mathrm{d}\mu\right)^{1/p} \star \left(\int_{\mathcal{S},D} g^p \,\mathrm{d}\mu\right)^{1/p} \tag{11}$$

if and only if

$$\mathbf{S}((a \star b)^p, c)^{1/p} \leqslant \mathbf{S}(a^p, c)^{1/p} \star \mathbf{S}(b^p, c)^{1/p}$$

for $a, b \in [0, 1]$ and $c \in \mu(\mathcal{A} \cap D)$.

Daraby and Ghadimi [10] claim that the inequality (11) is satisfied if

$$S(a \star b, c) \leqslant (S(a, c) \star b) \land (a \star S(b, c)), \quad a, b, c \in [0, 1],$$
(12)

under the assumption of continuity of monotone measure μ (see [10], Theorem 3.1). We present a counterexample showing that this result is not true.

Counterexample 1. Put A = X = [0,1], s = 1, $T = S_L$, $a \star b = (a + b) \wedge 1$ and $f(x) = g(x) = 0.5\sqrt{x}$, $x \in [0,1]$, in Theorem 3.1 from [10]. Clearly, f and g are comonotone. Let μ be the Lebesgue measure. Due to the property $a \star b = b \star a$, the condition (12) is satisfied if and only if

$$S_L((a+b) \wedge 1, c) \leq (S_L(a, c) + b) \wedge 1$$

for all $a, b, c \in [0, 1]$. Since $S_L \leq 1$, it suffices to show that $S_L((a + b) \wedge 1, c) \leq S_L(a, c) + b$. In fact, if $a + b \leq 1$, then $S_L(a + b, c) \leq S_L(a, c) + b$ (see also [20], Remark 5.13 (*iii*)). Otherwise,

$$c \leq (a+c-1)_{+} + (1-a)_{+} = (a+c-1)_{+} - (a+b-1) + b \leq S_{L}(a,c) + b.$$

Easy computations show that

$$\int_{S_{L},X} f \, d\mu = \sup_{t \in [0,1]} \left(t + \mu \left(\{ f \ge t \} \right) - 1 \right)_{+} = \sup_{t \in [0,1]} \left(t - 4t^{2} \right)_{+} = 0.0625,$$
$$\int_{S_{L},X} (f \star g) \, d\mu = \sup_{t \in [0,1]} \left(t + \mu \left(\{ f \star g \ge t \} \right) - 1 \right)_{+} = \sup_{t \in [0,1]} \left(t - t^{2} \right)_{+} = 0.25.$$
Hence, $0.25 = \int_{S_{L},X} (f \star g) \, d\mu > \int_{S_{L},X} f \, d\mu \star \int_{S_{L},X} g \, d\mu = 0.125.$

Now we focus on the subadditivity property of the generalized upper Sugeno integral (2), that is,

$$\int_{\circ,X} (f+g) \,\mathrm{d}\mu \leqslant \int_{\circ,X} f \,\mathrm{d}\mu + \int_{\circ,X} g \,\mathrm{d}\mu,\tag{13}$$

as this property is very important for applications. Let us recall that +-associativity is equivalent to comonotonicity, see Example 5.

Corollary 3. Let Y = [0, m] or Y = [0, m) for $0 < m \leq \infty$ and let $\circ: Y^2 \to Y$ be a nondecreasing operator such that $0 \circ y = y \circ 0 = 0$ for all y. The functional (2) is subadditive for comonotone functions $f, g: X \to Y$ such that $f + g \in Y$ if and only if $(a + b) \circ c \leq (a \circ c) + (b \circ c)$ for $a, b \in Y$, $a + b \in Y$ and $c \in \mu(\mathcal{A}) = Y$.

It follows from Corollary 3 that both the Sugeno integral (7) and the Shilkret integral (8) are subadditive for comonotone functions while the opposite-Sugeno integral $\int_{S_L,D} f d\mu$ [16] is not.

Corollary 4. Let $\circ = S \in \mathfrak{S}$. Then the subadditivity property (13) is fulfilled for any monotone measure μ such that $\mu(X) \leq 1$ and comonotone functions $f, g: X \to [0, 1]$ such that $f + g \in [0, 1]$ if and only if

$$S(a+b,c) \leqslant S(a,c) + S(b,c) \tag{14}$$

for all $a, b, c \in [0, 1], a + b \in [0, 1].$

Borzová-Molnárová et al. [4] showed that the inequality (14) is satisfied for each semicopula with concave horizontal sections $x \mapsto S(x, y)$. An example is the Marshall–Olkin semicopula $S_{\alpha,\beta}(x,y) = (x^{1-\alpha}y) \wedge (xy^{1-\beta})$, where $\alpha, \beta \in [0,1]$. Observe that if $f = \mathbb{1}_A$ and $g = \mathbb{1}_B$ for $A \cup B = X$ and $A \cap B = \emptyset$, then the inequality (13) is of the form $\mu(X) \leq \mu(A) + \mu(B)$ for any semicopula S. Thus, the seminormed fuzzy integral is not subadditive if $\mu(A) + \mu(B) < \mu(X)$.

We say that $\mu: \mathcal{A} \to Y$ is *subadditive* if it is a monotone measure and $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for all $A, B \in \mathcal{A}$. The class of subaditive measures is quite wide and includes

the following monotone measures: λ -measure of Sugeno for $\lambda \in (-1/\mu(X), 0)$ (see [33], Definition 4.3); the plausibility measure [33]; the coherent measure $\mu(A) = \sup_{\mathsf{P} \in \mathcal{P}} \mathsf{P}(A)$, where \mathcal{P} is a set of probability measures [14]; the possibility measure $\mu(A) = \sup_{x \in A} \psi(x)$, where $\psi: X \to Y$ [33], the distortion measure $\mu(A) = g(\mathsf{P}(A))$, where P is probability measure and $g: [0,1] \to Y$ is such that $g(x+y) \leq g(x) + g(y)$ [29] and uncertain measure [23], among others.

Theorem 2. Suppose Y = [0,m] or Y = [0,m) with $0 < m \leq \infty$ and suppose $\circ: Y^2 \to Y$ is a nondecreasing operator such that $x \circ (y + z) \leq (x \circ y) + (x \circ z)$ for all $x, y, z \in Y$ such that $y + z \in Y$. Suppose also that $(ax) \circ y \leq a^q (x \circ y)^r$ for some q, r > 0 and for all $x, y, z \in Y, a > 1$ such that $ax \in Y$. Then for any p > 0, any subadditive measure μ and any measurable functions $f, g: X \to \mathbb{R}$ such that $|f + g|^p, |f|^p, |g|^p \in Y$, we have

$$\left(\int_{\circ,X} |f+g|^p \,\mathrm{d}\mu\right)^{1/(pq+1)} \leqslant \left(\int_{\circ,X} |f|^p \,\mathrm{d}\mu\right)^{r/(pq+1)} + \left(\int_{\circ,X} |g|^p \,\mathrm{d}\mu\right)^{r/(pq+1)}.$$
 (15)

Proof. Without loss of generality, assume that $\int_{\circ,X} |f|^p d\mu + \int_{\circ,X} |g|^p d\mu < \infty$. Clearly, $\{|f + g| \ge t^{1/p}\} \subset \{|f| \ge \lambda t^{1/p}\} \cup \{|g| \ge (1 - \lambda)t^{1/p}\}$ for $t \in Y$ and $\lambda \in (0, 1)$. Thus, by the subadditivity of μ and monotonicity of \circ , we have

$$t \circ \mu\big(\{|f+g|^p \ge t\}\big) \leqslant t \circ \Big(\mu\big(\{|f|^p \ge \lambda^p t\}\big) + \mu\big(\{|g|^p \ge (1-\lambda)^p t\}\big)\Big).$$

From the assumptions on \circ , we get

$$\begin{split} \int_{\circ,X} |f+g|^p \, \mathrm{d}\mu &\leqslant \sup_{t \in Y} \left\{ t \circ \mu \left(\{ |f|^p \geqslant \lambda^p t \} \right) \right\} + \sup_{t \in Y} \left\{ t \circ \mu \left(\{ |g|^p \geqslant (1-\lambda)^p t \} \right) \right\} \\ &\leqslant \sup_{y \in \lambda^p Y} \left\{ \frac{y}{\lambda^p} \circ \mu \left(\{ |f|^p \geqslant y \} \right) \right\} + \sup_{y \in (1-\lambda)^p Y} \left\{ \frac{y}{(1-\lambda)^p} \circ \mu \left(\{ |g|^p \geqslant y \} \right) \right\} \\ &\leqslant \lambda^{-pq} \left(\int_{\circ,X} |f|^p \, \mathrm{d}\mu \right)^r + (1-\lambda)^{-pq} \left(\int_{\circ,X} |g|^p \, \mathrm{d}\mu \right)^r, \end{split}$$

where $\lambda^p Y = \{\lambda^p y \colon y \in Y\} \subset Y$. If $\int_{o,X} |f|^p d\mu = 0$ or $\int_{o,X} |g|^p d\mu = 0$, we take the limit as λ approaches 0 or 1, respectively. Otherwise, we obtain (15) by minimizing the right-hand side with respect to λ .

Corollary 5. Let Y = [0,1], $Y = [0,\infty)$ or $Y = [0,\infty]$. If μ is subadditive, then for all measurable functions $f, g: X \to \mathbb{R}$ and p > 0 we have

$$\left((S) \int_X |f+g|^p \, \mathrm{d}\mu \right)^{1/(p+1)} \leqslant \left((S) \int_X |f|^p \, \mathrm{d}\mu \right)^{1/(p+1)} + \left((S) \int_X |g|^p \, \mathrm{d}\mu \right)^{1/(p+1)},$$

$$\left((N) \int_X |f+g|^p \, \mathrm{d}\mu \right)^{1/(p+1)} \leqslant \left((N) \int_X |f|^p \, \mathrm{d}\mu \right)^{1/(p+1)} + \left((N) \int_X |g|^p \, \mathrm{d}\mu \right)^{1/(p+1)}$$

where $|f + g|^p$, $|f|^p$, $|g|^p \in Y$ and the integrals are defined, respectively, by (7) and (8).

The next result deals with a modified Shilkret integral and follows from Theorem 2 and the inequality $(x + y)^s \leq x^s + y^s$ for $x, y \geq 0$ and 0 < s < 1.

Corollary 6. Let $a \circ_q b = (ab)^q$ with 0 < q < 1 and let Y = [0, 1] or $Y = [0, \infty)$. For any subadditive measure μ and any measurable functions $f, g: X \to \mathbb{R}$, we get

$$\left(\int_{\circ_{q,X}} |f+g|^p \,\mathrm{d}\mu\right)^{1/p} \leqslant \left(\int_{\circ_{q,X}} |f|^p \,\mathrm{d}\mu\right)^{1/p} + \left(\int_{\circ_{q,X}} |g|^p \,\mathrm{d}\mu\right)^{1/p},$$

where p = 1/(1-q), and $|f+g|^p$, $|f|^p$, $|g|^p \in Y$.

Simple calculations show that

$$\left(\int_{\circ_{q,X}} |f|^{p} \,\mathrm{d}\mu\right)^{1/p} = \sup_{t \in Y} \left\{ t^{q} \mu \left(\{|f| \ge t\}\right)^{q/p} \right\},$$

so this functional is similar to a quasi-norm in the Lorentz type capacity spaces [8].

Now, we analyze the subadditivity of the Shilkret integral. Recall that a monotone measure μ is *maxitive* if for all disjoint sets $A, B \in \mathcal{A}$ we have

$$\mu(A \cup B) = \mu(A) \lor \mu(B). \tag{16}$$

Observe that μ is maximize if and only if (16) holds for all $A, B \in \mathcal{A}$. In fact, if μ is maximize and $A \cap B \neq \emptyset$, then $\mu(A \cup B) = \mu(A) \lor \mu(C)$ and $\mu(A \cup B) = \mu(D) \lor \mu(B)$, where $C = B \setminus A$ and $D = A \setminus B$. This implies that $\mu(A \cup B) = \mu(A) \lor \mu(B) \lor \mu(C) \lor \mu(D) = \mu(A) \lor \mu(B)$, so (16) is satisfied. Clearly, any maximize measure is subadditive.

The following result can be found in [7] (see also [30] pp. 112-113 and [8] Theorem 4.2).

Theorem 3. Let Y = [0, 1], $Y = [0, \infty)$ or $Y = [0, \infty]$. The Shilkret inegral (8) is subadditive for all measurable functions $f, g: X \to Y$ if and only if the monotone measure μ is maxitive.

Proof. First, observe that μ is maxitive if and only if (16) holds for all $A, B \in \mathcal{A}$. In fact, if μ is maxitive and $A \cap B \neq \emptyset$, then $\mu(A \cup B) = \mu(A) \lor \mu(C)$ and $\mu(A \cup B) = \mu(D) \lor \mu(B)$, where $C = B \setminus A$ and $D = A \setminus B$. This implies that $\mu(A \cup B) = \mu(A) \lor \mu(B) \lor \mu(C) \lor \mu(D) = \mu(A) \lor \mu(B)$, so (16) is satisfied. Denote the Shilkret integal for short by $\mathbf{I}(f)$.

" \Leftarrow " We follow the proof of [7, 30]. If $\mathbf{I}(f) = \mathbf{I}(g) = 0$, then $\mathbf{I}(f + g) = 0$ as $\mu(\{f + g \ge t\}) \leqslant \mu(\{f \ge t/2\}) + \mu(\{g \ge t/2\}) = 0$ for all t > 0. Therefore, we assume that $0 < \mathbf{I}(f) + \mathbf{I}(g) < \infty$, without loss of generality. By maximizity of μ , we have

$$t\mu(\{f+g \ge t\}) \leqslant t\mu(\{f \ge \lambda t\} \cup \{g \ge (1-\lambda)t\})$$
$$= t\mu(\{f \ge \lambda t\}) \lor t\mu(\{g \ge (1-\lambda)t\})$$

with $\lambda = \mathbf{I}(f)/(\mathbf{I}(f) + \mathbf{I}(g))$. Hence,

$$\mathbf{I}(f+g) \leqslant \left((\mathbf{I}(f)/\lambda) \lor \left(\mathbf{I}(g)/(1-\lambda) \right) = \mathbf{I}(f) + \mathbf{I}(g) \right)$$

"⇒" Suppose μ is not maximum, i.e. $\mu(A \cup B) > \mu(A) \lor \mu(B)$ for some disjoint sets $A, B \in \mathcal{A}$. Thus, there exists $\lambda \in (0, 1)$ such that $\lambda \mu(A \cup B) > \mu(A) \lor \mu(B)$. Putting $f = \mathbb{1}_A + \lambda \mathbb{1}_B, g = (1 - \lambda) \mathbb{1}_B$, we get

$$\mathbf{I}(f) + \mathbf{I}(g) = \left((\lambda \mu(A \cup B)) \lor \mu(A) \right) + (1 - \lambda)\mu(B)$$

$$< \lambda \mu(A \cup B) + (1 - \lambda)\mu(A \cup B) = \mathbf{I}(f + g),$$

so the Shilkret integral is not subadditive.

Subadditivity of the Sugeno integral will be examined in the next section.

3 Results for generalized lower Sugeno integral

The generalized lower Sugeno integral of a measurable function $f: X \to Y$ on a set $D \in \mathcal{A}$ with respect to a monotone measure μ and nondecreasing operator $\circ: Y \times \mu(\mathcal{A}) \to [0, \infty]$ is defined as

$$\int_{\circ,D} f \,\mathrm{d}\mu := \inf_{t \in Y} \left\{ t \circ \mu \left(D \cap \{f > t\} \right) \right\}.$$
(17)

Observe that the functional (17) is the universal integral in the sense of Definition 2.5 in [21] if $a \circ 0 = a$ and $0 \circ b = b$ for all $a \in Y$ and $b \in \mu(\mathcal{A})$. Putting $\circ = \lor$ in (17) we obtain the lower Sugeno integral [24]

$$(S) \oint_D f \,\mathrm{d}\mu := \inf_{t \in Y} \left\{ t \lor \mu \left(D \cap \{f > t\} \right) \right\}.$$

$$(18)$$

Mimicking the proof of Theorem 5 in [19] and Theorem 9.1 in [33] one can show that for any $Y = [0, m] \subset [0, \infty]$ the integral (18) is equal to the Sugeno integral (7)

$$(S) \oint_D f \,\mathrm{d}\mu = (S) \int_D f \,\mathrm{d}\mu.$$
(19)

Open problem 2. Does there exist a pair of operators $(\nabla, \triangle) \neq (\lor, \land)$ such that for all $f: X \to Y$

$$\oint_{\nabla,D} f \,\mathrm{d}\mu = \int_{\Delta,D} f \,\mathrm{d}\mu ? \tag{20}$$

We say that measurable functions $f, g: X \to Y$ are μ -subadditive for an operator $\nabla: \mu(\mathcal{A})^2 \to \mu(\mathcal{A})$ and a set $D \in \mathcal{A}$ if for all $a, b \in Y$

$$\mu\Big(D\cap\big(\{f>a\}\cup\{g>b\}\big)\Big)\leqslant\mu\big(D\cap\{f>a\}\big)\nabla\mu\big(D\cap\{g>b\}\big).$$

Observe that μ -subadditivity implies that $x \lor y \leqslant x \lor y$ for all $x, y \in \mu(\mathcal{A} \cap D)$.

Now, we present several examples of μ -subadditive functions.

Example 7. Any comonotone functions f, g are μ -subadditive with respect to an operator ∇ such that $x \vee y \leq x \nabla y$ for all $x, y \in Y$. For instance, any t-semiconorm S^{*} on Y = [0, 1] has this property (see [20]).

Example 8. Recall that μ is submodular if $\mu(A \cup B) \leq \mu(A) + \mu(B) - \mu(A \cap B)$ for all $A, B \in \mathcal{A}$. Let D = X, $x \forall y = 1 - (1 - x)(1 - y)$ for $x, y \in Y = [0, 1]$ and let μ be a submodular and monotone measure. Functions f, g are μ -subadditive if f, g are positive quadrant dependent [18], that is, $\mu(\{f > t\} \cap \{g > s\}) \geq \mu(\{f > t\})\mu(\{g > s\})$ for all $t, s \in Y$.

Example 9. Put $x \forall y = x + y$ for $x, y \in Y = [0, \infty]$. Then any functions f, g are μ -subadditive for a subadditive measure μ on X.

Suppose $\star, \Diamond: Y^2 \to Y$, and $\circ_i: Y \times \mu(\mathcal{A}) \to Y$, i = 1, 2, 3, are nondecreasing and \Diamond is right-continuous. Suppose also that $\phi_i: Y \to Y$ is an increasing function and $\phi_i(Y) = Y$ for i = 1, 2, 3.

Theorem 4. Assume that for $a, b \in Y$ and $c, d \leq \mu(D)$, we have

$$\phi_1^{-1}(\phi_1(a \star b) \circ_1 (c \nabla d)) \leqslant \phi_2^{-1}(\phi_2(a) \circ_2 c) \diamond \phi_3^{-1}(\phi_3(b) \circ_3 d).$$
(21)

If f, g are μ -subadditive for ∇ and D, then

$$\phi_1^{-1}\left(\oint_{\circ_1,D}\phi_1(f\star g)\,\mathrm{d}\mu\right)\leqslant\phi_2^{-1}\left(\oint_{\circ_2,D}\phi_2(f)\,\mathrm{d}\mu\right)\diamond\phi_3^{-1}\left(\oint_{\circ_3,D}\phi_3(g)\,\mathrm{d}\mu\right).\tag{22}$$

Proof. By the monotonicity of \star and μ , for any $D \in \mathcal{A}$ we obtain

$$\mu \big(D \cap \{ f \star g > a \star b \} \big) \leq \mu \Big(D \cap \big(\{ f > a \} \cup \{ g > b \} \big) \Big).$$

From μ -subadditivity of f, g and from the fact that $b \mapsto a \circ_1 b$ is a nondecreasing function we get

$$\phi_1(a \star b) \circ_1 \mu \left(D \cap \left\{ \phi_1(f \star g) > \phi_1(a \star b) \right\} \right)$$

$$\leqslant \phi_1(a \star b) \circ_1 \left(\mu \left(D \cap \left\{ \phi_2(f) > \phi_2(a) \right\} \right) \nabla \mu \left(D \cap \left\{ \phi_3(g) > \phi_3(b) \right\} \right) \right).$$
(23)

By (21) and (23)

$$\phi_{1}^{-1} \Big(\phi_{1}(a \star b) \circ_{1} \mu \Big(D \cap \{ \phi_{1}(f \star g) > \phi_{1}(a \star b) \} \Big) \Big) \\ \leqslant \phi_{2}^{-1} \Big(\phi_{2}(a) \circ_{2} \mu \Big(D \cap \{ \phi_{2}(f) > \phi_{2}(a) \} \Big) \Big) \Diamond \phi_{3}^{-1} \Big(\phi_{3}(b) \circ_{3} \mu \Big(D \cap \{ \phi_{3}(g) > \phi_{3}(b) \} \Big) \Big).$$

$$(24)$$

Since ϕ_1^{-1} is increasing, we have

$$\phi_1^{-1}\left(\int_{\circ_1,D} \phi_1(f \star g) \,\mathrm{d}\mu\right) \\ \leqslant \phi_2^{-1}\left(\phi_2(a) \circ_2 \mu \left(D \cap \{\phi_2(f) > \phi_2(a)\}\right)\right) \Diamond \phi_3^{-1}\left(\phi_3(b) \circ_3 \mu \left(D \cap \{\phi_3(g) > \phi_3(b)\}\right)\right)$$

for all $a, b \in Y$. Taking the infimum over $a \in Y$, we get

$$\phi_1^{-1}\left(\oint_{\circ_1,D}\phi_1(f\star g)\,\mathrm{d}\mu\right) \leqslant \phi_2^{-1}\left(\oint_{\circ_2,D}\phi_2(f)\,\mathrm{d}\mu\right) \diamondsuit \phi_3^{-1}\left(\phi_3(b)\circ_3\mu\left(D\cap\{\phi_3(g)>\phi_3(b)\}\right)\right).$$
Proceeding similary with the infimum in $b\in Y$, we obtain (22).

Proceeding similary with the infimum in $b \in Y$, we obtain (22).

Example 10. We know from Example 7 that any comonotone functions $f, g: X \to Y$ are μ -subadditive with $\nabla = \vee$. Put $Y = [0, \infty]$. If

$$(a \star b) \lor \left(\phi_1^{-1}(c) \lor \phi_1^{-1}(d)\right) \leqslant \left(a \lor \phi_2^{-1}(c)\right) \star \left(b \lor \phi_3^{-1}(d)\right),$$
(25)

then for all $D \in \mathcal{A}$ we get

$$\phi_1^{-1}\left((S) \oint_D \phi_1(f \star g) \,\mathrm{d}\mu\right) \leqslant \phi_2^{-1}\left((S) \oint_D \phi_2(f) \,\mathrm{d}\mu\right) \star \phi_3^{-1}\left((S) \oint_D \phi_3(g) \,\mathrm{d}\mu\right). \tag{26}$$

The inequality (25) is satisfied for any operator \star such that $a \star b \ge a \lor b$ and functions $\phi_1 \ge \phi_i, i = 2, 3$. Indeed, combining $a_1 \star a_2 \le (a_1 \lor \phi_2^{-1}(b_1)) \star (a_2 \lor \phi_3^{-1}(b_2))$ with

$$\phi_1^{-1}(b_1) \lor \phi_1^{-1}(b_2) \leqslant \phi_1^{-1}(b_1) \star \phi_1^{-1}(b_2) \leqslant (a_1 \lor \phi_2^{-1}(b_1)) \star (a_2 \lor \phi_3^{-1}(b_2))$$

yields (25). From (19) and (26) we can get a generalization of Theorem 3.1 in [35].

Example 11. Let Y = [0, 1], D = X and $\mu(X) = 1$. Put $x \bigtriangledown y = x + y - xy$ and $x \star y = y$ $x \diamond y = (x+y) \land 1$, where $x, y \in Y$. If $\circ_i = \lor$ and $\phi_i(x) = x$, i = 1, 2, 3, then the condition (21) takes the form

$$((a+b)\wedge 1)\vee(c+d-cd)\leqslant (a\vee c+b\vee d)\wedge 1,$$
(27)

 $a, b, c, d \in Y$. Since $(a + b) \land 1 \leq a + b \leq (a \lor c) + (b \lor d)$ and $c + d - cd \leq (a \lor c + b \lor d)$, the inequality (27) is true for all a, b, c, d. Hence, if $f, g: X \to [0, 1]$ are positive quadrant dependent functions with respect to a submodular and monotone measure μ on X (see Example 8), then

$$(S) \oint_X (f+g) \wedge 1 \,\mathrm{d}\mu \leqslant (S) \oint_X f \,\mathrm{d}\mu + (S) \oint_X g \,\mathrm{d}\mu.$$
(28)

Example 12. Set $Y = [0, \infty]$ and $\nabla = \star = \Diamond = +$. Let μ be a subadditive measure, $\circ_i = \lor$ and $\phi_i(x) = x$ for all *i*. Then

$$(S) \oint_{X} (f+g) \,\mathrm{d}\mu \leqslant (S) \oint_{X} f \,\mathrm{d}\mu + (S) \oint_{X} g \,\mathrm{d}\mu$$
(29)

for all $f, g: X \to Y$.

Next, we prove that the subadditivity property of Sugeno integral (7) with Y = [0, 1] or $Y = [0, \infty)$ is equivalent to subadditivity of a finite measure μ .

Theorem 5. If μ is subadditive, then

$$(S)\int_{X}(f+g)\,\mathrm{d}\mu \leqslant (S)\int_{X}f\,\mathrm{d}\mu + (S)\int_{X}g\,\mathrm{d}\mu \tag{30}$$

for all $f, g: X \to Y$. Moreover, if (30) holds for all measurable functions $f, g: X \to Y$ and $\mu(X) < \infty$, then μ is subadditive.

Proof. The inequality (30) follows immediately from (29) and (19). Moreover, let $f = a \mathbb{1}_A$, $g = a \mathbb{1}_B$, where $a \ge 0$, $A, B \in \mathcal{A}$. From (30) and the finite and monotone measure μ , we have

$$\mu(A \cup B) = a \land \mu(A \cup B) \leqslant (a \land \mu(A)) + (a \land \mu(B)) \leqslant \mu(A) + \mu(B)$$

for $a \ge \mu(A \cup B)$, which completes the proof.

The assumption $\mu(X) = \infty$ in Theorem 5 cannot be omitted. Indeed, if μ is a subadditive measure such that $\mu(X) = \infty$ and $\mu(A), \mu(B) < \infty$ for some A, B, such that $A \cup B = X$, then the inequality (30) is not true for $f = a \mathbb{1}_A, g = a \mathbb{1}_B$ and $a > \max(\mu(A), \mu(B))$.

Now, we show that from the Minkowski-Hölder type inequality for integral (2) one can obtain an inequality of the same type for integral (17) and vice versa. Suppose Y = [0, m], $0 < m \leq \infty$. Let $h: Y \to Y$ be a decreasing function such that h(Y) = Y, h(0) > 0and h(m) = 0. For instance, h(x) = 1 - x for Y = [0, 1] and h(x) = 1/x for $Y = [0, \infty]$ under convention that $1/0 = \infty$ and $1/\infty = 0$. Suppose μ_h is a monotone measure on (X, \mathcal{A}) defined as $\mu_h(A) = h^{-1}(\mu(X \setminus A))$. For a given operator $\circ: Y^2 \to Y$ let us define the operator

$$a \circ_h b = h^{-1} (h(a) \circ h(b)), \quad a, b \in Y.$$

For any measurable function $f: X \to Y$, we have

$$h^{-1}\left(\int_{\circ,X} h(f) \,\mathrm{d}\mu\right) = h^{-1}\left(\inf_{y \in Y} \left\{ h(y) \circ \mu(\{h(f) > h(y)\}) \right\} \right)$$
$$= \inf_{y \in Y} \left\{ h^{-1}\left(h(y) \circ \mu(\{f \leqslant y\})\right) \right\} = \int_{\circ_h,X} f \,\mathrm{d}\mu. \tag{31}$$

Applying the formula (31) and Theorem 1 with $\star = \Diamond$ and $\phi_i(x) = x$ for all *i* gives the following Corollary.

Corollary 7. Assume $\circ: Y^2 \to Y$ is nondecreasing, $m \circ y = y \circ m = m$ for all $y \in Y = [0, m]$ and $f, g: X \to Y$ are \star -associated. The following inequality is satisfied

$$h^{-1}\left(\oint_{\circ,X} h(f \star g) \,\mathrm{d}\mu\right) \leqslant h^{-1}\left(\oint_{\circ,X} h(f) \,\mathrm{d}\mu\right) \star h^{-1}\left(\oint_{\circ,X} h(g) \,\mathrm{d}\mu\right)$$

if and only if $(a \star b) \circ_h c \leq (a \circ_h c) \star (b \circ_h c)$ for all $a, b, c \in Y$.

Example 13. From the well-known inequality $(x+y)/(1+x+y) \leq (x/(1+x)) + (y/(1+y))$ for $x, y \geq 0$, it follows that for all $a, b, c \geq 0$

$$((a+b)^{-1}+c^{-1})^{-1} \leq (a^{-1}+c^{-1})^{-1}+(b^{-1}+c^{-1})^{-1}$$

with $1/0 = \infty$ and $1/\infty = 0$. This implies that the necessary and sufficient condition of Corollary 7 is satisfied for $Y = [0, \infty]$, $h(x) = x^{-1}$ and $\star = \circ = +$. Thus, for any comonotone functions $f, g: X \to Y$, we have

$$\left(\oint_{+,X} 1/(f+g) \,\mathrm{d}\mu \right)^{-1} \leqslant \left(\oint_{+,X} 1/f \,\mathrm{d}\mu \right)^{-1} + \left(\oint_{+,X} 1/g \,\mathrm{d}\mu \right)^{-1}.$$

The next result is an immediate consequence of Theorem 4 with $\phi_i(x) = x$ for all *i* and the formula

$$\oint_{\circ_h, X} f \,\mathrm{d}\mu_h = h^{-1} \bigg(\int_{\circ, X} h(f) \,\mathrm{d}\mu \bigg).$$

Corollary 8. Assume that $f, g: X \to Y$ are μ_h -subadditive for ∇ , operator \star is nondecreasing and right-continuous and \circ is nondecreasing. Assume also that $(a \star b) \circ_h (c \nabla d) \leq (a \circ_h c) \star (b \circ_h d)$ for all $a, b, c, d \in Y$. Then

$$h^{-1}\Big(\int_{o,X} h(f \star g) \,\mathrm{d}\mu\Big) \leqslant h^{-1}\Big(\int_{o,X} h(f) \,\mathrm{d}\mu\Big) \star h^{-1}\Big(\int_{o,X} h(g) \,\mathrm{d}\mu\Big).$$

Example 14. Suppose $\mu(A) = 1/\mu_h(X \setminus A)$ for $A \in \mathcal{A}$. From Example 12, formula (19) and Corollary 8 for h(x) = 1/x, $\circ_h = \lor$ and $\star = +$, it follows that for any measurable functions $f, g: X \to [0, \infty]$, we have

$$\left((S) \int_X 1/(f+g) \,\mathrm{d}\mu \right)^{-1} \leqslant \left((S) \int_X 1/f \,\mathrm{d}\mu \right)^{-1} + \left((S) \int_X 1/g \,\mathrm{d}\mu \right)^{-1},$$

with (S) \int_X being the Sugeno integral (7) for $Y = [0, \infty]$ and $1/\infty = 0, 1/0 = \infty$.

4 Application

As an application of the results of this paper, we provide new metrics in the space of \mathcal{A} measurable functions $f: X \to \mathbb{R}$ defined on a fuzzy space (X, \mathcal{A}, μ) . First, let us recall some facts. Taking $\circ = +$ and $Y = [0, \infty]$ in (17), we get the functional

$$d_F(\mathsf{X},\mathsf{Y}) = \inf_{\varepsilon \ge 0} \left\{ \varepsilon + \mu \left(\{ |\mathsf{X} - \mathsf{Y}| > \varepsilon \} \right) \right\}$$

on the space $L^0(X)$ of all random variables defined on a probability space (X, \mathcal{A}, μ) . This functional was proposed by Fréchet [15] in order to metrize the convergence in measure μ (see also [6], p. 356, and [11], pp. 101 – 104). The integral (17) with $\circ = \lor$ was introduced by Ky Fan [13]. He proved that $L^0(X)$ with the metric

$$d_{KF}(\mathsf{X},\mathsf{Y}) = \inf \left\{ \varepsilon \ge 0 \colon \mu \big(\{ |\mathsf{X} - \mathsf{Y}| > \varepsilon \} \big) \leqslant \varepsilon \right\}$$

is a complete space. By (19) we have

$$d_{KF}(\mathbf{X}, \mathbf{Y}) = (S) f_X |\mathbf{X} - \mathbf{Y}| \, \mathrm{d}\mu = (S) \int_X |\mathbf{X} - \mathbf{Y}| \, \mathrm{d}\mu.$$

Li [22] extended Ky Fan's result to cover the case of any continuous from below, finite and subadditive measure μ .

Now we are ready to introduced new metrics. Given p > 0, let $Y = [0, \infty]$ and $\circ: Y^2 \to Y$ be a non-decreasing operator such that $x \circ (y+z) \leq (x \circ y) + (x \circ z)$ and $(ax) \circ y \leq a^p (x \circ y)$ for $x, y, z \in Y$ and a > 1. We also assume that if $1 \circ x \leq y$ for 0 < y < 1, then $x \leq y$. For instance, $x \circ y = x^p \wedge y^u$ or $x \circ y = x^p y^u$, where $0 < u \leq 1$. Suppose μ is a subadditive measure and put

$$D_{\circ,p}(f,g) = \left(\int_{\circ,X} |f-g|^p \,\mathrm{d}\mu\right)^{1/(p^2+1)}.$$
(32)

As special cases we get

$$D_{\wedge,1}(f,g) = \left((S) \int_X |f-g| \,\mathrm{d}\mu \right)^{1/2}, \quad D_{\cdot,1}(f,g) = \left((N) \int_X |f-g| \,\mathrm{d}\mu \right)^{1/2}. \tag{33}$$

Denote by \mathcal{L}^p_{\circ} the class of measurable functions $f: X \to \mathbb{R}$ such that $D_{\circ,p}(f,0) < \infty$. Let $f \sim g$ mean that $\mu(\{|f-g|>0\}) = 0$ and let L^p_{\circ} be the set of the equivalence classes in \mathcal{L}^p_{\circ} determined by the equivalence relation \sim . If [f] is the equivalence class containing f, define $d_{\circ,p}(f,g) = D_{\circ,p}([f],[g])$.

Theorem 6. Suppose $\circ: Y^2 \to Y$ is left-continuous in the second argument. If μ is subadditive and continuous from below, then $(L^p_{\circ}, d_{\circ,p})$ is a complete metric space.

To prove Theorem 6 we need the monotone convergence theorem and Fatou's lemma for the integral (2). We recall that μ is *null-additive* if $\mu(A) = 0$ implies $\mu(A \cup B) = \mu(B)$ for every $B \in \mathcal{A}$. Observe that if μ is subadditive, then it is also null-additive.

Lemma 1 (Monotone convergence). Let $\circ: Y^2 \to Y$ be left-continuous in the second argument. If μ is a continuous from below, null-additive and monotone measure and if (f_n) is a sequence of functions $f_n: X \to Y$ which is nondecreasing and converges to f on $A^c = X \setminus A$ with $\mu(A) = 0$, then $\lim_{n \to \infty} \int_{o, X} f_n \, \mathrm{d}\mu = \int_{o, X} f \, \mathrm{d}\mu$.

Proof. Measure μ is null-additive, so $\int_{\circ,X} g \, d\mu = \int_{\circ,A^c} g \, d\mu$ for any g. The rest of the proof is similar to that of Lemma 14 in [9].

Lemma 2 (Fatou). Suppose $\circ: Y^2 \to Y$ is left-continuous in the second argument, $f_n: X \to Y$ for all n. If μ is a continuous from below, null-additive and monotone measure and $f(x) = \lim_{n \to \infty} f_n(x)$ for all $x \in A^c$ with $\mu(A) = 0$, then $\int_{\circ, X} f \, d\mu \leq \liminf_{n \to \infty} \int_{\circ, X} f_n \, d\mu$.

Proof. The proof follows from Lemma 1 and standard arguments (see [17], Lemma 1.20). \Box

Proof of Theorem 6. Assume $c \circ d > 0$ for some c, d > 0; otherwise the result is trivial. We will show that $x \circ y > 0$ for all x, y > 0. In fact, suppose that $x \circ y = 0$ for some x, y > 0. Then $x \circ t \leq x \circ y = 0$ for all $0 \leq t \leq y$, so from the subadditivity of $t \mapsto x \circ t$ it easily follows that $x \circ t = 0$ for all $t \in Y$. Next, by the monotonicity of \circ , we have $s \circ t \leq x \circ t = 0$ for all $0 \leq s \leq x$ and $(ax) \circ t \leq a^p(x \circ t) = 0$ for any a > 1. Hence so $s \circ t = 0$ for all $s, t \in Y$, a contradiction.

Next, suppose $d_{\circ,p}(f,g) = 0$. Hence, $\mu(\{|f-g| \ge t\}) = 0$ for all t > 0. Since μ is continuous from below, we have $\mu(\{|f-g| > 0\}) = 0$, so $f \sim g$. Clearly $d_{\circ,p}$ is symmetric and it follows from Theorem 2 for r = 1 and q = p that $d_{\circ,p}$ satisfies the triangle inequality. The proof of the completeness is a modified version of that of Lemma 1.31 in [17]. Given a Cauchy sequence (f_n) , let $(f_{n(k)})$ be a subsequence such that $(d_{\circ,p}(f_{n(k+1)}, f_{n(k)}))^{p^2+1} \leq 4^{-kp}$. Put $A_k = \{x \in X : |f_{n(k+1)}(x) - f_{n(k)}(x)|^p \ge 2^{-k}\}$. Then $2^{-k} \circ \mu(A_k) \leq 4^{-kp}$ by the definition of $d_{\circ,p}$. From the property $(2x) \circ y \leq 2^p(x \circ y)$ we get

$$1 \circ \mu(A_k) \leqslant (2^p)^k \left(2^{-k} \circ \mu(A_k) \right) \leqslant 2^{-kp}.$$

By the assumption on \circ , $\mu(A_k) \leq 2^{-kp}$ for all k. Set $A = \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} A_k$. Since μ is subadditive, we have

$$\mu(A) \leqslant \mu\left(\bigcup_{k=i}^{\infty} A_k\right) \leqslant \sum_{k=i}^{\infty} \mu(A_k) \to 0 \text{ as } i \to \infty,$$

so $\mu(A) = 0$. Let $x \in A^c = X \setminus A$. Since $|f_{n(k+1)}(x) - f_{n(k)}(x)| < 2^{-k/p}$ for all large enough k, we have

$$\sup_{r \ge k} |f_{n(r)}(x) - f_{n(k)}(x)| \le \sum_{r=k}^{\infty} |f_{n(r+1)}(x) - f_{n(r)}(x)| \le \frac{2^{-k/p}}{1 - 2^{-1/p}}.$$

Thus, $((f_{n(k)}(x))_{k=1}^{\infty})$ is a Cauchy sequence and $f_{n(k)}(x) \to f(x)$ for all $x \in A^c$, where f is some measurable function (if $A \neq \emptyset$, define f on A arbitrarily). We recall that any subadditive measure μ is also null-additive. By Lemma 2, we get

$$d_{\circ,p}(f,f_n) \leqslant \liminf_{k \to \infty} d_{\circ,p}(f_{n(k)},f_n) \leqslant \sup_{m \ge n} d_{\circ,p}(f_m,f_n) \to 0, \ n \to \infty,$$

as (f_n) is a Cauchy sequence. This shows that $f_n \to f$ in metric $d_{\circ,p}$.

Put $||f|| = (N) \int_X |f| d\mu$ and denote by L_N^1 the space of all functions $f: X \to \mathbb{R}$ such that $||f|| < \infty$.

Corollary 9. If μ is maximize, then L_N^1 is a Banach space with the norm $\|\cdot\|$.

Proof. Any maximum measure is subadditive. Observe that $||f - g|| = D_{.,1}(f,g)^2$ (see (33)) and ||cf|| = |c|||f|| for $c \in \mathbb{R}$. The result follows immediately from Theorems 3 and 6.

The next theorem is a counterpart of Theorem 2.21 in [5]

Theorem 7. Adopt the assumptions of Theorem 2 with $Y = [0, \infty]$ and some p > 0. If $f_n, f \in L^p_{\circ}$ for all n and $\lim_{n \to \infty} d_{\circ,p}(f_n, f) = 0$, then $\int_{\circ, X} f_n^p d\mu \to \int_{\circ, X} f^p d\mu$ as $n \to \infty$.

Proof. From Theorem 2 we get $d_{\circ,p}(f_n, 0) \leq d_{\circ,p}(f_n, f) + d_{\circ,p}(f, 0)$ and $d_{\circ,p}(f, 0) \leq d_{\circ,p}(f_n, f) + d_{\circ,p}(f_n, 0)$. This implies

$$\left| \left(\int_{\circ,X} f_n^p \,\mathrm{d}\mu \right)^{1/(p^2+1)} - \left(\int_{\circ,X} f^p \,\mathrm{d}\mu \right)^{1/(p^2+1)} \right| \leqslant d_{\circ,p}(f_n,f),$$

which completes the proof.

Theorem 7 gives a partial answer to the open problem 2.22 in [5] as there exist discontinuous and subadditive measures.

References

- H. Agahi, R. Mesiar, Y. Ouyang, General Minkowski type inequalities for Sugeno integrals, Fuzzy Sets and Systems 161 (2010) 708–715.
- [2] H. Agahi, R. Mesiar, On Cauchy–Schwarz's inequality for Choquet-like integrals without the comonotonicity condition, Soft Computing 19 (2015) 1627–1634.
- [3] B. Bassan, F. Spizzichino, Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes, Journal of Multivariate Analysis 93 (2005) 313–339.
- [4] J. Borzová-Molnárová, L. Halčinová, O. Hutník, The smallest semicopula-based universal integrals I: Properties and characterizations, Fuzzy Sets and Systems 271 (2015) 1–17.
- [5] J. Borzová-Molnárová, L. Halčinová, O. Hutník, The smallest semicopula-based universal integrals II: Convergence theorems, Fuzzy Sets and Systems 271 (2015) 18–30.
- [6] N.L. Carothers, Real Analysis, Cambridge University Press, 2000.
- [7] M.E. Cattaneo, On maxitive integration, Technical Report 147, Department of Statistics University of Munich (2013) http://www.stat.uni-muenchen.de

- [8] J. Cerdá, Lorentz capacity spaces, Contemporary Mathematics 445 (2007) 45–59.
- [9] A. Chateauneuf, M. Grabisch, A. Rico, Modeling attitudes toward uncertainty through the use of the Sugeno integral, Journal of Mathematical Economics 44 (2008) 1084– 1099.
- [10] B. Daraby, F. Ghadimi, General Minkowski type and related inequalities for seminormed fuzzy integrals, Sahand Communications in Mathematical Analysis 1 (2014) 9–20.
- [11] N. Dunford, J.T. Schwartz, Linear Operators, Part I, Wiley-Interscience, New York, 1988.
- [12] F. Durante, C. Sempi, Semicopulæ, Kybernetika 41 (2005) 315–328.
- [13] K. Fan, Entfernung zweier zufälligen Grössen und die Konvergenz nach Wahrscheinlichkeit, Mathematische Zeitschrift 49 (1944) 681–683.
- [14] H. Föllmer, A. Schied, Stochastic Finance. An Introduction In Discrete Time, De Gruyter, Berlin, 2011.
- [15] M. Fréchet, Sur divers modes de convergence d'une suite de fonctions d'une variable, Bulletin of the Calcutta Mathematical Society 11 (1919-20) 187–206.
- [16] H. Imaoka, On a subjective evaluation model by a generalized fuzzy integral, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 5 (1997) 517–529.
- [17] O. Kallenberg, Foundations of Modern Probability, Springer, Berlin, 2002.
- [18] M. Kaluszka, A. Okolewski, M. Boczek, On Chebyshev type inequalities for generalized Sugeno integrals, Fuzzy Sets and Systems 244 (2014) 51–62.
- [19] A. Kandel, W.J. Byatt, Fuzzy sets, fuzzy algebra, and fuzzy statistics, Proceedings of the IEEE 66 (1978) 1619–1639.
- [20] E.P. Klement, R. Mesiar, E. Pap, Triangular norms, Kluwer Academic Publishers, Dordrecht, 2000.
- [21] E.P. Klement, R. Mesiar, E. Pap, A universal integral as common frame for Choquet and Sugeno integral, IEEE Transactions on Fuzzy Systems 18 (2010) 178–187.
- [22] G. Li, A metric on space of measurable functions and the related convergence, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 20 (2012) 211–222.

- [23] B. Liu, Uncertainty Theory, 4th Edition, Springer, 2015.
- [24] T. Murofushi, A note on upper and lower Sugeno integrals, Fuzzy Sets and Systems 138 (2003) 551–558.
- [25] Y. Ouyang, R. Mesiar, On the Chebyshev type inequality for seminormed fuzzy integral, Applied Mathematics Letters 22 (2009) 1810–1815.
- [26] Y. Ouyang, R. Mesiar, H. Agahi, An inequality related to Minkowski type for Sugeno integrals, Information Sciences 180 (2010) 2793–2801.
- [27] E. Pap (Ed.), Handbook of Measure Theory, Elsevier Science, Amsterdam, 2002.
- [28] H. Román-Flores, A. Flores-Franulič, Y. Chalco-Cano, The fuzzy integral for monotone functions, Applied Mathematics and Computation 185 (2007) 492–498.
- [29] L. Rüschendorf, Mathematical Risk Analysis: Dependence, Risk Bounds, Optimal Allocations and Portfolios, Springer Science and Business Media, Berlin, 2013.
- [30] N. Shilkret, Maxitive measure and integration, Indagationes Mathematicae 33 (1971) 109–116.
- [31] F. Suárez García, P. Gil Álvarez, Two families of fuzzy integrals, Fuzzy Sets and Systems 18 (1986) 67–81.
- [32] M. Sugeno, Theory of Fuzzy Integrals and its Applications, Ph.D. Dissertation, Tokyo Institute of Technology, 1974.
- [33] Z. Wang, G. Klir, Generalized Measure Theory, Springer, New York, 2009.
- [34] Ch. Wu, X. Rena, C. Wu, A note on the space of fuzzy measurable functions for a monotone measure, Fuzzy Sets and Systems 182 (2011) 2–12.
- [35] L. Wu, J. Sun, X. Ye, L. Zhu, Hölder type inequality for Sugeno integral, Fuzzy Sets and Systems 161 (2010) 2337–2347.