Clustering and Hitting Times of Threshold Exceedances and Applications

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Abstract: We investigate exceedances of the process over a sufficiently high threshold. The exceedances determine the risk of hazardous events like climate catastrophes, huge insurance claims, the loss and delay in telecommunication networks. Due to dependence such exceedances tend to occur in clusters. The cluster structure of social networks is caused by dependence (social relationships and interests) between nodes and possibly heavy-tailed distributions of the node degrees. A minimal time to reach a large node determines the first hitting time. We derive an asymptotically equivalent distribution and a limit expectation of the first hitting time to exceed the threshold u_n as the sample size n tends to infinity. The results can be extended to the second and, generally, to the kth (k > 2) hitting times. Applications in large-scale networks such as social, telecommunication and recommender systems are discussed.

Keywords: first hitting time; rare events; exceedance over threshold; cluster of exceedances; extremal index; application.

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1 Introduction

Let $\{X_n\}_{n\geq 1}$ be a stationary sequence with marginal distribution function F(x) and $M_n=\max\{X_1,...,X_n\}$. We investigate rare events, namely, exceedances of the sequence over a sufficiently high threshold u. Due to dependence such exceedances tend to occur in clusters. Such clusters of rare events and the asymptotic distributions of the cluster and inter-cluster sizes have been widely studied due to numerous applications, see Ancona-Navarrete and

Tawn (2000), Beirlant et al. (2004), Ferro and Segers (2003), Markovich (2014), Markovich (2016a), Robert (2009), Robert (2013), Roberts et al. (2006), Robinson et al. (2000) among others. There are three approaches in the cluster size study, namely, the blocks method, the runs method and the inter-exceedance times method. The first two methods define the cluster as a block of data with at least one exceedance over the threshold or the clusters are blocks of data with some number of exceedances which are separated by at least a fixed number of observations running under the threshold, respectively, Smith and Weissman (1994), Weissman and Novak (1998). Following the inter-exceedance times approach proposed in Ferro and Segers (2003) we define the cluster as a conglomerate of consecutive exceedances over the threshold between two consecutive non-exceedances. Our main objective is to study the distribution of the first hitting time to exceed the threshold u.

Let us consider the inter-cluster size

$$T_1(u) = \min\{j \ge 1 : M_{1,j} \le u, X_{j+1} > u | X_1 > u\},\tag{1}$$

i.e. the number of inter-arrivals of observations running under the threshold between two consecutive exceedances, where $M_{1,j} = \max\{X_2,...,X_j\}$, $M_{1,1} = -\infty$. Let

$$T^*(u) = \min\{j+1 \ge 1 : M_j \le u, X_{j+1} > u\}$$

be the first hitting time corresponding to the threshold u. We get

$$P\{T^*(u) = j+1\} = P\{M_i \le u, X_{i+1} > u\},\tag{2}$$

$$j = 0, 1, 2, ..., M_0 = -\infty.$$

Let $T_T^*(u)$ be the first hitting time in the time interval [0,T]. Let $\{Y_n\}_{n\geq 1}$ be a stationary sequence of inter-arrival times between consecutive observations of the $\{X_n\}$ and $S_j = \sum_{i=1}^{j-1} Y_i$ denotes the time interval between arrivals of X_1 and X_j . Then we have

$$P\{T_T^*(u) = j+1\} = P\{M_i \le u, X_{i+1} > u, S_{i+1} \le T\}.$$
(3)

Similarly, we determine the probability of the k consecutive hitting times $T^{**}(u)$ by

$$P\{T^{**}(u) = k\} = P\{M_{i_1} \le u, X_{i_1+1} > u, M_{i_1+1, i_2} \le u, X_{i_2+1} > u, \dots, M_{i_{k-1}+1, i_k} \le u, X_{i_k+1} > u\},$$

$$i_j = 0, 1, 2, ...; j = 1, 2, ..., k.$$

The necessity to evaluate the distribution, quantiles and the mean of the first hitting time is arising in many applications. In social networks it is important to compare sampling strategies (Avrachenkov et al. (2012, 2015); Lee et al. (2012)) like random walks, Metropolis-Hastings Markov chains, Page Ranks and others with regard to how quickly they allow to reach a node with a large degree, that is the number of links with other nodes. In Markovich (2015) it is proposed to compare sampling techniques by the mean first hitting time that is illustrated on the real data of social networks. It is important to investigate the first hitting time of significant nodes since it allows us to disseminate advertisements or to collect opinions more effectively within clusters surrounding such nodes. It can be helpful also for recommender systems with collaborative filtering, in which the system recommends to a user some item or product that has been rated by previous users, Linyuan Lü et al.

(2012). A similar problem occurs in telecommunication peer-to-peer networks, namely to find a node with a large number of peers, Dán, G. and Fodor (2009); Markovich (2013). Our concept is also relevant in other areas of operations research and inventory control. For instance, the first hitting time can be important to analyze the customer churn that has a huge impact on companies, Mahajan et al. (2016). Considering the customer impatience in multi-server queues Choudhury and Medhi (2011) and the customer waiting time in the queue Zhao and Gilbert (2015), the first hitting time indicates the moment when the waiting time exceeds a threshold and hence, the customer may leave the queue. Following Lättilä and Hilmola (2012) the forecasting of exceedances of industrial production can be a driving factor for the development of sea ports. The kth hitting time is important in Internet to find the k-top web sites that are significant with regard to some topic.

The measure of the dependence between the rare events is expressed by the extremal index. The notion of the extremal index is determined in Leadbetter et al. (1983), p.53.

Definition 1: The stationary sequence $\{X_n\}_{n\geq 1}$ is said to have extremal index $\theta\in[0,1]$ if for each $0<\tau<\infty$ there is a sequence of real numbers $u_n=u_n(\tau)$ such that

$$\lim_{n \to \infty} n(1 - F(u_n)) = \tau \quad \text{and}$$
 (4)

$$\lim_{n \to \infty} P\{M_n \le u_n\} = e^{-\tau\theta} \tag{5}$$

hold.

The extremal index θ of $\{X_n\}$ relates to the first hitting time $T^*(u_n)$, Roberts et al. (2006). Really, since u_n is selected according to (4) it follows that $P\{X_n>u_n\}$ is asymptotically equivalent to 1/n. Notice, that $P\{M_k\leq u_n\}=P\{T^*(u_n)>k\}$. Hence, substituting τ by (4) we get from (5)

$$P\{T^*(u_n)/n > k/n\} \sim e^{-\theta k P\{X_n > u_n\}} \sim e^{-\theta k/n},$$

$$\lim_{n \to \infty} P(T^*(u_n)/n > x) = e^{-\theta x}$$

for positive x. It follows

$$\lim_{n \to \infty} E(T^*(u_n)/n) = 1/\theta. \tag{6}$$

This implies, the smaller θ , the longer it takes to reach an observation with a large value. The result is then interesting for processes which have $\theta=0$, e.g., for the Metropolis Markov chain Roberts et al. (2006) and the Lindley process with subexponential step distribution Asmussen (2000). A mixture of i.i.d. non-ergodic sequences with $\theta=0$ is given in Theorem 4 by Doukhan et al. (2015).

Using achievements regarding the limit geometric-like distribution of $T_1(x_{\rho_n})$ derived in (Theorem 2, Markovich (2014), Markovich (2016a)), where the $(1-\rho_n)$ th quantile x_{ρ_n} of $\{X_n\}$ is taken as u_n , we derive in Section 2 a limit distribution of the first hitting time and its expectation that specifies (6). The achievements are similarly extended to the second hitting time, Section 3.

Theorem 1 (that is Theorem 2 in Markovich (2014)) is based on the mixing condition proposed in Ferro and Segers (2003)

$$\alpha_{n,q}(u) = \max_{1 \le k \le n-q} \sup |P(B|A) - P(B)| = o(1), \quad n \to \infty,$$
 (7)

where for real u and integers $1 \le k \le l$, $\mathcal{F}_{k,l}(u)$ is the σ -field generated by the events $\{X_i > u\}$, $k \le i \le l$ and the supremum is taken over all $A \in \mathcal{F}_{1,k}(u)$ with P(A) > 0 and $B \in \mathcal{F}_{k+a,n}(u)$ and k, q are positive integers.

To formulate the theorem we need the following partition of the interval [1, j]

$$1 = k_{n,0}^* \le k_{n,1}^* \le k_{n,2}^* \le k_{n,3}^* \le k_{n,4}^* \le k_{n,5}^* = j, \quad j \to \infty,$$
 (8)

where positive integers $\{k_{n,i}^*\}$ are such that

$$\{k_{n,i-1}^* = o(k_{n,i}^*), i \in \{1, 2, ..., 5\}\}. \tag{9}$$

Roughly speaking, the partition is required to split the conditional probability of the maximum $M_{1,j}$ in $P\{T_1(u)=j\}=P\{M_{1,j}\leq u,X_{j+1}>u|X_1>u\}$ into the product of independent probabilities of partial maxima $M_{1,k_{n,1}^*}$, $M_{k_{n,2}^*,k_{n,3}^*}$ and $M_{k_{n,4}^*,j}$. The independence follows from mixing conditions (10), (11). The statement (12) is obtained from Theorem 2.1 and Corollary 2.3 of O'Brien (1987).

Theorem 1: Let $\{X_n\}_{n\geq 1}$ be a stationary process with the extremal index θ . Let $\{x_{\rho_n}\}$ be a sequence of quantiles of X_1 of the levels $\{1-\rho_n\}$, that satisfies the conditions (4) and (5) if u_n is replaced by x_{ρ_n} . Let positive integers $\{k_{n,i}^*\}$, $i=\overline{0,5}$, be as in (8) and (9), respectively, $\Delta_{n,i}=k_{n,i}^*-k_{n,i-1}^*$, $q_{n,i}^*=o(\Delta_{n,i})$, $i\in\{1,2,...,5\}$, be such that for each $\varepsilon>0$ there exist n_ε and $j=j_0(n_\varepsilon)$ such that for all $n>n_\varepsilon$ and $j>j_0(n_\varepsilon)$

$$\alpha_n^*(x_{\rho_n}) = \max\{\alpha_{k_{n,4}^*, q_{n,1}^*}; \alpha_{k_{n,3}^*, q_{n,2}^*}; \alpha_{\Delta_{n,3}, q_{n,3}^*}; \alpha_{j+1-k_{n,2}^*, q_{n,4}^*}; \alpha_{j+1-k_{n,1}^*, q_{n,5}^*}\} < \varepsilon$$
(10)

and

$$\alpha_{j+1,k_{n,4}^*-k_{n,1}^*}/\rho_n < \varepsilon \tag{11}$$

hold, where $\alpha_{n,q} = \alpha_{n,q}(x_{\rho_n})$ is determined by (7). Then for the same n and j it holds

$$|P\{T_1(x_{\rho_n}) = j\}/(\theta^2 \rho_n (1 - \rho_n)^{(j-1)\theta}) - 1| < \varepsilon.$$
(12)

The theorem implies that the probability $P\{T_1(x_{\rho_n})=j\}$ is close to the geometric form corrupted by extremal index θ for sufficiently large n and j.

The paper is organized as follows. In Section 2 we derive the limit distribution and expectation of the first hitting time to exceed a sufficiently high threshold. The limit distribution of the second hitting time is obtained in Section 3. In Section 4 examples of first hitting time distributions are obtained for different processes including real data. Conclusions are given in Section 5. Proofs are presented in the Appendix.

2 Distribution and expectation of the first hitting time

For all n and j sufficiently large one can rewrite (12) in a geometric form as

$$\left| \frac{c_n P\{T_1(x_{\rho_n}) = j\}}{\eta_n (1 - \eta_n)^{j-1}} - 1 \right| < \varepsilon, \tag{13}$$

where $c_n = \eta_n / \left(\theta^2 \left(1 - (1 - \eta_n)^{1/\theta}\right)\right)$, $0 < \eta_n < 1$, using the replacement $(1 - \rho_n)^{\theta} = 1 - \eta_n$. We shall use (13) to prove the next theorem.

Theorem 2: Let all conditions of Theorem 1 be satisfied. Then for the same n and j as in Theorem 1 we get

$$\left| \frac{P\{T^*(x_{\rho_n}) = j\}}{\psi_{j-1}(n)} - 1 \right| < \varepsilon,$$
 (14)

where

$$\psi_{j-1}(n) = \frac{\theta^2 \rho_n^2 (1 - \rho_n)^{\theta(j-1)}}{1 - (1 - \rho_n)^{\theta}}.$$
(15)

From (4)

$$\rho_n \sim \tau/n \quad \text{and} \quad (1 - \rho_n)^{\theta} = 1 - \theta \rho_n + o(\rho_n)$$
(16)

hold as $n \to \infty$. Expressions (14) and (15) imply that for any positive ε there exists n_{ε} such that for $n > n_{\varepsilon}$ and $j > j_0(n_{\varepsilon})$ the probability of the first hitting time has a geometric distribution with probability $\theta \rho_n$, i.e.

$$\left|\frac{P\{T^*(x_{\rho_n})=j\}}{\theta\rho_n(1-\theta\rho_n)^{j-1}}-1\right|<\varepsilon.$$

Together with (12) it implies that for sufficiently large n and j it holds

$$P\{T^*(x_{\rho_n}) = j\} \approx \theta P\{T_1(x_{\rho_n}) = j\}.$$

Lemma 3: Let the conditions of Theorem 1 be satisfied and for some $\beta > 0$

$$\sup_{x} E((T^*(x_{\rho_n}))^{1+\beta})/\Lambda_n < \infty \tag{17}$$

holds. Then it follows

$$|ET_{j_0}^*(x_{\rho_n})/(\Lambda_n\rho_n) - 1| < \varepsilon, \tag{18}$$

where $j_0 = o(n)$, $ET_{j_0}^*(x_{\rho_n}) = \sum_{j=j_0+1}^{\infty} j P\{T^*(x_{\rho_n}) = j\}$,

$$\Lambda_n = \frac{\theta^2 \rho_n}{(1 - (1 - \rho_n)^{\theta})^3} (1 - \rho_n)^{\theta j_0} \left(j_0 (1 - (1 - \rho_n)^{\theta}) + 1 \right). \tag{19}$$

The expression (18) specifies the rate of convergence in (6).

Remark 1: The condition (17) provides a uniform convergence of the series $\sum_{j=1}^{\infty} jP\{T^*(x_{\rho_n})=j\}/\Lambda_n$ by n. The condition is fulfilled particularly for $T^*(x_{\rho_n})$ corresponding to the ARMAX process, see Section 4.1. From (25) we have $\sup_n E((T^*(x_{\rho_n}))^{1+\beta})/\Lambda_n < \sup_n E((T^*(x_{\rho_n}))^2)/\Lambda_n \sim \sup_n (2-\theta\rho_n)(1-\rho_n)^{1-\theta(j_0+1)}/(\theta(1+j_0\theta\rho_n)) < \infty$ for $\rho_n \sim \tau/n$.

Let us turn to (3). If $\{X_i\}$ and $\{Y_i\}$ are mutually independent then

$$P\{T_T^*(u) = j+1\} = P\{M_j \le u, X_{j+1} > u\}P\{S_{j+1} \le T\}$$
$$= P\{T^*(u) = j+1\}P\{\sum_{i=1}^j Y_i \le T\}$$

follows. From (14) and (15) we get

$$|P\{T_T^*(x_{\rho_n}) = j+1\}/(\psi_j(n)(1-P\{\sum_{i=1}^j Y_i > T\})) - 1| < \varepsilon$$

for any $\varepsilon > 0$ and $n > n_{\varepsilon}$ and $j > j_0(n_{\varepsilon})$. Assuming Y_i 's are iid regularly varying random variables with tail index $\alpha \geq 0$ we have

$$P\{\sum_{i=1}^{j} Y_i > T\} \sim jP\{Y_i > T\} \sim jT^{-\alpha}$$

for $j \ge 1$ as $T \to \infty$, see Lemma 3.1, Jessen and Mikosch (2006). The condition $1 - jT^{-\alpha} > 0$ is provided by $T > j_0^{1/\alpha}$ since $j > j_0$. Then the lemma follows.

Lemma 4: Let the conditions of Theorem 1 be satisfied. Let $\{X_i\}$ and $\{Y_i\}$ in (3) be mutually independent and $\{Y_i\}_{i\geq 1}$ be iid regularly varying random variables with tail index $\alpha\geq 0$ and $T>j_0^{1/\alpha}$ holds. Then for the same n and j as in Theorem 1 we get

$$|P\{T_T^*(x_{\rho_n}) = j+1\}/(\psi_j(n)(1-jT^{-\alpha})-1) < \varepsilon.$$

3 Distribution of the second hitting time

Let us denote the second hitting time of u_n as $T^{**}(u_n)$. The probability to hit u_n twice is determined by

$$P\{T^*(x_{\rho_n}) = j, T^{**}(x_{\rho_n}) = j + m\}$$

$$= P\{M_{i-1} \le u_n, X_i > u_n, M_{i,i+m-1} \le u_n, X_{j+m} > u_n\}, \quad m = 1, 2, \dots$$
(20)

Lemma 5: Let the conditions of Theorem 1 be satisfied. Then for the same n and j as in Theorem 1 we have

$$\left| \frac{P\{T^*(u_n) = j, T^{**}(u_n) = j + m\}}{P\{\chi = j\}P\{\chi = m\}} - 1 \right| < \varepsilon,$$

where χ is a geometrically distributed random variable with probability $\rho_n\theta$.

Similarly the statement can be extended to the probability of the kth hitting time, i.e. the minimal time to find k large nodes of the network. Random walks used in social networks as sampling may return to the same nodes with some positive probability. This may reduce the number of distinct nodes in the sample and particularly ones which degrees exceed the threshold. The degrees of repeated nodes may not exceed the threshold and hence, do not impact on the probability to reach k different large nodes. Moreover, the degrees of repeated nodes may change over time. These problems are out of scope of this paper.

4 Examples

4.1 ARMAX process

Let us obtain the distribution of the first hitting time of the ARMAX process. The latter process is determined as

$$X_t = \max\{\alpha X_{t-1}, (1-\alpha)Z_t\}, \quad t \in \mathbb{Z},$$

where $0 \le \alpha < 1$, and $\{Z_t\}$ are iid standard Fréchet distributed r.v.s with the distribution function $F(x) = \exp{(-1/x)}$, x > 0. The r.v. X_t has the same distribution assuming $X_0 = Z_0$. The extremal index is equal to $\theta = 1 - \alpha$, Ancona-Navarrete and Tawn (2000). Using that

$$P\{X_i \le x_\rho\} = 1 - \rho = q = e^{-1/x_\rho} \tag{21}$$

and

$$P\{\alpha Z_i \le x_\rho\} = e^{-\alpha/x_\rho} = (1-\rho)^\alpha = q^\alpha$$
(22)

we derive in Section 6.4 the following.

Proposition 6: For the ARMAX and MM processes we have

$$P\{T^*(x_\rho) = j\} = (1 - (1 - \rho)^{\theta})(1 - \rho)^{\theta(j-2)+1},\tag{23}$$

$$ET^*(x_\rho) = (1 - \rho)^{1 - \theta} / (1 - (1 - \rho)^{\theta})$$
(24)

and

$$E(T^*(x_\rho))^2 = (1-\rho)^{1-\theta} (1 + (1-\rho)^\theta) / (1 - (1-\rho)^\theta)^2.$$
(25)

4.2 MM process

We obtain the distribution of the first hitting time of the MM process. This process is determined by the formula

$$X_t = \max_{i=0,\dots,m} \{\alpha_i Z_{t-i}\}, \quad t \in \mathbf{Z},$$

where $\{\alpha_i\}$ are nonnegative constants such that $\sum_{i=0}^m \alpha_i = 1$ and $\{Z_t\}$ are iid standard Fréchet distributed r.v.s. The distribution of X_t is also standard Fréchet. The extremal index of the process is determined by $\theta = \max_i \{\alpha_i\}$, Ancona-Navarrete and Tawn (2000). Assuming $\alpha_0 \geq \alpha_1 \geq ... \geq \alpha_m$ we derive in Section 6.5 that the distribution of the first hitting time is the same as for the ARMAX process.

In Figure 1 the comparison of the exact distribution of $T^*(x_\rho)$ for the ARMAX and MM processes and the model obtained in Theorem 2 is shown. The model (15) is valid for sufficiently large n and j. This corresponds to ρ_n close to zero and high quantiles x_{ρ_n} using as thresholds u_n . Thus the model approximates the distribution (23) better for small ρ and large j.

The comparison of the mean first hitting time (24) and the theoretical model obtained in Lemma 3 is shown in Figure 2. The difference is observed only for $j_0=0$ and when ρ is close to 1. It should be noted that we consider $\theta=0.1$ corresponding to a large local dependence in the extremes of the process $\{X_n\}$. $\theta=1$ corresponds to independent observations.

4.3 AR(1) process

We consider the AR(1) process with uniform noise, Chernick et al. (1991). For a fixed integer $r \ge 2$ let ϵ_n , $n \ge 1$ be iid r.v.s with $P\{\epsilon_1 = k/r\} = 1/r$, $k = 0, 1, \ldots, r-1$. The process is defined by

$$X_i = (1/r)X_{i-1} + \epsilon_i, \quad i > 1$$
 and $X_0 \sim U(0, 1)$

with X_0 independent of the ϵ_j . Since $X_0 \sim U(0,1)$ then $X_1 \sim U(0,1)$ holds. The extremal index of AR(1) is $\theta = 1 - 1/r$.

Proposition 7: For the AR(1) process we have

$$P\{T^*(u_n) = j\} = \begin{cases} 1 - \theta, & j = 1\\ (1 - \theta)^j (u_n - j\theta(1 - u_n)), & 2 \le j \le j_0\\ (1 - \theta)^{j_0 + 2} (u_n - j\theta(1 - u_n)), j_0 < j \le m - 1, \end{cases}$$
(26)

where $j_0 = [\ln n/(2 \ln r)]$ and m satisfies the inequality

$$-\frac{\ln(1-u_n)}{\ln(r)} - 1 < m - 1 \le -\frac{\ln(1-u_n)}{\ln(r)}.$$
(27)

Selecting $u_n = 1 - x/n$, x > 0 we get $u_n - j\theta(1 - u_n) = 1 - (x/n)(1 + j\theta)$ that is positive for sufficiently large n.

The proof is given in Section 6.6.

Remark 2: The mixing conditions (10) and (11) of Theorem 1 are fulfilled for the ARMAX and the AR(1) processes if $j > j_0(n)$ holds, where $j_0(n) \to \infty$ as $n \to \infty$, and for the MM process if j > m and $\alpha_0 \ge \alpha_1 \ge ... \ge \alpha_m$ hold, Markovich (2016b).

4.4 Real data

We consider two real data sets of the Enron email and DBPL networks presented in Leskovec and Krevl (2014) and investigated in Markovich (2015). The sets contain node degrees. In Markovich (2015) it was found that the both data sets are heavy-tailed distributed and their extremal index θ was calculated by intervals estimator proposed by Ferro and Segers (2003). Typically, it may be assumed that the node degrees are regularly varying distributed. In Figure 3 the model from Lemma 3 of the mean first hitting time against j_0 is shown. The j_0 indicates the truncated expectation $ET_{j_0}^*(x_\rho)$. The theoretical model is valid for any sampling technique (a random walk, Markov chain) that satisfies the mixing condition (10)-(11) for $j > j_0$, where $j_0 = j_0(n)$ is sufficiently large. The latter condition is equivalent to the j-dependence. The j-dependence may be checked in practice by an autocorrelation function (ACF) (see, e.g., Markovich and Krieger (2010)). Since both data sets have infinite variance according to Markovich (2015) it is better to use the special sample ACF for heavy-tailed data recommended in Davis and Resnick (1985), i.e.

$$\widetilde{\rho}(j) = \sum_{t+1}^{n-j} X_t X_{t+j} / \sum_{t=1}^n X_t^2$$
(28)

at lag j. This ACF is not centralized by the sample average \overline{X} in contrast to the classical sample ACF. Moreover, this estimate may behave in a very unpredictable way if one uses the class of non-linear processes in the sense that $\widetilde{\rho}(j)$ may converge in distribution to a non-degenerate random variable depending on j. For linear processes it converges in distribution to a constant depending on j, Davis and Resnick (1985). From Figure 4 one may conclude that the DBPL data are short-range dependent since its ACF decreases after $j\approx 50$ as far as the Enron data are not. This may indirectly indicate that the DBPL and Enron data determine linear and non-linear processes, respectively.

Everything what we need for our model are the extremal index θ and the quantile threshold $1-\rho$. We take $\rho=0.05$ that corresponds to 95% quantile x_ρ of an underlying data set taken as the threshold. We may conclude from Figure 3 that the mean minimal time required to reach a node with degree larger than $u=x_\rho$ is longer for the DBPL data than for the Enron data.

5 Conclusions

We have obtained the limit distribution and expectation of the first hitting time for processes which satisfy the mixing conditions of Theorem 1. The latter are fulfilled particularly for Markov chains represented by the ARMAX, the MM and the AR(1) processes. Exact distributions of the first hitting time for the latter processes are obtained. Markov chains used in social networks as sampling techniques can be compared with regard to the quantiles and expectation of the first hitting time. The presented research can be particularly useful for such comparison of sampling strategies. The results are extended to the second hitting time.

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6 Appendix

6.1 Proof of Theorem 2

It follows from (2) that

$$P\{T^*(u_n) = j+1\} = P\{M_j \le u_n, X_{j+1} > u_n\}$$

= $P\{M_j \le u_n\} - P\{M_{j+1} \le u_n\}.$ (29)

Following Ferro and Segers (2003) we get alternatively for $n \ge 1$

$$\begin{split} P\{T_1(u_n) > j\} &= P\{M_{1,j+1} \le u_n | X_1 > u_n\} \\ &= \left(P\{M_{1,j+1} \le u_n\} - P\{M_{j+1} \le u_n\}\right) / P\{X_1 > u_n\} \\ &= \left(P\{M_j \le u_n\} - P\{M_{j+1} \le u_n\}\right) / P\{X_1 > u_n\} \\ &= P\{T^*(u_n) = j+1\} / P\{X_1 > u_n\}. \end{split}$$

Thus, we get

$$P\{T^*(x_{\rho_n}) = j+1\} = P\{X_1 > x_{\rho_n}\} \cdot P\{T_1(x_{\rho_n}) > j\}$$
$$= \rho_n \sum_{i=j+1}^{\infty} P\{T_1(x_{\rho_n}) = i\}.$$

From (13) we obtain

$$P\{T^*(x_{\rho_n}) = j+1\} = \frac{\rho_n}{c_n} \sum_{i=j+1}^{\infty} c_n P\{T_1(u_n) = i\}$$

$$< (1+\varepsilon) \frac{\rho_n}{c_n} \sum_{i=j+1}^{\infty} \eta_n (1-\eta_n)^{i-1} = (1+\varepsilon)\psi_{j-1}(n),$$

$$P\{T^*(x_{\rho_n}) = j+1\} > (1-\varepsilon)\psi_{j-1}(n),$$
(30)

where $\psi_{j-1}(n)$ is determined by (15). Since $\psi_{j-1}(n) \to 0$ as $n \to \infty$ holds, the series in (30) converges uniformly by all $n > n_{\varepsilon}$ by Weierstrass' theorem.

6.2 Proof of Lemma 3

Let us consider the expectation of the first hitting time

$$ET^*(x_{\rho_n}) = \sum_{j=1}^{\infty} jP\{T^*(x_{\rho_n}) = j\}.$$

From (15) we get

$$\sum_{j=j_0+1}^{\infty} j\psi_{j-1}(n) = \sum_{j=j_0+1}^{\infty} j\frac{\theta^2 \rho_n^2 (1-\rho_n)^{\theta(j-1)}}{1-(1-\rho_n)^{\theta}} = \Lambda_n \rho_n,$$

where Λ_n is determined by (19). Due to (16) we have

$$\Lambda_n \sim (1 - \theta \rho_n)^{j_0} (j_0 \theta \rho_n + 1) / (\theta \rho_n^2) \sim \exp(-\tau \theta j_0 / n) / (\theta \rho_n^2) \to \infty$$

as $n \to \infty$. Let us denote

$$a_j(n) = jP\{T^*(x_{\rho_n}) = j\}$$

and

$$S_k(n) = \sum_{j=1}^{k-1} a_j(n)/\Lambda_n, \quad r_k(n) = \sum_{j=k}^{\infty} a_j(n)/\Lambda_n.$$

We have to prove that $S(n)=\sum_{j=1}^\infty a_j(n)/\Lambda_n$ converges uniformly by n. For this purpose, we shall prove that

$$\lim_{k \to \infty} \sup_{n} r_k(n) = 0.$$

The latter follows from

$$\sup_{n} r_{k}(n) = \sup_{n} \sum_{j=k}^{\infty} \frac{jk^{\beta} P\{T^{*}(x_{\rho_{n}}) = j\}}{k^{\beta} \Lambda_{n}} \le \frac{1}{k^{\beta}} \sup_{n} \frac{E(T^{*}(x_{\rho_{n}}))^{1+\beta}}{\Lambda_{n}}$$

and the assumption (17). It remains to prove that

$$\lim_{n \to \infty} S_{j_0}(n)/\rho_n = 1,\tag{31}$$

were $S_{j_0}(n) = \sum_{j=j_0+1}^{\infty} a_j(n)/\Lambda_n$.

Using the replacement $(1 - \rho_n)^{\theta} = 1 - \eta_n$ from (14), (15) and (19) we get for any $\varepsilon > 0$ that it holds

$$S_{j_0}(n) < \frac{(1+\varepsilon)(1-(1-\rho_n)^{\theta})^2}{(1-\rho_n)^{\theta j_0}(j_0(1-(1-\rho_n)^{\theta})+1)} \sum_{j=j_0+1}^{\infty} j\rho_n(1-\rho_n)^{(j-1)\theta}$$
$$= \frac{(1+\varepsilon)\eta_n\rho_n}{(1-\eta_n)^{j_0}(j_0\eta_n+1)} \sum_{j=j_0+1}^{\infty} j\eta_n(1-\eta_n)^{j-1}.$$

Similarly, one can get

$$S_{j_0}(n) > \frac{(1-\varepsilon)\eta_n \rho_n}{(1-\eta_n)^{j_0} (j_0 \eta_n + 1)} \sum_{j=j_0+1}^{\infty} j \eta_n (1-\eta_n)^{j-1}.$$

Since

$$\sum_{j=j_0+1}^{\infty} j\eta_n (1-\eta_n)^{j-1} = \frac{(1-\eta_n)^{j_0}}{\eta_n} (j_0\eta_n + 1)$$

and ε is arbitrary then (31) and thus, (18) follows.

6.3 Proof of Lemma 5

By (1) and the stationarity of $\{X_n\}$ we obtain

$$P\{T_{1}(u_{n}) = n\} = P\{M_{1,n} \leq u_{n}, X_{n+1} > u_{n} | X_{1} > u_{n}\}$$

$$= (P\{M_{1,n} \leq u_{n}, X_{n+1} > u_{n}\} - P\{M_{n} \leq u_{n}, X_{n+1} > u_{n}\}) / P\{X_{1} > u_{n}\}$$

$$= (P\{M_{n-1} \leq u_{n}, X_{n+1} > u_{n}\} - P\{M_{n} \leq u_{n}, X_{n+1} > u_{n}\}) / P\{X_{1} > u_{n}\}$$

$$= (P\{T^{*}(u_{n}) = n\} - P\{T^{*}(u_{n}) = n+1\}) / P\{X_{1} > u_{n}\}.$$
(32)

From (20) we get due to stationarity

$$\begin{split} &P\{T^*(u_n)=j,T^{**}(u_n)=j+m\}\\ &=P\{M_{j-1}\leq u_n,M_{j,j+m-1}\leq u_n,X_{j+m}>u_n\}-P\{M_{j+m-1}\leq u_n,X_{j+m}>u_n\}\\ &=P\{M_{j+m-2}\leq u_n,X_{j+m-1}>u_n\}-P\{T^*(u_n)=j+m\}\\ &=P\{T^*(u_n)=j+m-1\}-P\{T^*(u_n)=j+m\}\\ &=P\{T_1(u_n)=j+m-1\}P\{X_1>u_n\}. \end{split}$$

The last two lines are obtained from (2) and (32). Then using (13) and denoting $\varphi_{i+m-2}(n) = \rho_n \eta_n (1 - \eta_n)^{j+m-2}/c_n$ one can rewrite

$$|P\{T^*(x_{\rho_n}) = j, T^{**}(x_{\rho_n}) = j + m\}/\varphi_{j+m-2}(n) - 1| < \varepsilon.$$

Since it holds

$$\varphi_{j+m-2}(n) \sim \theta^2 \rho_n^2 (1-\rho_n)^{(j+m-2)\theta} \sim \theta \rho_n (1-\theta \rho_n)^{j-1} \theta \rho_n (1-\theta \rho_n)^{m-1}$$

the statement of the lemma follows.

6.4 Proof of Proposition 6 for an ARMAX process

From the definition of the ARMAX process we obtain the distribution of the first hitting time. It holds

$$P\{T^*(u) = j\} = P\{M_{j-1} \le u, X_j > u\}$$

$$= P\{X_1 \le u, ..., X_{j-1} \le u, X_j > u\}$$

$$= P\{X_1 \le u, (1 - \alpha)Z_2 \le u, ..., (1 - \alpha)Z_{j-1} \le u, \max\{\alpha X_{j-1}, (1 - \alpha)Z_j\} > u\},$$

since $X_{i+1} \leq u, i=1,...,j-2$ leads to $\alpha X_i \leq u$ and $(1-\alpha)Z_{i+1} \leq u$ and together with $X_i \leq u$ it implies both $X_i \leq u$ and $(1-\alpha)Z_{i+1} \leq u$ due to $0 < \alpha < 1$. For an independent sequence $\{X_t\}$ $\alpha = 0$ holds.

Let us consider $\max\{\alpha X_{j-1}, (1-\alpha)Z_j\} > u$. Supposing $\alpha X_{j-1} > u$ contradicts $X_{j-1} \le u$. Hence, it follows $(1-\alpha)Z_j > u$ and it holds

$$P\{T^*(u) = i\} = P\{X_1 \le u, (1 - \alpha)Z_2 \le u, ..., (1 - \alpha)Z_{i-1} \le u, (1 - \alpha)Z_i > u\}.$$

Taking the $(1 - \rho)$ -level quantile x_{ρ} as u and using (21) and (22) we get

$$P\{T^*(x_\rho) = j\} = (1 - \rho)(1 - \rho)^{(1-\alpha)(j-2)}(1 - (1-\rho)^{1-\alpha})$$

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We shall obtain $ET^*(x_\rho)$ for the ARMAX process. Denoting $1 - \eta = (1 - \rho)^{\theta}$ we get

$$ET^*(x_\rho) = \sum_{j=1}^{\infty} jP\{T^*(x_\rho) = j\} = (1-\rho)(1-(1-\rho)^{\theta})\sum_{j=1}^{\infty} j(1-\rho)^{\theta(j-2)}$$
$$= (1-\eta)^{1/\theta-1}\sum_{j=1}^{\infty} j\eta(1-\eta)^{j-1} = (1-\eta)^{1/\theta}/(\eta(1-\eta)).$$

Similarly, one can get (25).

6.5 Proof of Proposition 6 for a MM process

From the definition of the MM process we get

$$P\{T^*(u) = j\}$$

$$= P\{\max_{i=0,\dots,m} \{\alpha_i Z_{1-i}\} \le u, \dots, \max_{i=0,\dots,m} \{\alpha_i Z_{j-1-i}\} \le u, \max_{i=0,\dots,m} \{\alpha_i Z_{j-i}\} > u\}$$
(33)

Assuming $\alpha_0 \ge \alpha_1 \ge ... \ge \alpha_m$ we obtain that the right-hand side of (33) is equal to

$$P\{\alpha_m Z_{1-m} \leq u, ..., \alpha_0 Z_1 \leq u, \alpha_0 Z_2 \leq u, ..., \alpha_0 Z_{j-1} \leq u, \max_{i=0,...m} \{\alpha_i Z_{j-i}\} > u\}.$$

Let us consider the event $\{\max_{i=0,\dots,m} \{\alpha_i Z_{j-i}\} > u\}$. This is equivalent to $\{\alpha_0 Z_j > u\}$. $u, \alpha_1 Z_{j-1} \le u, ..., \alpha_m Z_{j-m} \le u$.

Really, suppose $\alpha_m Z_{j-m} > u$ holds. But this is in contradiction with $\alpha_{m-1} Z_{j-m} \leq u$ in (33). Furthermore, $\alpha_1 Z_{j-1} > u$ contradicts $\alpha_0 Z_{j-1} \le u$ etc. Summarizing we obtain

$$\begin{split} P\{T^*(u) = j\} &= P\{\alpha_m Z_{1-m} \leq u, ..., \alpha_0 Z_1 \leq u, \alpha_0 Z_2 \leq u, ..., \alpha_0 Z_{j-1} \leq u, \\ \alpha_m Z_{j-m} \leq u, ..., \alpha_1 Z_{j-1} \leq u, \alpha_0 Z_j > u\} \\ &= P\{\alpha_m Z_{1-m} \leq u, ..., \alpha_0 Z_1 \leq u, \alpha_0 Z_2 \leq u, ..., \alpha_0 Z_{j-1} \leq u, \alpha_0 Z_j > u\}. \end{split}$$

Hence, from (21) and (22) it follows

$$P\{T^*x_{\rho}) = j\} = q^{\alpha_m + \dots + \alpha_0 + \alpha_0(j-2)}(1 - q^{\alpha_0}) = (1 - (1 - \rho)^{\alpha_0})(1 - \rho)^{1 + \alpha_0(j-2)}.$$

Thus, (23) follows.

6.6 Proof of Proposition 7

To prove (26) we use (29) and results obtained in Chernick (1981) and Chernick et al. (1991). For j = 1 we have

$$P\{T^*(u_n) = j\} = P\{M_0 \le u_n, X_1 > u_n\} = P\{\epsilon_1 = (r-1)/r\} = 1 - \theta,$$

since $X_1 > u_n$ implies $\epsilon_1 = (r-1)/r$ for sufficiently large n and r < n/x. Really, suppose $\epsilon_1 \leq (r-2)/r$ holds. Then we get $X_1 = 1/rX_0 + \epsilon_1 \leq 1 - 1/r$. This contradicts to $X_1 > 1$

 $u_n = 1 - x/n$ for r < n/x.

From Lemma 2.5 in Chernick et al. (1991) it follows that the event $X_{j+1} > u_n = 1 - x/n$, x > 0 with $1 \le j \le j_0$ and $j_0 = [\ln n/(2 \ln r)]$ leads to

$$\epsilon_2 = \epsilon_3 = \dots = \epsilon_{j+1} = (r-1)/r \tag{34}$$

for all n sufficiently large. From another side, from Lemma 2.6 in Chernick et al. (1991) it follows that for all n sufficiently large, the event $X_{j+1} > u_n$ for $j > j_0$ leads to

$$\epsilon_t = (r-1)/r, \quad t = j - j_0, ..., \ j+1.$$
 (35)

If (r-1)x < n holds, we get by formula (4.3) in Chernick (1981)

$$P\{M_j \le u_n\} = 1 - \frac{(j+1)r - j}{r}(1 - u_n)$$
(36)

and if $j \le m-1$ holds, where m is the integer for which $1-r^m(1-u_n) < 0$ and $1-r^{m-1}(1-u_n) \ge 0$ (i.e. (27)) hold.

Thus, (29) can be rewritten as

$$P\{T^*(u_n) = j+1\} = P\{M_i \le u_n\}P\{X_{i+1} > u_n\}.$$

From (34) and (35) we get

$$P\{X_{j+1} > u_n\} = \begin{cases} P\{\epsilon_t = (r-1)/r, \ t = 2, ..., \ j+1\}, & \text{if } 2 \le j \le j_0, \\ P\{\epsilon_t = (r-1)/r, \ t = j-j_0, ..., \ j+1\}, & \text{if } j_0 < j \le m-1 \end{cases}$$
$$= \begin{cases} \prod_{t=2}^{j+1} P\{\epsilon_t = (r-1)/r\}, & \text{if } 2 \le j \le j_0, \\ \prod_{t=j-j_0}^{j+1} P\{\epsilon_t = (r-1)/r\}, & \text{if } j_0 < j \le m-1 \end{cases}$$

Then the statement follows from (36).

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Figure 1 The distribution (23) of the first hitting time of the ARMAX and MM processes and the model (15) with $\theta=0.1$ for j=5 (top) and j=20 (bottom) against ρ , where ρ close to zero corresponds to a high quantile x_{ρ} as the threshold u.

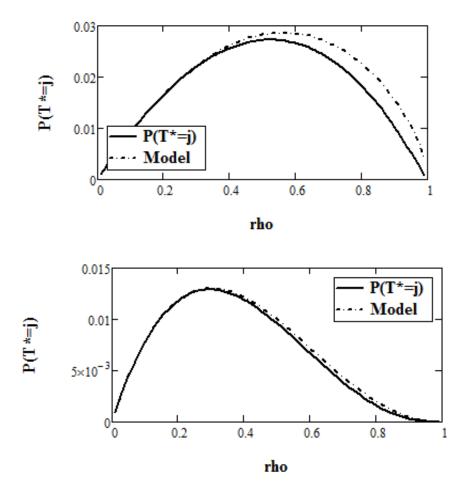


Figure 2 The mean first hitting time (24) of the ARMAX and MM processes and the model $\Lambda_n \rho_n$ based on (18) and (19) with $\theta=0.1$ for $j_0=0$ and $j_0=5$ against ρ .

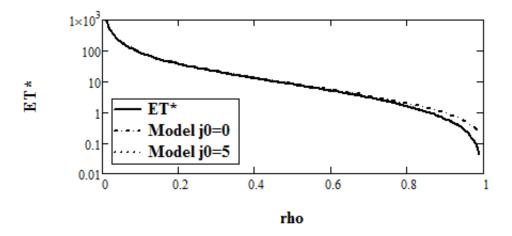


Figure 3 The model $\Lambda_n \rho_n$ of the mean first hitting time calculated by (18) and (19) of the Enron and DBPL data sets with $\theta=0.22$ and $\theta=0.15$, respectively, for $\rho=0.05$ against j_0 .

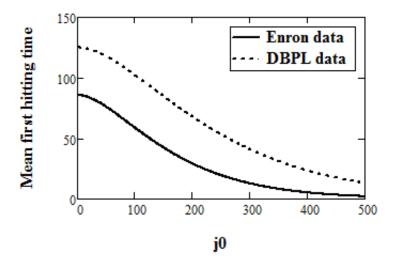


Figure 4 The sample autocorrelation function (28) of the Enron and DBPL data sets.

