
A batch arrival retrial queuing system for essential and optional services with server breakdown and Bernoulli vacation

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Abstract: This paper studies a general retrial $M^X/G/1$ queue with an additional phase of second optional service and Bernoulli vacation where breakdowns occur randomly at any instant while servicing the customers. If an arriving batch finds that the server is busy in providing either first essential service (FES)/second optional service (SOS) or on vacation then arriving batch enters an orbit called *retrial queue*. Otherwise, one customer from arriving batch starts to be served by the server while the rest join the orbit. The vacation times and service times of both first essential and second optional services are assumed to be general distributed while the retrial times are exponential distributed. Introducing supplementary variables and by employing embedded Markov chain technique, we derive some important performance measures of the system such as average orbit size, average queue size, mean waiting time, expected lengths of busy period, etc. Numerical results have been facilitated to illustrate the effect of different parameters on several performance measures.

Keywords: retrial queue; bulk arrival; second optional service; SOS, Bernoulli vacation; supplementary variable; embedded Markov chain; server breakdown; queue size.

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1 Introduction

Retrial queue is characterised by the fact that any arriving batch who finds the server busy has to leave the service area and enters the virtual pool of blocked customers considered as *orbit* to ask his request after some random amount of time. Such types of retrial queuing system has potential applications in telephone switching systems, computers and telecommunications networks, packet switching networks, call centres, etc. Moreover, in past literature, retrial queue was studied in different frame works under three types of policies, i.e.,

- 1 classical retrial policy
- 2 constant retrial policy
- 3 linear retrial policy.

In classical retrial policy, the intervals between successive repeated attempts are exponentially distributed with rate $n\theta$ (say), when the number of the customer in retrial

group is n . On the other hand, the intervals between successive repeated attempts are exponentially distributed with retrial rate $\nu(1 - \delta_{n,0})$ and $\nu(1 - \delta_{n,0}) + n\theta$ for constant and linear retrial policy, respectively, where n is the orbit size, $\delta_{i,j}$ represents Kronecker's delta function, θ is the retrial rate per customer and ν can be considered as the rate of the server providing service during the idle state.

The purpose of present investigation is to analyse the M/G/1 retrial queue with an additional phase of second optional service (SOS) and server breakdown by incorporating the concepts of

- 1 bulk arrival
- 2 vacation
- 3 choice for the server to go for a vacation in both phases of service according to Bernoulli vacation.

The rest of the paper is organised as follows. The survey of the previous relevant literature is presented in Section 2. In Section 3, we provide the brief description of the mathematical model. By using embedded Markov chain technique, we obtain the limiting distribution of the queue size at random and departure epochs in next Section 4. Further, in Section 5, the joint distribution of the number of the customers of the server states in the retrial group and queue size distribution has been discussed. Various queuing performance measures have been derived in Section 6. Further, we deduce some special cases of the model by setting appropriate parameters with previous existing works in Section 7. In the next Section 8, the stochastic decomposition property is discussed. Numerical results have also been given in Section 9 to verify the analytical results established in previous sections. Finally, in Section 10, the paper comes to end with some concluding remarks.

2 Survey of literature

For an early part notable contributions on retrial queue, we refer the book by Falin and Templeton (1997). A few applications of retrial queue have been discussed in the survey paper of Kulkarni and Liang (1997). An M/G/1 queuing system with two phases of heterogeneous service has been examined by Atencia and Moreno (2005), Choudhury (2007), and Boualem et al. (2009) in different frame-works. Recently, Falin (2010a) studied a single server batch arrival retrial queue and applied embedded Markov chain technique to obtain the joint distribution for the number of the customers in the queue as well as for orbit.

In most of the queuing literature, we often meet the situations wherein the server may breakdown; such queuing problem is termed as queue with server breakdown. The assumptions of perfect reliable server are unrealistic in most of the congestion scenarios including in the area of computer and communication networks, flexible manufacturing system, production system, inventory and many others. According to the queuing and reliability point of view, the queue theorists are interested to develop the repairable service station. The worth nothing contributions in this area can be seen in the works of Ke (2005), Atencia et al. (2008), Choudhury and Deka (2009), Falin (2010b), Jain and Bhargava (2010). They have obtained the joint distribution for the number of the customers in the queue in the retrial group.

Several contributions are available on M/G/1 queuing system in which the server may provide a second phase of optional service. Madan (2000) considered the classical M/G/1 queuing system in which the server provides the first essential service (FES) to all the arriving customers whereas some of them receive SOS. The steady state analysis of an M/G/1 queue with repeated attempts and additional second phase of service has been done in different frameworks by Artelejo and Choudhury (2004), Choudhury (2008), and Choudhury and Tadj (2009).

3 Model description

Consider a single server retrial queuing system with batch arrivals. The queuing model is formulated by using stochastic process. The whole system works under some assumptions of the arrival process, service process, repair process, retrial policy, and vacation mechanism given below:

- a *Arrival process:* The primary customers arrive in batches according to Poisson process with rate λ . It is assumed that at every arrival epoch, a batch of k primary units arrives with probability c_k . Furthermore, the generating function for the batch size distribution is

$$C(z) = \sum_{k=1}^{\infty} c_k z^k$$

which follows that $E(X) = C'(1)$ and $E(X^2) = C''(1) + c'(1)$.

- b *Service process:* The system has a single server who provides preliminary FES denoted by B_1 to all arriving customers one by one according to first in first out (FIFO) discipline. As soon as the FES of a customer is completed, the server may provide SOS denoted by B_2 with probability p to only those customers who opt for it otherwise leaves the system with the complementary probability $\bar{p} = (1 - p)$.
- c *Repair process:* While the server is in working state, i.e., providing FES or SOS, it may breakdown at any time with failure rate α_1 during FES and α_2 during SOS. As soon as server breakdown occurs, it is immediately sent for repairing where repair time denoted by R_1 for FES and R_2 for SOS. After repairing, the server renders remaining service of the customers of both of the phases (FES or SOS) and such service time are cumulative and is known as generalised service times.
- d *Retrial policy:* If the server is busy in providing FES/SOS or under repair or on vacation at the arrival epoch, then all the arriving customers join the orbit. On the other hand, if the server is free, then one of the customer from arriving batch begins its service and the others join a retrial group to seek its service again and again under linear retrial policy, classical retrial policy, constant retrial policy, with rate $\theta_n = \nu(1 - \delta_{n,0}) + n\theta$, $\theta_n = n\theta$, $\theta_n = \nu(1 - \delta_{n,0})$, respectively, till he finds the server free.
- e *Vacation mechanism:* After each service completion epochs, the server either takes a vacation of random length given by V_1 (V_2) with probability σ_1 (σ_2) during FES (SOS) or may decide to serve new customer with the complementary probability

$\bar{\sigma}_1(\bar{\sigma}_2)$. Further, we assume that the vacation time is iid random variable which is independent of the input process.

In Table 1, we define some notations used for probability distribution functions (PDF), and their Laplace Steiljes transforms (LST) and k^{th} ($k \geq 1$) moments.

Table 1 Notations for distribution functions

<i>Time</i>	<i>PDF</i>	<i>LST</i>	<i>kth moments</i>
FES	$B_1(t)$	$b_1^*(s)$	$b_1^{(k)}$
SOS	$B_2(t)$	$b_2^*(s)$	$b_2^{(k)}$
Repair during FES	$R_1(t)$	$r_1^*(s)$	$r_1^{(k)}$
Repair during SOS	$R_2(t)$	$r_2^*(s)$	$r_2^{(k)}$
Vacation during FES	$V_1(t)$	$v_1^*(s)$	$v_1^{(k)}$
Vacation during SOS	$V_2(t)$	$v_2^*(s)$	$v_2^{(k)}$
i^{th} ($i = 1, 2$) phase generalised service time	$G_i(t)$	$G_i^*(s)$	$g_i^{(k)}$
Modified Service	$B(t)$	$B^*(s)$	$b^{(k)}$

We describe the state of the system at time t by employing stochastic process $X(t) = \{C(t), M(t), \delta(t)\}$, where $C(t)$ takes values 0, 1, 2, or 3 according to whether the server is idle, busy with first phase of generalised service time (including both service time and repair time during FES), busy with second phase of generalised service time (including both service time and repair time during SOS) or on vacation at time t . Let $M(t)$ denotes the number of customers in the orbit at time t . If $C(t) \in \{1, 2\}$, then $\delta(t)$ represents the corresponding elapsed service time.

The time required by a customer to complete a service cycle is defined as a modified service time B which is given by

$$B = \begin{cases} G_1, & \text{with probability } \bar{p}\bar{\sigma}_1 \\ G_1 + V_1, & \text{with probability } \bar{p}\sigma_1 \\ G_1 + G_2, & \text{with probability } p\bar{\sigma}_2 \\ G_1 + G_2 + V_2, & \text{with probability } p\sigma_2 \end{cases} \quad (1)$$

4 Embedded Markov chain

For analysis purpose, we employ the concept of modified service time which was first introduced by Keilson and Servi (1986) for $GI/G/1$ queuing system and subsequently used by Keilson and Servi (1987, 1989) and others for investigating $M/G/1$ queuing systems in different frameworks. Let B_n be the number of the customers arriving during the n^{th} total service time. Let τ_n be the time instant at which n^{th} service completion occurs. Let us consider the sequence $N_n = N(\tau_n^+)$ which is embedded Markov renewal process of the continuous time Markov process $Y(t)$. Then

$$(N_n / N_{n-1} = j) = \begin{cases} j-1+B_n & \text{with probability } \frac{\theta_j}{\lambda+\theta_j} \\ j+B_n & \text{with probability } \frac{\lambda}{\lambda+\theta_j} \end{cases} \quad (2)$$

4.1 The limiting distribution

By using the classical criteria based on mean drifts (see Artelejo and Gomez-Corral, 1997; Sennott et al., 1983), it can be proved that the sequence $\{N_n\}_{n=1}^{\infty}$ is positive recurrent to guarantee that the limiting probabilities $\pi_j = \lim_{n \rightarrow \infty} P\{N_n = j\}$, $j \geq 0$; exist and are positive. Let $l_n^{(1)}$, l_n , $h_n^{(1)}$ and h_n be the probabilities that ‘ n ’ customers arrive during the time intervals (G_1) , $(G_1 + G_2)$, $(G_1 + V_1)$, and $(G_1 + G_2 + V_2)$, respectively. These probabilities are defined as

$$l_n^{(1)} = l_{1,n}; \quad l_n = \sum_{i=0}^n l_{1,2,n-i}; \quad h_n^{(1)} = \sum_{i=0}^n l_i^{(1)} m_{n-i}^{(1)} \quad \text{and} \quad h_n = \sum_{i=0}^n l_i m_{n-i}.$$

Also, for $i = 1, 2$

$$G_i^*(\theta) = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-\theta t} e^{-\alpha_i t} \left[\frac{(\alpha_i t)^n}{n!} \right] [r_i^*(\theta)]^n dB_i(t) = b_i^*(\theta + \alpha_i(1 - r_i^*(\theta))) \quad (3)$$

$$l_{i,n} = \begin{cases} \int_0^{\infty} e^{-\lambda t} dG_i(t), & \text{if } j = 0 \\ \sum_{n=1}^{\infty} \int_0^{\infty} \frac{e^{-\lambda t} (\lambda t)^n c_j^{(n)} dG_i(t)}{n!}, & \text{if } n \geq 1 \end{cases} \quad (4)$$

and

$$m_{i,n} = \begin{cases} \int_0^{\infty} e^{-\lambda t} dV_i(t), & \text{if } n = 0 \\ \sum_{n=1}^{\infty} \int_0^{\infty} \frac{e^{-\lambda t} (\lambda t)^n c_n^{(t)} dV_i(t)}{n!}, & \text{if } n \geq 1 \end{cases} \quad (5)$$

Here $\{c_n^{(t)}\}_{n=0}^{\infty}$ denotes the k^{th} convolution of the sequence $\{c_n\}_{n=0}^{\infty}$.

4.2 Transition matrix

The one step transition matrix is $P = (p_{ij})$, where $p_{ij} = Pr(N_{n+1} = j / N_n = i)$ associated with Markov chain $\{N_n\}_{n=1}^{\infty}$ is obtained as

$$p_{ij} = \begin{cases} \frac{\theta_j}{\lambda + \theta_j} \left[\frac{\bar{p}\bar{\sigma}_1 l_0^{(1)} + \bar{p}\sigma_1 h_0^{(1)}}{+ p\bar{\sigma}_2 l_0 + p\sigma_2 h_0} \right], & \text{if } i \geq 1, j = i-1 \\ \frac{\lambda}{\lambda + \theta_j} \sum_{n=1}^{j-i+1} c_n \left[\frac{\bar{p}\bar{\sigma}_1 l_{j-i+1-n}^{(1)} + \bar{p}\sigma_1 h_{j-i+1-n}^{(1)}}{+ p\bar{\sigma}_2 l_{j-i+1-n} + p\sigma_2 h_{j-i+1-n}} \right] \\ + \frac{\theta_j}{\lambda + \theta_j} \left[\frac{\bar{p}\bar{\sigma}_1 l_{j-i+1}^{(1)} + \bar{p}\sigma_1 h_{j-i+1}^{(1)}}{+ p\bar{\sigma}_2 l_{j-i+1} + p\sigma_2 h_{j-i+1}} \right], & \text{if } 0 \leq i \leq j \end{cases} \quad (6)$$

Thus, using $\pi = \pi P$, the Kolmogorov equations associated with Markov chain is given by

$$\begin{aligned} \pi_n = & \sum_{i=0}^n \frac{\lambda \pi_i}{\lambda + \theta_i} \sum_{j=1}^{n-i+1} c_j \left[\frac{\bar{p}\bar{\sigma}_1 l_{n-i+1-j}^{(1)} + \bar{p}\sigma_1 h_{n-i+1-j}^{(1)}}{+ p\bar{\sigma}_2 l_{n-i+1-j} + p\sigma_2 h_{n-i+1-j}} \right] \\ & + \sum_{i=1}^{n+1} \frac{\theta_i \pi_i}{\lambda + \theta_i} \left[\frac{\bar{p}\bar{\sigma}_1 l_{n-i+1}^{(1)} + \bar{p}\sigma_1 h_{n-i+1}^{(1)}}{+ p\bar{\sigma}_2 l_{n-i+1} + p\sigma_2 h_{n-i+1}} \right], \quad n \geq 0 \end{aligned} \quad (7)$$

4.2.1 Generating functions

Now, we define the following generating functions:

$$\begin{aligned} \pi(z) &= \sum_{n=0}^{\infty} \pi_n z^n, & \psi(z) &= \sum_{n=0}^{\infty} \frac{\pi_n z^n}{(\lambda + \theta_j)}, & L(z) &= \sum_{n=0}^{\infty} l_n z^n, \\ H(z) &= \sum_{n=0}^{\infty} h_n z^n, & C(z) &= \sum_{n=0}^{\infty} c_n z^n, & L_i(z) &= \sum_{n=0}^{\infty} l_n^{(i)} z^n, \\ H_i(z) &= \sum_{n=0}^{\infty} h_{i,n} z^n, & M_i(z) &= \sum_{n=0}^{\infty} m_{i,n} z^n, & K_i(z) &= \sum_{n=0}^{\infty} k_{i,n} z^n \\ D_i(z) &= \sum_{n=0}^{\infty} d_{i,n} z^n, & \forall i &= 1, 2. \\ P_i(z) &= \sum_{n=0}^{\infty} P_{i,n} z^n, & \forall i &= 0, 1, 2, 3. \end{aligned}$$

It is noted that

$$\pi(z) = \theta z \psi'(z) + (\lambda + \nu) \psi(z) - \lambda^{-1} \nu \pi_0 \quad (8)$$

Also

$$\begin{aligned} L_i(z) &= b_i^*(A_i(z)), & M_i(z) &= v_i^*(a(z)); & \forall i &= 1, 2 \\ L(z) &= L_1(z)L_2(z); & H_1(z) &= L_1(z)M_1(z); & H(z) &= L(z)M_2(z) \end{aligned}$$

where $A_i(z) = a(z) + \alpha_i(1 - r_i^*(a(z)))$ and $a(z) = \lambda(1 - C(z))$.

On multiplying equation (7) by appropriate powers of z and then summing it over $n \geq 0$; and doing some algebraic manipulations, we get

$$\pi(z) = \frac{P(z)}{z} \left[\theta z \psi'(z) + [(\lambda - \lambda(1 - C(z)) + \nu) \psi(z) - \nu \lambda^{-1} \pi_0] \right] \quad (9)$$

Using equations (8) and (9), we obtain

$$\theta z \psi'(z) + \psi(z) \left[\nu - \frac{\lambda[z - C(z)P(z)]}{[P(z) - z]} \right] = \nu \pi_0 \lambda^{-1} \quad (10)$$

and

$$\pi(z) = \frac{a(z)P(z)}{[P(z) - z]} \psi(z) \quad (11)$$

Using the normalising condition $\pi(1) = 1$, equation (11) yields

$$\psi(1) = \frac{1 - \rho}{\lambda E(X)} \quad (12)$$

where

$$\begin{aligned} P(z) &= [\bar{p}\bar{\sigma}_1 L_1(z) + \bar{p}\sigma_1 H_1(z) + p\bar{\sigma}_2 L(z) + p\sigma_2 H(z)] \\ \rho &= \rho_1 (1 + \alpha_1 r_1^{(1)}) + p\rho_2 (1 + \alpha_2 r_2^{(1)}) + \lambda E(X) [\bar{p}\sigma_1 v_1^{(1)} + p\sigma_2 v_2^{(1)}]; \\ \rho_i &= \lambda E(X) b_i^{(1)}, \quad \forall i = 1, 2 \end{aligned}$$

5 Joint distribution and queue size distribution

In this section, we are interested to study the limiting behaviour of the process $X(t) = \{C(t), M(t), \delta(t)\}$ as $t \rightarrow \infty$. The stationary probabilities are given by

$$P_{i,j} = \lim_{t \rightarrow \infty} P\{(C(t), M(t)) = (i, j)\}, \quad (i, j) \in \{0, 1, 2, 3\} \times Z^+.$$

Since the arrival stream is Poisson, it follows from Burke's theorem (see Cooper, 1981) that the stationary probabilities $\{P_{ij}\}$ exist and are positive under the same conditions of the limiting probabilities $\{\pi_j\}_{j=0}^\infty$ of the embedded Markov chain $\{N_n\}_{n=1}^\infty$; i.e., iff $\rho < 1$.

Theorem 1: The marginal generating function for the stationary queue size distribution is given as

1 Linear retrial policy when $\nu > 0$ and $\theta > 0$,

$$\begin{aligned} \psi(z) &= z^{-\nu/\theta} \exp \left\{ \frac{\lambda}{\theta} \int_z^1 \frac{(y - C(y)P(y)) dy}{(P(y) - y)y} \right\} \\ &\times \left[\frac{(1 - \rho)}{\lambda E(X)} - \frac{\nu \pi_0}{\lambda \theta} \int_z^1 u^{\nu/\theta - 1} \exp \left\{ \left(\frac{-\lambda}{\theta} \right) \int_u^1 \frac{(y - C(y)P(y)) dy}{(P(y) - y)y} \right\} du \right] \end{aligned} \quad (13a)$$

and

$$\pi_0 = \frac{\theta(1-\rho)}{\nu E(X)} \left[\int_0^1 u^{\nu/\theta-1} \exp \left\{ \left(\frac{-\lambda}{\theta} \right) \int_u^1 \frac{(y-C(y)P(y))}{(P(y)-y)} \frac{dy}{y} \right\} du \right]^{-1} \quad (13b)$$

and

$$\begin{aligned} \rho &= \rho_1 (1 + \alpha_1 r_1^{(1)}) + p \rho_2 (1 + \alpha_2 r_2^{(1)}) + \lambda E(X) (\bar{p} \sigma a_1 v_1^{(1)} + p \sigma_2 v_2^{(1)}), \\ \rho_i &= \lambda E(X) b_i^{(1)} \end{aligned}$$

2 Classical retrial policy when $\nu = 0$ and $\theta > 0$,

$$\psi(z) = \frac{(1-\rho)}{\lambda E(X)} \exp \left\{ \left(\frac{-\lambda}{\theta} \right) \int_z^1 \frac{(y-C(y)P(y))}{(P(y)-y)} \frac{dy}{y} \right\} \quad (14a)$$

and

$$\pi_0 = \frac{(1-\rho)}{E(X)} \left[\exp \left\{ \left(\frac{-\lambda}{\theta} \right) \int_0^1 \frac{(y-C(y)P(y))}{(P(y)-y)} \frac{dy}{y} \right\} \right] \quad (14b)$$

3 Constant retrial policy when $\nu > 0$ and $\theta = 0$

$$\psi(z) = \frac{\nu \pi_0}{\lambda \left[\nu - \frac{\lambda(z-C(z)P(z))}{(P(z)-z)} \right]} \quad (15a)$$

and

$$\pi_0 = \frac{(1-\rho)}{E(X)} \left[1 + \frac{\lambda}{\nu} \right] - \frac{\lambda}{\nu} \quad (15b)$$

Proof: For proof see Appendix A.I.

Remark 1: It should be noted that the limiting probabilities $\{\pi_j\}_{j=0}^\infty$ can also be solved recursively using equation (7) and the expression for π_0 can be determined from above theorem in case of linear retrial policy, classical retrial policy, and constant retrial policy, respectively.

5.1 Embedded Markov renewal process

Here, as we note that $\{X(t); t \geq 0\}$ is a Markov regenerative process with the embedded Markov renewal process $\{N_n\}_{n=1}^\infty$, so we may use some classical results established in Cinlar (1975).

Denote

- $\psi_n(i, j)$ the expected amount of time spent by the process $X(t)$; $t \geq 0$ in the state (i, j) during an interval between two successive total completion epochs given that at the beginning of this interval, the number of customers in orbit was n
- ψ_n the expectation of the interval between two successive total completion epochs given that at the beginning of this interval, the number of customers in orbit was n .

Then, we have

$$P_{i,j} = \frac{\sum_{n=0}^{\infty} \pi_n \psi_n(i, j)}{\sum_{n=0}^{\infty} \pi_n \psi_n}; \quad 0 \leq i \leq 1, j \geq 0 \quad (16)$$

For this model, we observe that

$$\psi_n = \frac{1}{\lambda + \theta_n} + (1 + \alpha_1 r_1^{(1)}) b_1^{(1)} + p(1 + \alpha_2 r_2^{(1)}) b_2^{(1)} + \bar{p} \sigma_1 v_1^{(1)} + p \sigma_2 v_2^{(1)} \quad (17)$$

and

$$\sum_{n=0}^{\infty} \psi_n \pi_n = [\lambda E(X)]^{-1}.$$

Lemma 1: The limiting probabilities of the stationary queue size distribution are as follows

$$P_{0,j} = \frac{\lambda E(X) \pi_j}{\lambda + \theta_j}, \quad j \geq 0 \quad (18)$$

$$P_{1,j} = \lambda E(X) \left[\sum_{n=0}^j \frac{\lambda (1 - \delta_{j,n-1}) \sum_{m=1}^{j-n+1} c_m k_{1,j-n+1-m}}{\lambda + \theta_n} + \sum_{n=1}^{j+1} \frac{\theta_n k_{1,j-n+1}}{\lambda + \theta_n} \right], \quad j \geq 0 \quad (19)$$

$$P_{2,j} = \lambda E(X) p \left[\sum_{n=0}^j \frac{\lambda \pi_i (1 - \delta_{j,n-1}) \sum_{k=1}^{j-n+1} c_k \sum_{m=1}^{j-n+1-k} l_{1,m} k_{2,j-n+1-k-m}}{\lambda + \theta_n} + \sum_{n=1}^{j+1} \frac{\theta_n \pi_i \sum_{m=1}^{j-n+1} l_{1,m} k_{2,j-n-m+1}}{\lambda + \theta_n} \right], \quad j \geq 0 \quad (20)$$

$$\begin{aligned}
P_{3,j} = \lambda E(X) & \left[\sum_{n=0}^j \frac{\lambda \bar{p} \sigma_1 (1 - \delta_{j,n-1}) \pi_i \sum_{k=1}^{j-n+1} c_k \sum_{m=1}^{j-n+1-k} l_{1,m} d_{1,j-n+1-k-m}}{\lambda + \theta_n} \right. \\
& + \sum_{n=0}^j \frac{\lambda p \sigma_2 \pi_i (1 - \delta_{j,n-1}) \sum_{k=1}^{j-n+1} c_k \sum_{m=1}^{j-n+1-k} l_m d_{2,j-n+1-k-m}}{\lambda + \theta_n} \\
& + \sum_{n=1}^{j+1} \frac{\theta_n \bar{p} \sigma_1 \pi_i \sum_{m=1}^{j-n+1} l_{1,m} d_{1,j-n+1-m}}{\lambda + \theta_n} \\
& \left. + \sum_{n=1}^{j+1} \frac{\theta_n p \sigma_2 \pi_i \sum_{m=1}^{j-n+1} l_m d_{2,j-n+1-m}}{\lambda + \theta_n} \right], \quad j \geq 0
\end{aligned} \tag{21}$$

where

$$k_{i,n} = \begin{cases} \int_0^{\infty} e^{-\lambda t} [1 - G_i(t)] dt, & \text{if } n = 0, \forall i = 1, 2 \\ \sum_{n=1}^{\infty} \int_0^{\infty} \frac{e^{-\lambda t} (\lambda t)^n c_n^{(k)} [1 - G_i(t)]}{n!} dt, & \text{if } n \geq 1, \forall i = 1, 2 \end{cases}$$

and

$$d_{i,n} = \begin{cases} \int_0^{\infty} e^{-\lambda t} [1 - V_i(t)] dt, & \text{if } n = 0, \forall i = 1, 2 \\ \sum_{n=1}^{\infty} \int_0^{\infty} \frac{e^{-\lambda t} (\lambda t)^k c_n^{(k)} [1 - V_i(t)]}{n!} dt, & \text{if } n \geq 1, \forall i = 1, 2 \end{cases}$$

Proof: For proof, see Appendix A. II.

Theorem 2: The partial generating functions for the idle, busy with first phase of modified service, busy with second phase of modified service and vacation states of the server, respectively are obtained as

$$P_0(z) = \lambda E(X) \psi(z) \tag{22}$$

$$P_1(z) = \frac{(1 - b_1^*(A_1(z)))}{(P(z) - z)} P_0(z) \tag{23}$$

$$P_2(z) = \frac{p(1-b_2^*(A_2(z)))b_1^*(A_1(z))}{(P(z)-z)} P_0(z) \quad (24)$$

$$P_3(z) = \frac{\left[\bar{p}\sigma_1 b_1^*(A_1(z)) [1-v_1^*(a(z))] + p\sigma_2 b_1^*(A_1(z)) b_2^*(A_2(z)) [1-v_2^*(a(z))] \right]}{(P(z)-z)} P_0(z) \quad (25)$$

Proof: For proof see Appendix A.III.

Theorem 3: The probability generating function (PGF) of the number of customers in the orbit is given by

$$R(z) = \frac{(1-z)P_0(z)}{[P(z)-z]} \quad (26)$$

Proof: For proof see Appendix A.IV.

Theorem 4: The stationary distribution of the total number of customers in the system is given by

$$Q(z) = \frac{(1-z)P_0(z)P(z)}{[P(z)-z]} \quad (27)$$

Proof: For proof see Appendix A.V.

6 Performance measures

We derive some performance measures in terms of steady state probabilities in order to investigate the behaviour of unreliable $M^X/G/1$ retrial queuing system. The probabilities of the server being in different states, mean orbit length, mean system length, mean waiting time of the customer and mean busy period are obtained as follows:

a *States of the server:* The long run probabilities of the server being in different states are obtained as follows:

- The long run probability of the server being idle is obtained as

$$P(I) = \lim_{z \rightarrow 1} P_0(z) = (1 - \rho) \quad (28)$$

- The long run probability of the server being on first phase of generalised service is

$$P(FG) = \lim_{z \rightarrow 1} P_1(z) = \lambda E(X) b_1^{(1)} (1 + \alpha_1 r_1^{(1)}) \quad (29)$$

- The long run probability of the server being on second phase of generalised service is

$$P(SG) = \lim_{z \rightarrow 1} P_2(z) = p \lambda E(X) b_2^{(1)} (1 + \alpha_2 r_2^{(1)}) \quad (30)$$

- The long run probability of the server being on vacation is

$$P(V) = \lim_{z \rightarrow 1} P_3(z) = \lambda E(X) (\bar{p}\sigma_1\nu_1^{(1)} + p\sigma_2\nu_2^{(1)}) \quad (31)$$

- b *Queue length:* The average queue lengths of the customers in the orbit as well as in the system are obtained as follows:

- 1 The expected number of customers in the orbit is given by

$$\begin{aligned} E(N_o) &= \left. \frac{d}{dz} R(z) \right|_{z=1} \\ &= \frac{E(X^2)}{E(X)} + \frac{\psi'(1)}{(1-\rho)} + \frac{P''(1)}{2(1-\rho)} \end{aligned} \quad (32)$$

- If $\nu > 0$ and $\theta > 0$, then

$$\psi'(1) = [\theta]^{-1} [\nu(\pi_0 E(X) + \rho - 1) + \lambda(\rho + E(X) - 1)] \quad (33)$$

- If $\nu = 0$ and $\theta > 0$, then

$$\psi'(1) = \lambda[\theta]^{-1}(\rho + E(X) - 1) \quad (34)$$

- If $\nu > 0$ and $\theta = 0$, then

$$\psi'(1) = \frac{\lambda[(1-\rho)(E(X^2) + 2E(X)\rho) + P''(1)E(X)]}{2(1-\rho)[\nu - \lambda(E(X) + \rho - 1)(1-\rho)^{-1}]} \quad (35)$$

where

$$P''(1) = \bar{p}\bar{\sigma}_1 L_1''(1) + \bar{p}\sigma_1 H_1''(1) + p\bar{\sigma}_2 L''(1) + p\sigma_2 H''(1)$$

$$L''(1) = L_1''(1) + 2L_1'(1)L_2'(1) + L_2''(1), H''(1) = L''(1) + 2M_2'(1)L'(1) + M_2''(1)$$

$$L'(1) = \rho_1(1 + \alpha_1 r_1^{(1)}) + \rho_2(1 + \alpha_2 r_2^{(1)})$$

$$L_i''(1) = (A_i'(1))^2 b_i^{(2)} - A_i''(1) b_i^{(1)}, M_i''(1) = (a'(1))^2 \nu_i^{(2)} - a''(1) \nu_i^{(1)} \quad \forall i = 1, 2$$

$$L_i'(1) = \rho_i(1 + \alpha_i r_i^{(1)}), M_i'(z) = \lambda E(X) \nu_i^{(1)}, A_i'(1) = -\lambda E(X)(1 + \alpha_i r_i^{(1)}) \quad \forall i = 1, 2$$

$$A_i''(1) = -\lambda E(X^2) - \alpha_i [\lambda^2 E(X)^2 r_i^{(2)} + \lambda E(X^2) r_i^{(1)}], \quad \forall i = 1, 2$$

$$a'(1) = -\lambda E(X), a''(1) = -\lambda E(X^2)$$

2 The expected number of the customers in the system is given by

$$E(N_S) = \frac{d}{dz} Q(z) \Big|_{z=1} = \rho + E(N_O) \quad (36)$$

c *Average waiting time:* Waiting time is an important characteristic of any queuing system. The mean time of a customer that he spends in the system can be obtained by using Little's formula, given by

$$W = \frac{E(N_S)}{\lambda} = \frac{[\rho + E(N_O)]}{\lambda} \quad (37)$$

d *Mean busy period:* The mean busy period can be calculated by utilising well known result of alternating renewal process, given as (cf. Choudhury and Deka, 2009)

$$E(T_b) = \frac{(\pi_0^{-1} - 1)}{\lambda} \quad (38)$$

where the value of π_0 can be put from theorem 1 in case of linear retrial policy/classical retrial policy/constant retrial policy.

Remark 2: It is noticed that the value of $\psi'(1)$ can be easily computed by using equation (10) in case of linear retrial policy, classical retrial policy, constant retrial policy, given in equations (33), (34), (35), respectively.

7 Special cases

By assigning appropriate parameter values, we can deduce some special cases of the models in order to verify our results with the existing results.

Case 1: unreliable M/G/1 retrial queuing system with FES and SOS

In this case, we set $E(X) = 1$, $E(X^2) = 0$ and $\sigma_1, \sigma_2 = 0$, so that equation (32) becomes

$$E(N_O) = \frac{[\lambda\rho + v(\pi_0 + \rho - 1)]}{\theta(1 - \rho)} + \frac{P''(1)}{2(1 - \rho)} \quad (39)$$

where

$$P''(1) = \lambda \left[\rho_1 \left\{ \alpha_1 r_1^{(2)} + 2(1 + \alpha_1 r_1^{(1)})^2 b_R^{(1)} \right\} + p\rho_2 \left\{ \alpha_2 r_2^{(2)} + 2(1 + \alpha_2 r_2^{(1)})^2 b_R^{(2)} \right\} \right. \\ \left. + 2\rho_1\rho_2 (1 + \alpha_1 r_1^{(1)})(1 + \alpha_2 r_2^{(1)}) \right]$$

and

$$b_R^{(i)} = \frac{b_i^{(2)}}{2b_i^{(1)}}, \rho = \rho_1 (1 + \alpha_1 r_1^{(1)}) + p\rho_2 (1 + \alpha_2 r_2^{(1)}).$$

The above results coincide with those obtained by Choudhury and Deka (2009).

Case 2: M/G/1 retrial queue under Bernoulli vacation along with two phases of essential service

On setting $E(X) = 1$, $E(X^2) = 0$, $\alpha_i = 0$, $r_i^{(1)}, r_i^{(2)} = 0$, $\forall i = 1, 2$, $p = 1$ and $\sigma_1 = 0$, then equation (32) reduces to

$$E(N_o) = \frac{[\lambda\rho + v(\pi_o + \rho - 1)]}{\theta(1-\rho)} + \frac{P''(1)}{2(1-\rho)} \quad (40)$$

where

$$P''(1) = \lambda^2 \left[b_1^{(2)} + b_2^{(2)} + 2b_1^{(1)}b_2^{(2)} + p \left\{ v_1^{(2)} + 2v_1^{(2)} (b_1^{(1)} + b_2^{(1)}) \right\} \right]$$

and

$$\rho = \lambda (b_1^{(1)} + b_2^{(1)} + \sigma_2 v_2^{(1)}).$$

This case coincides with the model developed by Choudhury (2008).

Case 3: M^X/G/1 retrial queue with Bernoulli vacation schedule and two phase service

In this case, setting $\alpha_i = 0$, $r_i^{(1)}, r_i^{(2)} = 0$, $\forall i = 1, 2$, $p = 1$ and $\sigma_1 = 0$, equation (32) gives

$$E(N_o) = \frac{\rho E(X^2)}{2(1-\rho)E(X)} + \frac{[\lambda E(X) + \lambda(\rho - 1)]}{\theta(1-\rho)} + \frac{P''(1)}{2(1-\rho)} \quad (41)$$

where

$$P''(1) = [\lambda E(X)]^2 \left[b_1^{(2)} + b_2^{(2)} + 2b_1^{(1)}b_2^{(2)} + p \left\{ v_1^{(2)} + 2v_1^{(2)} (b_1^{(1)} + b_2^{(1)}) \right\} \right]$$

and

$$\rho = \lambda E(X) [b_1^{(1)} + b_2^{(1)} + \sigma_2 v_2^{(1)}].$$

In particular case, if Bernoulli admission control is assumed to be zero, above result coincides with the results obtained by Choudhury (2007).

Case 4: M/G/1 queue with repeated attempts with SOS along with FES and without vacation

Substituting, $E(X) = 1$, $E(X^2) = 0$, $\alpha_i = 0$, $r_i^{(1)}, r_i^{(2)} = 0$, $\forall i = 1, 2$ and $\sigma_1 = \sigma_2 = 0$ in equation (32), we get

$$E(N_o) = \frac{\lambda\rho}{\theta(1-\rho)} + \frac{P''(1)}{2(1-\rho)} \quad (42)$$

where

$$P''(1) = \lambda^2 (b_1^{(2)} + pb_2^{(2)} + 2pb_1^{(1)}b_2^{(1)})$$

and

$$\rho = \lambda (b_1^{(1)} + pb_2^{(1)}).$$

The above results match with those obtained by Artelejo and Choudhury (2004).

Case 5: $M^x/G/1$ queue with classical retrial policy

If we set, $\alpha_i = 0$, $r_i^{(1)}, r_i^{(2)} = 0$, $\forall i = 1, 2$, $\sigma_1 = 0$, $\sigma_2 = 0$ and $p = 0$, then equation (32) yields

$$E(N_o) = \frac{\rho E(X^2)}{2(1-\rho)E(X)} + \frac{\lambda[E(X) + (\rho-1)]}{\theta(1-\rho)} + \frac{[\lambda E(X^1)]^2 b_1^{(2)}}{2(1-\rho)} \quad (43)$$

where

$$\rho = \lambda E(X)b_1^{(1)}.$$

This model is consistent with that of Falin and Templeton (1997).

8 Stochastic decomposition property

In this section, we present stochastic decomposition property of the system size distribution. The stochastic decomposition property for the system size has been established for the retrial queuing systems with optional service, vacation and unreliable server. The stochastic decomposition law for retrial queues has also been studied by Yang and Templeton (1987) and Fuhrmann and Cooper (1985).

The average number of customers (L) in the system can be expressed as the sum of two independent random variables, one of which is the average number of the customers (L_1) in the $M^x/G/1$ queuing system with optional service and vacation subject to server breakdown and the other one is the average number of repeated customers (M_1) given that the server is idle, i.e., $L = L_1 + M_1$. Thus,

$$\begin{aligned} \pi(z) &= \left[\frac{(1-\rho)(1-C(z))P(z)}{[P(z)-z]} \right] \left[\frac{\lambda\psi(z)}{(1-\rho)} \right] \\ &\equiv \eta(z)\Omega(z) \end{aligned} \quad (44)$$

where

$$\eta(z) = \left[\frac{(1-\rho)(1-C(z))P(z)}{[P(z)-z]} \right]$$

and

$$\Omega(z) = \left[\frac{\lambda\psi(z)}{(1-\rho)} \right].$$

The first fraction of equation (44), $\eta(z)$ is the PGF of the system size distribution at departure epoch of the $M^X/G/1$ queuing system with optional service and vacation subject to server breakdown whereas the second fraction of equation (44), $\Omega(z)$ denotes the PGF of the number of the blocked customers given that the system is idle.

9 Numerical illustration

In this section, we present numerical illustration in order to verify the implementation of analytical results. To develop computational program, software MATLAB has used and the results are displayed in Tables 2–3 and Figures 1(a)–1(c) to Figures 8(a)–8(c). We examine the effects of some system parameters such as arrival rate (λ), service rates (μ_1, μ_2), failure rates (α_1, α_2), repair rates (β_1, β_2), vacation rates (ν_1, ν_2), retrial rate (θ), probabilities p and σ_1 on the average system size ($E(N_S)$). We consider (a) $M^X/E_2/1$ model and (b) $M^X/\gamma/1$ model for computational purpose. We set default parameters as $\lambda = 0.7, \theta = 0.2, \nu = 0.4, \alpha_1 = 0.6, \alpha_2 = 0.7, \beta_1 = 1.9, \beta_2 = 1.6, p = 0.4, \mu_1 = 1.5, \mu_2 = 2, \sigma_1 = 0.3, \sigma_2 = 0.2, \nu_1 = 0.3, \nu_2 = 0.4$ and $E(X) = 1$ for Tables 2–3 and Figures 1(a)–1(c) to Figures 8(a)–8(c).

Table 2 Effect of p, λ, μ_1 and ν_1 on long run probabilities of the server states

α_1	β_1	σ_1	$p = .1$				$p = .9$			
			$P(I)$	$P(FG)$	$P(SG)$	$P(V)$	$P(I)$	$P(FG)$	$P(SG)$	$P(V)$
.1	1	.3	0.902	0.054	0.003	0.040	0.894	0.054	0.027	0.023
.2	1	.3	0.804	0.108	0.006	0.081	0.789	0.108	0.055	0.047
.3	1	.3	0.706	0.162	0.009	0.122	0.683	0.162	0.083	0.070
.4	1	.3	0.608	0.216	0.012	0.162	0.578	0.216	0.110	0.094
.5	1	.3	0.510	0.270	0.015	0.203	0.472	0.270	0.138	0.117
.7	1	.3	0.314	0.378	0.021	0.285	0.261	0.378	0.194	0.165
.7	2	.3	0.503	0.189	0.021	0.285	0.451	0.189	0.194	0.165
.7	3	.3	0.567	0.126	0.021	0.285	0.514	0.126	0.194	0.165
.7	4	.3	0.598	0.094	0.021	0.285	0.546	0.094	0.194	0.165
.7	5	.3	0.617	0.075	0.021	0.285	0.565	0.075	0.194	0.165
.7	1	.3	0.314	0.378	0.021	0.285	0.261	0.378	0.194	0.165
.7	1	.4	0.381	0.378	0.021	0.217	0.269	0.378	0.194	0.157
.7	1	.5	0.422	0.378	0.021	0.177	0.273	0.378	0.194	0.153
.7	1	.6	0.449	0.378	0.021	0.150	0.276	0.378	0.194	0.150
.7	1	.7	0.468	0.378	0.021	0.130	0.279	0.378	0.194	0.147

Table 3 Effect of p , α_1 , β_1 and σ_1 on long run probabilities of the server states

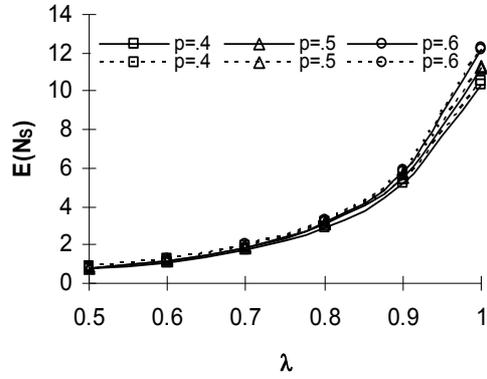
α_1	β_1	σ_1	$p = .1$				$p = .9$			
			$P(I)$	$P(FG)$	$P(SG)$	$P(V)$	$P(I)$	$P(FG)$	$P(SG)$	$P(V)$
.1	1.9	.3	0.482	0.210	0.021	0.285	0.430	0.210	0.194	0.165
.2	1.9	.3	0.472	0.221	0.021	0.285	0.419	0.221	0.194	0.165
.3	1.9	.3	0.461	0.231	0.021	0.285	0.409	0.231	0.194	0.165
.4	1.9	.3	0.451	0.242	0.021	0.285	0.398	0.242	0.194	0.165
.5	1.9	.3	0.440	0.252	0.021	0.285	0.388	0.252	0.194	0.165
.5	1	.3	0.393	0.300	0.021	0.285	0.340	0.300	0.194	0.165
.5	2	.3	0.443	0.250	0.021	0.285	0.390	0.250	0.194	0.165
.5	3	.3	0.460	0.233	0.021	0.285	0.407	0.233	0.194	0.165
.5	4	.3	0.468	0.225	0.021	0.285	0.415	0.225	0.194	0.165
.5	5	.3	0.473	0.220	0.021	0.285	0.420	0.220	0.194	0.165
.5	1.9	.3	0.440	0.252	0.021	0.285	0.388	0.252	0.194	0.165
.5	1.9	.4	0.350	0.252	0.021	0.375	0.378	0.252	0.194	0.175
.5	1.9	.5	0.260	0.252	0.021	0.465	0.368	0.252	0.194	0.185
.5	1.9	.6	0.170	0.252	0.021	0.555	0.358	0.252	0.194	0.195
.5	1.9	.7	0.080	0.252	0.021	0.645	0.348	0.252	0.194	0.205

Tables 2 and 3 display the effect of λ , μ_1 , ν_1 , α_1 , β_1 and σ_1 for the different values of p on various performance characteristics. From Tables 2–3, it is noted that $P(I)$ decreases on increasing the values of λ , α_1 and σ_1 while increases with other system parameters for both the values of p . On the other hand, $P(FG)$ show the increasing (decreasing) trend with increasing values of λ , $\alpha_1(\mu, \beta_1)$. As we increase the values of λ , increasing pattern is followed by $P(SG)$ and $P(V)$. It is also seen that $P(V)$ decreases (increases) for increasing the values of ν_1 (σ_1) for both the values of p . On the contrary, $P(FG)$ and $P(SG)$ remain constant for ν_1 and σ_1 . Moreover, $P(SG)$ and $P(V)$ almost remain constant for the increasing values of μ_1 , α_1 , β_1 and p .

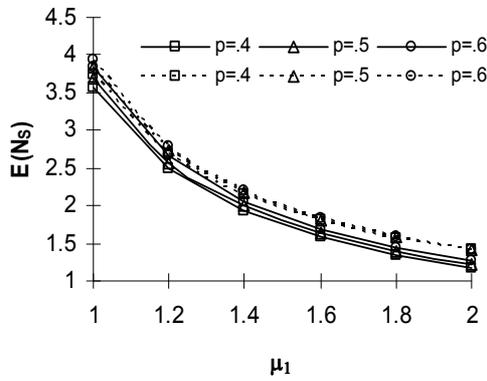
In Figures 1(a)–1(c) to Figures 8(a)–8(c), we display the effect of various parameters on average system size ($E(N_S)$) for $M^X/E_2/1$ and $M^X/\gamma/1$ models for different values of p , σ_1 , θ and $E(X)$. We choose the continuous (discrete) lines correspond to $M^X/E_2/1$ ($M^X/\gamma/1$) model. It is seen from Figures 1(a)–1(c) to Figures 2(a)–2(c) that $E(N_S)$ first increases (decreases) slowly then after sharply for increasing values of $\lambda(\mu_1$ and $\mu_2)$. It is also observed that the average system size increases sharply with respect to probabilities p and σ_1 .

Figures 3(a)–3(c) to Figures 6(a)–6(c) illustrate the graphs for the average system size for different values of p and σ_1 by varying α_1 , β_1 , ν_1 , α_2 , β_2 and ν_2 , respectively. It is claimed from these figures that $E(N_S)$ shows the gradual increasing trend on increasing the values of the failure rates, i.e., α_1 and α_2 . Further, as we increase the values of repair rates (β_1 and β_2) and vacation rates (ν_1 and ν_2) the decreasing pattern has been observed for $E(N_S)$ for the different values of p and σ_1 . The results are more prominent for $M^X/\gamma/1$ model rather than that of $M^X/E_2/1$ model.

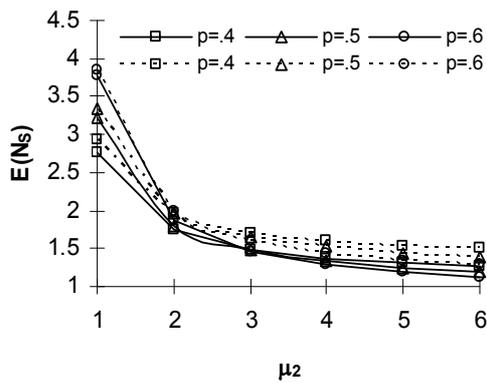
Figure 1 The effect of (a) λ , (b) μ_1 , and (c) μ_2 on $E(N_S)$ for the different values of p



(a)



(b)



(c)

Figure 2 The effect of (a) λ , (b) μ_1 , and (c) μ_2 on $E(N_s)$ for the different values of σ_i

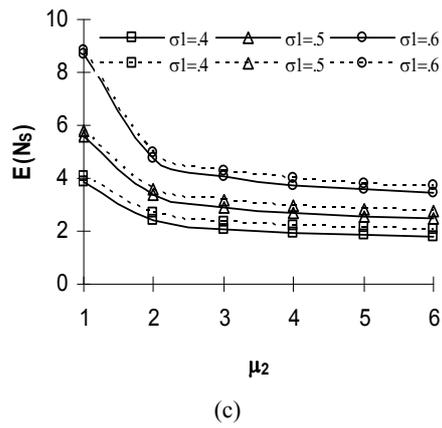
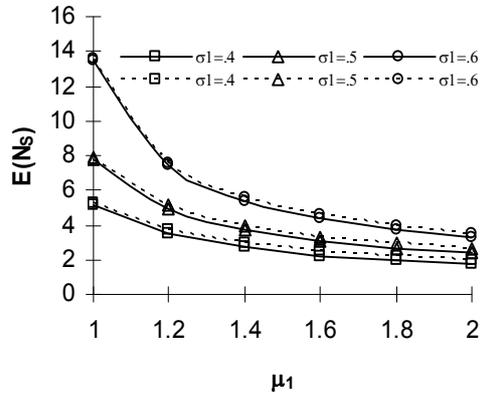
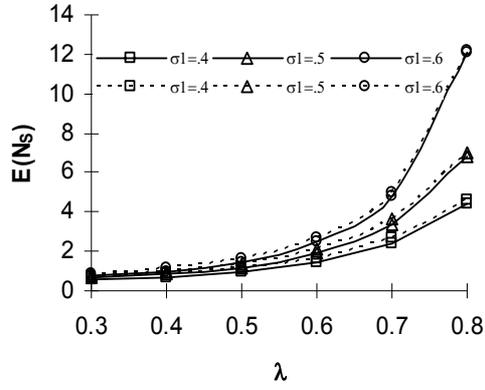
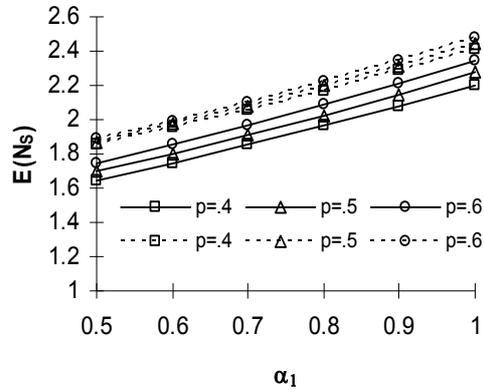
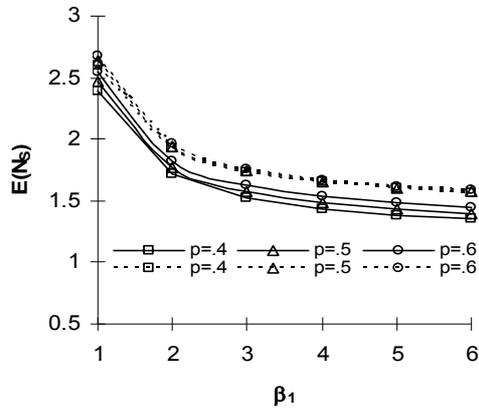


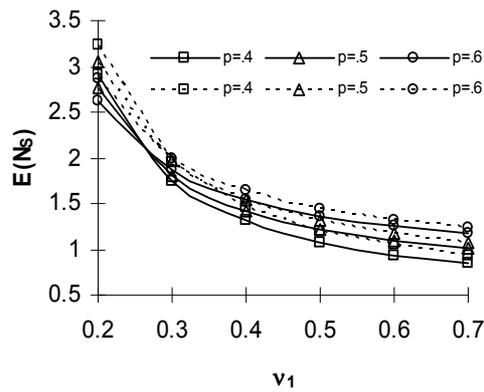
Figure 3 The effect of (a) α_1 , (b) β_1 , and (c) ν_1 on $E(N_S)$ for the different values of p



(a)

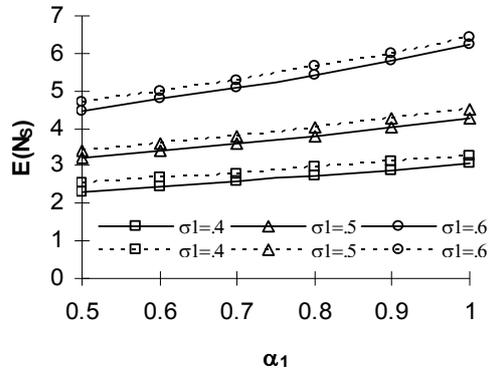


(b)

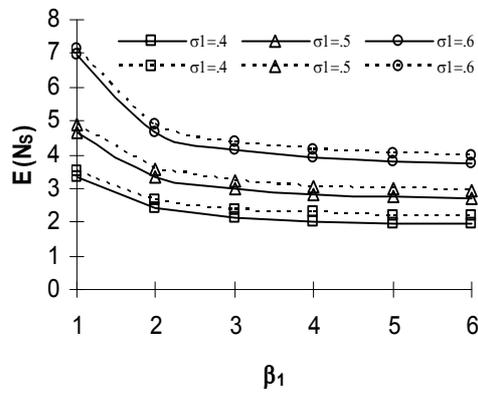


(c)

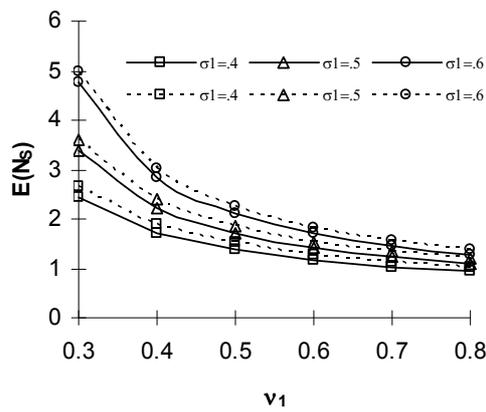
Figure 4 The effect of (a) α_1 , (b) β_1 , and (c) ν_1 on $E(N_S)$ for the different values of σ_1



(a)

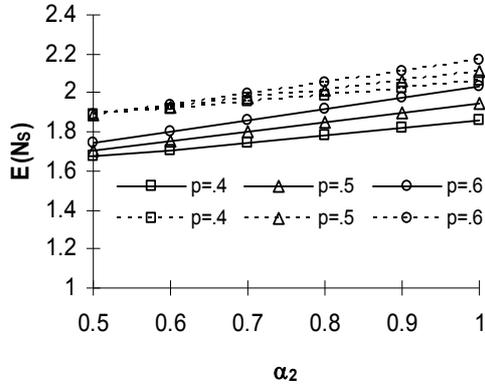


(b)

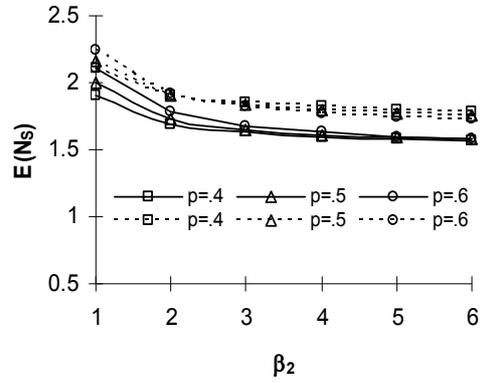


(c)

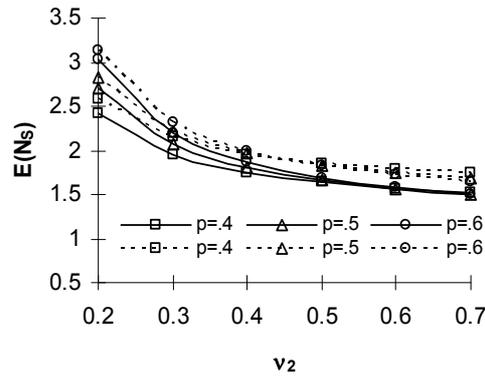
Figure 5 The effect of (a) α_2 , (b) β_2 , and (c) ν_2 on $E(N_S)$ for the different values of p



(a)

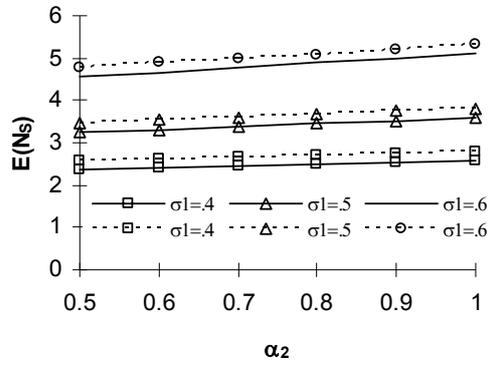


(b)

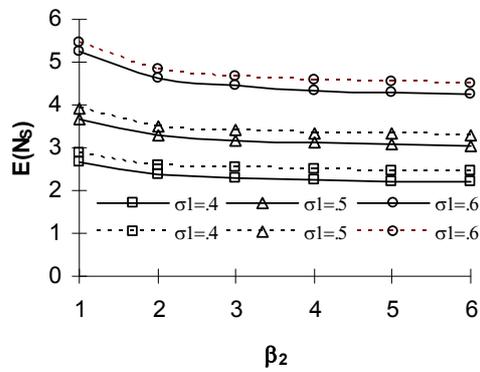


(c)

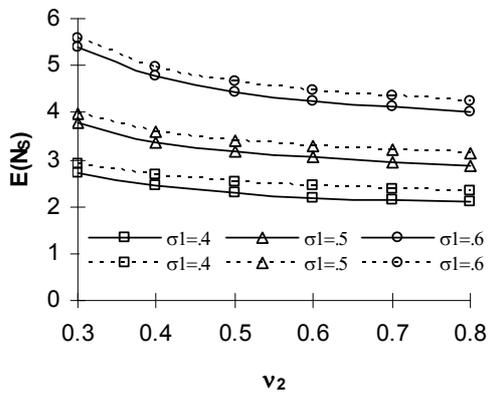
Figure 6 The effect of (a) α_2 , (b) β_2 , and (c) ν_2 on $E(N_S)$ for the different values of σ_1



(a)

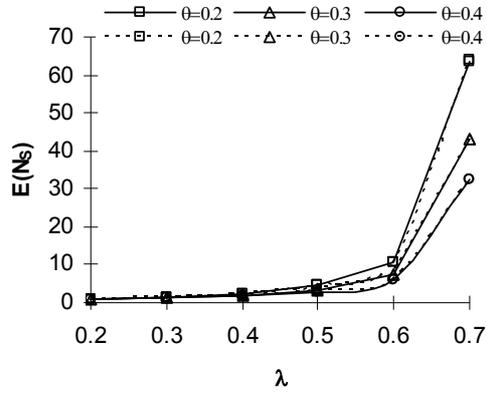


(b)

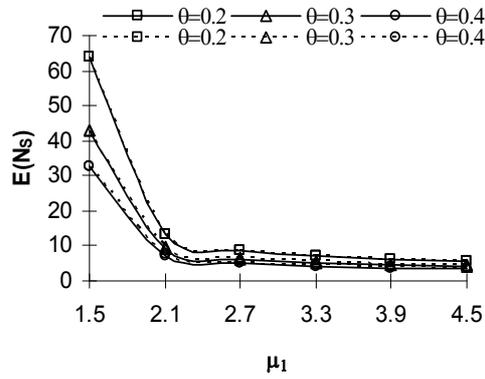


(c)

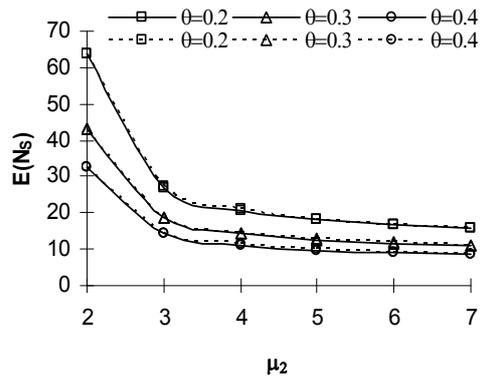
Figure 7 The effect of (a) λ , (b) μ_1 , and (c) μ_2 on $E(N_S)$ for the different values of θ



(a)

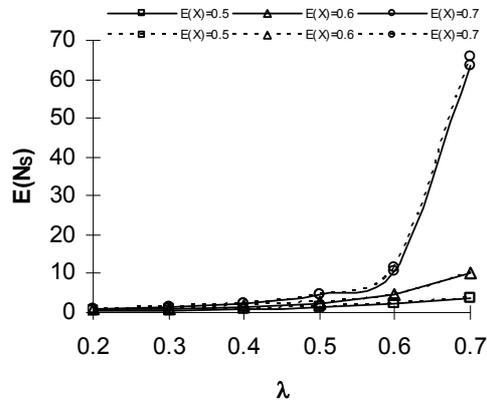


(b)

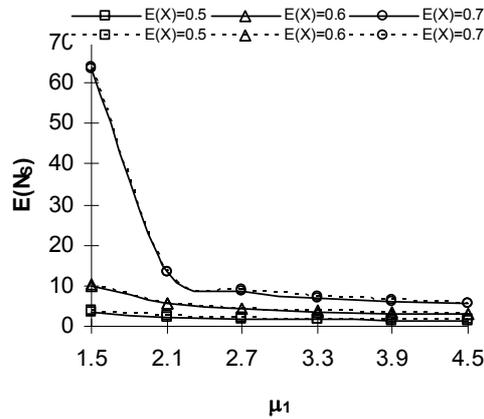


(c)

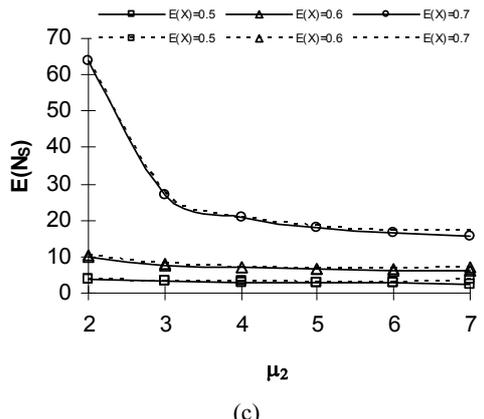
Figure 8 The effect of (a) λ , (b) μ_1 , and (c) μ_2 on $E(N_S)$ for the different values of $E(X)$



(a)



(b)



(c)

From Figures 7(a)–7(c) to Figures 8(a)–8(c), it can be easily seen that on increasing (decreasing) the values of $\lambda(\mu_1$ and $\mu_2)$, average system size first increases (decreases) slowly then after sharply. On the other hand, average system size shows the increasing trend for the increasing values of retrial rate and batch size respectively, which is quite obvious.

From the numerical results summarised in the form of tables and graphs, we overall conclude the following observations:

- As we expect, the average system size increases on increasing the arrival rate and failure rates; but decreases with service rates, repair rates and vacation rates, which tally with the real life situations.
- Our analysis advises to the system analysts and the decision makers that the grade of service can be improved by controlling some sensitive parameters such as arrival rate and failure rate.

10 Conclusions

The performance analysis of bulk arrival retrial queue with random service interruption and Bernoulli vacation is presented in the present investigation. The provision of Bernoulli vacation and SOS make our model more versatile in real congestion situations. In this investigation, vacation models considered may be helpful for the queuing systems wherein the server may intend to utilise the idle time for rest or doing other tasks. The queuing model developed by incorporating many features simultaneously including

- 1 bulk arrival
- 2 retrial
- 3 unreliable server
- 4 vacation,

makes our results applicable to more versatile and real life congestion situations encountered in a variety of congestion problems ranging from day to day as well as industrial queues such as computer and communication systems, distribution and service sectors, production and manufacturing systems, etc.

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Appendices

A.I Proof of Theorem 1

First we consider the case of linear retrial policy. On solving the differential equation (10), we get the generating function of the stationary queue size distribution given in equation (13a). Then by putting $z = 0$ in equation (13a), we can determine the value of π_0 as given in equation (13b).

Now, for the case of classical retrial policy the differential equation (10) directly provides the result given in equation (14a).

Using, the relation $\pi_0 = \lambda\psi(0)$, the value of π_0 can be easily obtained as given in equation (14b).

Finally, in the case of constant retrial policy, on substituting $\theta = 0$ in equation (10), we have the result as given in equation (15a). Now, in order to obtain the value of π_0 as given in equation (15b), we take limit $z \rightarrow 1$ in equation (15a) and apply L-Hospital rule once.

A.II Proof of Lemma 1

We observe that

$$\psi_n(0, j) = \frac{1}{\lambda + \theta_j} \delta_{j,n}, \quad n \geq 0, j \geq 0 \quad (\text{A.1})$$

where

$$\delta_{j,n} = \begin{cases} 1, & \text{if } n = j \\ 0, & \text{if } n \neq j \end{cases}$$

Then utilising equation (A.1) in equation (16), we obtain the limiting probability $P_{0,j}$ as given in equation (18).

Next, we suppose that a modified service time ends leaving n customers in the orbit. For the customer who receives the next service, we may distinguish two cases according to the origin. For this case, we assume that this customer is a primary one then his FES starts at time (say) $t = 0$. It is observed that the time interval $(t, t + \Delta t)$ contributes to $\psi_n(1, j)$ if

- 1 the FES has not been completed before time t with probability $[1 - G_1(t)]$
- 2 $j - n$ primary customers arrive during $(0, t]$. Thus,

$$\psi_n(1, j) = \frac{\lambda(1 - \delta_{j,n-1}) \sum_{m=1}^{j-n+1} c_m k_{1,j-n+1-m}}{\lambda + \theta_n} + \frac{\theta_n k_{1,j-n+1}}{\lambda + \theta_n} \quad (\text{A.2})$$

Now, using equations (A.2) and (16), we get equation (19).

If the service of the customer proceeds from the orbit, then it can be analysed analogously from the second term in the right hand side of expression (A.2).

Other quantities $\psi_n(2, j)$ and $\psi_n(3, j)$ can be obtained in the similar manner as

$$\psi_n(2, j) = \frac{\lambda p (1 - \delta_{j,n-1}) \sum_{k=1}^{j-n+1} c_k \sum_{m=1}^{j-n+1-k} l_{1,m} k_{2,j-n+1-k-m}}{\lambda + \theta_n} + \frac{\theta_n p \sum_{m=1}^{j-n+1} l_{1,m} k_{2,j-n-m+1}}{\lambda + \theta_n} \quad (\text{A.3})$$

$$\begin{aligned} \psi_n(3, j) = & \frac{\lambda \bar{p} \sigma_1 (1 - \delta_{j,n-1}) \sum_{k=1}^{j-n+1} c_k \sum_{m=1}^{j-n+1-k} l_{1,m} d_{1,j-n+1-k-m}}{\lambda + \theta_n} \\ & + \frac{\lambda p \sigma_2 (1 - \delta_{j,n-1}) \sum_{k=1}^{j-n+1} c_k \sum_{m=1}^{j-n+1-k} l_m d_{2,j-n+1-k-m}}{\lambda + \theta_n} \\ & + \frac{\theta_n \bar{p} \sigma_1 \sum_{m=1}^{j-n+1} l_{1,m} d_{1,j-n+1-m}}{\lambda + \theta_n} + \frac{\theta_n p \sigma_2 \sum_{m=1}^{j-n+1} l_m d_{2,j-n+1-m}}{\lambda + \theta_n} \end{aligned} \quad (\text{A.4})$$

Hence, we obtain equations (20) and (21) by using the results given in above equations (A.3) and (A.4), respectively in equation (16).

A.III Proof of Theorem 2

On multiplying equations (18) to (21) by appropriate powers of z and then summing over j for $j = 0, 1, 2, \dots$, we get equations (22) to (25).

A.IV Proof of Theorem 3

The partial generating functions obtained in Theorem 2 are summed up to give the distribution of the number of customers in the orbit. Then,

$$R(z) = P_0(z) + P_1(z) + P_2(z) + P_3(z) \quad (\text{A.5})$$

After doing some algebraic manipulations, we get result given in equation (26).

A.V Proof of Theorem 4

The stationary distribution of the total number of customers in the system is

$$P_n = P_{0,n} + (1 - \delta_{n,0})(P_{1,n-1} + P_{2,n-1} + P_{3,n-1}), \quad n \geq 0 \quad (\text{A.6})$$

Then the corresponding generating function $Q(z)$ can be easily obtained by multiplying above equation with appropriate powers of z and then summing over n for $n = 0, 1, 2, \dots$, as

$$Q(z) = P_0(z) + z[P_1(z) + P_2(z) + P_3(z)] \quad (\text{A.7})$$

Substituting the values of $P_1(z)$, $P_2(z)$, $P_3(z)$ from equations (23) to (25), respectively in equation (A.7), we obtain the desired results as given in equation (27).