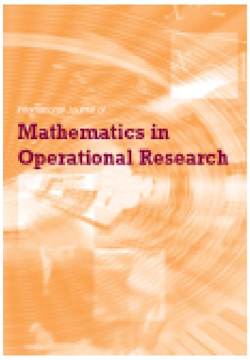

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Optimisation on a constrained integrated supply chain for multiple steel products with investment and freight cost discount

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Abstract: This paper intends to help the reader to understand a new inventory control model for steel products. This model introduces a steel marketing supply chain between a manufacturer and a merchant with multiple steel products. The purpose of this work is to influence a price discount on backorder for the merchant, ordering cost reduction and setup cost reduction, proceeding with the cogitation of discount on freight cost. We hold the freight cost to be a function of the order's quantity, and it is to be in the format of all-unit-discount costs. We thus assimilate freight costs into the system comprehensively and straightforwardly and have worked out the optimal solution procedures for working with the posed inventory model. A numerical example and sensitivity analyses are also carried out to demonstrate the study's implementations and performances.

Keywords: multiple steel products; investment; price discount; freight cost.

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1 Prologue

Roads, trains, other infrastructure, appliances, and buildings are all constructed with iron and steel. Steel skeletons support most big modern constructions, such as stadiums and skyscrapers, bridges, and airports. Steel is used for reinforcement even in concrete structures. It is also widely used in large appliances and automobiles. Despite its increasing popularity, aluminium remains the most common material for vehicle bodywork. Steel is utilised in a wide range of other building materials, including bolts, nails, and screws, as well as various home items and culinary utensils. Shipbuilding, pipelines, mining, offshore construction, aerospace, white goods (e.g., washing machines), heavy equipment like bulldozers, office furniture, steel wool, tool, and armour in the shape of personal vests or vehicle armour are some of the other typical uses.

Nowadays, even steel trading prefers to become an integral part of the supply chain, rather than an entity, to increase the trade's aggressive quantity. In agreement with that, one of the best research topics in the field of supply chain management (SCM) is still the development of joint economic lot size (JELS) models.

In economy, there is a sliding scale of green transport depending on the sustainability of the choice. Green motors are more fuel-efficient, but most effective in evaluation with standard vehicles, and they still contribute to traffic congestion and road crashes. well-patronised open transport networks based on conventional diesel buses utilise much less fuel per traveler than private motors, and are generally safer and use less road space than private vehicles. Green public transport vehicles including electric trains, trams and electric buses combine the advantages of green vehicles with those of sustainable transport choices. Other transport choices in the olden days are cycling and other human-powered vehicles, and animal powered transport which are scream to natural environment. In those days, the foremost common green transport choice, with the least environmental impact is walking. But this could be impossible in the modern world today. Therefor we move on to the electric vehicle technology which exceptionally promote sustainable transportation.

Electric vehicle technology (especially non-battery-based vehicles, fuel cell vehicles, ...) has the potential to diminish transport CO_2 emissions, depending on the encapsulated vitality of the vehicle and the source of the electricity. The essential sources of electricity now utilised in most countries (coal, gas and oil) imply that until world electricity production changes substantially, private electric cars will result in the same or higher production of CO_2 than petrol equivalent vehicles. Battery-based electric vehicles may or may not be better in terms of green house gases (GHGs) emissions then fossil-fuel-based vehicles depending on several factors, such as battery type, capacity of the battery, life hope of the battery and so forth.

Sustainable transport, arising from the concept of sustainable development, aims to provide accessibility for all to help and to meet the basic daily mobility which needs consistent with human and ecosystem health, but to constrain GHG emissions (for example, decoupling mobility from oil dependence and LDV use). An electric truck is an electric vehicle fueled by batteries intended to convey payload. Electric trucks have been around in special application for over a hundred years, however more as of late the approach of lithium particle batteries has empowered the scope of electric trucks to increment for a few hundred miles, making them more extensive consistency and to decrease the GHGs emissions. Cui et al. (2021) proposed a cost model of lead time reduction through the method of piecewise accumulation, and then designed a cost optimisation strategy by the comparison between the lead time reduction cost and the shortage cost. Tharani and Uthayakumar (2020) worked with a stock consignment model by introducing lead time and ordering reduction under two different investments.

In recent years, supply chain strategy has evolved much recognition from researchers (see Lee et al., 2007) in the context of price discounts on backorder and ordering cost reduction. In general, though, the literature of this kind is limited to single echelons. Eydi and Bakhtiari (2017) developed a multi-product model for evaluating and choosing two layers of suppliers incorporating environmental factors. An integrated inventory system of ordering cost reduction and permissible delay in payments was analysed by Huang (2010). Esmaeili et al. (2018) built a two-echelon supply chain with incomplete supplier service-level data under the (R, Q) policy. Pourmehdi et al. (2020) had developed a multi-objective linear mathematical model under uncertainty to optimise a steel sustainable closed-loop supply chain. Hoque (2021) developed a single-manufacturer multi-retailer integrated model by extending the ideas of two single-vendor single-buyer integrated models, where Normal distribution of lead times of delivering equal- and/or unequal-sized batches of a lot.

On an optimised seller-buyer supply chain system, Sarkar and Majumder (2013) dealt with the setup cost reduction of the vendor. A stochastic demand inventory model with some preservation methods for deteriorating items was made by Soni and Chauhan (2018). Sarkar et al. (2015) had explored manufacturing system with setup cost reduction, quality improvement under free distribution approach, and service level constraint. Mohanty et al. (2018) had done work on an imperfect production-inventory system with trade credit and variable setup cost. Uthayakumar and Kumar (2018) contributed work on a constrained integrated multi-product inventory model on a mixture of distribution. Uthayakumar and Tharani (2017) collaborated with different manufacturing setups on rework.

Lin et al. (2012) proposed a new system of inventory management for fresh agricultural products and investigated the impact of inflation and payment delays on the inventory model's Weibull distribution. Xie and Zhang (2011) developed a fresh agricultural product inventory management model based on the expected reduction in inventory costs. Acosta et al. (2018) had provided a quantitative model for resource management in agricultural supply chains that takes perishability into account. Lesmono et al. (2020) executed a multi-item perishable inventory model with deterministic demands, return and all-units discount.

Later, Kang and Kim (2010) developed an inventory model with various transportation rate functions. Priyan and Uthayakumar (2014) addressed a vendor-buyer inventory system with ordering cost reduction, transport cost, and price discount on backorder with trade credit and uncertain quantity. Kaviyani-Charati and Kargar (2021) had dealt two real case studies along with their optimisation models which are provided to show how the demand rate changes over time. Moghaddam et al. (2019) had designed a mathematical model for the reverse supply chain of perishable goods, taking into account the sustainable production system.

2 Catalyst to the research

Many manufacturers and merchants would be fond of generating a long-term collaborative agreement and getting a balanced contentedness point of supply-demand to minimise the cost function and enhance overall quality in this aggressive trading market of prevailing assets. Thus, nowadays, both manufacturers and merchants have a combined attempt between the commodities for achievements.

In existing days, merchants wish to stay back for backorders for products. Accordingly, stimulating the merchant to stay back for backorders is valuable to the manufacturer's organisation. Many consumers give a discounted price upon this stockout product to save backorders from piling up in today's world. This overture will pique the merchant's interest in the desirable products remaining in the store. Today by granting price discounts, many authorities produce high customer reliability.

In many developing countries, because of the insufficient resources of the prospering population and automobiles, the freight expense for production process concerns to move on the goods at the right time is very abundant. Many merchants/retailers have also received their goods to minimise freight costs from nearby manufacturers/suppliers. Nowadays, thus, the cost of freight plays a critical role in the overall functional cost. Today, the rate of freight is disturbed by the transmitting opinions and the desired size of shipping. As a result, manufacturers/suppliers reverse discounts for the freight cost of plentiful ordered commodities in prosperous countries to entice the customer. Because of the uncertain circumstances and inflation, inventory costs grow as a significant part of the production authority. Therefore, certain commodities pay out additional funds to equip the employees in order to curtail inventory costs. In that manner, small-scale authorities in India pay out an extra added fund by way of an expenditure, resulting in a decrease in the inventory model's overall total cost.

When the manufacturers fail to supply the items ordered by the merchants, as well as when the merchants open a price compromise on stockout products to guarantee higher backorders and expend more time in order to mitigate the ordering expense to promote consumer competition, the concessions are reversed for the freight cost of abundant commodities.

3 Notations and assumptions

3.1 Notations

We require the following notations and assumptions to establish the mathematical configuration for the present system. When there is some requirement, additional notations and assumptions will be added.

3.1.1 Decision variables

A_{Bi}	Ordering cost of the merchant per order acquired by the merchant for the product i (rupees/product).
A_{Vi}	Setup cost of the manufacturer per setup acquired by the manufacturer for the product i (rupees/product).
π_{xi}	Merchant's price discount on backorder to the customer for the i^{th} product.
Q_i	Merchant's order quantity of the product i (kg).
n	Number of lots for all products which are all shipped in one manufacturing cycle from the manufacturer to the merchant.
L	Lead time for delivering a lot from the manufacturer to the merchant for all products.

3.1.2 Parameters

D_i	Merchant's average rate of demand for the i^{th} product.
P_i	Manufacturer's production rate for the i^{th} product.
m	Number of products.
A_{B0i}	Original ordering cost for the i^{th} product.
A_{V0i}	Original setup cost for the i^{th} product.
h_{Vi}	Manufacturer's stock holding cost for the i^{th} product per unit per unit time.
h_{Bi}	Merchant's stock holding cost for the i^{th} product per unit per unit time.
r_i	Merchant's reorder point of the i^{th} product.
τ_i	Merchant's compartmental annual oppurtunity cost of the capital.
α_i	Manufacturer's compartmental annual oppurtunity cost of the capital.
π_{0i}	Marginal profit for the i^{th} product (i.e., cost of lost demand).
β_i	Fraction of the shortage that will be backordered at the merchant's end, $0 \leq \beta_i < 1$.
β_{0i}	Upper bound of the backorder ratio, $0 \leq \beta_{0i} < 1$.

3.1.3 Random variables

X_i Lead time demand rate of the merchant for the i^{th} product.

3.1.4 Functions and operators

$E(\cdot)$ Mathematical expectation.

x^+ Maximum value of x and 0 (i.e., $x^+ = \max\{x, 0\}$).

$f(x_i)$ Probability density function of X_i with finite mean D_iL and standard deviation.

$\sigma_i\sqrt{L}$ Where σ_i denotes the standard deviation of the demand per unit time.

$R(L)$ Lead time crashing cost per cycle.

3.2 Assumptions

Some of the strategic characteristics become fundamental and the following hypothesis need to be considered:

1. The framework deals with multiple steel products with the cooperation of a manufacturer and a merchant. In different corporate entities, the manufacturer and the merchant are excited about providing a collaborative inventory system. Therefore, all members intend to minimise the integrated expected total cost (ETC) in the joint strategy. The merchant prefers a continuous review inventory strategy, and the order is retained once the inventory volume exceeds the reorder point r_i.
2. The production rate of the manufacturer and the merchant's demand rate are constant.
3. The merchant pays sustainable freight cost which depends on the merchant's order quantity.
4. The reorder level of the i^{th} product is $r_i =$ (expected demand during lead time of the i^{th} product $= D_iL$)+ (safety stock of the i^{th} product $= k_i\sigma_i\sqrt{L}$) where k_i is the safety factor for the i^{th} product.
5. The rate of production should be greater than the rate of demand.
6. Shortages and backorders are considered.
7. The time horizon is infinite.
8. L lead time consists of N components that are mutually independent. The j^{th} component has a normal a_j duration and c_l crashing cost per unit time. We rearrange c_l for convenience, so that $c_1 < c_2 < ... < c_N$. Starting from the first component, the lead time components are crashed one at a time so they have the lowest unit crashing cost, and then the second component, and so on. Let $L_0 = \sum_{j=1}^{N} b_j$, and L_j be the length of lead time with components $1, 2, ..., j$ crashed to their minimum duration, then L_j can be expressed as

$L_j = L_0 - \sum_{l=1}^{i}(b_l - a_l)$, $i = 1, 2, ..., m$ and the lead time crashing cost $R(L)$ per cycle is given by $R(L) = c_j(L_{j-1} - L) + \sum_{l=1}^{j-1} c_l(b_l - a_l)$, $L \in [L_j, L_{j-1}]$.

9 $I(A_{Bi})$ is an investment for the merchant's ordering cost reduction and it can be expressed as follows:

$$I(A_{Bi}) = \frac{1}{\delta_i} \ln\left(\frac{A_{B0i}}{A_{Bi}}\right) \qquad 0 < A_{Bi} \leq A_{B0i} \tag{1}$$

where δ represents the decrease in the percentage of A_{Bi} per dollar increment in $I(A_{Bi})$.

10 The capital investment $I(A_{Vi})$, in reducing manufacturer's setup cost is logarithmic function of the setup cost A_{Vi}. That is,

$$I(A_{Vi}) = \frac{1}{\gamma_i} \ln\left(\frac{A_{V0i}}{A_{Vi}}\right) \qquad 0 < A_{Vi} \leq A_{V0i} \tag{2}$$

11 The merchant endeavours price discount to his customers as a deficit, precisely, the backorder ratio, β_i is variable during the stockout time and in proposition to the price discount endeavoured by the merchant per unit, π_{xi}, that is,

$$\beta_i = \frac{\beta_{0i}\pi_{xi}}{\pi_{0i}} \tag{3}$$

where $0 \leq \beta_{0i} < 1$ and $0 \leq \pi_{xi} \leq \pi_{0i}$.

4 Manufacturer-merchant inventory model without freight cost

In an integrated manufacturer-merchant model for multiple steel products, we work with the single-setup-multiple-delivery (SSMD) policy, i.e., if the merchant orders Q_i quantity for i^{th} product of non-defective products, then the manufacturer produces nQ_i quantity where n is any positive integer, and Q_i quantity is transferred n times to the merchant. The manufacturer produces the nQ_i quantity in one manufacturing period. The manufacturer's setup cost per setup for the i^{th} product is A_{Vi}. For the i^{th} product, the manufacturer's production rate per unit time is Pi. As the demand rate is D_i, the utilisation time of the produced quantity Q_i is Q_i/D_i. Thus, the time for utilising all nQ_i products is nQ_i/D_i. Furthermore, we have the integrated inventory model with ordering and setup cost reductions and shortages under partial backlog. As the inventory model follows (s, S) policy, the fraction β_i of shortages is back-ordered in the next replenishment.

The lead time demand X_i for i^{th} product follows normal distribution with mean D_iL and standard deviation $\sigma_i\sqrt{L}$. The expected shortages at the end of the cycle for i^{th} product is $E(X_i - r_i)^+$ where r_i is the reorder point. As β_i is backorder ratio, the expected amount of the backorder per cycle is $\beta_iE(X_i - r_i)^+$ and the lost sale is $(1 - \beta_i)E(X_i - r_i)^+$. The expected inventory level before receipt of an order is $r_i - D_iL + (1 - \beta_i)E(X_i - r_i)^+$. The expected inventory level after the delivery of quantity Q_i is $Q_i + (r_i - D_iL) + (1 - \beta_i)E(X_i - r_i)^+$. Thus the average inventory level at any time in the cycle is $\frac{Q_i}{2} + r_i - D_iL + (1 - \beta_i)E(X_i - r_i)^+$.

From the manufacturer to the merchant, the freight cost per unit is irrespective of the quantity ordered. Therefore, the freight cost is ignored in this section of the ETC for simplification. The joint expected total cost (JETC) is proportional to the manufacturer's total expected cost, and the merchant's total expected cost.

4.1 Manufacturer's ETC

The expected total setup cost per unit time of the manufacturer for i^{th} product is formulated as

$$\alpha_i I(A_{Vi}) + \frac{A_{Vi}D_i}{nQ_i} \tag{4}$$

Subject to the constraint

$$0 < A_{Vi} \leq A_{V0i}$$

The total holding cost per unit time of the manufacturer for i^{th} product can be jotted as

$$\frac{h_{Vi}Q_i}{2}\left[n\left(1-\frac{D_i}{P_i}\right)-1+\frac{2D_i}{P_i}\right] \tag{5}$$

We also considered the same ETC per unit time for the manufacturer, as in Sarkar and Majumder (2013). Thus, for all m products, the ETC of the manufacturer is the sum of the setup cost and holding cost. Now we have the total expected cost for the manufacturer to be represented as

$$ETC_V = \sum_{i=1}^{m}\left(\alpha_i I(A_{Vi}) + \frac{A_{Vi}D_i}{nQ_i} + \frac{h_{Vi}Q_i}{2}\left[n\left(1-\frac{D_i}{P_i}\right)-1+\frac{2D_i}{P_i}\right]\right) \tag{6}$$

Subject to the constraint

$$0 < A_{Vi} \leq A_{V0i}$$

4.2 Merchant's ETC

Merchant's expected total ordering cost per unit time for i^{th} product can be carved as

$$\tau_i I(A_{Bi}) + \frac{A_{Bi}D_i}{Q_i} \tag{7}$$

Subject to the constraint

$$0 < A_{Bi} \leq A_{B0i}$$

The expected holding cost of the merchant per unit time for i^{th} product is given as

$$h_{Bi}\left[\frac{Q_i}{2}+r_i-D_iL\right]+h_{Bi}\left[1-\frac{\beta_{0i}\pi_{xi}}{\pi_{0i}}\right] \tag{8}$$

For all m products, the ETC of the merchant per unit time is equal to the sum of ordering cost, holding cost, shortage cost, and crashing cost of lead time. Therefore, for the merchant, the ETC is expressed as

$$ETC_B = \sum_{i=1}^{m} \Bigg(\tau_i I(A_{Bi}) + \frac{A_{Bi} D_i}{Q_i} + h_{Bi} \left[\frac{Q_i}{2} + r_i - D_i L \right] + h_{Bi} \left[1 - \frac{\beta_{0i}\pi_{xi}}{\pi_{0i}} \right] E(X_i - r_i)^+ + \frac{D_i}{Q_i} \left[\frac{\beta_{0i}\pi_{xi}^2}{\pi_{0i}} + \pi_{0i} - \beta_{0i}\pi_{xi} \right] E(X_i - r_i)^+ + \frac{D_i}{Q_i} R(L) \Bigg) \tag{9}$$

Subject to the constraints

$$0 < A_{Bi} \leq A_{B0i}$$
$$0 < \pi_{xi} \leq \pi_{0i}$$

where $E(X_i - r_i)^+ = \int_{r_i}^{\infty} (x_i - r_i) \mathrm{d}F(x_i)$

4.3 JETC

Thus the JETC per unit time for all m products is exhibited as

$$JETC = \sum_{i=1}^{m} \Bigg(\alpha_i I(A_{Vi}) + \frac{A_{Vi} D_i}{nQ_i} + \frac{h_{Vi} Q_i}{2} \left[n \left(1 - \frac{D_i}{P_i} \right) - 1 + \frac{2D_i}{P_i} \right] \tau_i I(A_{Bi}) + \frac{A_{Bi} D_i}{Q_i} + h_{Bi} \left[\frac{Q_i}{2} + r_i - D_i L \right] + h_{Bi} \left[1 - \frac{\beta_{0i}\pi_{xi}}{\pi_{0i}} \right] E(X_i - r_i)^+ + \frac{D_i}{Q_i} \left[\frac{\beta_{0i}\pi_{xi}^2}{\pi_{0i}} + \pi_{0i} - \beta_{0i}\pi_{xi} \right] E(X_i - r_i)^+ + \frac{D_i}{Q_i} R(L) \Bigg) \tag{10}$$

Subject to the constraints

$$0 < A_{Vi} \leq A_{V0i}$$
$$0 < A_{Bi} \leq A_{B0i}$$
$$0 < \pi_{xi} \leq \pi_{0i}$$

Therefore, the following nonlinear programming model can be used to reduce setup costs, price discount on backorder, purchasing cost reduction without incurring freight costs:

$$\min JETC = \sum_{i=1}^{m} \Bigg(\alpha_i I(A_{Vi}) + \frac{A_{Vi} D_i}{nQ_i} + \frac{h_{Vi} Q_i}{2} \left[n \left(1 - \frac{D_i}{P_i} \right) - 1 + \frac{2D_i}{P_i} \right] \tau_i I(A_{Bi}) + \frac{A_{Bi} D_i}{Q_i} + h_{Bi} \left[\frac{Q_i}{2} + r_i - D_i L \right]$$

$$+ h_{Bi}\left[1 - \frac{\beta_{0i}\pi_{xi}}{\pi_{0i}}\right] E(X_i - r_i)^+ + \frac{D_i}{Q_i}\left[\frac{\beta_{0i}\pi_{xi}^2}{\pi_{0i}}\right.$$
$$\left. + \ \pi_{0i} - \beta_{0i}\pi_{xi}\right] E(X_i - r_i)^+ + \frac{D_i}{Q_i}R(L)\Bigg) \tag{11}$$

Subject to the constraint

$$0 < A_{Vi} \leq A_{V0i}$$
$$0 < A_{Bi} \leq A_{B0i}$$
$$0 < \pi_{xi} \leq \pi_{0i}$$

4.4 Solution procedure

A constrained nonlinear programming problem is the problem framed in the previous section. We briefly relax the constraints and ignore the integer n to solve the restricted problem. For fixed Q_i, A_{Bi}, A_{Vi}, π_{xi}, L and n the $JETC$ can be verified to be a convex function of n with an optimal value n^* through the following conditions:

$$\begin{cases} JETC(Q_i, A_{Vi}, A_{Bi}, \pi_{xi}, L, n^*) \leq JETC(Q_i, A_{Vi}, A_{Bi}, \pi_{xi}, L, n^* - 1) \\ JETC(Q_i, A_{Vi}, A_{Bi}, \pi_{xi}, L, n^*) \leq JETC(Q_i, A_{Vi}, A_{Bi}, \pi_{xi}, L, n^* + 1) \end{cases}$$

For a fixed positive integer n, we take the partial derivatives of $JETC$ for i^{th} product with respect to Q_i, A_{Bi}, A_{Vi}, π_{xi} and L to acquire optimal solution.

$$\frac{\partial JETC}{\partial Q_i} = -\frac{A_{Bi}D_i}{Q_i^2} - \frac{A_{Vi}D_i}{nQ_i^2} + \frac{h_{Bi}}{2} - \frac{D_i}{Q_i^2}\left[\frac{\beta_{0i}\pi_{xi}^2}{\pi_{0i}} + \pi_{0i}\right.$$
$$\left. - \ \beta_{0i}\pi_{xi}\right]\sigma_i\sqrt{L}\Psi_i(k_i) - \frac{D_i}{Q_i^2}R(L) + \frac{h_{Vi}}{2}\left[n\left(1 - \frac{D_i}{P_i}\right)\right.$$
$$\left. - \ 1 + \frac{2D_i}{P_i}\right] \tag{12}$$

$$\frac{\partial JETC}{\partial A_{Bi}} = -\frac{\tau_i}{\delta_i A_{Bi}} + \frac{D_i}{Q_i} \tag{13}$$

$$\frac{\partial JETC}{\partial A_{Vi}} = -\frac{\alpha_i}{\gamma_i A_{Bi}} + \frac{D_i}{nQ_i} \tag{14}$$

$$\frac{\partial JETC}{\partial \pi_{xi}} = -\frac{h_{Bi}\beta_{0i}}{\pi_{0i}}\sigma_i\sqrt{L}\Psi_i(k_i) + \left[\frac{2D_i\beta_{0i}\pi_{xi}}{Q_i\pi_{0i}} - \frac{D_i\beta_{0i}}{Q_i}\right]\sigma_i\sqrt{L}\Psi_i(k_i) \tag{15}$$

$$\frac{\partial JETC}{\partial L} = \frac{h_{Bi}k_i\sigma_i}{2\sqrt{L}} + \frac{h_{Bi}}{2\sqrt{L}}\left(1 - \frac{\beta_{0i}\pi_{xi}}{\pi_{0i}}\right)\sigma_i\Psi_i(k_i)$$
$$+ \frac{D_i}{2\sqrt{L}Q_i}\left[\frac{\beta_{0i}\pi_{xi}^2}{\pi_{0i}} + \pi_{0i} - \beta_{0i}\pi_{xi}\right]\sigma_i\Psi_i(k_i) - \frac{D_i}{Q_i}c_j \tag{16}$$

By analysing the second order sufficient conditions (SOSC) for a minimum value, it can be easily confirmed that $JETC$ is not a convex function of (Q_i, A_{Bi}, A_{Vi}, π_{xi}, L, n).

However, for fixed Q_i, A_{Bi}, A_{Vi}, π_{xi} and n, $JETC$ is noted to be a concave function in $L \in [L_j, L_{j-1}]$, because

$$\frac{\partial^2 JETC}{\partial L^2} = -\frac{3h_{Bi}k_i\sigma_i}{4L\sqrt{L}} - \frac{3h_{Bi}}{4L\sqrt{L}}\left(1 - \frac{\beta_{0i}\pi_{xi}}{\pi_{0i}}\right)\sigma_i\Psi_i(k_i) - \frac{3D_i}{4L\sqrt{L}Q_i}\left[\frac{\beta_{0i}\pi_{xi}^2}{\pi_{0i}} + \pi_{0i} - \beta_{0i}\pi_{xi}\right]\sigma_i\Psi_i(k_i) < 0 \tag{17}$$

Therefore, for fixed (Q_i, A_{Bi}, A_{Vi}, π_{xi}), at the interval's end points, the minimum JETC per unit time exists $L \in [L_j, L_{j-1}]$. Conversely, for fixed n and $L \in [L_j, L_{j-1}]$, $JETC$ can be proved to be a convex function of (Q_i, A_{Bi}, A_{Vi}, π_{xi}). Thus, for fixed n and $L \in [L_j, L_{j-1}]$, the minimum value of $JETC$ occurs at the point (Q_i, A_{Bi}, A_{Vi}, π_{xi}) which satisfies $\frac{\partial JETC}{\partial Q_i} = 0$, $\frac{\partial JETC}{\partial A_{Bi}}$, $\frac{\partial JETC}{\partial A_{Vi}}$ and $\frac{\partial JETC}{\partial \pi_{xi}}$ simultaneously. Solving these three equations, yields

$$Q_i = \sqrt{\frac{2D_i[A_{Bi} + \frac{1}{n}A_{Vi} + (\beta_{0i}\pi_{xi}^2 + \pi_{0i} - \beta_{0i}\pi_{xi})\sigma_i\sqrt{L}\Psi_i(k_i) + R(L)]}{h_{Bi} + h_{Vi}\left[n\left(1 - \frac{D_i}{P_i}\right) - 1 + \frac{2D_i}{P_i}\right]}} \tag{18}$$

$$A_{Bi} = \frac{\tau_i Q_i}{\delta_i D_i} \tag{19}$$

$$A_{Vi} = \frac{n\alpha_i Q_i}{\gamma_i D_i} \tag{20}$$

$$\pi_{xi} = \frac{Q_i\pi_{0i}}{2D_i}\left[\frac{D_i}{Q_i} + \frac{h_{bi}}{\pi_{0i}}\right] \tag{21}$$

Theorem 4.1: If we denote the optimum values as Q_i^*, A_{Bi}^*, A_{Vi}^*, and π_{xi}^*, then for fixed n and $L \in [L_j, L_{j-1}]$, the JETC function $JETC$ has a global minimum at (Q_i^*, A_{Bi}^*, A_{Vi}^*, π_{xi}^*) acquired from equations (19), (20), (21) and (21) with the relaxed constraints such as $0 < A_{Vi} \leq A_{V0i}$, $0 < A_{Bi} \leq A_{B0i}$ and $0 < \pi_{xi} \leq \pi_{0i}$ and also provided the following condition which is to be satisfied

$$\frac{D_i^2\alpha_i}{Q_i^4\gamma_i A_{Vi}^2} > \frac{2D_i}{Q_i^3}\left[A_{Bi} + \frac{1}{n}A_{Vi} + (\beta_{0i}\pi_{xi}^2 + \pi_{0i} - \beta_{0i}\pi_{xi})\sigma_i\sqrt{L}\Psi_i(k_i) + R(L)\right]\frac{\tau_i\alpha_i}{\delta_i\gamma_i A_{Bi}^2 A_{Vi}^2} + \frac{D_i^2\tau_i}{n^2 Q_i^4\delta_i A_{Bi}^2} \tag{22}$$

Proof: See Appendix. □

Now we consider the constraints $0 < A_{Vi} \leq A_{V0i}$, $0 < A_{Bi} \leq A_{B0i}$ and $0 < \pi_{xi} \leq \pi_{0i}$. From equations (19), (20), (21) and (21) we note that A_{Bi}, A_{Vi}, π_{xi} are positive as the parameters Q_i, D_i, P_i, β_{0i}, π_{0i}, σ_i, $\Psi_i(k_i)$, $R(L)$, h_{Bi}, h_{Vi}, τ_i, γ_i, α_i, δ_i are positive. Moreover, if $A_{Vi}^* < A_{V0i}$, $A_{Bi}^* < A_{B0i}$, and $\pi_{xi}^* < \pi_{0i}$ then (Q_i^*, A_{Bi}^*, A_{Vi}^*, π_{xi}^*) is an interior optimal for given $L \in [L_j, L_{j-1}]$. However, if $A_{Vi}^* < A_{V0i}$, $A_{Bi}^* < A_{B0i}$, and $\pi_{xi}^* \geq \pi_{0i}$ consequently the merchant shall decide against introducing the price discount on backorder, as he generates opinions to acquire benefits. Therefore, if the price discount on backorder, π_{xi} is greater than the marginal benefit π_{0i} then the merchant may decide against introducing the price discount on backorder. That is, $\pi_{xi}^* = \pi_{0i}$.

On the other hand,if $A_{Vi}^* \geq A_{V0i}$, $A_{Bi}^* < A_{B0i}$, and $\pi_{xi}^* < \pi_{0i}$ then investing in reducing setup costs is impractical; in this case, the optimal $A_{Vi}^* = A_{V0i}$. And also if $A_{Vi}^* < A_{V0i}$, $A_{Bi}^* \geq A_{B0i}$, and $\pi_{xi}^* < \pi_{0i}$ it is then difficult to produce any investment to modify the existing ordering cost level; in this situation, the optimal $A_{Bi}^* = A_{B0i}$.

On the other hand, if $A_{Vi}^* \geq A_{V0i}$, $A_{Bi}^* < A_{B0i}$, and $\pi_{xi}^* \geq \pi_{0i}$ then investing in reducing setup costs is also impractical; in this case, the optimal $A_{Vi}^* = A_{V0i}$ and since the merchant creates opinions in order to receive benefits which means $\pi_{xi}^* = \pi_{0i}$, the merchant can decide not to overturn the price discount on backorder. And also if $A_{Vi}^* < A_{V0i}$, $A_{Bi}^* \geq A_{B0i}$, and $\pi_{xi}^* \geq \pi_{0i}$ it is then impossible to produce some investment to modify the present level of ordering costs; in this case, the optimal $A_{Bi}^* = A_{B0i}$ and since the merchant creates opinions in order to receive benefits which means $\pi_{xi}^* = \pi_{0i}$, the merchant can decide not to overturn the price discount on backorder. If $A_{Vi}^* \geq A_{V0i}$, $A_{Bi}^* \geq A_{B0i}$, and $\pi_{xi}^* < \pi_{0i}$ it is then impossible to produce some investment in reducing setup cost then; in this situation, the optimal $A_{Vi}^* = A_{V0i}$ and difficult to generate any investment to modify the present level of ordering costs $A_{Bi}^* = A_{B0i}$.

On the other hand, if $A_{Vi}^* \geq A_{V0i}$, $A_{Bi}^* \geq A_{B0i}$, and $\pi_{xi}^* \geq \pi_{0i}$ then we should not make any investments to adjust the new buying and configuration costs, and we should also avoid overturning the backorder price discount. The ideal ordering cost and setup cost for this special case are, respectively, the original ordering cost and setup cost, namely, $A_{Bi}^* = A_{B0i}$ and $A_{Vi}^* = A_{V0i}$ and the optimal price discount on backorder is $\pi_{xi}^* = \pi_{0i}$.

4.5 Algorithm

The following algorithm is created to detect the optimum values depending on the objective function's convexity behaviour concerning the decision variables.

Step 1 For each $L \in [L_j, L_{j-1}]$, determine the $JETC(Q_i^*, A_{Bi}^*, A_{Vi}^*, \pi_{xi}^*)$ by proceeding the following steps.

Step 2 Set $n = 1$.

Step 3 Determine the values $\hat{Q}_i$, $\hat{A}_{Bi}$, $\hat{A}_{Vi}$, $\hat{\pi}_{xi}$ and the corresponding total cost $JETC(\hat{Q}_i, \hat{A}_{Bi}, \hat{A}_{Vi}, \hat{\pi}_{xi})$ by using the sub-algorithm (Subsection 4.5.1).

Step 4 Compute $\min_{j=1,2,\ldots,N} JETC(\hat{Q}_i, \hat{A}_{Bi}, \hat{A}_{Vi}, \hat{\pi}_{xi})$. If $JETC(Q_i^*, A_{Bi}^*, A_{Vi}^*, \pi_{xi}^*) = \min_{j=1,2,\ldots,N} JETC(\hat{Q}_i, \hat{A}_{Bi}, \hat{A}_{Vi}, \hat{\pi}_{xi})$ then set $JETC(Q_i^*, A_{Bi}^*, A_{Vi}^*, \pi_{xi}^*)$ as the optimal solution for fixed n.

Step 5 Raise the value of n by $n + 1$, and do the steps 2 and 3 again to gain the optimal solution $JETC(Q_i^*, A_{Bi}^*, A_{Vi}^*, \pi_{xi}^*)$.

Step 6 If the previous $JETC$ for $n - 1$ is greater than the current $JETC$ for n then proceed to step 4 otherwise move on to step 6.

Step 7 Now the $JETC$ for $n - 1$ is optimal solution for the presented model.

4.5.1 Sub-algorithm

Step 1 Reiterate the steps from steps 1.1 to 1.3 in the sub-algorithm until no change emerges in the values of Q_i A_{Bi}, A_{Vi} and π_{xi}. Represent the solution by $\tilde{Q}_i$ $\tilde{A}_{Bi}$, $\tilde{A}_{Vi}$ and $\tilde{\pi}_{xi}$.

Step 1.1 Initiate with $A_{Vi} = A_{V0i}$, $A_{Bi} = A_{B0i}$ and $\pi_{xi} = \pi_{0i}$.

Step 1.2 Now with these values compute Q_i in the equation (19).

Step 1.3 Find the values of A_{Vi}, A_{Bi} and π_{xi} by substituting the value of Q_i in the equations (20), (21) and (21).

Step 2 Compare A_{Vi}, A_{Bi} and π_{xi} with A_{V0i}, A_{B0i} and π_{0i} respectively.

1 If $\tilde{A}_{Vi} < A_{V0i}$, $\tilde{A}_{Bi} < A_{B0i}$, and $\tilde{\pi}_{xi} < \pi_{0i}$ then the solution got through step 1 is optimal for the given L. Then compute $JETC$ by following step 4.

2 If $\tilde{A}_{Vi} \geq A_{V0i}$, $\tilde{A}_{Bi} < A_{B0i}$, and $\tilde{\pi}_{xi} < \pi_{0i}$ then keep $\tilde{A}_{Vi} = A_{V0i}$ for finding the optimal values for the given L. Then compute $JETC$ by following step 4. Otherwise follow step 3.

3 If $\tilde{A}_{Vi} < A_{V0i}$, $\tilde{A}_{Bi} \geq A_{B0i}$, and $\tilde{\pi}_{xi} < \pi_{0i}$ then keep $\tilde{A}_{Bi} = A_{B0i}$ for finding the optimal values for the given L. Then compute $JETC$ by following step 4. Otherwise follow step 3.

4 If $\tilde{A}_{Vi} < A_{V0i}$, $\tilde{A}_{Bi} < A_{B0i}$, and $\tilde{\pi}_{xi} \geq \pi_{0i}$ then keep $\tilde{\pi}_{xi} = \pi_{0i}$ for finding the optimal values for the given L. Then compute $JETC$ by following step 4. Otherwise follow step 3.

5 If $\tilde{A}_{Vi} \geq A_{V0i}$, $\tilde{A}_{Bi} \geq A_{B0i}$, and $\tilde{\pi}_{xi} < \pi_{0i}$ then keep $\tilde{A}_{Vi} = A_{V0i}$ and $\tilde{A}_{Bi} = A_{B0i}$ for finding the optimal values for the given L. Then compute $JETC$ by following step 4. Otherwise follow step 3.

6 If $\tilde{A}_{Vi} \geq A_{V0i}$, $\tilde{A}_{Bi} < A_{B0i}$, and $\tilde{\pi}_{xi} \geq \pi_{0i}$ then keep $\tilde{A}_{Vi} = A_{V0i}$ and $\tilde{\pi}_{xi} = \pi_{0i}$ for finding the optimal values for the given L. Then compute $JETC$ by following step 4. Otherwise follow step 3.

7 If $\tilde{A}_{Vi} < A_{V0i}$, $\tilde{A}_{Bi} \geq A_{B0i}$, and $\tilde{\pi}_{xi} \geq \pi_{0i}$ then keep $\tilde{A}_{Bi} = A_{B0i}$ and $\tilde{\pi}_{xi} = \pi_{0i}$ for finding the optimal values for the given L. Then compute $JETC$ by following step 4. Otherwise follow step 3.

8 If $\tilde{A}_{Vi} \geq A_{V0i}$, $\tilde{A}_{Bi} \geq A_{B0i}$, and $\tilde{\pi}_{xi} \geq \pi_{0i}$ then keep $\tilde{A}_{Vi} = A_{V0i}$, $\tilde{A}_{Bi} = A_{B0i}$ and $\tilde{\pi}_{xi} = \pi_{0i}$ for finding the optimal values for the given L. Then compute $JETC$ by following step 4. Otherwise follow step 3.

4.6 Numerical analysis

In order to help our model verification, we implemented the more realistic numerical example here. Using the MATLAB programming program, the solution for the given example is acquired. Consider a system of inventory with parameters equivalent to

Uthayakumar and Kumar (2018), and also we have considered the safety factor k = 0.845 for all the three products.

Table 1 Merchant's parameters

i^{th} *product*	D_i *kg/product*	A_{Bi} *rupees/product*	h_{Bi} *rupees/product*	π_{0i} *rupees/product*	σ_i	τ_i *rupees/product*	δ_i	β_{0i}
1	600	200	25	150	7	0.1	0.003	0.2
2	1,000	300	35	250	8	0.2	0.005	0.7
3	800	250	30	200	7.5	0.15	0.004	0.5

Table 2 Manufacturer's parameters

i^{th} *product*	P_i *kg/product*	A_{Vi} *rupees/product*	h_{Vi} *rupees/product*	α_i *rupees/product*	γ_i
1	2,000	1,500	20	0.1	1/18,000
2	2,500	2,500	30	0.2	1/16,000
3	2,300	2,000	25	0.15	1/17,000

Table 3 Lead time data

Component – j	*Duration (days)*		*Unit crashing cost (rupees/day)*
	Normal – b_j	*Minimum – a_j*	
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Table 4 Summarised lead time data

Lead time (week)	$R(L)$
8	0
6	5.6
4	22.4
3	57.4

Table 5 Optimal values without ordering and setup cost reduction and price discounts (see online version for colours)

L	n	Q	$JETC_i$ *(i = 1, 2, 3)*	$JETC$
8	1	[257 341 303]	[9,189 18,772 13,625]	41,586
	2	[154 214 186]	[8,922 17,842 13,107]	39,870
	3	[115 164 141]	[9,231 18,262 13,499]	40,993
6	1	[254 337 300]	[9,043 18,469 13,409]	40,921
	2	[152 210 183]	[8,731 17,435 12,818]	38,984
	3	[113 160 138]	[9,003 17,771 13,151]	39,925
4	1	[252 333 297]	[8,892 18,135 13,176]	40,203
	2	[149 206 179]	[8,537 16,986 12,508]	38,031
	3	[111 156 135]	[8,772 17,230 12,779]	38,780
3	1	[252 332 296]	[8,856 18,002 13,097]	39,955
	2	[150 205 179]	[8,505 16,819 12,417]	37,741
	3	[111 155 134]	[8,744 17,033 12,676]	38,453

Figure 1 Convex graph on optimal values with investments and price discount (see online version for colours)

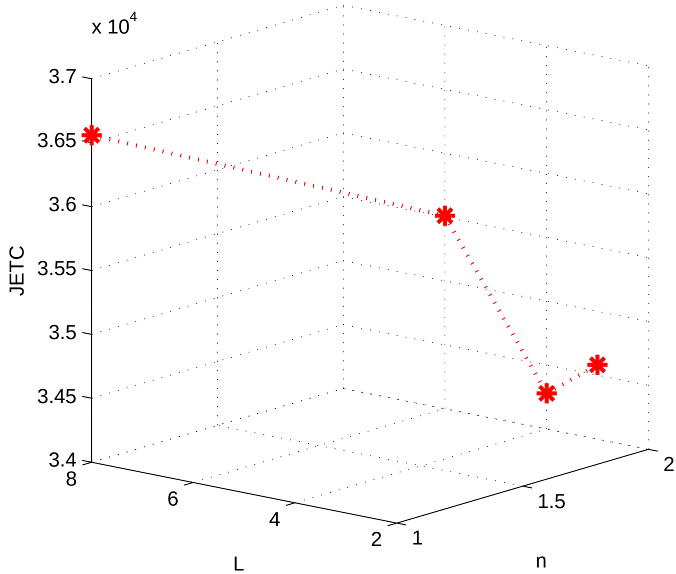

Figure 2 Sensitivity on the parameters such as D_i, A_{Bi}, A_{Vi}, π_{xi} (see online version for colours)

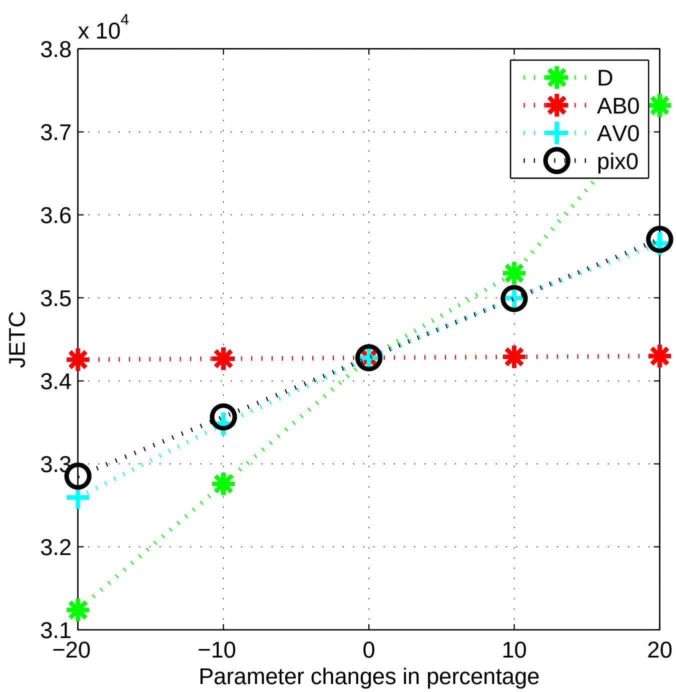

Figure 3 Sensitivity on the parameters such as D_i, A_{Bi}, A_{Vi}, π_{xi} (see online version for colours)

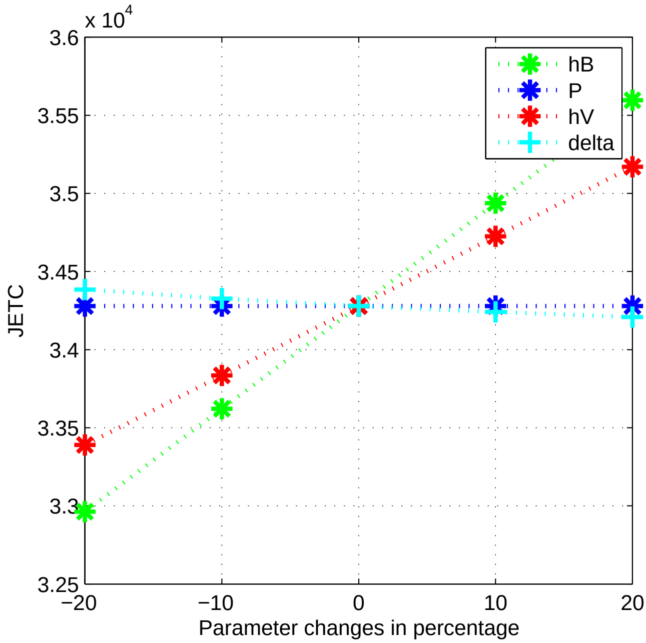

Figure 4 Sensitivity on the parameters such as D_i, A_{Bi}, A_{Vi}, π_{xi} (see online version for colours)

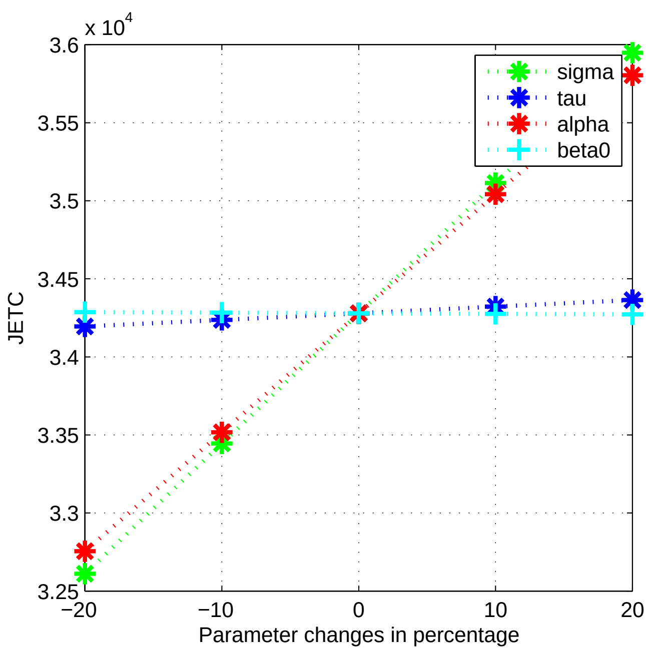

Table 6 Optimal values with ordering and setup cost reduction and price discounts (see online version for colours)

L	n	m	Q	A_B	A_V	π_x	$JETC_i$ $(i = 1, 2, 3)$	$JETC$
8	1	1	162	9.0000	486.0000	78.3750	8,172	36,559
		2	199	7.9600	636.8000	128.4825	16,340	
		3	185	8.6719	589.6875	103.4688	12,047	
	2	1	110	6.1111	660.0000	77.2917	8,180	36,606
		2	145	5.8000	928.0000	127.5375	16,369	
		3	130	6.0938	828.7500	102.4375	12,057	
	3	1	87	4.8333	783.0000	76.8125	8,451	37,836
		2	118	4.7200	1,132.800	127.0650	16,922	
		3	104	4.8750	994.5000	101.95	12,463	
6	1	1	157	8.7222	471.0000	78.2708	7,980	35,626
		2	192	7.6800	614.4000	128.36	15,899	
		3	179	8.3906	570.5625	103.3563	11,747	
	2	1	105	5.8333	630.0000	77.1875	7,955	35,510
		2	138	5.5200	883.2000	127.4150	15,852	
		3	124	5.8125	790.5000	102.3250	11,703	
	3	1	83	4.6111	747.0000	76.7292	8,186	36,559
		2	112	4.4800	1,075.200	126.9600	16,323	
		3	99	4.6406	946.7000	101.8563	12,050	
4	1	1	152	8.4444	456.0000	78.1667	7,783	34,620
		2	183	7.3200	585.6000	128.2025	15,412	
		3	172	8.0625	548.2500	103.2250	11,425	
	2	1	101	5.6111	606.0000	77.1040	7,814	34,279
		2	131	5.2400	838.4000	127.2925	15,352	
		3	118	5.5312	752.2500	102.2125	11,312	
	3	1	80	4.4444	720.0000	76.6667	7,901	35,123
		2	106	4.2400	1,017.600	126.855	15,630	
		3	94	4.4063	898.9000	101.7625	11,592	
3	1	1	153	8.5000	459.0000	78.1875	7,750	34,801
		2	181	7.2400	579.2000	128.1675	15,223	
		3	171	8.0156	545.0625	103.2063	11,828	
	2	1	102	5.6667	612.0000	77.1250	7,380	34,582
		2	130	5.2000	832.0000	127.275	15,013	
		3	118	5.5312	752.2500	102.2125	12,189	
	3	1	80	4.4444	720.0000	76.6667	7,883	34,690
		2	105	4.2000	1,008.000	126.8375	15,355	
		3	94	4.4063	898.9000	101.7625	11,452	

4.7 Sensitivity analysis

The sensitivity analysis is accomplished by adjusting each parameters by –40%, –20%, +20%, +40%. One parameter can be modified in a moment of time and all the others are left intact. Tables list the outcomes.

Table 7 Sensitivity of $JETC$

Parameter	*–20%*	*–10%*	*+10%*	*+20%*
P_i	34,280	34,280	34,280	34,280
D_i	31,239	32,759	35,298	37,320
A_{Bi}	34,255	34,268	34,290	34,300
A_{Vi}	32,595	33,484	34,999	35,656
h_{Bi}	32,963	33,621	34,938	35,596
h_{Vi}	33,390	33,835	34,725	35,170
π_{xi}	32,852	33,566	34,992	35,704
σ_i	32,611	33,445	35,114	35,948
τ_i	34,195	34,237	34,322	34,364
δ_i	34,385	34,327	34,241	34,209
α_i	32,755	33,517	35,042	35,804
β_{0i}	34,287	34,283	34,276	34,272

The following managerial phenomena was achieved on the basis of our numerical results:

1 As there is an increase in demand, the merchant's order cost, the merchant's holding cost, and standard deviation, we have a dramatic increase in the total expected joint cost, as seen in Table 7.

2 If there is an increase in parameters such as merchant's ordering cost, manufacturer's holding cost, merchant's price discount on backorder, and opportunity cost to both manufacturer and merchant, there is a moderate increase in the joint total expected cost.

3 If there is a rise in the manufacturer's production rate, there are no variations in the total expected joint cost.

4 If the percentage decrease in the merchant's ordering cost changes, there is a slight change in the joint total expected cost.

5 While comparing Tables 5 and 6 we acquire the lowest joint total expected cost in table 6 with the consideration of price discounts and investments.

5 Manufacturer-merchant inventory model with freight cost

Sustainable transport came into utilise as a consistent follow-on from sustainable advancement, and is utilised to depict modes of transport, and systems of transport planning, which are steady with more extensive concerns of sustainability. Instead of using fuels for transportation, we have used electricity for running the truck, etc. Though we have used up electricity for transportation there will some transportation emission.

In today's greatly assertive situation, the freight cost plays a vital role in overall dealing cost. Accordingly, freight cost is accommodated with the ETC function; first, that should be capable of identifying freight cost functions that imitate the reality. In most integrated models, freight costs are handled only indirectly by a fixed setup or ordering cost function and are thus expected to depend on the shipment scale. We also found the freight expense in this section as an explicit function of order quantity. We implement the following all-unit-discount freight cost structure, equivalent to Priyan and Uthayakumar (2014).

Range	*Unit tranportation cost*
$0 \le Q_i < M_1$	C_0
$M_1 \le Q_i < M_2$	C_1
$M_2 \le Q_i < M_3$	C_2
$\vdots$	$\vdots$
$M_{b-1} \le Q_i < M_b$	C_{b-1}
$M_b \le Q_i$	C_b

with the condition that $C_0 < C_1 < ... < C_{b-1} < C_b$.

For the procurement of abundant goods, the manufacturer opens the freight cost concession to the customer. Here, we have studied that the cost of freight relys upon the merchant's Q_i. Owing to its way, we presume that once the merchant places the order Q_i units, the manufacturer instantly provides the concession. The amount Q_i is then purchased from the merchant to receive a discount from the manufacturer on freight costs. The freight cost per unit time for a given shipment of lot size $Q_i \in [M_l, M_{l+1})$ is equivalent to $\frac{C_1 Q_i}{Q_i/D} = C_1 D$, which is determined by splitting the freight cost per order cycle by the order cycle length. We depict the freight cost as follows:

$$Trans\ cost = \begin{cases} C_0 D_i & Q_i \in [0, M_1) \\ C_1 D_i & Q_i \in [M_1, M_2) \\ C_2 D_i & Q_i \in [M_2, M_3) \\ \vdots & \vdots \\ C_{bi-1} D_i & Q_i \in [M_{bi-1}, M_{bi}) \\ C_{bi} D_i & Q_i \in [M_{bi}, \infty) \end{cases}$$

Therefore, in addition to the freight costs for this integrated supply chain system, including price discount on backorder, setup costs, and order cost reduction, the joint estimated average expense per unit period is denoted by

$$\begin{aligned} JETC_{trans} &= JETC + \sum_{i=1}^{m} trans\ cost_i \\ &= \sum_{i=1}^{m} \left(\alpha_i I(A_{Vi}) + \frac{A_{Vi} D_i}{n Q_i} + \frac{h_{Vi} Q_i}{2} \left[n \left(1 - \frac{D_i}{P_i} \right) - 1 \right. \right. \\ &\quad \left. + \frac{2 D_i}{P_i} \right] \tau_i I(A_{Bi}) + \frac{A_{Bi} D_i}{Q_i} + h_{Bi} \left[\frac{Q_i}{2} + r_i - D_i L \right] \\ &\quad + h_{Bi} \left[1 - \frac{\beta_{0i} \pi_{xi}}{\pi_{0i}} \right] E(X_i - r_i)^+ + \frac{D_i}{Q_i} \left[\frac{\beta_{0i} \pi_{xi}^2}{\pi_{0i}} + \pi_{0i} \right. \\ &\quad \left. \left. - \beta_{0i} \pi_{xi} \right] E(X_i - r_i)^+ + \frac{D_i}{Q_i} R(L) \right) + \sum_{i=1}^{m} trans\ cost_i \end{aligned} \tag{23}$$

Subject to the constraints

$0 < A_{Vi} \leq A_{V0i}$
$0 < A_{Bi} \leq A_{B0i}$
$0 < \pi_{xi} \leq \pi_{0i}$

5.1 Solution procedure

For a given range of $[M_g, M_{g+1})$, $[L_j, L_{j-1}]$ and n value, this expected total joint cost with freight expense is convex in Q_i, A_{Bi}, A_{Vi} and π_{xi} which is found by the equations (19), (20), (21) and (21) and is identical for this case also as the freight cost is a constant which depends on $[M_g, M_{g+1})$. Likewise, $JETC_{trans}(Q_i, A_{Bi}, A_{Vi}, \pi_{xi}, L, n)$ is a concave function of $L \in [L_j, L_{j-1}]$ for fixed $(Q_i, A_{Bi}, A_{Vi}, \pi_{xi}, n)$ and given range $[M_g, M_{g+1})$.

Without picking up the freight cost to an account and setting the bound where this optimal shipment lot size decreases, we calculate the optimal results of ordering cost, setup cost, price discount on backorder, shipment lot size, lead time, and the total number of shipments using the given algorithm (Subsection 4.5). Later, recognising that both the freight and inventory-related costs such as ordering costs, setup costs, and price discount on backorder increase on the left of this lot size, we analyse in comparison to the estimated total cost in addition to freight costs at this lot size and more than this lot size at all reorder points. Based on the above discussion, we would like to find the optimal results using the freight expense algorithm.

5.2 Algorithm

Step 1 Proceed the algorithm (Subsection 4.5) and acquire the values of $(Q_i^*, A_{Bi}^*, A_{Vi}^*, \pi_{xi}^*, L^*, n^*)$.

Step 2 Next check whether $Q_i \geq M_{bi}$ if yes then take the above acquired values as optimal one and find the corresponding $JETC_{trans}$ using equations (23). If no proceed to the following step.

Step 3 For each $L \in [L_j, L_{j-1}]$, determine the $JETC(Q_i^*, A_{Bi}^*, A_{Vi}^*, \pi_{xi}^*)$ by proceeding the following steps.

Step 4 Set $n = 1$.

Step 5 Determine the values $\hat{Q}_i$, $\hat{A}_{Bi}$, $\hat{A}_{Vi}$, $\hat{\pi}_{xi}$ and the corresponding total cost $JETC(\hat{Q}_i, \hat{A}_{Bi}, \hat{A}_{Vi}, \hat{\pi}_{xi})$ by using the sub-algorithm (Subsection 5.2.1).

Step 6 Compute $\min_{j=1,2,\ldots,N} JETC(\hat{Q}_i, \hat{A}_{Bi}, \hat{A}_{Vi}, \hat{\pi}_{xi})$. If $JETC(Q_i^*, A_{Bi}^*, A_{Vi}^*, \pi_{xi}^*) = \min_{j=1,2,\ldots,N} JETC(\hat{Q}_i, \hat{A}_{Bi}, \hat{A}_{Vi}, \hat{\pi}_{xi})$ then set $JETC(Q_i^*, A_{Bi}^*, A_{Vi}^*, \pi_{xi}^*)$ as the optimal solution for fixed n.

Step 7 Increase the value of n by $n + 1$, and do the steps 2 and 3 again to gain the optimal solution $JETC(Q_i^*, A_{Bi}^*, A_{Vi}^*, \pi_{xi}^*)$.

Step 8 If the previous $JETC$ for $n - 1$ is greater than the current $JETC$ for n then proceed to step 4 otherwise move on to step 6.

Step 9 Now the $JETC$ for $n - 1$ is optimal solution for the presented model.

5.2.1 Sub-algorithm

Step 1 Find the values of Q_i, A_{Bi}, A_{Vi} and π_{xi} using the similar procedure presented in sub-algorithm (Subsection 4.5.1) and also let h_i be the highest range index such that $M_{hi} \leq Q_i$. Denote the solution by $\tilde{Q}_i$ $\tilde{A}_{Bi}$, $\tilde{A}_{Vi}$ and $\tilde{\pi}_{xi}$.

Step 2 Now set $Q_i = M_{hi+1}$.

Step 3 Using this determine the values of A_{Vi}, A_{Bi} and π_{xi}.

Step 4 Compare A_{Vi}, A_{Bi} and π_{xi} with A_{V0i}, A_{B0i} and π_{0i} respectively.

1. If $\tilde{A}_{Vi} < A_{V0i}$, $\tilde{A}_{Bi} < A_{B0i}$, and $\tilde{\pi}_{xi} < \pi_{0i}$ then the solution got through step 1 is optimal for the given $Q_i = M_{hi+1}$. Then compute $JETC$ by following step 6 in algorithm (Subsection 5.2).
2. If $\tilde{A}_{Vi} \geq A_{V0i}$, $\tilde{A}_{Bi} < A_{B0i}$, and $\tilde{\pi}_{xi} < \pi_{0i}$ then keep $\tilde{A}_{Vi} = A_{V0i}$ for finding the optimal values for the given $Q_i = M_{hi+1}$. Then compute $JETC$ by following step 6 in algorithm (Subsection 5.2). Otherwise follow step 5 in algorithm (Subsection 5.2).
3. If $\tilde{A}_{Vi} < A_{V0i}$, $\tilde{A}_{Bi} \geq A_{B0i}$, and $\tilde{\pi}_{xi} < \pi_{0i}$ then keep $\tilde{A}_{Bi} = A_{B0i}$ for finding the optimal values for the given $Q_i = M_{hi+1}$. Then compute $JETC$ by following step 6 in algorithm (Subsection 5.2). Otherwise follow step 5 in algorithm (Subsection 5.2).
4. If $\tilde{A}_{Vi} < A_{V0i}$, $\tilde{A}_{Bi} < A_{B0i}$, and $\tilde{\pi}_{xi} \geq \pi_{0i}$ then keep $\tilde{\pi}_{xi} = \pi_{0i}$ for finding the optimal values for the given $Q_i = M_{hi+1}$. Then compute $JETC$ by following step 6 in algorithm (Subsection 5.2). Otherwise follow step 5 in algorithm (Subsection 5.2).
5. If $\tilde{A}_{Vi} \geq A_{V0i}$, $\tilde{A}_{Bi} \geq A_{B0i}$, and $\tilde{\pi}_{xi} < \pi_{0i}$ then keep $\tilde{A}_{Vi} = A_{V0i}$ and $\tilde{A}_{Bi} = A_{B0i}$ for finding the optimal values for the given $Q_i = M_{hi+1}$. Then compute $JETC$ by following step 6 in (Subsection 5.2). Otherwise follow step 5 in (Subsection 5.2).
6. If $\tilde{A}_{Vi} \geq A_{V0i}$, $\tilde{A}_{Bi} < A_{B0i}$, and $\tilde{\pi}_{xi} \geq \pi_{0i}$ then keep $\tilde{A}_{Vi} = A_{V0i}$ and $\tilde{\pi}_{xi} = \pi_{0i}$ for finding the optimal values for the given $Q_i = M_{hi+1}$. Then compute $JETC$ by following step 6 in algorithm (Subsection 5.2). Otherwise follow step 5 in algorithm (Subsection 5.2).
7. If $\tilde{A}_{Vi} < A_{V0i}$, $\tilde{A}_{Bi} \geq A_{B0i}$, and $\tilde{\pi}_{xi} \geq \pi_{0i}$ then keep $\tilde{A}_{Bi} = A_{B0i}$ and $\tilde{\pi}_{xi} = \pi_{0i}$ for finding the optimal values for the given $Q_i = M_{hi+1}$. Then compute $JETC$ by following step 6 in algorithm (Subsection 5.2). Otherwise follow step 5 in algorithm (Subsection 5.2).
8. If $\tilde{A}_{Vi} \geq A_{V0i}$, $\tilde{A}_{Bi} \geq A_{B0i}$, and $\tilde{\pi}_{xi} \geq \pi_{0i}$ then keep $\tilde{A}_{Vi} = A_{V0i}$, $\tilde{A}_{Bi} = A_{B0i}$ and $\tilde{\pi}_{xi} = \pi_{0i}$ for finding the optimal values for the given $Q_i = M_{hi+1}$. Then compute $JETC$ by following step 6 in algorithm (Subsection 5.2). Otherwise follow step 5 in algorithm (Subsection 5.2).

5.3 Numerical illustration

We use the datum which is discussed in the subsection of Section 4 with the freight cost structure as follows:

Range	*Unit tranportation cost*
$0 \leq Q_i < 50$	[0.2 0.4 0.3]
$50 \leq Q_i < 100$	[0.1 0.25 0.125]
$100 \leq Q_i < 150$	[0.08 0.17, 0.12]
$150 \leq Q_i$	[0.07 0.14 0.1]

Using the algorithm presented in Subsection 5.2, find the optimal solution with the freight cost.

Table 8 Optimal values with ordering, setup cost reduction, price discounts and freight cost for $L = 8$ and $L = 6$ (see online version for colours)

L	n	m	Q	A_B	A_V	π_x	$JETC_i$ (i = 1, 2, 3)	$JETC$
$L = 8$	1	1	162	9.0000	486.0000	78.3750	8,214	36,821
		2	199	7.9600	636.8000	128.4825	16,480	
		3	185	8.6719	589.6875	103.4688	12,127	
	2	1	110	6.1111	660.0000	77.2917	8,228	3,6920
		2	145	5.8000	928.0000	127.5375	16,539	
		3	130	6.0938	828.7500	102.4375	12,153	
	2	1	150	6.4111	660.0700	77.2919	8,222	36,868
		2	150	5.8200	928.2100	127.5385	16,509	
		3	150	6.1038	828.7700	102.4475	12,137	
	3	1	87	4.8333	783.0000	76.8125	8,526	38,177
		2	118	4.7200	1,132.800	127.0650	17,092	
		3	104	4.8750	994.5000	101.95	12,559	
	3	1	100	4.8037	783.2000	76.8125	8,499	38,104
		2	150	4.0700	1,132.802	127.0650	17,062	
		3	150	4.7050	994.5100	101.95	12,543	
	3	1	150	4.8037	783.2000	76.8125	8,493	38,098
		2	150	4.0700	1,132.802	127.0650	17,062	
		3	150	4.7050	994.5100	101.95	12,543	
$L = 6$	1	1	157	8.7222	471.0000	78.2708	8,022	35,888
		2	192	7.6800	614.4000	128.36	16,039	
		3	179	8.3906	570.5625	103.3563	11,827	
	2	1	105	5.8333	630.0000	77.1875	8,003	35,824
		2	138	5.5200	883.2000	127.4150	16,022	
		3	124	5.8125	790.5000	102.3250	11,799	
	2	1	150	5.8333	630.0000	77.1875	7,997	35,772
		2	150	5.5200	883.2000	127.4150	15,992	
		3	150	5.8125	790.5000	102.3250	11,783	
	3	1	83	4.6111	747.0000	76.7292	8,261	36,964
		2	112	4.4800	1,075.200	126.9600	16,493	
		3	99	4.6406	946.7000	101.8563	12,210	
	3	1	100	4.6111	747.0000	76.7292	8,234	36,843
		2	150	4.4800	1,075.200	126.9600	16,463	
		3	100	4.6406	946.7000	101.8563	12,146	
	3	1	150	4.6111	747.0000	76.7292	8,228	36,821
		2	150	4.4800	1,075.200	126.9600	16,463	
		3	150	4.6406	946.7000	101.8563	12,130	

Table 9 Optimal values with ordering, setup cost reduction, price discounts and freight cost for L = 4 and L = 3 (see online version for colours)

L	n	m	Q	A_B	A_V	π_x	$JETC_i$ (i = 1, 2, 3)	$JETC$
$L = 4$	1	1	152	8.4444	456.0000	78.1667	7,825	34,882
		2	183	7.3200	585.6000	128.2025	15,552	
		3	172	8.0625	548.2500	103.2250	11,505	
	2	1	101	5.6111	606.0000	77.1040	7,762	34,593
		2	131	5.2400	838.4000	127.2925	15,423	
		3	118	5.5312	752.2500	102.2125	11,408	
	2	1	150	5.6111	606.0000	77.1040	7,756	34,541
		2	150	5.2400	838.4000	127.2925	15,393	
		3	150	5.5312	752.2500	102.2125	11,392	
	3	1	80	4.4444	720.0000	76.6667	7,976	35,528
		2	106	4.2400	1,017.600	126.855	15,800	
		3	94	4.4063	898.9000	101.7625	11,752	
	3	1	100	4.4444	720.0000	76.6667	7,949	35,407
		2	150	4.2400	1,017.600	126.855	15,770	
		3	100	4.4063	898.9000	101.7625	11,688	
	3	1	150	4.4444	720.0000	76.6667	7,943	35,385
		2	150	4.2400	1,017.600	126.855	15,770	
		3	150	4.4063	898.9000	101.7625	11,672	
$L = 3$	1	1	153	8.5000	459.0000	78.1875	7,792	35,063
		2	181	7.2400	579.2000	128.1675	15,363	
		3	171	8.0156	545.0625	103.2063	11,908	
	2	1	102	5.6667	612.0000	77.1250	7,428	34,896
		2	130	5.2000	832.0000	127.275	15,183	
		3	118	5.5312	752.2500	102.2125	12,285	
	2	1	150	5.6667	612.0000	77.1250	7,422	34,844
		2	150	5.2000	832.0000	127.275	15,153	
		3	150	5.5312	752.2500	102.2125	12,269	
	3	1	80	4.4444	720.0000	76.6667	7,958	35,095
		2	105	4.2000	1,008.000	126.8375	15,525	
		3	94	4.4063	898.9000	101.7625	11,612	
	3	1	100	4.4444	720.0700	76.7667	7,931	34,974
		2	150	4.2000	1,008.2000	126.8153	15,495	
		3	100	4.4063	898.9100	101.7625	11,548	
	3	1	150	4.4444	720.0900	76.9671	7,925	34,952
		2	150	4.2000	1,008.2000	126.8153	15,495	
		3	150	4.4063	898.9100	101.7625	11,532	

From Tables 8 and 9, we have observed that the joint total expected cost has increased with freight cost adoption.

6 Conclusions

In this study, for multiple steel products, we have specifically implemented two integrated supply chain models without and with freight costs. Mathematical modelling and solution procedure with algorithms are provided to find the optimal results by considering investments and price discounts on backorders. We then run the numerical examples to illustrate the fact and provide some managerial insights with sensitivity analysis. We have concluded that the model with price discount and investment will be more better than the model without price discount and investment. Even after including freight cost, the JETC is much lesser than the JETC in the model without freight cost, price discount and investment.

In many ways, the model described can be generalised. The retailer can provide the remaining unpaid balance with a permissible delay in payment. We could often extend the model of multiple steel products from a single manufacturer to multiple merchants.

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Appendix

Proof of Theorem 1

Proof: For the given values of L and m, the Hessian matrix H is as follows

$$H = \begin{bmatrix} \frac{\partial^2 JETC}{\partial Q_i^2} & \frac{\partial^2 JETC}{\partial Q_i \partial A_{Bi}} & \frac{\partial^2 JETC}{\partial Q_i \partial A_{Vi}} & \frac{\partial^2 JETC}{\partial Q_i \partial \pi_{xi}} \\ \frac{\partial^2 JETC}{\partial A_{Bi} \partial Q_i} & \frac{\partial^2 JETC}{\partial A_{Bi}^2} & \frac{\partial^2 JETC}{\partial A_{Bi} \partial A_{Vi}} & \frac{\partial^2 JETC}{\partial A_{Bi} \partial \pi_{xi}} \\ \frac{\partial^2 JETC}{\partial A_{Vi} \partial Q_i} & \frac{\partial^2 JETC}{\partial A_{Vi} \partial A_{Bi}} & \frac{\partial^2 JETC}{\partial A_{Vi}^2} & \frac{\partial^2 JETC}{\partial A_{Vi} \pi_{xi}} \\ \frac{\partial^2 JETC}{\partial \pi_{xi} \partial Q_i} & \frac{\partial^2 JETC}{\partial \pi_{xi} \partial A_{Bi}} & \frac{\partial^2 JETC}{\partial \pi_{xi} \partial A_{Vi}} & \frac{\partial^2 JETC}{\partial \pi_{xi}^2} \end{bmatrix}$$

Now we have the following

$$\frac{\partial^2 JETC}{\partial Q_i^2} = \frac{2D_i}{Q_i^3} \left[A_{Bi} + \frac{1}{n} A_{Vi} + (\beta_{0i}\pi_{xi}^2 + \pi_{0i} - \beta_{0i}\pi_{xi})\sigma_i \sqrt{L}\Psi_i(k_i) + R(L) \right]$$

$$\frac{\partial^2 JETC}{\partial A_{Bi}^2} = \frac{\tau_i}{\delta_i A_{Bi}^2}$$

$$\frac{\partial^2 JETC}{\partial A_{Vi}^2} = \frac{\alpha_i}{\gamma_i A_{Vi}^2}$$

$$\frac{\partial^2 JETC}{\partial \pi_{xi}^2} = \frac{2D_i\beta_{0i}}{Q_i\pi_{0i}} \sigma_i \sqrt{L}\Psi_i(k_i)$$

Since $JETC$ is continuous we have

$$\frac{\partial^2 JETC}{\partial Q_i \partial A_{Bi}} = \frac{\partial^2 JETC}{\partial A_{Bi} \partial Q_i} = -\frac{D_i}{Q_i^2}$$
$$\frac{\partial^2 JETC}{\partial Q_i \partial A_{Vi}} = \frac{\partial^2 JETC}{\partial A_{Vi} \partial Q_i} = -\frac{D_i}{nQ_i^2}$$
$$\frac{\partial^2 JETC}{\partial Q_i \partial \pi_{xi}} = \frac{\partial^2 JETC}{\partial Q_i \partial \pi_{xi}} = \frac{D_i}{Q_i^2}\left(\beta_{0i} - \frac{2\beta_{0i}\pi_{xi}}{\pi_{0i}}\right)$$
$$\frac{\partial^2 JETC}{\partial A_{Vi} \partial \pi_{xi}} = \frac{\partial^2 JETC}{\partial \pi_{xi} \partial A_{Vi}} = 0$$
$$\frac{\partial^2 JETC}{\partial A_{Bi} \partial \pi_{xi}} = \frac{\partial^2 JETC}{\partial \pi_{xi} \partial A_{Bi}} = 0$$
$$\frac{\partial^2 JETC}{\partial A_{Bi} \partial A_{Vi}} = \frac{\partial^2 JETC}{\partial A_{Vi} \partial A_{Bi}} = 0$$

As the values of $\frac{\partial^2 JETC}{\partial A_{Vi}\partial \pi_{xi}}$, $\frac{\partial^2 JETC}{\partial \pi_{xi}\partial A_{Vi}}$, $\frac{\partial^2 JETC}{\partial A_{Bi}\partial \pi_{xi}}$, $\frac{\partial^2 JETC}{\partial \pi_{xi}\partial A_{Bi}}$, $\frac{\partial^2 JETC}{\partial A_{Bi}\partial A_{Vi}}$, $\frac{\partial^2 JETC}{\partial A_{Vi}\partial A_{Bi}}$ are all zero, the Hessian matrix H can rewritten as

$$H = \begin{bmatrix} \frac{\partial^2 JETC}{\partial Q_i^2} & \frac{\partial^2 JETC}{\partial Q_i \partial A_{Bi}} & \frac{\partial^2 JETC}{\partial Q_i \partial A_{Vi}} & \frac{\partial^2 JETC}{\partial Q_i \partial \pi_{xi}} \\ \frac{\partial^2 JETC}{\partial A_{Bi} \partial Q_i} & \frac{\partial^2 JETC}{\partial A_{Bi}^2} & 0 & 0 \\ \frac{\partial^2 JETC}{\partial A_{Vi} \partial Q_i} & 0 & \frac{\partial^2 JETC}{\partial A_{Vi}^2} & 0 \\ \frac{\partial^2 JETC}{\partial \pi_{xi} \partial Q_i} & 0 & 0 & \frac{\partial^2 JETC}{\partial \pi_{xi}^2} \end{bmatrix}$$

Then we have to find the principal minors of the Hessian matrix. The principal minors of $|H|$ are

$$|H_{11}| = \frac{2D_i\beta_{0i}}{Q_i\pi_{0i}}\sigma_i\sqrt{L}\Psi_i(k_i) > 0$$
$$|H_{22}| = \left(\frac{\alpha_i}{\gamma_i A_{Vi}^2}\right)\left(\frac{2D_i\beta_{0i}}{Q_i\pi_{0i}}\sigma_i\sqrt{L}\Psi_i(k_i)\right) > 0$$
$$|H_{33}| = \left(\frac{\tau_i\alpha_i}{\delta_i\gamma_i A_{Bi}^2 A_{Vi}^2}\right)\left(\frac{2D_i\beta_{0i}}{Q_i\pi_{0i}}\sigma_i\sqrt{L}\Psi_i(k_i)\right) > 0$$
$$\begin{aligned} |H_{44}| = &\left(\frac{2D_i\beta_{0i}}{Q_i\pi_{0i}}\right)\sigma_i\sqrt{L}\Psi_i(k_i)\left(\frac{D_i^2\alpha_i}{Q_i^4\gamma_i A_{Vi}^2} - \frac{2D_i}{Q_i^3}\left[A_{Bi} + \frac{1}{n}A_{Vi} + (\beta_{0i}\pi_{xi}^2\right.\right. \\ &\left.\left. + \pi_{0i} - \beta_{0i}\pi_{xi})\sigma_i\sqrt{L}\Psi_i(k_i) + R(L)\right]\frac{\tau_i\alpha_i}{\delta_i\gamma_i A_{Bi}^2 A_{Vi}^2} - \frac{D_i^2\tau_i}{n^2 Q_i^4\delta_i A_{Bi}^2}\right) \\ &+ \left[\frac{2D_i\beta_{0i}\pi_{xi}}{Q_i\pi_{0i}} - \frac{D_i\beta_{0i}}{Q_i}\right]^2\left(\frac{\tau_i\alpha_i}{\delta_i\gamma_i A_{Bi}^2 A_{Vi}^2}\right)\sigma_i^2 L\Psi_i^2(k_i) > 0 \end{aligned}$$

Therefore, from the principal minors it is clearly seen that the Hessian matrix H is positive definite at point $(Q_i,\, A_{Bi},\, A_{Vi},\, \pi_{xi})$.

The proof is completed. □