Reward Processes and Performance Simulation in Supermarket Models with Different Servers

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Abstract

Supermarket models with different servers become a key in modeling resource management of stochastic networks, such as, computer networks, manufacturing systems, transportation networks and healthcare systems. While the different servers always make analysis of such a supermarket model more interesting, difficult and challenging. This paper provides a novel method for analyzing the supermarket models with different servers through a multi-dimensional continuous-time Markov reward process. Firstly, some utility functions are constructed for the routine selection mechanism according to the queue lengths, the service rates, and the probability of individual preference. Secondly, applying the state jump points of the continuous-time Markov reward process, some segmented stochastic integrals of the random reward function are established by means of an event-driven technique. Based on this, the mean of the random reward function in a finite time interval is computed, and the mean of the discounted random reward function in an infinite time interval can also be calculated. Finally, some simulation experiments are given to indicate how the expected queue length of each server depends on some key parameters of this supermarket model.

Keywords: Supermarket model; Routine selection mechanism; Markov reward process; Random reward function; Stochastic integral; Event-driven technique.

1 Introduction

Randomized load balancing, where a job is assigned to a server from a small subset of randomly chosen servers, is very simple to implement, and can surprisingly deliver better performance (for example, reducing collisions, waiting times, and backlogs) in a number of applications, such as, data centers, capacity allocation, hash tables, distributed memory machines, path selection, and task scheduling. The supermarket model is a dynamic randomized load balancing method, and its original idea may be inspired by operation mechanism of supermarket checkout in a large supermarket. Because the supermarket model has simple operations, quick response, dynamical real-time management, and many other advantages, it has been regarded as one of the most effective technologies in the study of large-scale stochastic networks with resource management and task scheduling.

During the last two decades considerable attention has been paid to studying the supermarket models through queueing theory as well as Markov processes. Since a simple supermarket model was discussed by Mitzenmacher [31], Vvedenskaya et al. [44] and Turner [42, 43], subsequent papers have been published on this theme, among which, see, Vvedenskaya and Suhov [45], Mitzenmacher et al. [32], Graham [10, 11], Luczak and Norris [26], Luczak and McDiarmid [24, 25], Brightwell and Luczak [5], Bramson et al. [2, 3, 4], Li and Lui [21, 22], Li et al. [23, 18, 20, 19] and Li [15, 16]. For the fast Jackson networks (or the supermarket networks), readers may refer to Martin and Suhov [29], Martin [30] and Suhov and Vvedenskaya [38]. On the other hand, Janssen [13] applied the discrete-time Markov reward processes as well as the discrete-time Markov decision processes to the study of supermarket models with N identical servers. The stability of more general supermarket models was discussed by Foss and Chernova [9], Bramson [1] and MacPhee et al. [28].

There are some successful research on various Markov reward processes, important examples include Reibman et al. [34], Ciardo et al. [7], Qureshi and Sanders [33], Telek et al. [40], de Souza e Silva and Gail [8], Telek and Rácz [41], Telek et al. [39], Li and Cao [17], Stefanov [36], Stenberg et al. [37], and two books by Cao [6] and Li [14].

Little work has been done on analysis of the supermarket models with different servers, which is more difficult and challenging due to high complexity and percipient subjectivity of designing a fair routine selection mechanism with respect to the different servers. Specifically, a practical understanding can indicate that such a routine selection mechanism may depend on the queue lengths, on the service rates, on the probability of individual preference and so forth. Janssen [13] described a simple intuitive outline of discussing the supermarket model with different servers, and demonstrated that analysis of the supermarket model with different servers will be an interesting and difficult topic in the future research. Based on this, Li et al. [19] provided a birth-death reward process for the supermarket model with different servers, and established a system of functional reward equations which can be solved by a value iterative algorithm. It is worth noting that this paper uses a more general Markov reward process to set up the segmented stochastic integrals of the random reward function in the supermarket model with different servers by means of an event-driven technique, which is shown to be useful for performance simulation of a more general large-scale stochastic system. In addition to this, we would like to remark two key points: (1) Although the mean-field theory is an effective method in the study of supermarket models with the same servers (e.g., see Vvedenskaya et al. [44], Li et al. [18] and Li and Lui [22]), the complicated routine selection mechanism with respect to the different servers makes setting up the systems of mean-field equations more difficult. To our best knowledge, up to now no paper has applied the mean-field theory to the study of supermarket models with different servers. (2) The generating functions are always classical and effective for performance evaluation of many practical stochastic systems, but they are not convenient to deal with a multi-dimensional problem, and are also very difficult to analyze a system of nonlinear equations.

The main contributions of this paper are twofold. The first one is to describe a supermarket model with different servers, in which the arrival and service processes are given in a detailed discussion, and the reward value at each state is chosen from some practical points of view. We show that the arrival process of this supermarket model is very complicated due to a routine selection mechanism that depends on the queue lengths, on the service rates, on the probability of individual preference and so forth. Also, it is seen that the routine selection mechanism is very different from that in the supermarket model with same servers, where our construction of this routine selection mechanism is based on the utility functions so that the subjective behavior of customers is also covered in the routine selection mechanism. The second one is to set up a multi-dimensional continuous-time Markov reward process, and provide a segmented stochastic integral for expressing the random reward function in a finite time interval through an event-driven technique. Furthermore, we calculate the mean of the discounted reward function in an infinite time interval. Based on this, we give a simple discussion on optimal criterions for designing the supermarket model with different servers. Also, we provide some simulation experiments to indicate how the expected queue length of each server depends on some key parameters of this supermarket model.

The remainder of this paper is organized as follows. In Section 2, we first describe a supermarket model with M different servers. Then we construct a routine selection mechanism that depends on the queue lengths, on the service rates, on the probability of individual preference and so forth. In Section 3, we set up an M-dimensional continuous-time Markov reward process, and provide a segmented stochastic integral for expressing the random reward function in a finite time interval through an event-driven technique. In Section 4, applying the segmented stochastic integral, we compute the mean of the random reward function in a finite time interval. In Section 5, we compute the mean of the discounted reward function in an infinite time interval. Based on this, we provide two optimal criterions for designing the supermarket model with different servers. In Section 6, we provide some simulation experiments to indicate how the expected queue length of each server depends on some key parameters of this supermarket model. Some concluding remarks are given in Section 7.

2 Supermarket Model Description

In this section, we first describe a supermarket model with M different servers. Then we construct a routine selection mechanism that depends on the queue lengths, on the service rates, on the probability of individual preference and so forth.

In the supermarket model, there are M different servers whose waiting rooms are all infinite. The service times in each server are i.i.d. and are exponential, and also the service rates of the M different servers are denoted as $\mu_1, \mu_2, \ldots, \mu_M$, respectively. The arrivals of customers are a Poisson process with arrival rate λ. Because the servers are different, it is a key to optimize the service ability of this supermarket model through designing a better routine selection mechanism. In fact, designing such a better routine selection mechanism will become not only complicated but also subjective due to the difference of the M servers. The physical structure of this supermarket model is shown in Figure 1.

In what follows we will provide a detailed description for how to construct such a better routine selection mechanism. Notice that our method for constructing the routine selection mechanism is intuitive and heuristic according to some practical points of view.

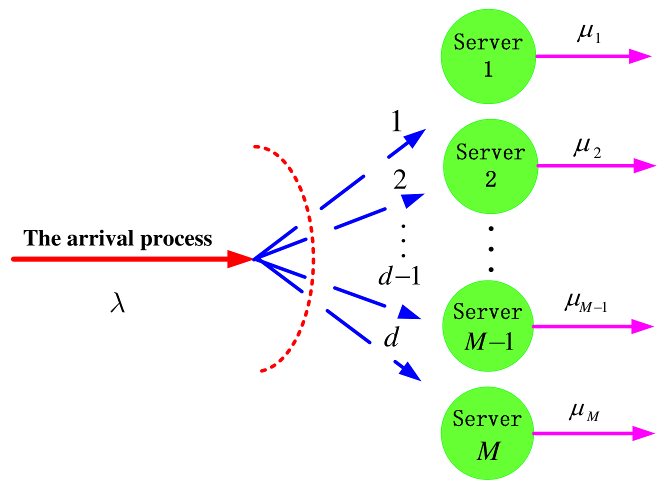

Figure 1: A physical illustration of the supermarket models with different servers

From Figure 1, it is seen that for the M different servers, each arriving customer joins a server (or queue) according to a suitable routine selection mechanism. From a practical point of view, each arriving customer chooses one server based on at least three crucial factors: (1) Choosing one server with the largest service rate, (2) choosing one server with the shortest queue length, and (3) choosing one server with the maximal probability of individual preference.

We write

$$x = (x_1, x_2, \ldots, x_M),$$

which is the vector of the queue lengths in the M servers;

$$g = (g_1, g_2, \ldots, g_M),$$

which is a probability vector of individual preference for choosing one of the M servers. In general, the individual preference is based on the priori knownledge, and the present feeling etc.; and

$$\mu = (\mu_1, \mu_2, \ldots, \mu_M).$$

It is worth noting that the two vectors g and μ are always inherent in the system, but the vector x of queue lengths can change dynamically according to a customer arrival or a service completion.

Based on the above analysis, let $\Delta_i(x) = f(x_i, \mu_i, g_i)$ be a routine selection function which represents the measurement of choosing the ith server for $i = 1, 2, \ldots, M$, where $f(x_i, \mu_i, g_i)$ satisfies three conditions: (1) $f(x_i, \mu_i, g_i)$ is increasing for $x_i \geq 0$, (2) $f(x_i, \mu_i, g_i)$ is decreasing for $\mu_i > 0$, and (3) $f(x_i, \mu_i, g_i)$ is decreasing for $g_i \in (0, 1]$.

We assume that if

$$\Delta_{i_0}(x) = \min_{1 \leq i \leq M} \{\Delta_i(x)\},$$

then the arriving customer joins the i_0th server among the M servers. It indicates that an arriving customer likes the server with the minimal value in the set of routine selection functions

$$\mathbf{\Delta} = \{\Delta_1(x), \Delta_2(x), \ldots, \Delta_{M-1}(x), \Delta_M(x)\}.$$

From the routine selection function, now we further describe the routine selection mechanism as follows:

The routine selection mechanism: Each arriving customer chooses $d \geq 1$ servers independently and uniformly at random from the M servers, and joins the server with the smallest number in $\mathbf{\Delta}_d = \left\{\Delta_{i_1}(x), \Delta_{i_2}(x), \ldots, \Delta_{i_{d-1}}(x), \Delta_{i_d}(x)\right\}$, where the d selected servers are denoted as Servers $i_1, i_2, \ldots, i_d$. If there is a tie, servers with the smallest number in $\mathbf{\Delta}_d$ will be chosen randomly. All customers in any server will be served in the first come first service (FCFS) manner. We assume that all the random variables defined for the arrival and service processes are independent of each other.

In what follows we provide some useful interpretation for each element in the set $\mathbf{\Delta} = \{\Delta_1(x), \Delta_2(x), \ldots, \Delta_{M-1}(x), \Delta_M(x)\}$ of routine selection functions.

Interpretation one: $\Delta_i(x) = f(x_i, \mu_i, g_i)$ has some useful forms

Note that $f(x_i, \mu_i, g_i)$ needs to satisfy the above three monotone conditions for each element in one of the three vectors x, μ and g, thus such a function $f : \mathbf{N}^+ \times (0, +\infty) \times (0, 1] \to \mathbf{R}^+$ can be chosen easily, where $\mathbf{N}^+ = \{0, 1, 2, \ldots\}$ and $\mathbf{R}^+ = [0, +\infty)$. To that end, we give some examples to indicate how to construct such a function $f(x_i, \mu_i, g_i)$ as follows:

(1) A tandem-type decision-making method

For the three decision variables x_i, μ_i and g_i, we set up a tandem-type decision-making structure as $x_i \cdot \frac{1}{\mu_i} \cdot \frac{1}{g_i}$, thus it is seen from a normalization that the routine selection function is given by

$$\Delta_i(x) = \frac{1 + \frac{x_i}{\mu_i g_i}}{\sum\limits_{j=1}^{M} \left[1 + \frac{x_j}{\mu_j g_j}\right]}, \quad i = 1, 2, \ldots, M.$$

(2) A weighted-type decision-making method

For the three decision variables x_i, μ_i and g_i, we take a weighted-type decision-making structure as $\beta_1 x_i + \beta_2 \frac{1}{\mu_i} + \beta_3 \frac{1}{g_i}$, where the weighted coefficients satisfy that $\beta_k \geq 0$ and $\beta_1 + \beta_2 + \beta_3 = 1$, thus the routine selection function is given by

$$\Delta_i(x) = \frac{1 + \beta_1 x_i + \beta_2 \frac{1}{\mu_i} + \beta_3 \frac{1}{g_i}}{\sum\limits_{j=1}^{M} \left[1 + \beta_1 x_j + \beta_2 \frac{1}{\mu_j} + \beta_3 \frac{1}{g_j}\right]}, \quad i = 1, 2, \ldots, M.$$

Interpretation two: There exist multiple minimal elements in $\boldsymbol{\Delta}_d$

For $\boldsymbol{\Delta}_d = \left\{\Delta_{i_1}(x), \Delta_{i_2}(x), \ldots, \Delta_{i_{d-1}}(x), \Delta_{i_d}(x)\right\}$, set

$$\Re_{\min}(d) = \left\{ i_0 : \Delta_{i_0}(x) = \min_{1 \leq k \leq d} \{\Delta_{i_k}(x)\} \right\}.$$

Then we have two cases: (1) $\Re_{\min}(d)$ contains only one element, and (2) $\Re_{\min}(d)$ contains multiple elements. For the former, the routine selection of the arriving customer is simple for choosing Server i_0; while for the latter, the routine selection of the arriving customer has a little complicated, for example, a simple mode is taken as that if there is a tie, servers with the smallest number in $\Re_{\min}(d)$ will be chosen randomly, e.g., see Vvedenskaya et al. [44] and Mitzenmacher [31].

To use more information in the set $\Re_{\min}(d)$, we may set up some new routine selection ways. If there is a tie (that is, $\Re_{\min}(d)$ contains multiple elements), then servers with the smallest number in $\Re_{\min}(d)$ may be chosen by means of other ways, for example, either

(1) for all the different elements in $\Re_{\min}(d)$, the arriving customer joins the server with the biggest service rate;

(2) for all the different elements in $\Re_{\min}(d)$, the arriving customer joins the server with the shortest queue length;

(3) for all the different elements in $\Re_{\min}(d)$, the arriving customer joins the server with the maximal probability of individual preference; or

(4) some hybrid combination from the above (1), (2) and (3).

In this paper, we will not discuss the above four cases, which are interesting and will be studied in our future work.

Interpretation three: Useful relations between our above model and the ordinary supermarket model

On the one hand, when $\mu_1 = \mu_2 = \cdots = \mu_M = \mu$ and $g_1 = g_2 = \cdots = g_M = \frac{1}{M}$, it is seen that

$$\Delta_i(x) = f(x_i, \mu_i, g_i) = f\left(x_i, \mu, \frac{1}{M}\right),$$

which shows that the routine selection of the arriving customer only depends on the vector $x = (x_1, x_2, \ldots, x_M)$, hence the arriving customer joins the server with the shortest queue length, e.g., see Vvedenskaya et al. [44]. On the other hand, we remark that the probability vector $g = (g_1, g_2, \ldots, g_M)$ of individual preference can give rise to the study of modern supermarket business or network economy.

In the supermarket model with different servers, data collection and analysis is also a key task. Therefore, it is interesting that the routine selection mechanism can be designed from a data-based practical point of view. This will motivate statistical analysis of supermarket models with different servers from many real areas.

3 A Markov Reward Process

In this section, we set up an M-dimensional continuous-time Markov reward process, and provide a segmented stochastic integral for expressing the random reward function in a finite time interval through an event-driven technique.

In order to set up a continuous-time Markov reward process, we need to discuss the arrival and service processes, both of which lead to the state jumps of this Markov reward process. At the same time, we choose a suitable reward value at each state in this supermarket model.

(1) Analysis of the arrival processes

In this supermarket model, the arrival process of customers is a Poisson process with arrival rate λ. Each arriving customer chooses d servers independently and uniformly at random from the M servers, and joins one server with the smallest number in the set $\boldsymbol{\Delta}_d = \left\{\Delta_{i_1}(x), \Delta_{i_2}(x), \ldots, \Delta_{i_{d-1}}(x), \Delta_{i_d}(x)\right\}$. If there is a tie, servers with the smallest number in the set Δ_d will be chosen randomly.

In order to express the routine selection mechanism of each arriving customer, we need to introduce an *ascending* function $\sigma : [0,1]^M \to [0,1]^M$ as follows:

$$\sigma\left(\mathbf{\Delta}^{(x)}\right) = \left(\Delta_{k_1}(x), \Delta_{k_2}(x), \ldots, \Delta_{k_M}(x)\right)$$

for $\mathbf{\Delta}^{(x)} = (\Delta_1(x), \Delta_2(x), \ldots, \Delta_M(x))$, where

$$0 \le \Delta_{k_1}(x) \le \Delta_{k_2}(x) \le \ldots \le \Delta_{k_M}(x) \le 1. \tag{1}$$

For the ascending function $\sigma\left(\mathbf{\Delta}^{(x)}\right)$, it is necessary to explain the order numbers k_i for $1 \le i \le M$. Note that k_i denotes the k_ith element of the original order number vector $\mathbf{\Delta}^{(x)}$. For example, if $\mathbf{\Delta}^{(x)} = (1/3, 1/2, 1/6)$, then $\sigma\left(\mathbf{\Delta}^{(x)}\right) = (1/6, 1/3, 1/2)$. It is obvious that $\Delta_{k_1}(x) = 1/6$ and $k_1 = 3$; $\Delta_{k_2}(x) = 1/3$ and $k_2 = 1$; and $\Delta_{k_3}(x) = 1/2$ and $k_3 = 2$. In general, for these order numbers before and after sorting, we provide their corresponding relation in Figure 2.

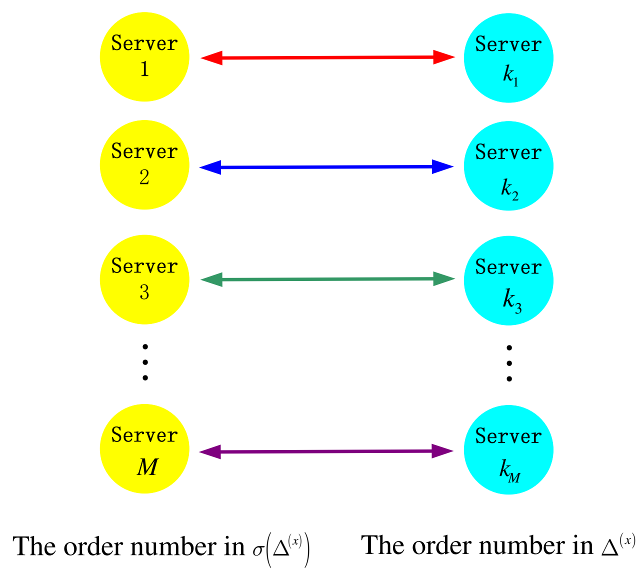

Figure 2: The order relation before and after sorting the M servers

Based on the ascending function with the sorting process, it is a key how to describe the arrivals of customers at each server in this supermarket model. It is worthwhile to note that Janssen [13] gave an effective method for analyzing the ascending function as well as the arrival processes at the M different servers. Here, we provide a detailed description for the Janssen's method as follows.

For a sorted vector x with $0 \le x_1 \le x_2 \le \cdots \le x_M$, it follows from (3.6) and (3.7) in Janssen [13] that the probability that the arriving customer first randomly selects d

servers from M servers, and then enters the ith server (that is, the ith shortest queue is also in the d selected servers) is given by

$$k(M,i,d)=\begin{cases} d\frac{(M-i)!(M-d)!}{(M-i-d+1)!(M)!}, & 1\le i\le M-d+1,\\ 0, & M-d+2\le i\le M,\end{cases} \tag{2}$$

and specifically, we may randomly give a sort for these servers whose queue lengths are equal. At the same time, Lemma 3.2.1 in Janssen [13] proved that for $1 \le d \le M$,

$$\sum_{i=1}^{M} k(M,i,d)=\sum_{i=1}^{M-d+1} k(M,i,d)=1. \tag{3}$$

Now, we explain the probability $k(M,i,d)$ for sorted vector x with $0 \le x_1 \le x_2 \le \cdots \le x_M$.

As seen from Figure 3, notice that the arriving customer first randomly selects d servers from the M servers, and enters one server with the shortest queue length among the d selected servers (if there is a tie, then servers with the shortest queue length will be chosen randomly), thus the routine selection mechanism is converted to the probability $k(M,i,d)$ of entering the ith server for $1 \le i \le M$. Therefore, $\lambda k(M,i,d)$ is the arrival rate that the customers arrive at the server with the ith shortest queue length among the M servers.

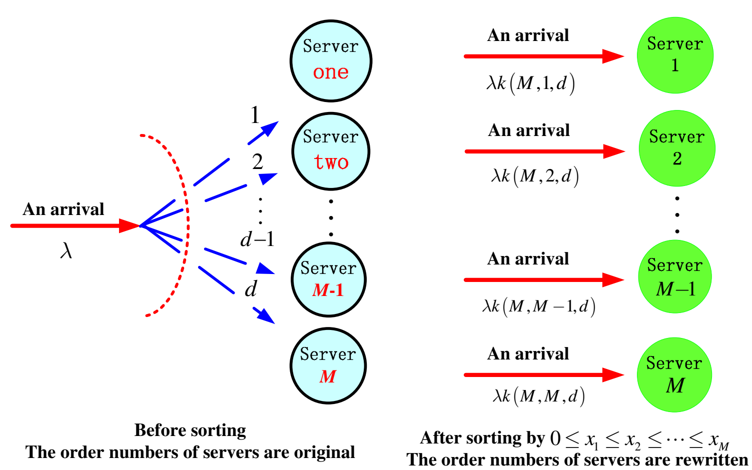

Figure 3: Some interpretation on the probability $k(M,i,d)$

For the ascending function $\sigma\left(\mathbf{\Delta}^{(x)}\right)$ which is similar to the sorted vector x with $0 \le x_1 \le x_2 \le \cdots \le x_M$, it is easy to see that the Janssen's method still work. Thus, for the ith element in $\sigma\left(\mathbf{\Delta}^{(x)}\right)$ (that is, the k_ith element in $\mathbf{\Delta}^{(x)}$, this corresponds to the k_ith server in this supermarket model), using (3) we obtain

$$k(M,k_i,d)=\begin{cases} d\frac{(M-k_i)!(M-d)!}{(M-k_i-d+1)!(M)!}, & 1\leq k_i\leq M-d+1,\\ 0, & M-d+2\leq k_i\leq M. \end{cases}$$

Obviously, we also have

$$\sum_{k_i=1}^{M}k(M,k_i,d)=\sum_{k_i=1}^{M-d+1}k(M,k_i,d)=1.$$

According to the probability $k(M,k_i,d)$, it is clear that the arrivals of customers at the k_ith server is a Poisson process with arrival rate $\lambda k(M,k_i,d)$ for $i=1,2,\ldots,M$. Hence, the Poisson arrival rate at the k_ith server is given by

$$\lambda k(M,k_i,d)=\begin{cases} \lambda d\frac{(M-k_i)!(M-d)!}{(M-k_i-d+1)!(M)!}, & 1\leq k_i\leq M-d+1,\\ 0, & M-d+2\leq k_i\leq M. \end{cases} \tag{4}$$

(2) Analysis of the service processes

Analysis of the service processes is simpler than that of the above arrival processes in this supermarket model. Let $\mathbf{1}_{\{x_i>0\}}$ be an indicator function of the event: $\{x_i>0\}$, that is,

$$\mathbf{1}_{\{x_i>0\}}=\begin{cases} 1, & x_i>0,\\ 0, & x_i=0. \end{cases}$$

The service rate of the ith server may be written as $\mu_i\mathbf{1}_{\{x_i>0\}}$, because the server is idle when there is no customer (i.e., $x_i=0$) in this server.

(3) Choosing a suitable reward value at each state

Note that $\Delta_{k_M}(x)\geq\Delta_{k_1}(x)$, it is obvious that if the value $\left[\Delta_{k_M}(x)-\Delta_{k_1}(x)\right]/\Delta_{k_M}(x)$ is bigger, then the customers in the M servers are not distributed well. On the contrary, if the value $\left[\Delta_{k_M}(x)-\Delta_{k_1}(x)\right]/\Delta_{k_M}(x)$ is smaller, then the customers in the M servers are load balanced very well. Thus, our purpose of designing and optimizing this supermarket model is to make the value $\left[\Delta_{k_M}(x)-\Delta_{k_1}(x)\right]/\Delta_{k_M}(x)$ as small as possible. At the same time, it is easy to see that

$$\min_{d;\lambda;\mu_k,1\leq k\leq M}\left\{\Delta_{k_M}(x)\right\}-\max_{d;\lambda;\mu_k,1\leq k\leq M}\left\{\Delta_{k_1}(x)\right\}\leq\Delta_{k_M}(x)-\Delta_{k_1}(x).$$

Based on the above analysis, we may choose two different reward values at state x as follows:

$$r_{\min}(x):=\Delta_{k_1}(x), \tag{5}$$

and

$$r_{\max}(x) := \Delta_{k_M}(x). \tag{6}$$

Notice that we use the two reward values: $r_{\min}(x)$ and $r_{\max}(x)$, to be able to provide a better observation on performance of this supermarket model, which will be studied in Subsection 5.2.

In the remainder of this section, we introduce a useful continuous-time Markov process, which will be used to give performance computation and performance simulation in the supermarket model with different servers.

Let $X_k(t)$ be the number of customers in the kth server of this supermarket model at time $t \geq 0$, and

$$\mathbf{X}(t) = (X_1(t), X_2(t), \ldots, X_M(t)).$$

Obviously, $\{\mathbf{X}(t) : t \geq 0\}$ is an M-dimensional continuous-time Markov process on the space state $\Omega = \{x = (x_1, x_2, \ldots, x_M) : x_k \geq 0, 1 \leq k \leq M\}$.

Let $r(x)$ be a real function for $x \in \Omega$, and $r(x)$ denote a reward value of this Markov process $\{\mathbf{X}(t) : t \geq 0\}$ at state x. Based on this, we define a random reward function as

$$\Phi(t) = \int_0^t r(\mathbf{X}(\xi))\, \mathrm{d}\xi, \tag{7}$$

which is a stochastic integral, e.g., see Chapter 10 in Li [14] for more details.

In what follows we propose an event-driven technique to deal with the random reward function $\Phi(t)$. To this end, we denote by $\eta_1, \eta_2, \eta_3, \ldots, \eta_n$ the n successive state jump points of the Markov process $\{\mathbf{X}(t) : t \geq 0\}$ in the finite time interval $[0, t]$, it is clear that

$$0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}. \tag{8}$$

Note that $\eta_k = \eta_k^-$, and η_k is a state jump time of Markov process $\{\mathbf{X}(t) : t \geq 0\}$, thus it is helpful for understanding the stochastic integral $\int_0^t r(\mathbf{X}(\xi))\mathrm{d}\xi$ under an interval decomposition as follows:

$$[0, t] = [0, \eta_1^-) \cup [\eta_1, \eta_2^-) \cup [\eta_2, \eta_3^-) \cdots \cup [\eta_{n-1}, \eta_n^-) \cup [\eta_n, t],$$

it follows from (7) and (8) that

$$\Phi(t) = \int_0^{\eta_1^-} r(\mathbf{X}(\xi))\, \mathrm{d}\xi + \sum_{j=1}^{n-1} \int_{\eta_j}^{\eta_{j+1}^-} r(\mathbf{X}(\xi))\, \mathrm{d}\xi + \int_{\eta_n}^{t} r(\mathbf{X}(\xi))\, \mathrm{d}\xi, \tag{9}$$

which is a segmented stochastic integral for expressing the random reward function $\Phi(t)$. Note that this segmented stochastic integrals will be useful in our later study.

4 Computation of the Expected Reward Function

In this section, we use an event-driven technique to compute the mean of the random reward function in a finite time interval, where our computation is based on the above segmented stochastic integral, which is expressed through the successive state jump points generated by either customer arrivals or service completions.

From (a) in Figure 4, let $\{\mathcal{N}(t) : t \geq 0\}$ be a Poisson process with parameter $\omega = \lambda + \mu_1 + \mu_2 + \cdots + \mu_M$. Then for $k \geq 0$

$$\mathbf{p}_k(t) = P\{\mathcal{N}(t) = k\} = e^{-\omega t}\frac{(\omega t)^k}{k!}.$$

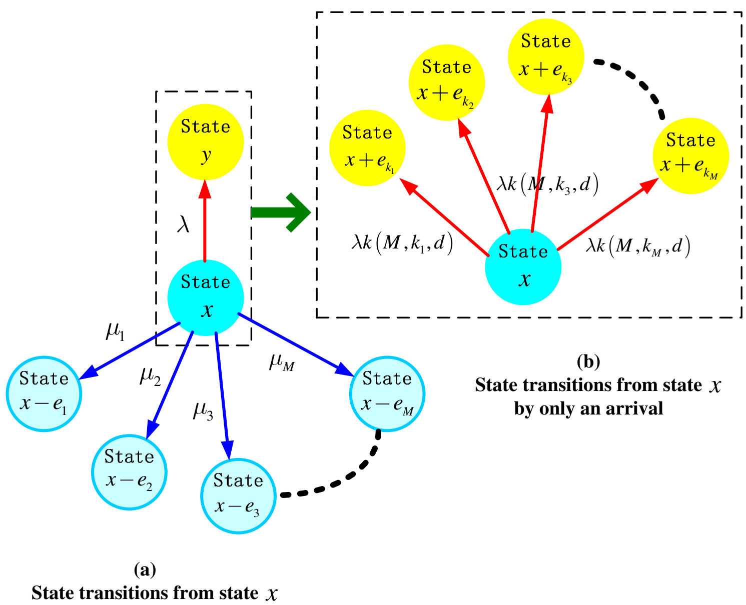

Figure 4: State transitions and associated rates at State x

We assume that the random sequence $\{Y_k : k \geq 1\}$ is i.i.d. and is exponential with mean $1/\omega$. Let $\eta_n = \sum_{k=1}^{n} Y_k$. Then $\mathcal{N}(t) = \sup\{n : \eta_n \leq t\}$, and $0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}$. From Section 2.3 in Ross [35], it is easy to see that

$$P\{\eta_1 \leq s \mid \mathcal{N}(t) = 1\} = P\{Y_1 \leq s \mid \mathcal{N}(t) = 1\} = \frac{s}{t}.$$

Let the n-dimensional probability distribution be

$$F(s_1, s_2, \ldots, s_n) = P\{\eta_1 \leq s_1, \eta_2 \leq s_2, \ldots, \eta_n \leq s_n\}$$

and the n-dimensional probability density function

$$f\left(s_1, s_2, \ldots, s_n\right) = \frac{\partial^n}{\partial s_1 \partial s_2 \cdots \partial s_n} F\left(s_1, s_2, \ldots, s_n\right).$$

Then it follows from Theorem 2.3.1 in Ross [35] that

$$f\left(s_1, s_2, \ldots, s_n\right) = \frac{n!}{t^n}, \quad 0 < s_1 < s_2 < \cdots < s_n < t.$$

At the same time, Theorem 2.3.1 in Ross [35] demonstrates that given that $\mathcal{N}(t) = n$, the n arrival times $\eta_1, \eta_2, \ldots, \eta_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$. Thus, using the condition: $0 < \eta_1 < \eta_2 < \cdots < \eta_n < t$, we obtain

$$E\left[\eta_1\right] = E\left[\eta_2 - \eta_1\right] = \cdots = E\left[\eta_n - \eta_{n-1}\right] = E\left[t - \eta_n\right] = \frac{t}{n+1}. \tag{10}$$

It is seen from (a) and (b) in Figure 4 that for $k \geq 1$, the Markov process $\{\mathbf{X}(t) : t \geq 0\}$ transits to State $\mathbf{X}(\eta_k)$ from State $\mathbf{X}\left(\eta_k^-\right)$ (i.e., a state jump), where State $\mathbf{X}(\eta_k)$ may be either State $\mathbf{X}\left(\eta_k^-\right) - e_j$ due to a service completion by Server j for $1 \leq j \leq M$, or State $\mathbf{X}\left(\eta_k^-\right) + e_{k_i}$ due to a customer arrival at Server k_i with the routine selection mechanism for $1 \leq i \leq M$. Note that $\mathbf{X}\left(\eta_1^-\right) = \mathbf{X}(0) = x$ and $\mathbf{X}\left(\eta_k^-\right) = \mathbf{X}(\eta_{k-1})$ for $2 \leq k \leq n$, thus we have

$$\begin{aligned}
\mathbf{X}(\eta_k) &\in \left\{\mathbf{X}\left(\eta_k^-\right) - e_j : 1 \leq j \leq M\right\} \cup \left\{\mathbf{X}\left(\eta_k^-\right) + e_{k_i} : 1 \leq i \leq M\right\} \\
&= \left\{\mathbf{X}(\eta_{k-1}) - e_j : 1 \leq j \leq M\right\} \cup \left\{\mathbf{X}(\eta_{k-1}) + e_{k_i} : 1 \leq i \leq M\right\}.
\end{aligned}$$

Let A_n be the nth inter-arrival time of the Poisson process with arrival rate λ, and $S_n^{(k)}$ the exponential service time with service rate μ_k of the nth customer in Server k. Then $\{A_n\}$ and $\left\{S_n^{(k)}\right\}$ are all i.i.d for $1 \leq k \leq M$. In this case, we write that $A = A_1$ and $S^{(k)} = S_1^{(k)}$ for $1 \leq k \leq M$. Based on these random variables A and $S^{(k)}$ for $1 \leq k \leq M$, we can express the random events of the Markov process $\{\mathbf{X}(t) : t \geq 0\}$ at time η_k as follows:

(1) An arrival at time η_k

In this case, we need the sufficient condition

$$A < \min_{1 \leq k \leq M} \left\{S^{(k)}\right\}.$$

It is easy to compute that

$$a = P\left\{A < \min_{1 \leq k \leq M} \left\{S^{(k)}\right\}\right\} = \frac{\lambda}{\lambda + \mu_1 + \mu_2 + \cdots + \mu_M}.$$

(2) A service completion in Server j for $1 \leq j \leq M$

In this case, we need the sufficient condition

$$S^{(j)} < \min \left\{ A, \min_{\substack{k \neq j \\ 1 \leq k \leq M}} \left\{ S^{(k)} \right\} \right\}.$$

We can that

$$b^{(j)} = P \left\{ S^{(j)} < \min \left\{ A, \min_{\substack{k \neq j \\ 1 \leq k \leq M}} \left\{ S^{(k)} \right\} \right\} \right\} = \frac{\mu_j}{\lambda + \mu_1 + \mu_2 + \cdots + \mu_M}.$$

Now, we compute the conditional mean $E_x [\Phi(t)]$, where $E_x [\bullet] = E [\bullet \mid \mathbf{X}(0) = x]$. We have

$$\begin{aligned} E[\Phi(t) \mid \mathbf{X}(0) = x] =& \sum_{n=0}^{\infty} P\{\mathcal{N}(t) = n\} E[\Phi(t) \mid \mathbf{X}(0) = x, \mathcal{N}(t) = n] \\ =& P\{\mathcal{N}(t) = 0\} E[\Phi(t) \mid \mathbf{X}(0) = x, \mathcal{N}(t) = 0] \\ &+ \sum_{n=1}^{\infty} P\{\mathcal{N}(t) = n\} E[\Phi(t) \mid \mathbf{X}(0) = x, \mathcal{N}(t) = n], \end{aligned} \tag{11}$$

Since $\mathcal{N}(t) = 0$, it is clear that $\eta_1 > t$, this gives

$$E[\Phi(t) \mid \mathbf{X}(0) = x, \mathcal{N}(t) = 0] = E\left[\int_0^t r(\mathbf{X}(\xi)) \, \mathrm{d}\xi \mid \mathbf{X}(0) = x, \eta_1 > t\right] = r(x)\, t. \tag{12}$$

For $n \geq 1$, notice that the event $\{\mathcal{N}(t) = n\}$ is the same as the event $\{0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}\}$, thus we obtain

$$\begin{aligned} & E[\Phi(t) \mid \mathbf{X}(0) = x, \mathcal{N}(t) = n] \\ &= E\left[\int_0^t r(\mathbf{X}(\xi)) \, \mathrm{d}\xi \mid \mathbf{X}(0) = x, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}\right] \\ &= E\left[\int_0^{\eta_1^-} r(\mathbf{X}(\xi)) \, \mathrm{d}\xi \mid \mathbf{X}(0) = x, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}\right] \\ &+ \sum_{k=1}^{n-1} E\left[\int_{\eta_k}^{\eta_{k+1}^-} r(\mathbf{X}(\xi)) \, \mathrm{d}\xi \mid \mathbf{X}(0) = x, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}\right] \\ &+ E\left[\int_{\eta_n}^{t} r(\mathbf{X}(\xi)) \, \mathrm{d}\xi \mid \mathbf{X}(0) = x, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}\right] \end{aligned} \tag{13}$$

To compute (13), we may observe some useful relations as follows:

(1) If $\xi \in [0, \eta_1^-)$ and $\mathbf{X}(0) = x$, then $\mathbf{X}(\xi) = x$ for $\xi \in [0, \eta_1^-)$.

(2) For $1 \leq j \leq n-1$, if $\xi \in [\eta_j, \eta_{j+1}^-)$ and $\mathbf{X}(\eta_j) = y$, then $\mathbf{X}(\xi) = y$ for $\xi \in [\eta_j, \eta_{j+1}^-)$.

(3) If $\xi \in [\eta_n, t]$ and $\mathbf{X}(\eta_n) = z$, then $\mathbf{X}(\xi) = z$ for $\xi \in [\eta_n, t]$.

Based on the above useful relations, together with (10), we obtain

$$\begin{aligned}
&E\left[\int_0^{\eta_1^-} r(\mathbf{X}(\xi))\,\mathrm{d}\xi \mid \mathbf{X}(0) = x, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}\right] \\
&= r(x)\, E\left[\eta_1^-\right] = r(x)\, E[\eta_1] = r(x)\frac{t}{n+1},
\end{aligned}$$

for $1 \leq k \leq n-1$

$$\begin{aligned}
&E\left[\int_{\eta_k}^{\eta_{k+1}^-} r(\mathbf{X}(\xi))\,\mathrm{d}\xi \mid \mathbf{X}(0) = x, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}\right] \\
&= E\left[r(\mathbf{X}(\eta_k)) \mid \mathbf{X}(0) = x, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}\right] \cdot E\left[\eta_{k+1}^- - \eta_k\right] \\
&= E\left[r(\mathbf{X}(\eta_k)) \mid \mathbf{X}(0) = x, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}\right] \cdot E\left[\eta_{k+1} - \eta_k\right] \\
&= \frac{t}{n+1} E\left[r(\mathbf{X}(\eta_k)) \mid \mathbf{X}(0) = x, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}\right]
\end{aligned}$$

and

$$\begin{aligned}
&E\left[\int_{\eta_n}^{t} r(\mathbf{X}(\xi))\,\mathrm{d}\xi \mid \mathbf{X}(0) = x, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}\right] \\
&= E\left[r(\mathbf{X}(\eta_n)) \mid \mathbf{X}(0) = x, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}\right] \cdot E\left[t - \eta_n\right] \\
&= \frac{t}{n+1} E\left[r(\mathbf{X}(\eta_n)) \mid \mathbf{X}(0) = x, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}\right].
\end{aligned}$$

We write that for $1 \leq k \leq n$

$$\Re_k = E\left[r(\mathbf{X}(\eta_k)) \mid \mathbf{X}(0) = x, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}\right].$$

It follows from (11), (12) and (13) that

$$E\left[\Phi(t) \mid \mathbf{X}(0) = x\right] = r(x)\, t e^{-\omega t} + \sum_{n=1}^{\infty} e^{-\omega t}\frac{(\omega t)^n}{n!} \cdot \frac{t}{n+1}\left[r(x) + \sum_{k=1}^{n} \Re_k\right]. \tag{14}$$

Clearly, it is a key to compute the functions: $\Re_k$ for $1 \leq k \leq n$.

Now, we use (14) to compute the conditional mean $E[\Phi(t) \mid \mathbf{X}(0) = x]$ of the random reward function $\Phi(t)$ through an event-driven technique. To this end, our computation is decomposed in the following three steps:

Step one: Compute $\Re_1 = E\left[r(\mathbf{X}(\eta_1)) \mid \mathbf{X}(0) = x, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}\right]$

It is seen from (a) and (b) in Figure 4 that the Markov process $\{\mathbf{X}(t) : t \geq 0\}$ transits to a state $\mathbf{X}(\eta_1)$ from the initial state x, where the state $\mathbf{X}(\eta_1)$ may be either State $x - e_j$

due to a service completion by Server j for $1 \leq j \leq M$, or State $x+e_{k_i}$ due to a customer arrival at Server k_i for $1 \leq i \leq M$. Using the routine selection mechanism, we have

$$\mathbf{X}(\eta_1) \in \{x - e_j : 1 \leq j \leq M\} \cup \{x + e_{k_i} : 1 \leq i \leq M\}.$$

From (a) and (b) in Figure 4, it is seen that the computation of $\Re_1$ is decomposed into two parts: One by an arrival, and another by a service completion. Thus we obtain

$$\Re_1 = \sum_{i=1}^{M} r(x + e_{k_i}) \cdot a \cdot k(M, k_i, d) + \sum_{j=1}^{M} r(x - e_j) \cdot b^{(j)} \mathbf{1}_{\{x_j > 0\}}, \tag{15}$$

where $a \cdot k(M, k_i, d)$ is the probability that an arriving customer joins Server k_i, and $b^{(j)} \mathbf{1}_{\{x_j > 0\}}$ is the probability that a service is completed in Server j.

Step two: Compute $\Re_2 = E[r(\mathbf{X}(\eta_2)) \mid \mathbf{X}(0) = x, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}]$

It is seen from (a) and (b) in Figure 4 that the Markov process $\{\mathbf{X}(t) : t \geq 0\}$ transits to a state $\mathbf{X}(\eta_2)$ from a state $\mathbf{X}(\eta_1)$ in the set

$$\{x - e_j : 1 \leq j \leq M\} \cup \{x + e_{k_i} : 1 \leq i \leq M\},$$

hence we have

$$\mathbf{X}(\eta_2) = \begin{cases} x - e_j + e_{k_m}, & \text{if an arrival occurs in Server } k_m \text{ at time } \eta_2, \\ x + e_{k_i} + e_{k_m}, & \text{if an arrival occurs in Server } k_m \text{ at time } \eta_2, \\ x - e_j - e_l & \text{if a service is completed in Server } l \text{ at time } \eta_2, \\ x + e_{k_i} - e_l & \text{if a service is completed in Server } l \text{ at time } \eta_2, \end{cases}$$

thus we have

$$\begin{aligned} \mathbf{X}(\eta_2) \in & \{x - e_j + e_{k_m} : 1 \leq j, m \leq M\} \cup \{x + e_{k_i} + e_{k_m} : 1 \leq i, m \leq M\} \\ & \cup \{x - e_j - e_l : 1 \leq j, l \leq M\} \cup \{x + e_{k_i} - e_l : 1 \leq i, l \leq M\}. \end{aligned}$$

Based on the above analysis, it is seen from (a) and (b) in Figure 4 that

$$\begin{aligned} \Re_2 = & \left\{ \sum_{m=1}^{M} \sum_{j=1}^{M} r(x - e_j + e_{k_m}) \cdot b^{(j)} \mathbf{1}_{\{x_j > 0\}} \cdot ak(M, k_m, d) \right. \\ & \left. + \sum_{m=1}^{M} \sum_{i=1}^{M} r(x + e_{k_i} + e_{k_m}) \cdot ak(M, k_i, d) \cdot ak(M, k_m, d) \right\} \\ & + \left\{ \sum_{l=1}^{M} \sum_{j=1}^{M} r(x - e_j - e_l) \cdot b^{(j)} \mathbf{1}_{\{x_j > 0\}} \cdot b^{(l)} \mathbf{1}_{\{(x - e_j)_l > 0\}} \right. \\ & \left. + \sum_{l=1}^{M} \sum_{i=1}^{M} r(x + e_{k_i} - e_l) \cdot ak(M, k_i, d) \cdot b^{(l)} \mathbf{1}_{\{(x + e_{k_i})_l > 0\}} \right\}. \end{aligned} \tag{16}$$

Table 1: The order number of servers with either an arrival or a service completion

State jump points	**Server number by arrival**	**Server number by service**
η_1	k_{i_1}	j_1
η_2	k_{i_2}	j_2
$\vdots$	$\vdots$	$\vdots$
η_n	k_{i_n}	j_n

Step three: Compute $\Re_k = E\left[r\left(\mathbf{X}\left(\eta_k\right)\right) \mid \mathbf{X}\left(0\right) = x, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < t < \eta_{n+1}\right]$ for $3 \leq k \leq n$

From the above two special computations, here we will further develop the event-driven technique to calculate the conditional mean of the random reward function.

For the general term $\Re_k$, our computation is more complicated than that in the above two special cases. To that end, we need to introduce some notation to record the order number of the server with either an arrival or a service completion at each of the state jump points η_k for $k = 1, 2, \ldots, n$. Observing the two expressions (15) and (16), the order numbers of the servers need to relate to the state jump points η_k for $k = 1, 2, \ldots, n$. For simplicity of description, it is necessary to list some notation in Table 1, the purpose of which is to express the state jump points and associated useful information.

For simplification of description, when deriving some conditional means involved below, we introduce a convention notation: $E_Y\left[X\right] = E\left[E\left[X|Y\right]\right]$ (that is, a deterministic value), where X and Y are two random variables.

From Steps one and two, it is easy to see that $\Re_k$ depends on the k successive samples for the states $\mathbf{X}\left(\eta_m\right)$ for $m = 1, 2, \ldots, k-1$. To describe the states $\mathbf{X}\left(\eta_k^-\right)$, we express the successive state jumps as follows: $\mathbf{X}\left(0\right) \underset{\rightarrow}{\times} \mathbf{X}\left(\eta_1\right) \underset{\rightarrow}{\times} \mathbf{X}\left(\eta_2\right) \underset{\rightarrow}{\times} \cdots \underset{\rightarrow}{\times} \mathbf{X}\left(\eta_{k-1}\right)$, where $A \underset{\rightarrow}{\times} B$ denote the Cartesian product from the set A to the set B. Since $\mathbf{X}\left(\eta_k^-\right) = \mathbf{X}\left(\eta_{k-1}\right)$ and our computation depends on the $k-1$ successive samples for the states $\mathbf{X}\left(\eta_m\right)$ for $m = 1, 2, \ldots, k-1$, we set $\mathbf{X}\left(\eta_k^-\right) = \mathbf{X}\left(0\right) \underset{\rightarrow}{\times} \mathbf{X}\left(\eta_1\right) \underset{\rightarrow}{\times} \mathbf{X}\left(\eta_2\right) \underset{\rightarrow}{\times} \cdots \underset{\rightarrow}{\times} \mathbf{X}\left(\eta_{k-1}\right)$, hence the first $k-1$ samples $\mathbf{X}\left(0\right) \underset{\rightarrow}{\times} \mathbf{X}\left(\eta_1\right) \underset{\rightarrow}{\times} \mathbf{X}\left(\eta_2\right) \underset{\rightarrow}{\times} \cdots \underset{\rightarrow}{\times} \mathbf{X}\left(\eta_{k-1}\right)$ is used to record our previous computational process. Therefore, we obtain

$$\Re_k = E_{\mathbf{X}(0) \underset{\rightarrow}{\times} \mathbf{X}(\eta_1) \underset{\rightarrow}{\times} \mathbf{X}(\eta_2) \underset{\rightarrow}{\times} \cdots \underset{\rightarrow}{\times} \mathbf{X}(\eta_{k-1})}\left[r\left(\mathbf{X}\left(\eta_k\right)\right)\right].$$

In this case, we need to represent the initial state $\mathbf{X}(0) \underset{\rightarrow}{\times} \mathbf{X}(\eta_1) \underset{\rightarrow}{\times} \mathbf{X}(\eta_2) \underset{\rightarrow}{\times} \cdots \underset{\rightarrow}{\times} \mathbf{X}(\eta_{k-1})$ by means of $\mathbf{X}(\eta_m) \in \{\bullet - e_{j_m} : 1 \leq j_m \leq M\} \cup \left\{\bullet + e_{k_{i_m}} : 1 \leq i_m \leq M\right\}$ for $1 \leq m \leq k-1$, thus we have

$$\mathbf{X}(0) \underset{\rightarrow}{\times} \mathbf{X}(\eta_1) \underset{\rightarrow}{\times} \mathbf{X}(\eta_2) \underset{\rightarrow}{\times} \cdots \underset{\rightarrow}{\times} \mathbf{X}(\eta_{k-1}) \in \Theta^{(k-1)},$$

where

$$\Theta^{(k-1)} = \Theta_0 \times \Theta_1 \times \Theta_2 \times \cdots \times \Theta_{k-1},$$

$$\begin{aligned}
\Theta_0 &= \{x\} \\
\Theta_1 &= \{\bullet - e_{j_1} : 1 \leq j_1 \leq M\} \cup \left\{\bullet + e_{k_{i_1}} : 1 \leq i_1 \leq M\right\}, \\
\Theta_2 &= \{\bullet - e_{j_2} : 1 \leq j_2 \leq M\} \cup \left\{\bullet + e_{k_{i_2}} : 1 \leq i_2 \leq M\right\}, \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
\Theta_{k-1} &= \left\{\bullet - e_{j_{k-1}} : 1 \leq j_{k-1} \leq M\right\} \cup \left\{\bullet + e_{k_{i_{k-1}}} : 1 \leq i_{k-1} \leq M\right\}.
\end{aligned}$$

To understand the elements in the set $\Theta^{(k-1)}$, we need the Cartesian product as follows:

$$\{A, B\} \times \{C, D\} = \{A \times C, A \times D, B \times C, B \times D\},$$

where A, B, C, D are four sets with finite elements.

In the set $\Theta^{(k-1)}$, the k elements are successively taken from the subsets $\Theta_0, \Theta_1, \Theta_2, \ldots,$ Θ_{k-1}, for example, $x \in \Theta_0$, $\bullet - e_{j_1} \in \Theta_1$, $\bullet + e_{k_{i_2}} \in \Theta_2$, $\ldots$, $\bullet - e_{j_{k-2}} \in \Theta_{k-2}$, $\bullet + e_{k_{i_{k-1}}} \in \Theta_{k-1}$. For the successive k elements, we have a simple computation through the following convention

$$\{x\}\{\bullet - e_{j_1}\} = x - e_{j_1},$$

$$\{x\}\{\bullet - e_{j_1}\}\left\{\bullet + e_{k_{i_2}}\right\} = x - e_{j_1} + e_{k_{i_2}},$$

$$\cdots\cdots\cdots$$

$$\{x\}\{\bullet - e_{j_1}\}\left\{\bullet + e_{k_{i_2}}\right\}\cdots\{\bullet - e_{j_{k-2}}\}\left\{\bullet + e_{k_{i_{k-1}}}\right\} = x - e_{j_1} + e_{k_{i_2}} \cdots - e_{j_{k-2}} + e_{k_{i_{k-1}}}.$$

Based on this, we can easily give a sample of the initial state $\mathbf{X}(0) \underset{\rightarrow}{\times} \mathbf{X}(\eta_1) \underset{\rightarrow}{\times} \mathbf{X}(\eta_2) \underset{\rightarrow}{\times} \cdots \underset{\rightarrow}{\times} \mathbf{X}(\eta_{k-1})$ in the set $\Theta^{(k-1)}$.

Now, we compute the conditional mean $E_{\mathbf{X}(0) \underset{\rightarrow}{\times} \mathbf{X}(\eta_1) \underset{\rightarrow}{\times} \mathbf{X}(\eta_2) \underset{\rightarrow}{\times} \cdots \underset{\rightarrow}{\times} \mathbf{X}(\eta_{k-1})}\left[r\left(\mathbf{X}(\eta_k)\right)\right]$ by means of an iterative algorithm as follows:

(a) For $m=1$, we have

$$E_{\mathbf{X}(0)}\left[r\left(\mathbf{X}\left(\eta_1\right)\right)\right]=\sum_{i=1}^{M} ak\left(M,k_i,d\right) r\left(x+e_{k_i}\right)+\sum_{i=1}^{M} b^{(j)}\mathbf{1}_{\{x_j>0\}} r\left(x-e_j\right).$$

(b) For $m=2$, we have

$$\begin{aligned}E_{\mathbf{X}(0)\underset{\rightarrow}{\times}\mathbf{X}(\eta_1)}\left[r\left(\mathbf{X}\left(\eta_2\right)\right)\right]=&\left\{\sum_{m=1}^{M}\sum_{j=1}^{M} r\left(x-e_j+e_{k_m}\right)\cdot b^{(j)}\mathbf{1}_{\{x_j>0\}}\cdot ak\left(M,k_m,d\right)\right.\\&\left.+\sum_{m=1}^{M}\sum_{i=1}^{M} r\left(x+e_{k_i}+e_{k_m}\right)\cdot ak\left(M,k_i,d\right)\cdot ak\left(M,k_m,d\right)\right\}\\&+\left\{\sum_{l=1}^{M}\sum_{j=1}^{M} r\left(x-e_j-e_l\right)\cdot b^{(j)}\mathbf{1}_{\{x_j>0\}}\cdot b^{(l)}\mathbf{1}_{\left\{(x-e_j)_l>0\right\}}\right.\\&\left.+\sum_{l=1}^{M}\sum_{i=1}^{M} r\left(x+e_{k_i}-e_l\right)\cdot ak\left(M,k_i,d\right)\cdot b^{(l)}\mathbf{1}_{\left\{\left(x+e_{k_i}\right)_l>0\right\}}\right\},\end{aligned}$$

(c) For $m=k\geq 3$, we take an element $y_{k-1}\in\Theta^{(k-1)}$, then

$$\begin{aligned}E_{y_{k-1}}\left[r\left(\mathbf{X}\left(\eta_k\right)\right)\right]=&\left[\sum_{i_k=1}^{M} r\left(y_{k-1}+e_{k_{i_k}}\right) f\left(y_{k-1}\right)\cdot ak\left(M,k_{i_k},d\right)\right.\\&\left.+\sum_{l=1}^{M} r\left(y_{k-1}-e_l\right) f\left(y_{k-1}\right)\cdot b^{(l)}\mathbf{1}_{\left\{(y_{k-1})_l>0\right\}}\right],\end{aligned}\tag{17}$$

and $f\left(y_{k-1}\right)$ is the probability that the state y_{k-1} occurs. It is necessary to provide some interpretation for the probability $f\left(y_{k-1}\right)$ by means of the following three examples:

(c-1) $f\left(y_0\right)=1$ due to $y_0=x$.

(c-2) If $y_1=x-e_{j_1}$, then $f\left(y_1\right)=b^{(j_1)}\mathbf{1}_{\left\{x_{j_1}>0\right\}}$; If $y_1=x+e_{k_{i_1}}$, then $f\left(y_1\right)=ak\left(M,k_{i_1},d\right)$.

(c-3) If $y_2=x-e_{j_1}-e_{j_2}$, then $f\left(y_2\right)=b^{(j_1)}\mathbf{1}_{\left\{x_{j_1}>0\right\}}b^{(j_2)}\mathbf{1}_{\left\{\left(x-e_{j_1}\right)_{j_2}>0\right\}}$; if $y_2=x+e_{k_{i_1}}-e_{j_2}$, then $f\left(y_2\right)=ak\left(M,k_{i_1},d\right)b^{(j_2)}\mathbf{1}_{\left\{\left(x+e_{k_{i_1}}\right)_{j_2}>0\right\}}$; and the other two can similarly be computed and both of them are omitted here.

Note that $y_{k-1} \in \Theta^{(k-1)}$, using (17) we obtain

$$
\begin{aligned}
& E_{\mathbf{X}(0) \underset{\rightarrow}{\times} \mathbf{X}(\eta_1) \underset{\rightarrow}{\times} \mathbf{X}(\eta_2) \underset{\rightarrow}{\times} \cdots \underset{\rightarrow}{\times} \mathbf{X}(\eta_{k-1})} \left[r\left(\mathbf{X}\left(\eta_k \right) \right) \right] \\
= & \sum_{y_{k-1} \in \Theta^{(k-1)}} \left[\sum_{i_k=1}^{M} r\left(y_{k-1} + e_{k_{i_k}} \right) f\left(y_{k-1} \right) \cdot ak\left(M, k_{i_k}, d \right) \right. \\
& \left. + \sum_{l=1}^{M} r\left(y_{k-1} - e_l \right) f\left(y_{k-1} \right) \cdot b^{(l)} \mathbf{1}_{\left\{ (y_k)_l > 0 \right\}} \right]. \qquad (18)
\end{aligned}
$$

Now, we further discuss the key computation of $f\left(y_{k-1}\right)$ whose purpose is to provide some new highlight on the calculation program.

Intuitively, the set of jump states: $\mathbf{X}\left(\eta_1\right) \underset{\rightarrow}{\times} \mathbf{X}\left(\eta_2\right) \underset{\rightarrow}{\times} \cdots \underset{\rightarrow}{\times} \mathbf{X}\left(\eta_k\right)$, can be decomposed into two subsets: One for an arrival and another for a service completion. Based on this, we record the order numbers for either the arrivals or the service completions, for example, if $\mathbf{X}\left(\eta_m\right)$ occurs at an arrival, then we record the order number as V_m; while if $\mathbf{X}\left(\eta_m\right)$ occurs at a service completion, then we record the order number as W_m. Therefore, the set of the order numbers is given by

$$
\{1, 2, 3, \ldots, k\} = \left\{ V_{i_1}, V_{i_2}, V_{i_3}, \ldots, V_{i_p} \right\} \cup \left\{ W_{j_1}, W_{j_2}, W_{j_3}, \ldots, W_{j_{k-p}} \right\},
$$

where $0 \leq p \leq k$. Specifically, if $p = 0$, then the set of the order numbers only contains the service completions; while if $p = k$, then the set of the order numbers only contains the arrivals.

Based on the two subsets $\left\{ V_{i_1}, V_{i_2}, V_{i_3}, \ldots, V_{i_p} \right\}$ and $\left\{ W_{j_1}, W_{j_2}, W_{j_3}, \ldots, W_{j_{k-p}} \right\}$, we obtain

$$
a^p \prod_{m=1}^{p} k(M, k_{i_{V_{im}}}, d) \cdot \prod_{h=1}^{k-p} b^{(j_{W_h})} \mathbf{1}_{\left\{ \left(x - \sum_{m=1}^{h-1} e_{j_{W_m}} + \sum_{V_s \leq W_h - 1} e_{k_{i_{V_s}}} \right)_{j_{W_h}} > 0 \right\}} \cdot r\left(x + \sum_{m=1}^{p} e_{k_{i_{V_{im}}}} - \sum_{h=1}^{k-p} e_{j_{W_h}} \right),
$$

where we have some convention on $\prod_{m=1}^{0} \bullet = 1$ and $\sum_{m=1}^{0} \bullet = 0$, and notice that $k_{i_{V_{im}}}$ depends on the state $y_{(V_{i_m}-1)}$. Thus we obtain

$$\begin{aligned}
&E_{\mathbf{X}(0)\underset{\rightarrow}{\times}\mathbf{X}(\eta_1)\underset{\rightarrow}{\times}\mathbf{X}(\eta_2)\underset{\rightarrow}{\times}\cdots\underset{\rightarrow}{\times}\mathbf{X}(\eta_{k-1})}\left[r\left(\mathbf{X}\left(\eta_k\right)\right)\right] \\
&=\sum_{p=0}^{k}\sum_{i_{V_{i_1}}=1}^{M}\cdots\sum_{i_{V_{i_p}}=1}^{M}\sum_{j_{W_1}=1}^{M}\cdots\sum_{j_{W_{k-p}}=1}^{M}a^p\prod_{m=1}^{p}k(M,k_{i_{V_{i_m}}},d) \\
&\times\prod_{h=1}^{k-p}b^{(j_{W_h})}\mathbf{1}_{\left\{\left(x-\sum_{m=1}^{h-1}e_{j_{W_m}}+\sum_{V_s\leq W_h-1}e_{k_{i_{V_s}}}\right)_{j_{W_h}}>0\right\}}\cdot r\left(x+\sum_{m=1}^{p}e_{k_{i_{V_{i_m}}}}-\sum_{h=1}^{k-p}e_{j_{W_h}}\right). \quad (19)
\end{aligned}$$

Similarly, from the two subsets $\left\{V_{i_1},V_{i_2},V_{i_3},\ldots,V_{i_p}\right\}$ and $\left\{W_{j_1},W_{j_2},W_{j_3},\ldots,W_{j_{k-p}}\right\}$ we obtain

$$f\left(y_k\right)=a^p\prod_{m=1}^{p}k(M,k_{i_{V_{i_m}}},d)\cdot\prod_{h=1}^{k-p}b^{(j_{W_h})}\mathbf{1}_{\left\{\left(x-\sum_{m=1}^{h-1}e_{j_{W_m}}+\sum_{V_s\leq W_h-1}e_{k_{i_{V_s}}}\right)_{j_{W_h}}>0\right\}}.$$

In the remainder of this section, we finally compute the conditional mean $E\left[\Phi\left(t\right)\ \mid\mathbf{X}\left(0\right)=x\right]$ of the stochastic integral $\Phi\left(t\right)$ according to the above steps one to three.

It follows from (14) that

$$\begin{aligned}
E\left[\Phi\left(t\right)\ \mid\mathbf{X}\left(0\right)=x\right]&=r\left(x\right)te^{-\omega t}+\sum_{n=1}^{\infty}e^{-\omega t}\frac{\left(\omega t\right)^n}{n!}\cdot\frac{t}{n+1} \\
&\times\left\{r\left(x\right)+\sum_{k=1}^{n}E_{\mathbf{X}(0)\underset{\rightarrow}{\times}\mathbf{X}(\eta_1)\underset{\rightarrow}{\times}\mathbf{X}(\eta_2)\underset{\rightarrow}{\times}\cdots\underset{\rightarrow}{\times}\mathbf{X}(\eta_{k-1})}\left[r\left(\mathbf{X}\left(\eta_k\right)\right)\right]\right\}, \quad (20)
\end{aligned}$$

where $E_{\mathbf{X}(0)\underset{\rightarrow}{\times}\mathbf{X}(\eta_1)\underset{\rightarrow}{\times}\mathbf{X}(\eta_2)\underset{\rightarrow}{\times}\cdots\underset{\rightarrow}{\times}\mathbf{X}(\eta_{k-1})}\left[r\left(\mathbf{X}\left(\eta_k\right)\right)\right]$ is given in (18) or (19).

5 A Markov Discounted Reward Process

In this section, we provide an effective method for computing the mean of the discounted random reward function in an infinite time interval. Based on this, we give a simple discussion on optimal criterions for designing the supermarket model with different servers.

In the infinite time interval $[0,+\infty)$, it is possible that $E\left[\Phi\left(+\infty\right)\mid\mathbf{X}\left(0\right)=x\right]=+\infty$. To avoid the infinite case, the random reward function is always taken as a discounted reward. Notice that $r\left(x\right)$ is a reward value of the M-dimensional Markov process $\left\{\mathbf{X}\left(t\right):t\geq0\right\}$ at state $x\in\Omega$, we define a discounted random reward function as

$$\Psi\left(\beta\right)=\int_0^{+\infty}e^{-\beta t}r\left(\mathbf{X}\left(t\right)\right)\mathrm{d}t, \quad (21)$$

where $\beta \geq 0$ is a discounted rate, and the discounted factor $e^{-\beta t}$ guarantees that $\Psi(\beta)$ is finite a.s..

If $\Psi(0)$ is finite a.s., then $\Psi(0) = E[\Phi(+\infty)]$ is an ordinary (non-discounted) random reward function, as studied in Section 4 with $t \to +\infty$.

Now, we provide a segmented stochastic integral for expressing the random reward function $\Psi(\beta)$, this will be useful in our following computation.

Let $\eta_1, \eta_2, \eta_3, \ldots$ be the successive state jump points of the M-dimensional Markov process $\{\mathbf{X}(t) : t \geq 0\}$ in the time interval $[0, +\infty)$, it is clear from the Poisson or exponential assumptions that

$$0 < \eta_1 < \eta_2 < \eta_3 < \cdots .$$

At the same time, the sequence: η_1, $\eta_{n+1} - \eta_n$ for $n \geq 1$, is i.d.d. and exponential with mean $1/\omega$. Note that the case with the time interval $[0, +\infty)$ is different from that in Section 4 with respect to analysis of the uniform distributions.

Note that

$$[0, +\infty) = [0, \eta_1^-) \cup [\eta_1, \eta_2^-) \cup [\eta_2, \eta_3^-) \cup \cdots ,$$

it follows from (21) that

$$\Psi(\beta) = \int_0^{\eta_1^-} e^{-\beta t} r(\mathbf{X}(t))\,\mathrm{d}t + \sum_{j=1}^{\infty} \int_{\eta_j}^{\eta_{j+1}^-} e^{-\beta t} r(\mathbf{X}(t))\,\mathrm{d}t. \tag{22}$$

Thus we obtain

$$\begin{aligned} E[\Psi(\beta) \mid \mathbf{X}(0) = x] &= E_x\left[\int_0^{\eta_1^-} e^{-\beta t} r(\mathbf{X}(t))\,\mathrm{d}t\right] + \sum_{k=1}^{\infty} E_{\mathbf{X}(\eta_k^-)}\left[\int_{\eta_k}^{\eta_{k+1}^-} e^{-\beta t} r(\mathbf{X}(t))\,\mathrm{d}t\right] \\ &= r(x)\, E_x\left[\int_0^{\eta_1^-} e^{-\beta t}\mathrm{d}t\right] + \sum_{k=1}^{\infty} E_{\mathbf{X}(\eta_k^-)}\left[r(\eta_k)\int_{\eta_k}^{\eta_{k+1}^-} e^{-\beta t}\mathrm{d}t\right]. \end{aligned} \tag{23}$$

Note that our following computation shows that $E[\Psi(\beta) \mid \mathbf{X}(0) = x]$ is not about the taken sequence $\{\eta_k : k \geq 1\}$.

Since $r(\mathbf{X}(t)) = r(x)$ for $t \in [0, \eta_1^-)$ and $r(\mathbf{X}(t)) = r(\eta_k)$ for $t \in [\eta_k, \eta_{k+1}^-)$, we need to compute $E\left[\int_0^{\eta_1^-} e^{-\beta t}\mathrm{d}t\right]$ and $E\left[\int_{\eta_k}^{\eta_{k+1}^-} e^{-\beta t}\mathrm{d}t\right]$ for $k \geq 1$.

It is easy to check that

$$E\left[\int_0^{\eta_1^-} e^{-\beta t}\mathrm{d}t\right] = \frac{1}{\beta + \lambda + \mu_1 + \mu_2 + \cdots + \mu_M}. \tag{24}$$

To compute $E\left[\int_{\eta_k}^{\eta_{k+1}^-} e^{-\beta t}\mathrm{d}t\right]$, let the random variable Γ be exponential with parameter $\lambda+\mu_1+\mu_2+\cdots+\mu_M$. Then we have

$$E\left[\int_{\eta_k}^{\eta_{k+1}^-} e^{-\beta t}\mathrm{d}t\right]=E^{(\Gamma)}\left[E^{(\eta_k)}\left[\int_{\eta_k}^{\eta_k+\Gamma} e^{-\beta t}\mathrm{d}t\right]\right], \tag{25}$$

where $E^{(Y)}[\bullet]$ denote such a mean with respect to the random variable Y. It is clear that η_k is a random variable with the Erlang distribution of order k as follows:

$$P\{\eta_k\leq y\}=1-\exp\{-(\lambda+\mu_1+\mu_2+\cdots+\mu_M)\,y\}\sum_{j=0}^{k-1}\frac{[(\lambda+\mu_1+\mu_2+\cdots+\mu_M)\,y]^j}{j!}.$$

Hence it follows from (25) that

$$E\left[\int_{\eta_k}^{\eta_{k+1}^-} e^{-\beta t}\mathrm{d}t\right]=\int_0^{+\infty}\int_0^{+\infty}\int_y^{y+x} e^{-\beta t}\mathrm{d}t\mathrm{d}P\{\Gamma\leq x\}\,\mathrm{d}P\{\eta_k\leq y\}. \tag{26}$$

Based on (25) and (26), we set

$$\theta_0(\beta)=E\left[\int_0^{\eta_1^-} e^{-\beta t}\mathrm{d}t\right]$$

and for $k\geq 1$

$$\theta_k(\beta)=E\left[\int_{\eta_k}^{\eta_{k+1}^-} e^{-\beta t}\mathrm{d}t\right].$$

Note that the sequence $\{\theta_n(\beta): n\geq 0\}$ can explicitly be determined by (25) and (26), although we omit some computational details.

It is easy to check that

$$E_{\mathbf{X}(0)}\left[\int_0^{\eta_1^-} e^{-\beta t} r(\mathbf{X}(t))\,\mathrm{d}t\right]=\frac{r(x)}{\beta+\lambda+\mu_1+\mu_2+\cdots+\mu_M}=\theta_0(\beta)\,r(x).$$

Now, we compute $E_{\mathbf{X}(\eta_k^-)}\left[\int_{\eta_k}^{\eta_{k+1}^-} e^{-\beta t} r(\mathbf{X}(t))\,\mathrm{d}t\right]$ by a similar method given in (18) as follows:

$$\begin{aligned}
&E_{\mathbf{X}(0)\underset{\rightarrow}{\times}\mathbf{X}(\eta_1)\underset{\rightarrow}{\times}\mathbf{X}(\eta_2)\underset{\rightarrow}{\times}\cdots\underset{\rightarrow}{\times}\mathbf{X}(\eta_{k-1})}\left[\int_{\eta_k}^{\eta_{k+1}^-} e^{-\beta t} r(\mathbf{X}(t))\,\mathrm{d}t\right]\\
=&\theta_k(\beta)\sum_{y_{k-1}\in\Theta^{(k-1)}}\left[\sum_{i_k=1}^{M} r\left(y_{k-1}+e_{k_{i_k}}\right)f(y_{k-1})\cdot ak(M,k_{i_k},d)\right.\\
&\left.+\sum_{l=1}^{M} r(y_{k-1}-e_l)\,f(y_{k-1})\cdot b^{(l)}\mathbf{1}_{\{(y_k)_l>0\}}\right].
\end{aligned} \tag{27}$$

It follows from (23), (24) and (27) that

$$E\left[\Psi\left(\beta\right)\ \mid \mathbf{X}\left(0\right)=x\right]=\theta_{0}\left(\beta\right)r\left(x\right)+\sum_{k=1}^{\infty}\theta_{k}\left(\beta\right)\sum_{y_{k-1}\in\Theta^{(k-1)}}\left[\sum_{i_{k}=1}^{M}r\left(y_{k-1}+e_{k_{i_{k}}}\right)f\left(y_{k-1}\right)\right.$$
$$\left.\times ak\left(M,k_{i_{k}},d\right)+\sum_{l=1}^{M}r\left(y_{k-1}-e_{l}\right)f\left(y_{k-1}\right)\cdot b^{(l)}\mathbf{1}_{\left\{(y_{k})_{l}>0\right\}}\right]. \tag{28}$$

It is seen from (28) that $E\left[\Psi\left(\beta\right)\ \mid \mathbf{X}\left(0\right)=x\right]$ is discounted by the β-sequence $\left\{\theta_{n}\left(\beta\right):n\geq 0\right\}$, which guarantees that $E\left[\Psi\left(\beta\right)\ \mid \mathbf{X}\left(0\right)=x\right]<+\infty$.

In the remainder of this section, we provide a simple discussion for optimal design of the supermarket model with different servers. Specifically, such an optimization may be realized through an event-driven technique with performance simulation as well as perturbation realization, e.g., see Cao [6] and Xia and Cao [46].

To realize an optimal design, the parameters of this supermarket model can be classified as three different groups: (1) The customer arrival parameters λ; and $g_{1},g_{2},\ldots,g_{M}$. (2) The customer service parameters M; d; $\mu_{1},\mu_{2},\ldots,\mu_{M}$. (3) The economic parameters $r\left(x\right)$ for $x\in\Omega$. In general, the customer arrival parameters are always fixed, given that the customer resource and environment are fixed; while the economic parameters are chosen in order that performance optimization of this supermarket model can be easy to be carried out. Based on this, our optimal design is to focus on taking the optimal service parameters: M; d; $\mu_{1},\mu_{2},\ldots,\mu_{M}$.

From a practical point of view of performance optimization, we take two different reward values: $r_{\min}(x):=\Delta_{k_{1}}(x)$, and $r_{\max}(x):=\Delta_{k_{M}}(x)$ for $x\in\Omega$, respectively. Thus, for $r\left(x\right)=r_{\min}(x)=\Delta_{k_{1}}(x)$ for $x\in\Omega$, we write

$$\mathbf{\Psi}\left(\beta,r_{\min}\right)=E\left[\Psi\left(\beta,r_{\min}\right)\ \mid \mathbf{X}\left(0\right)=x\right];$$

while for $r\left(x\right)=r_{\max}(x)=\Delta_{k_{M}}(x)$ for $x\in\Omega$, we set

$$\mathbf{\Psi}\left(\beta,r_{\max}\right)=E\left[\Psi\left(\beta,r_{\max}\right)\ \mid \mathbf{X}\left(0\right)=x\right].$$

Based on $E\left[\Psi\left(\beta\right)\ \mid \mathbf{X}\left(0\right)=x\right]$, using an event-driven technique with performance simulation as well as perturbation realization, we can obtain the optimal decision parameters M^{*}; d^{*}; $\mu_{1}^{*},\mu_{2}^{*},\ldots,\mu_{M}^{*}$ such that

$$\mathbf{\Psi}^{*}\left(\beta,r_{\min}\right)=\max\left\{\mathbf{\Psi}\left(\beta,r_{\min}\right)\right\}. \tag{29}$$

Similarly, we can also give the optimal decision parameters $M^{\diamond}$; $d^{\diamond}$; $\mu_1^{\diamond}, \mu_2^{\diamond}, \ldots, \mu_M^{\diamond}$ such that

$$\mathbf{\Psi}^{\diamond}(\beta, r_{\max}) = \min \{\mathbf{\Psi}(\beta, r_{\max})\}. \tag{30}$$

Furthermore, we can get the optimal decision parameters $M^{\bigtriangledown}$; $d^{\bigtriangledown}$; $\mu_1^{\bigtriangledown}, \mu_2^{\bigtriangledown}, \ldots, \mu_M^{\bigtriangledown}$ such that

$$\mathbf{L}^{\bigtriangledown}(\beta) = \min \{\mathbf{\Psi}(\beta, r_{\max}) - \mathbf{\Psi}(\beta, r_{\min})\}. \tag{31}$$

According to the above analysis, to design the supermarket model with different servers, it is seen from Equations (29) to (31) that here we provide two optimal criterions as follows:

Criterion one: This supermarket model is better when choosing some parameters such that $|\mathbf{\Psi}^{\diamond}(\beta, r_{\max}) - \mathbf{\Psi}^{*}(\beta, r_{\min})| < \delta_1$ for a given value $\delta_1 > 0$.

Criterion two: This supermarket model is better when choosing some parameters such that $\mathbf{L}^{\bigtriangledown}(\beta) < \delta_2$ for a given value $\delta_2 > 0$.

In general, the two optimal criterions can easily be implemented by means of the event-driven technique with performance simulation as well as perturbation realization, e.g., see Cao [6] and Xia and Cao [46] for more details.

6 Performance Simulation

In this section, we provide three simulation experiments whose purpose is to simply discuss how the expected queue length of each server depends on some key parameters: The choice number d, the service rate vector μ and the probability vector g of individual preference in the supermarket model with different servers.

In the three simulation experiments, we take the server number $M = 10$ and the arrival rate $\lambda = 10$.

Experiment one: In the supermarket model with different servers, we take that the choice number $d = 2$; the service rates of the 10 servers are listed as $\mu_1 = 1.1$, $\mu_2 = 1.2$, $\mu_3 = 1.3$, $\mu_4 = 1.4$, $\mu_5 = 1.5$, $\mu_6 = 1.6$, $\mu_7 = 1.7$, $\mu_8 = 1.8$, $\mu_9 = 1.9$ and $\mu_{10} = 2.0$, respectively; the probabilities of individual preference for the 10 servers are given by $g_1 = 0.10$, $g_2 = 0.20$, $g_3 = 0.30$, $g_4 = 0.05$, $g_5 = 0.05$, $g_6 = 0.02$, $g_7 = 0.10$, $g_8 = 0.03$, $g_9 = 0.10$ and $g_{10} = 0.05$, respectively. We simulate the expected queueing length for each

Table 2: The expected queue lengths in the 10 servers

Server number	**Expected queue lengths**	
One	0.6834	
Two	0.9454	
Three	1.0440	
Four	0.4318	
Five	0.4234	
Six	0.2894	
Seven	0.4864	
Eight	0.2793	
Nine	0.4319	
Ten	0.2640	

server by using the routine selection function

$$\Delta_i(x) = \frac{1 + \frac{x_i}{\mu_i g_i}}{\sum_{j=1}^{M} \left[1 + \frac{x_j}{\mu_j g_j}\right]}, \quad i = 1, 2, \ldots, 10.$$

The experimented results are shown in Table 1.

Experiment two: This experiment takes the different parameters: only the 10 service rates, from that in Experiment one. That is, the choice number $d = 2$; the service rates of the 10 servers are listed as $\mu_1 = 1$, $\mu_2 = 2$, $\mu_3 = 6$, $\mu_4 = 8$, $\mu_5 = 10$, $\mu_6 = 16$, $\mu_7 = 17$, $\mu_8 = 18$, $\mu_9 = 25$ and $\mu_{10} = 26$, respectively; the probabilities of individual preference for the 10 servers are given by $g_1 = 0.10$, $g_2 = 0.20$, $g_3 = 0.30$, $g_4 = 0.05$, $g_5 = 0.05$, $g_6 = 0.02$, $g_7 = 0.10$, $g_8 = 0.03$, $g_9 = 0.10$ and $g_{10} = 0.05$, respectively. We still simulate the expected queueing length for each server by using the routine selection function

$$\Delta_i(x) = \frac{1 + \frac{x_i}{\mu_i g_i}}{\sum_{j=1}^{M} \left[1 + \frac{x_j}{\mu_j g_j}\right]}, \quad i = 1, 2, \ldots, 10.$$

The experimented results are shown in Table 2. It is seen from Tables 1 and 2 that the

Table 3: The expected queue lengths in the 10 servers

Server number	**Expected queue lengths**	
One	0.3459	
Two	0.1656	
Three	0.0274	
Four	0.0158	
Five	0.0105	
Six	0.0042	
Seven	0.0038	
Eight	0.0034	
Nine	0.0018	
Ten	0.0017	

expected queue lengths of the M servers decrease, as the service rates of some servers increase.

Experiment three: Comparing with Experiments one and two, this experiment takes more different parameters. We take that the choice number $d = 3$; the service rates of the 10 servers are listed as $\mu_1 = 1$, $\mu_2 = 3$, $\mu_3 = 3$, $\mu_4 = 6$, $\mu_5 = 6$, $\mu_6 = 6$, $\mu_7 = 6$, $\mu_8 = 9$, $\mu_9 = 9$ and $\mu_{10} = 15$, respectively; the probabilities of individual preference for the 10 servers are given by $g_1 = 0.05$, $g_2 = 0.20$, $g_3 = 0.30$, $g_4 = 0.03$, $g_5 = 0.05$, $g_6 = 0.10$, $g_7 = 0.10$, $g_8 = 0.05$, $g_9 = 0.02$ and $g_{10} = 0.10$, respectively. We simulate the expected queueing length for each server by using the routine selection function

$$\Delta_i(x) = \frac{1 + \frac{x_i}{\mu_i g_i}}{\sum\limits_{j=1}^{M} \left[1 + \frac{x_j}{\mu_j g_j}\right]}, \quad i = 1, 2, \ldots, 10.$$

The experimented results are shown in Table 3. It is seen from Tables 1, 2 and 3 that the expected queue lengths of the M servers decrease largely, as the choice number d changes from 2 to 3. Therefore, "the power of two choices" is still kept well in the study of supermarket models with different servers.

Table 4: The expected queue lengths in the 10 servers

Server number	**Expected queue lengths**	
One	0.3447	
Two	0.0580	
Three	0.8598	
Four	0.0265	
Five	0.0265	
Six	0.0266	
Seven	0.0265	
Eight	0.0126	
Nine	0.0127	
Ten	0.0048	

7 Concluding Remarks

In this paper, we provide a novel method for analyzing the supermarket model with different servers through a multi-dimensional continuous-time Markov reward process, and develop an event-driven technique both for computing the mean of the random reward function in a finite time interval and for calculating the mean of the discounted random reward function in an infinite time interval. We indicate that the event-driven technique are useful in the study of supermarket models with different servers, and more generally, in the analysis of large-scale Markov reward processes. Notice that the supermarket model with different servers is an important tool to set up some basic relations between the system performance and the job routing rule, thus it can also help to design reasonable architecture to improve the performance and to balance the load in this supermarket model.

This paper provides a clear picture for how to use the event-driven technique to analyze multi-dimensional continuous-time Markov reward processes, which leads to performance analysis of the supermarket model with different servers. We illustrate that this picture is organized as three key parts: (1) Constructing a routine selection mechanism that depends on the queue lengths, on the service rates, on the probability of individual preference and so forth. (2) From the state jump points of the continuous-time Markov reward process, we set up some segmented stochastic integrals of the random reward function by means of an event-driven technique. Based on this, we compute the mean of the random reward function in a finite time interval, and also calculate the mean of the discounted random reward function in an infinite time interval. Therefore, the results of this paper give new highlight on understanding influence of the different servers on designing the routine selection mechanism and on performance computation of more general supermarket models. Along such a line, there are a number of interesting directions for potential future research, for example,

- analyzing non-Poisson inputs such as, renewal processes; and discussing non-exponential service time distributions, for example, general distributions, matrix-exponential distributions and heavy-tailed distributions;

- studying how to design a new routine selection mechanism with respect to key random factors, such as, the least workload, and the subjective behavior of customers;

- developing effective algorithms both for computing the means of the random reward functions and for solving the optimal problems in the study of supermarket models with different servers; and

- The event-driven technique is further developed for discussing the sample paths of continuous-time Markov reward processes, thus the results given in this paper may be very useful for performance simulation of more general supermarket models with different servers.

Up to now, we believe that a larger gap exists when dealing with either non-Poisson inputs or non-exponential service times in supermarket models with different servers, because the event-driven technique needs be established for being able to deal with more general Markov reward processes.

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