Tools

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AlgDiff: An open source toolbox for the design, analysis and discretisation of algebraic differentiators

AlgDiff: Eine Open-Source-Toolbox für den Entwurf, die Analyse und die Diskretisierung algebraischer Ableitungsschätzer

Abstract: Algebraic differentiators have attracted much interest in recent years. Their simple implementation as classical finite impulse response digital filters and systematic tuning guidelines may help to solve challenging problems, including, but not limited to, nonlinear feedback control, model-free control, and fault diagnosis. This contribution introduces the open source toolbox AlgDiff for the design, analysis and discretisation of algebraic differentiators.

Keywords: Numerical differentiation, orthogonal polynomials, linear filtering

Zusammenfassung: Algebraische Ableitungsschätzer haben in den letzten Jahren großes Interesse erlangt. Dank ihrer einfachen Implementierung als klassische digitale Filter mit endlicher Impulsantwort und der systematischen Parametrierungsansätze können sie zur Lösung anspruchsvoller regelungstechnischer Probleme beitragen. Dieser Beitrag stellt die Open-Source-Toolbox AlgDiff für den Entwurf, die Analyse und die Diskretisierung algebraischer Ableitungsschätzer vor.

Schlagwörter: Numerisches Differenzieren, orthogonale Polynome, lineare Filterung

1 Introduction

Numerical estimation of derivatives of measured signals is a long-standing and challenging problem because small perturbations in the measurements may yield significant errors in the estimates. Numerous approaches have been proposed to address this problem: frequency-domain digital filter design [1, 2], Tikhonov regularisation [3, 4], observers design [5–9], local least-squares fitting of data by polynomials [10, 11], and differentiation by integration methods [12–14]. Algebraic differentiators have been proposed in [15, 16] and further discussed and extended in [17–25], for example. A detailed introduction to these differentiation techniques, their historical evolution, and links to established methods in the literature are summarised in the free-access survey [26].

Algebraic differentiators may be interpreted as differentiation by integration methods and may be implemented as classical finite impulse response (FIR) digital filters. The design of these differentiators involves the judicious choice of up to five parameters, which can be challenging. For example, it has been pointed out in [15, 16] that accepting a slight but known delay in the estimates reduces the order of the approximation error. In [23], it has been observed that delay-free differentiators may exhibit intolerable gains in their amplitude spectra.

Analysing the effects of the parameters on the frequency-domain properties has been very useful in deriving systematic tuning guidelines. In [21], it has been shown that it is possible to approximate the differentiators as low-pass filters. The tuning parameters can then be determined by specifying the desired cutoff frequency and stopband slope. Further effects of the parameters on the frequency-domain properties have been discussed in [22, 24, 25], for example. The simple implementation of these differentiators and their good robustness with respect to measurement disturbances has motivated their wide use. An extensive list of applications including but not limited to parameter estimation, fault detection and feedback control has been published in [26].

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Fig. 1: Screenshot of the GUI for a specific parametrisation of algebraic differentiators.

A first version of an open-source toolbox implementing all necessary functions for the design, analysis, and discretisation of algebraic differentiators has been made available with the survey [26]. The toolbox is implemented in Python. Numerous examples show how it can be used in MATLAB. The toolbox [27] allows users to specify desired frequency-domain properties and estimating derivatives of signals in very few lines of code, for example. Time-domain and frequency-domain properties are automatically calculated and can be easily accessed. In the current contribution a step-by-step tutorial is provided for algebraic differentiators using this toolbox. Relevant properties are recalled and demonstrated. Discretisation effects and a special parametrisation for the exact annihilation of harmonic signals first discussed in [21] are also addressed using simple examples. The source code of the latter examples is made freely available in the toolbox repository for Python and MATLAB.

This tutorial-like article is structured as follows. Section 2 describes the main features of the toolbox. Section 3 recalls the basics of algebraic differentiators required for the remaining parts. Section 4 constitutes the main part of this work and recalls the properties of algebraic differentiators in the light of the toolbox. Example code snippets are used to show how the different functions can be used. A short conclusion is provided in Section 5. Appendices A to E summarise explicit mathematical expressions related to algebraic differentiators that are provided for the sake of completeness.

2 The toolbox AlgDiff

The toolbox contains an implementation in Python for all necessary functions for the design, analysis, and discretisation of algebraic differentiators. The toolbox is available under the BSD-3-Clause License, making it suitable for academic and industrial/commercial use. An interface to MATLAB is included in the package, and several use cases are documented. The repository also contains detailed documentation of the Python class and its methods. The prerequisites for using the toolbox are described in detail.

The repository also includes a graphical user interface (GUI) that further simplifies the design of algebraic differentiators. The GUI enables specifying desired time-domain and frequency-domain properties for the filters. Relevant properties like the approximation delay, the discretisation error, the cutoff frequency, and the filter window length are displayed. Numerical values for amplitude spectra, step responses, impulse responses, and filter coefficients can be exported. Different approaches can be used to discretise the differentiators. Measurement data can be

Table 1: Extract from [26]: Interpretations of $g = g_{N,T,\vartheta}^{(\alpha,\beta)}$ and the practical usage of ϵ	ach.
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Context	Interpretation	Practical usage
Approximation- theoretic	Polynomial approximation of derivative using a Hilbert reproducing kernel	Analysis of estimation properties and relation to established differentiation methods
	Linear time-invariant filtering of the sought derivative: $y^{(n)}$ g $\hat{y}^{(n)}$	Analysis of filter properties and parametrisation
Systems-theoretic	Linear time-invariant filtering of the measured signal: $y^{(n)}$ g $\hat{y}^{(n)}$	Discretisation and implementation

loaded into the GUI and derivatives can then be easily computed. Fig. 1 shows a screenshot of the GUI for a specific algebraic differentiator.

3 Background material on algebraic differentiators

This section provides a brief summary of the fundamentals of algebraic differentiators, which were originally developed in [15, 16]. A survey on the existing results, interpretations, relationships to established methods, tuning guidelines, and applications can be found in [26].

3.1 Derivative estimation

While the early works [15, 16] use differential algebraic manipulations and operational calculus to derive the differentiators, later ones such as [20-22, 25, 28] use time-domain derivations. This shall be briefly recalled.

Let $\mathcal{I}_t = [t - T, t] \subset \mathbb{R}, T > 0$, be an arbitrary closed interval and consider a Lebesgue integrable signal $y: \mathcal{I}_t \to$ \mathbb{R} . The derivatives up to a finite order $n < \min\{\alpha, \beta\} + 1$, with $\alpha, \beta \in \mathbb{R}$ and $\alpha, \beta > -1$, of y can be approximated using truncated generalised Fourier expansions of order N based on orthogonal Jacobi polynomials as

$$\hat{y}^{(n)}(t) = \int_{t-T}^{t} g^{(n)}(t-\tau)y(\tau)d\tau, \ g(\tau) = g_{N,T,\vartheta}^{(\alpha,\beta)}(\tau), \ (1)$$

with $g_{N,T,\vartheta}^{(\alpha,\beta)}$ defined in (A.7). In the following, the kernel $g_{N,T,\vartheta}^{(\alpha,\beta)}$ and the parameter T are called algebraic differentiator and window length, respectively. The scalars α and β parametrise the weight function of the Jacobi polynomials. Section 3.2 discusses their effects on the filter properties.

The estimate $\hat{y}^{(n)}$ corresponds to a delayed approximation of $y^{(n)}$ at time t and the delay, denoted by δ_t , can be parametrised by ϑ . It can be expressed as

$$\delta_t = \begin{cases} \frac{\alpha + 1}{\alpha + \beta + 2} T, & N = 0, \\ \frac{1 - \vartheta}{2} T, & N \neq 0. \end{cases}$$

If ϑ is a zero of the polynomial $\mathbf{P}_{N+1}^{(\alpha,\beta)}$, which yields a small but known delay, the order of the approximation is increased, as first pointed out in [16] (see also [22, 26]). Alternatively, a delay-free approximation can be achieved for $\vartheta = 1$ or a prediction for $\delta_t < 0$ (see also [22, 26, 29]).

3.2 Interpretations

As discussed in [26], different interpretations can be attached to algebraic differentiators. Two important ones are recalled in Table 1. The approximation-theoretic interpretation stems from the time-domain derivation of the differentiators. It is helpful for the analysis of the approximation error and delays. The systems-theoretic interpretation is a direct consequence of the definition of the approximated derivative in (1): The estimate is the output of a linear time invariant filter where the kernel is the *n*-th order derivative of $g_{N,T,\vartheta}^{(\alpha,\beta)}$. Applying an integration by parts in (1) shows that the estimate can be interpreted



Fig. 2: Impulse and step responses of algebraic differentiators with parameters $\alpha, \beta \in \{1, 3, 6\}$, and N = 0.

as the output of a filter with kernel $g_{N,T,\vartheta}^{(\alpha,\beta)}$. The filter output, then, is the sought derivative. This interpretation will be used in the next section to design differentiators using the toolbox.

4 Algebraic differentiators in the light of AlgDiff

This section reviews the properties of algebraic differentiators in the time and frequency domains and shows how the toolbox can be used to analyse these filters. First, the continuous-time differentiators are considered. Then, discretisation issues are discussed and functions of the toolbox that can be used to ensure that the discrete-time filter preserves the properties of the continuous-time ones are introduced. Finally, the estimation of the derivative of an example signal is considered. The code snippets provided in this section are written in Python to adhere to the open-source philosophy of the package. However, the repository contains the corresponding MATLAB code.

4.1 Impulse and step responses

The impulse and step responses of an algebraic differentiator $g_{N,T,\vartheta}^{(\alpha,\beta)}$ have been derived in [22] and are summarised in Appendix C. Example 1 shows how the toolbox can be used to easily compute impulse responses, their derivatives and step responses of an algebraic differentiator.

Example 1 (Impulse and step responses)

The impulse and step responses of an algebraic differentiator with parameter $\alpha = 4$, $\beta = 4$, N = 0, and T = 0.1 can be computed using the toolbox as shown in Code snippet 1.

Create an instance of the class
diffA = AlgebraicDifferentiator(N=0, alpha)
=4., beta = 4., T = 0.1
Evaluate the impulse and step responses
for a given time range $t \in [-0.05, 0.05]$
t = np. linspace(-0.05, 0.1, 100)
Order of derivative of impulse res.
$order_der = 0$
Evaluate the impulse response
$g = diffA \cdot evalKernelDer(order_der, t)$
Evaluate the step response
$h = diffA \cdot get_stepResponse(t)$

Code snippet 1: Evaluation of the impulse and step responses of an algebraic differentiator.

Fig. 2 shows the impulse and step responses of algebraic differentiators for N = 0 and different values of α and β . It can be observed that increasing β and keeping α constant causes a stronger weighting of the impulse response towards 0. This increase implies a stronger weighting of more recent values of the sought derivative and corresponds to the reduction of the estimation delay. In contrast, increasing α and keeping β constant yields a stronger weighting of the impulse response in the direction of T, which leads to a stronger weighting of older values of the sought derivative and, thus, to an increase of the estimation delay. The effects of the parameters on the delay has been analysed analytically in [16, 20, 25, 26], for example. \Box

The analytical properties of the impulse response and its derivatives can be used to derive tuning guidelines for fault detection algorithms, for example. Approaches have been developed and experimentally validated in [21, 22, 28, 30], for example.



Fig. 3: Amplitude and phase spectra of the differentiator from Code snippet 2. The approximation (2) and the upper bound from [22, 25] are also given.

4.2 Frequency-domain properties

By recalling that the estimate of the *n*-th order derivative of y is a convolution product of the signal and the *n*-th order derivative of the differentiator $g_{N,T,\vartheta}^{(\alpha,\beta)}$, it follows that

$$\mathcal{F}\left\{\hat{y}^{(n)}\right\}(\omega) = (\iota\omega)^n \mathcal{G}_{N,T,\vartheta}^{(\alpha,\beta)}(\iota\omega)\mathcal{Y}(\omega), \quad \iota^2 = -1,$$

with $\mathcal{F}\left\{\hat{y}^{(n)}\right\}$, $\mathcal{G}_{N,T,\vartheta}^{(\alpha,\beta)}$, and \mathcal{Y} the Fourier transforms of $\hat{y}^{(n)}$, $g_{N,T,\vartheta}^{(\alpha,\beta)}$, and y, respectively. For the sake of completeness, the closed form of $\mathcal{G}_{N,T,\vartheta}^{(\alpha,\beta)}$ is recalled in (A.9). Thus, this numerical differentiation scheme falls within the framework of classical approaches for differentiation in the frequency domain, where a smoothing filter, in this case $g_{N,T,\vartheta}^{(\alpha,\beta)}$, is followed by an ideal differentiation operator $(\iota\omega)^n$. This property has first been observed in [23].

Analysing $\mathcal{G}_{N,T,\vartheta}^{(\alpha,\beta)}$ is thus helpful to derive tuning guidelines by specifying desired frequency-domain properties. The approximation

$$\left|\mathcal{G}_{N,T,\vartheta}^{(\alpha,\beta)}(\omega)\right| \approx \tilde{\mathcal{G}}_{N,T,\vartheta}^{(\alpha,\beta)}(\omega) = \begin{cases} 1, & |\omega| \le \omega_{\rm c}, \\ \left|\frac{\omega_{\rm c}}{\omega}\right|^{\mu}, & \text{otherwise,} \end{cases}$$
(2)

of the amplitude spectrum of an algebraic differentiator has been proposed in [21], where ω_c is defined in (A.11) and $\mu = 1 + \min\{\alpha, \beta\}$. It follows that $\mathcal{G}_{N,T,\vartheta}^{(\alpha,\beta)}$ can be interpreted as a low pass filter: The frequency ω_c and the set $|\omega| < \omega_c$ are called cutoff frequency and passband, respectively. The set $|\omega| > \omega_c$ is called stopband.

Tuning guidelines proposed in [21, 22] can then be inferred from (2): A desired cutoff frequency ω_c and a desired filter order, i.e., the stopband slope given by 20μ dB, can be specified. Then, the filter window length can be computed from (A.11).

It has been shown in [24-26] that increasing N increases the sensitivity to noise. It has also been shown

in [24] that the choice $\alpha = \beta$ yields differentiators with maximum robustness to noise. In addition, according to [22, 25], and [26], choosing $\alpha \neq \beta$ reduces stopband ripples in the amplitude spectrum.

Example 2 shows how a differentiator can be designed from desired frequency-domain properties. The computation of the amplitude and phase spectra is demonstrated. Example 3 compares and discusses the amplitude spectra of differentiators with and without delay.

Example 2 (Specifying frequency-domain properties)

An algebraic differentiator with an amplitude spectrum asymptotically decreasing by 40 dB per decade, i.e., min{ α, β } = 1, and a cutoff frequency $\omega_c = 100 \text{ rad s}^{-1}$ can be designed as described in Code snippet 2. The example also shows how the toolbox can be used to compute the amplitude spectrum, its approximation (2), and the resulting window length T. To increase the robustness with respect to corrupting noises the choices N = 0 and $\alpha = \beta$ are made. Additionally, upper and lower bounds for $\mathcal{G}_{N,T,\vartheta}^{(\alpha,\beta)}$ can be computed. They have been discussed in [22, 25, 26].

The amplitude spectrum, the approximation, the bounds, and the phase spectrum are given in Fig. 3. Note that for this parametrisation the amplitude spectrum has infinitely many zeros. Thus, its lower bound is equal to zero (see also [22]). \Box

Create an instance of the class diffA = AlgebraicDifferentiator(N=0, alpha =1., beta=1., T=None, wc=100) # Compute the amplitude spectrum of the differentiator omega = np.linspace(1,800,4*10**3) ampA, phaseA = diffA.get_ampAndPhaseFilter (omega)



Fig. 4: Amplitude and phase spectra of the differentiators with and without delay from Code snippet 3.



Code snippet 2: Design of an algebraic differentiator from desired frequency-domain properties.

Example 3 (Differentiators with and without delay)

As discussed in Section 3, algebraic differentiators can also be parametrised such that the estimated derivative is delayfree. This is done in Code snippet 3 for a differentiator with N = 1 and $\alpha = \beta = 1$. Its amplitude spectrum is also computed. For a comparison a differentiator with delay but the same parameters α , β , and N is also considered.

Fig. 4 shows the amplitude spectra of both differentiators from the latter example. The delay-free differentiator shows a significant overshoot. Thus, the resulting estimated derivative might not be usable. The decrease in the estimation quality of these differentiators has been remarked in [31]. The overshoots in the amplitude spectra of delay-free differentiators have first been observed in [23]. \Box

```
# Create an instance of the class for a
    differentiator with delay
diffA = AlgebraicDifferentiator(N=1,alpha
    =1.,beta=1.,T=None,wc=100)
# Create an instance of the class for a
    differentiator without delay
diffB = AlgebraicDifferentiator(N=1,alpha
    =1.,beta=1.,T=None,wc=100)
diffB.set_theta(1,False)
# Compute amplitude spectrum of the
    differentiator
```

omega = np.linspace (1, 1000, 8*10**2)

ampA, phaseA = diffA.get_ampAndPhaseFilter(
 omega)
ampB, phaseB = diffB.get_ampAndPhaseFilter(
 omega)

Code snippet 3: Comparison of amplitude spectra for differentiators with and without delay.

It has been observed in [21, 22] that the Fourier transform $\mathcal{G}_{N,T,\vartheta}^{(\alpha,\beta)}$ of an algebraic differentiator with N = 0 and $\alpha = \beta$ simplifies to

$$\mathcal{G}_{0,T,\vartheta}^{(\alpha,\alpha)}(\omega) = \mathrm{e}^{-\iota \frac{\omega T}{2}} \Gamma\left(\alpha + \frac{3}{2}\right) \left(\frac{4}{\omega T}\right)^{\alpha + \frac{1}{2}} \mathrm{J}_{\alpha + \frac{1}{2}}\left(\frac{\omega T}{2}\right),$$

where Γ denotes the Gamma Function and $J_{\alpha+\frac{1}{2}}$ is the Bessel function of the first kind and order $\alpha + 1/2$. These special functions are defined in [32], for example. Thus, the amplitude spectrum exhibits an infinite number of positive zeros. Moreover, the differentiator provides phase linearity. By carefully choosing the parameters α and T such that the zeros of $\mathcal{G}_{0,T,\vartheta}^{(\alpha,\alpha)}$ coincide with a specific frequency, the exact annihilation of disturbing harmonics can be performed. This parametrisation has been used in experimental studies in [21, 22, 29, 30, 33]. Therein, the measured signals are corrupted by an unmodelled mechanical eigenmode of the system. Its influence on the control and estimation algorithms has been annihilated using the latter property. In [22], this parametrisation has been used to approximate a matched filter. The following example demonstrates this special parametrisation.

Example 4 (Exact annihilation of frequencies)

Assume that the signal the derivative of which has to be estimated is corrupted by a harmonic signal of known frequency ω_0 . Choosing the window length T such that $\omega_0 T = 2j_k$, with j_k the k-th zero of the Bessel function of the first kind and order $\alpha + 1/2$ yields $\mathcal{G}_{0,T,\vartheta}^{(\alpha,\alpha)}(\omega_0) =$ 0, i.e., the effects of the corrupting signal are exactly annihilated. This is demonstrated in Code snippet 4 for $\omega_0 = 100 \text{ rad s}^{-1}$ and $\alpha = 2$.



Fig. 5: Amplitude and phase spectrum of the differentiator from Code snippet 4 for the annihilation of a harmonic disturbance with angular frequency ω_0 .



Code snippet 4: Parametrisation of a differentiator to annihilate a known frequency.

Fig. 5 shows the amplitude and phase spectra of this differentiator and confirms the exact annihilation of the disturbing harmonic with frequency ω_0 . However, choosing an appropriate zero and a numerical value for α remain as the degrees of freedom for every specific application. The works [21, 22, 29, 30, 33] have proposed an approach where these parameters are chosen such that a satisfying compromise between disturbance rejection and tolerable window length can be achieved. The size T of the window is a crucial parameter since it influences the estimation delay, the computational burden, and the memory requirements of the implemented differentiators. Available hardware may imply constraints on the latter properties. \Box

4.3 Discrete-time implementation

In most applications, the measured signal is available at discrete time instants only and the integral (1) must be approximated by an appropriate quadrature method. Thus, the estimates are outputs of digital FIR filters.

From the Nyquist–Shannon sampling theorem, it follows that when discretising a filter, the gain at the Nyquist frequency must be sufficiently small to reduce aliasing effects. As proposed in [22], a possibility to avoid these effects is to specify the weakest relative attenuation

$$k_{\mathrm{N,min}} = \frac{\omega_{\mathrm{N}} \left| \mathcal{G}_{N,T,\vartheta}^{(\alpha,\beta)}(\omega_{\mathrm{N}}) \right|}{\omega_{\mathrm{c}} \left| \mathcal{G}_{N,T,\vartheta}^{(\alpha,\beta)}(\omega_{\mathrm{c}}) \right|} \approx \left(\frac{\omega_{\mathrm{c}}}{\omega_{\mathrm{N}}} \right)^{\min\{\alpha,\beta\}+1-n_{\mathrm{max}}}$$

of an algebraic differentiator $g_{N,T,\vartheta}^{(\alpha,\beta)}$ at the Nyquist frequency $\omega_N = \pi/t_s$, where t_s and n_{\max} are the sampling period and the highest required derivative order, respectively (see also [26]). Example 5 demonstrates the design of a differentiator with a desired relative attenuation $k_{N,\min}$.

Example 5 (Specifying the relative attenuation $k_{N,\min}$) To achieve at least a desired relative attenuation of $k_{N,\min}$ for a given cutoff frequency ω_c , the parameters α and β have to be chosen such that

$$\min\{\alpha,\beta\} = \frac{\ln(k_{\rm N,min})}{\ln(\omega_c/\omega_{\rm N})} + n_{\rm max} - 1.$$

A differentiator is designed in Code snippet 5 for the approximation of a first order derivative with the cutoff frequency equal to 90 rad s^{-1} and a relative attenuation equal to $10^{-3} = -30 \text{ dB}$. The considered sampling period is equal to 10 ms, the parameters α and β are chosen equal, and N = 0. The resulting differentiator has a filter window length equal to 100 ms, i.e., 10 sampling periods. The parameters α and β are equal to 5.53. The estimation delay is equal to 5 sampling periods. \Box

```
# Specify cutoff frequency
wc = 90
# Specify relative attenuation
kn = 10**(-3)
```



Fig. 6: Amplitude of two continuous-time differentiators compared to those of discretised ones with and without normalisation.

```
# Specify sampling period and Nyquist
    frequency
ts
   = 0.01
wN = np.pi/ts
# Specify order of derivative
der_order = 1
# Compute parameter \alpha
alpha = np. log(kn)/np. log(wc/wN)+
    der_order-1
  Create an instance of the class with
#
    delay
diffA = AlgebraicDifferentiator(N=0, alpha)
    = alpha, beta = alpha,
T=None, wc=wc, ts=ts)
# Get window length
T = diffA \cdot get_T()
# Get delay
delay = diffA \cdot get\_delay()
```

Code snippet 5: Design of a differentiator with a specified relative attenuation.

In the following, it is assumed that the window length T is an integral multiple of the sampling period, i.e., $T = n_{\rm s} t_{\rm s}$, and equidistant sampling is considered for simplicity. The abbreviation $y_k = y(kt_{\rm s}), k \in \mathbb{N}$, for a sample of a signal y at time $kt_{\rm s}$ is used. Then, a discrete-time approximation

$$\hat{y}_{k+\theta}^{(n)} = \frac{1}{\Phi} \sum_{i=0}^{L-1} w_i y_{k-i}, \quad \Phi = \frac{t_s^n}{n!} \sum_{k=0}^{L-1} w_k (-k)^n, \quad (3)$$

of (1) can be achieved. Therein, θ , L, and w_i depend on the numerical integration method used as discussed in [22, 26, 28]. For example, the mid-point rule yields $\theta = \frac{1}{2}$, $w_i = t_s g_{i+\theta}^{(n)}$, and $L = n_s$. The normalisation factor Φ has been first introduced in [22] to guarantee that the DC component of the sought derivative is preserved. The effects of normalisation are demonstrated in the following example using the functions of the toolbox. **Example 6** (Effect of the normalisation factor) Per default, the toolbox implements the normalisation presented in (3). The code in Code snippet 6 demonstrates the effects of the normalisation by comparing the amplitude spectra of two differentiators: The first one is discretised with the normalisation and the second is without.

Create an instance of the class for a # differentiator with the normalisation ts = 0.01diffA = AlgebraicDifferentiator(N=1, alpha)=1., beta = 1.,T=10*ts, ts=ts) Create an instance of the class for a differentiator without the normalisationdiffN = AlgebraicDifferentiator (N=1, alpha)=1., beta = 1.,T=10*ts, ts=ts, corr=False) # Evaluate Fourier transform of the continuous-time and discrete-timedifferentiators for comparisons omega = np. linspace(0, 200, 1000)method = 'mid - point'd = 0 $ampA \quad C, phaseA \quad C = diffA$. get ampAndPhaseFilter(omega) $ampA \quad D, phaseA \quad D = diffA$. get_ampSpectrumDiscreteFilter(omega, d, method=method) $ampN_D, phaseN_D = diffN$. get_ampSpectrumDiscreteFilter(omega, d, method=method)

Code snippet 6: Discretisation of differentiators with and without normalisation.

Fig. 6 shows the spectra of two continuous-time differentiators discretised with and without the normalisation factor. It can be seen that the DC component is only preserved when the normalisation is used. \Box Algebraic differentiators and their tuning guidelines have been discussed in the continuous-time domain. Thus, the discretised ones have to preserve the specified properties. Since most tuning guidelines discussed here rely on the frequency-domain interpretation of the differentiators, the analyses of the discretisation effects can be performed by comparing the Fourier transform of the continuous and discrete differentiators. This can be done using the cost function

$$\mathcal{J} = \frac{\int_0^\Omega \left| \mathcal{F} \left\{ g^{(n)} - \hat{g}^{(n)} \right\} (\omega) \right|^2 \mathrm{d}\omega}{\int_0^\Omega \left| \mathcal{F} \left\{ g^{(n)} \right\} (\omega) \right|^2 \mathrm{d}\omega} \tag{4}$$

introduced in [22] (see also [26, 28]), where Ω has to be chosen according to the frequency interval of interest. In the latter cost function, $\omega \mapsto \mathcal{F}\left\{g^{(n)}\right\}(\omega)$ is the Fourier transform of the continuous-time differentiator $g = g_{N,T,\vartheta}^{(\alpha,\beta)}$. The discrete-time filter from (3) is denoted $\hat{g}^{(n)}$ and its Fourier transform is

$$\omega \mapsto \mathcal{F}\left\{\hat{g}^{(n)}\right\}(\omega) = \frac{1}{\Phi} \sum_{k=0}^{L-1} w_k \mathrm{e}^{-\iota \omega (k+\theta)t_{\mathrm{s}}}.$$
 (5)

The cost function \mathcal{J} compares the discretisation error with the noise amplification of the continuous-time differentiator (the reader is referred to [22, 26] for further details). Thus, it can be used to decide if the former can be neglected compared to the latter. The cost function can be easily computed using the functions in the toolbox as demonstrated in Example 7.

Example 7 (Discretisation error)

The cost function (4) is computed in Code snippet 7 for a differentiator with the parameters $\alpha = \beta = 7$, N = 0, and $T = 20t_s$.



The resulting cost \mathcal{J} is approximately equal to $1.6 \cdot 10^{-10}$. Thus, the discretisation effects can be neglected. The variation of \mathcal{J} with respect to the parameters of the differentiator has been analysed in [22, 28], for example. \Box

4.4 Estimation of derivatives using AlgDiff

Example 8 demonstrates the numerical differentiation of a given signal using the implemented functions and a differentiator with desired frequency-domain properties.

Example 8 (Numerical differentiation)

An algebraic differentiator with the cutoff frequency $\omega_c = 20 \,\mathrm{rad}\,\mathrm{s}^{-1}$ and a desired relative attenuation $k_{\rm N,min} = 10^{-3}$ shall be designed for the estimation of the first order derivative of a signal sampled with a sampling period equal to 20 ms. As in Example 2, the choice N = 0 and $\alpha = \beta$ is made to increase the robustness with respect to disturbances. The resulting differentiator has the parameters $\alpha = \beta = 3.35$ and a window length equal to $14t_s$. The estimation delay corresponds to $7t_s$. The cost function \mathcal{J} is equal to $7.9 \cdot 10^{-6}$. Thus, discretisation effects can be neglected. The parametrisation of the differentiator and the estimation of the derivative of a noisy signal $t \mapsto y(t) = \sin(t) + \eta(t)$ is shown in Code snippet 8. The additive disturbance η is drawn from a Gaussian distribution with standard deviation equal to 0.02 such that the signal to noise ratio is $SNR = \log_{10} \left(\frac{\sum_{i} y^2(t_i)}{\sum_{i} \eta^2(t_i)} \right) \approx 31 \, \text{dB}.$

The estimation results are shown in Fig. 7 and compared to the results of a simple forward difference method. For a comparison with more advanced methods the reader is referred to [25, 34, 35]. This example shows the advantages of algebraic differentiators and their systematic tuning guidelines. Furthermore, it highlights the user friendliness of the toolbox AlgDiff. \Box

```
# Design differentiator
ts = 0.02
kn = 10 * * (-3)
wN = np \cdot pi/ts
a = np \cdot log(kn)/np \cdot log(wc/wN)
diffA = AlgebraicDifferentiator(N=0,
    alpha=a, beta=a, T=None, ts=ts, wc=20)
J = diffA.get\_discretizationError(1, np.pi
    /ts)
# Define the signal
t = np. arange(0, 10, ts)
a0 = 1
w0 = 1
x = a0 * np.sin(w0 * t)
dx = a0 * w0 * np. \cos(w0 * t)
eta = np.random.normal(0, 0.02, len(t))
y = x + eta
# Estimate derivative
dyApp = diffA \cdot estimateDer(1, y)
```

Code snippet 8: Approximation of a derivative of a given signal using a differentiator designed with desired frequency-domain properties.



Fig. 7: Estimation of the first order derivative \dot{x} of a sinusoidal signal x using a measurement y using the forward-difference method (FD) and the algebraic differentiator (Alg.) designed in Code snippet 8.

5 Conclusion

This contribution discusses algebraic differentiators and the use of the toolbox AlgDiff. Properties of the differentiators are recalled and the use of the functions implemented in AlgDiff was shown. The design of differentiators with desired frequency-domain properties and the analysis of discretisation issues are addressed with AlgDiff and the results are discussed and illustrated. It is shown that with few lines of code, efficient differentiators can be designed and used for the numerical differentiation of signals. Further works where the systematic tuning of the differentiators are discussed in detail are [21, 22, 25, 26, 28, 30, 33] and [36], for example. An extensive list of applications and more examples for the tuning of the differentiators are provided in [26].

References

- C.-K. Chen and J.-H. Lee. "Design of high-order digital differentiators using L₁ error criteria". In: *IEEE Trans. Circuits Syst. II. Analog Digit. Signal Process.* 42.4 (1995), pp. 287–291. DOI: 10.1109/82.378044.
- [2] C. M. Rader and L. B. Jackson. "Approximating Noncausal IIR Digital Filters Having Arbitrary Poles, Including New Hilbert Transformer Designs, Via Forward/Backward Block Recursion". In: *IEEE Trans. on Circuits and Systems I: Regular Papers* 53.12 (2006), pp. 2779–2787. DOI: 10. 1109/TCSI.2006.883877.
- [3] D. A. Murio. The mollification method and the numerical solution of ill-posed problems. New York, Santa Barbara, London, Sydney, Toronto: John Wiley & Sons, 2011. DOI: 10.1002/9781118033210.
- [4] J. Cullum. "Numerical Differentiation and Regularization". In: SIAM J. Numer. Anal. 8.2 (1971), pp. 254–265. DOI: 10.1137/0708026.
- [5] A. Levant. "Robust exact differentiation via sliding mode technique". In: *Automatica* 34.3 (1998), pp. 379–384. DOI: 10.1016/S0005-1098(97)00209-4.

- [6] A. M. Dabroom and H. K. Khalil. "Numerical differentiation using high-gain observers". In: *Proc. of the 36th IEEE Conf. on Decision and Control.* Vol. 5. San Diego, CA, USA, 1997, pp. 4790–4795. DOI: 10.1109/CDC.1997. 649776.
- [7] A. M. Dabroom and H. K. Khalil. "Discrete-time implementation of high-gain observers for numerical differentiation". In: *Int. J. Control* 72.17 (1999), pp. 1523–1537. DOI: 10.1080/002071799220029.
- [8] A. Levant. "Higher-order sliding modes, differentiation and output-feedback control". In: *Int. J. Control* 76.9-10 (2003), pp. 924–941. DOI: 10.1080/0020717031000099029.
- Y. Chitour. "Time-varying high-gain observers for numerical differentiation". In: *IEEE Trans. Autom. Control* 47.9 (2002), pp. 1565–1569. DOI: 10.1109/TAC.2002.802740.
- [10] A. Savitzky and M. J. E. Golay. "Smoothing and Differentiation of Data by Simplified Least Squares Procedures." In: Anal. Chem. 36.8 (1964), pp. 1627–1639. DOI: 10.1021/ac60214a047.
- H. Madden. "Comments on the Savitzky-Golay Convolution Method for Least-Squares Fit Smoothing and Differentiation of Digital Data". In: *Anal. Chem.* 50.9 (1978), pp. 1383–86. DOI: 10.1021/ac50031a048.
- [12] E. Diekema and T. H. Koornwinder. "Differentiation by integration using orthogonal polynomials, a survey". In: J. Approx. Theory 164.5 (2012), pp. 637–667. ISSN: 0021-9045. DOI: 10.1016/j.jat.2012.01.003.
- [13] N. Cioranescu. "La généralisation de la première formule de la moyenne". In: *Enseign. Math.* 37 (1938), pp. 292–302.
- [14] C. Lanczos. Applied Analysis. Englewood Cliffs: Prentice Hall, Inc., 1956.
- [15] M. Mboup, C. Join, and M. Fliess. "A revised look at numerical differentiation with an application to nonlinear feedback control". In: *Proc. of the 15th Mediterranean Conf. on Control and Automation*. Athen, Greece, 2007. DOI: 10.1109/MED.2007.4433728.
- [16] M. Mboup, C. Join, and M. Fliess. "Numerical differentiation with annihilators in noisy environment". In: *Numer. Algorithms* 50.4 (2009), pp. 439–467. DOI: 10.1007/s11075-008-9236-1.
- [17] J. Reger and J. Jouffroy. "On algebraic time-derivative estimation and deadbeat state reconstruction". In: Proc. of the 48h IEEE Conf. on Decision and Control held jointly

with 2009 28th Chinese Control Conf. Shanghai, China, 2009, pp. 1740–1745. DOI: 10.1109/CDC.2009.5400719.

- [18] J. Reger and J. Jouffroy. "Algebraische Ableitungsschätzung im Kontext der Rekonstruierbarkeit (Algebraic Time-derivative Estimation in the Context of Reconstructibility)". In: *at-Automatisierungstechnik* 56.6 (2008). DOI: 10.1524/auto.2008.0711.
- [19] D. Y. Liu, O. Gibaru, and W. Perruquetti. "Convergence Rate of the Causal Jacobi Derivative Estimator". In: *Curves and Surfaces*. Ed. by J.-D. Boissonnat et al. Berlin, Heidelberg: Springer, 2010, pp. 445–455. DOI: 10.1007/ 978-3-642-27413-8_28.
- [20] D. Y. Liu, O. Gibaru, and W. Perruquetti. "Error analysis of Jacobi derivative estimators for noisy signals". In: *Numer. Algorithms* 58.1 (2011), pp. 53–83. DOI: 10.1007/ s11075-011-9447-8.
- [21] L. Kiltz and J. Rudolph. "Parametrization of algebraic numerical differentiators to achieve desired filter characteristics". In: *Proc. of the 52nd IEEE Conf. on Decision and Control.* Florence, Italy, 2013, pp. 7010–7015. DOI: 10.1109/CDC.2013.6761000.
- [22] L. Kiltz. "Algebraische Ableitungsschätzer in Theorie und Anwendung". Doctoral dissertation. Saarland University, Germany, 2017. DOI: 10.22028/D291-27034.
- [23] M. Mboup and S. Riachy. "A Frequency Domain Interpretation of the Algebraic Differentiators". In: *IFAC Proc. Volumes* 47.3 (2014), pp. 9147–9151. DOI: 10.3182/20140824-6-za-1003.02132.
- [24] M. Mboup and S. Riachy. "Frequency-domain analysis and tuning of the algebraic differentiators". In: Int. J. Control 91.9 (2018), pp. 2073–2081. DOI: 10.1080/00207179.2017. 1421776.
- [25] A. Othmane, J. Rudolph, and H. Mounier. "Systematic comparison of numerical differentiators and an application to model-free control". In: *Eur. J. Control* 62 (2021), pp. 113–119. ISSN: 0947-3580. DOI: 10.1016/j.ejcon.2021. 06.020.
- [26] A. Othmane, L. Kiltz, and J. Rudolph. "Survey on algebraic numerical differentiation: Historical developments, parametrization, examples, and applications". In: Int. J. Syst. Sci. 53.9 (2022). free access, pp. 1848–1887. DOI: 10.1080/00207721.2022.2025948.
- [27] A. Othmane. AlgDiff: A Python package with MATLAB coupling implementing all necessary tools for the design, analysis, and discretization of algebraic differentiators. Version 1.1.1. Available at https://github.com/aothmanecontrol/Algebraic-differentiators. 2023.
- [28] A. Othmane. "Contributions to numerical differentiation using orthogonal polynomials and its application to fault detection and parameter identification". Doctoral dissertation. Université Paris-Saclay, France, Saarland University, Germany, 2022. DOI: 10.22028/D291-38806.
- [29] L. Kiltz, M. Mboup, and J. Rudolph. "Fault diagnosis on a magnetically supported plate". In: *Proc. of the 1st International Conf. on Systems and Computer Science*. Lille, France, 2012. DOI: 10.1109/IConSCS.2012.6502453.
- [30] L. Kiltz et al. "Fault-tolerant control based on algebraic derivative estimation applied on a magnetically supported plate". In: *Control Eng. Pract.* 26 (2014), pp. 107–115. DOI: 10.1016/j.conengprac.2014.01.009.

- [31] M. Mboup. "Parameter estimation for signals described by differential equations". In: *Appl. Anal.* 88.1 (2009), pp. 29– 52. DOI: 10.1080/00036810802555441.
- [32] M. Abramowitz and I. A. Stegun, eds. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. New York: Dover, 1965.
- [33] L. Kiltz, M. Janocha, and J. Rudolph. "Algebraic estimation of impact times: Juggling a ball with a magnetically levitated plate". In: Proc. of the 2nd International Conf. on Systems and Computer Science. Villeneuve d'Ascq, France, 2013, pp. 145–149. DOI: 10.1109/IcConSCS.2013.6632038.
- [34] L. Sidhom. "Sur les différentiateurs en temps réel: algorithmes et applications". Doctoral dissertation. INSA de Lyon, France, 2011.
- [35] D. Y. Liu. "Analyse d'erreurs d'estimateurs des dérivées de signaux bruités et applications". Doctoral dissertation. Université des Sciences et Technologies de Lille - Lille 1, France, 2011.
- P. M. Scherer, A. Othmane, and J. Rudolph. "Combining model-based and model-free approaches for the control of an electro-hydraulic system". In: *Control Eng. Pract.* 133 (2023), p. 105453. DOI: 10.1016/j.conengprac.2023. 105453.

Appendices

A Jacobi polynomials

The Jacobi polynomial $P_N^{(\alpha,\beta)}$ can be defined as

$$P_N^{(\alpha,\beta)}(\tau) = \sum_{k=0}^N c_k^{(\alpha,\beta)} (\tau-1)^k,$$
$$c_k^{(\alpha,\beta)} = \frac{\Gamma(\alpha+N+1)\Gamma(\alpha+\vartheta+N+k+1)}{2^k N! \Gamma(\alpha+\beta+N+1)\Gamma(\alpha+k+1)},$$

where $\tau \in \mathcal{I}_P = [-1, 1], N \in \mathbb{N}, \alpha, \beta > -1$, and Γ is the Gamma function (see [32, Sections 6.1 & 22.2] for the definitions). The polynomials are orthogonal in the interval \mathcal{I}_P with respect to the weight function

$$w^{(\alpha,\beta)}(\tau) = \begin{cases} (1-\tau)^{\alpha}(1+\tau)^{\beta}, & \tau \in \mathcal{I}_P, \\ 0, & \text{otherwise.} \end{cases}$$
(A.6)

B Kernel of algebraic differentiators

The kernel $g_{N,T,\vartheta}^{(\alpha,\beta)}$ in (1) is defined as

$$g_{N,T,\vartheta}^{(\alpha,\beta)}(t) = \frac{2}{T} \sum_{k=0}^{N} \frac{\mathbf{P}_{k}^{(\alpha,\beta)}(\vartheta)}{\left\|\mathbf{P}_{k}^{(\alpha,\beta)}\right\|^{2}} \left(w^{(\alpha,\beta)} \cdot \mathbf{P}_{k}^{(\alpha,\beta)}\right) \circ (\theta_{T}(\tau)),$$
(A.7)

with $\theta_T(t) = 1 - 2t/T$ and $||x|| = \sqrt{\langle x, x \rangle}$ the norm induced by the inner product

$$\langle x, y \rangle = \int_{-1}^{1} w^{(\alpha, \beta)}(\tau) x(\tau) y(\tau) \mathrm{d}\tau.$$

C Impulse and step responses

The impulse and step responses of an algebraic differentiator $g_{N,T,\vartheta}^{(\alpha,\beta)}$ are by (A.7) and

$$h_{N,T,\vartheta}^{(\alpha,\beta)}(\tau) = \begin{cases} 0, & \text{for } \tau < 0\\ I_{\frac{\tau}{T}}(\bar{\alpha},\bar{\beta}) + f_{N,T,\vartheta}^{(\alpha,\beta)}(\tau), & \text{for } \tau \in [0,T] \\ 1 & \text{otherwise,} \end{cases}$$

respectively, with $\bar{\alpha} = \alpha + 1$, $\bar{\beta} = 1 + \beta$, I_{τ} the regularized incomplete Beta function defined in [32, Section 6.6], and

$$f_{N,T,\vartheta}^{(\alpha,\beta)}(\tau) = \sum_{k=1}^{N} \frac{\mathbf{P}_{k}^{(\alpha,\beta)}(\vartheta)}{2k \left\|\mathbf{P}_{k}^{(\alpha,\beta)}\right\|^{2}} \left(w^{(\bar{\alpha},\bar{\beta})} \cdot \mathbf{P}_{k-1}^{(\bar{\alpha},\bar{\beta})}\right) \circ (\theta_{T}(\tau)) \,.$$

D Fourier transform $\mathcal{G}_{N,T,\vartheta}^{(\alpha,\beta)}$

Closed forms of the Fourier transform $\mathcal{G}_{N,T,\vartheta}^{(\alpha,\beta)}$ of an algebraic differentiator $g_{N,T,\vartheta}^{(\alpha,\beta)}$ have been derived in [22, 29] and [24]. One possible representation of $\mathcal{G}_{N,T,\vartheta}^{(\alpha,\beta)}$ is

$$\mathcal{G}_{N,T,\vartheta}^{(\alpha,\beta)}(\omega) = \sum_{i=0}^{N} \frac{(\alpha+\beta+2i+1)\mathbf{P}_{i}^{(\alpha,\beta)}(\theta)}{\alpha+\beta+i+1} \times \sum_{k=0}^{i} (-1)^{i-k} {i \choose k} \mathbf{M}_{i,k}^{(\alpha,\beta)}(-\iota\omega T), \quad \iota^{2} = -1,$$
(A.9)

with

$$M_{i,k}^{(\alpha,\beta)}(z) = M(\alpha + i - k + 1, \alpha + \beta + i + 2, z),$$
 (A.10)

where $z \in \mathbb{C}$ and M is the confluent hypergeometric function defined in [32, Section 13.1].

E Cutoff frequency ω_{c}

Let $\mu = 1 + \min\{\alpha, \beta\}, \kappa = |\alpha - \beta|$. The cutoff frequency of an algebraic differentiator is

$$\omega_{\rm c} = \frac{1}{T} \left(\frac{q_{N,\vartheta}^{(\alpha,\beta,\sigma)}}{\Gamma(\mu+\kappa)} \right)^{\frac{1}{\mu}}, \qquad (A.11)$$

where

$$q_{N,\vartheta}^{(\alpha,\beta,\sigma)} = \begin{cases} \Gamma(\mu) \max\left\{ \left| r_{N,T}^{(\mu,0,\sigma)} \right|, s_{N,T}^{(\mu,0,\sigma)} \right\}, & \kappa = 0, \\ \Gamma(\mu+\kappa) \left| r_{N,T}^{(\mu,\kappa,\sigma)} \right|, & \kappa > 0, \end{cases}$$

$$\begin{split} \sigma &= \begin{cases} 1, & \alpha \leq \beta, \\ -1, & \alpha > \beta, \end{cases} \\ r_{N,\vartheta}^{(\mu,\kappa,\sigma)} &= \sum_{i=0}^{N} \frac{c_i^{(\mu,\kappa)}}{\Gamma(\mu+\kappa+i)} \mathbf{P}_i^{(\mu-1,\mu+\kappa-1)}(\sigma\vartheta), \\ s_{N,\vartheta}^{(\mu,\kappa,\sigma)} &= \sum_{i=0}^{N} (-1)^i \frac{c_i^{(\mu,\kappa)}}{\Gamma(\mu+i)} \mathbf{P}_i^{(\mu-1,\mu+\kappa-1)}(\sigma\vartheta), \\ c_i^{(\mu,\kappa)} &= (2\mu+\kappa+2i-1)\,\Gamma(2\mu+\kappa+i-1). \end{cases} \end{split}$$

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