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The Construction of the Coarse de Rham Complexes with Improved Approximation Properties

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Abstract — We present two novel coarse spaces (H^1- and $H(\text{curl})$-conforming) based on element agglomeration on unstructured tetrahedral meshes. Each H^1-conforming coarse basis function is continuous and piecewise-linear with respect to an original tetrahedral mesh. The $H(\text{curl})$-conforming coarse space is a subspace of the lowest order Nédélec space of the first type. The H^1-conforming coarse space exactly interpolates affine functions on each agglomerate. The $H(\text{curl})$-conforming coarse space exactly interpolates vector constants on each agglomerate. Combined with the $H(\text{div})$- and L_2-conforming spaces developed previously in [8], the newly constructed coarse spaces form a sequence (with respect to exterior derivatives) which is exact as long as the underlying sequence of fine-grid spaces is exact. The constructed coarse spaces inherit the approximation and stability properties of the underlying fine-grid spaces supported by our numerical experiments. The new coarse spaces, in addition to multigrid, can be used for upscaling of broad range of PDEs involving curl, div and grad differential operators.

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1. Introduction

We describe a systematic approach of constructing accurate "coarse or upscaled models or discretizations" that utilize coarse finite element spaces with guaranteed approximation properties constructed by specialized, element-based, algebraic multigrid (or AMGe) methods. The main goal is not necessarily building a hierarchy for an efficient multigrid solver, rather providing accurate coarse (upscaled) discretizations that can be used instead of a fine-resolution one which may turn out infeasible for repeated large-scale simulations.

The present paper completes a sequence of results originally motivated by the so-called element agglomeration algebraic multigrid (or AMGe) [4, 6, 7, 12, 13], see also [14]. The AMGe exploits the natural idea of agglomerating the elements of the original (fine) mesh in

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order to produce macro-elements that serve as coarse elements and then construct by some local energy minimization principle the associated coarse spaces, so that recursion is possible to apply. Originally, the AMGe approach was developed for SPD problems and eventually extended, in [10], to the entire de Rham sequence of finite element spaces with the purpose to develop coarse hierarchies for use in multigrid. As motivated in [14], the AMGe approach can successfully be applied to construct coarse spaces with guaranteed approximation properties (not only for multigrid purposes). More recently, based on the work in [10], the AMGe was generalized in [8] to mixed systems involving the lowest order $H(\mathrm{div})$-conforming Raviart–Thomas spaces to guarantee that the coarse spaces contain locally the vector constants. In the present work, we extend the approach from [8], concentrating on the approximation properties of the constructed coarse spaces, by completing the entire de Rham sequence of the respective lowest order finite element spaces. More specifically, we propose two novel coarse spaces, which utilize agglomerates of standard tetrahedral elements. Each agglomerate must meet certain topological requirements, but otherwise it can have arbitrary shape. The proposed coarse spaces are subspaces of, respectively, the space of continuous piecewise-linear functions and the lowest order Nédélec space associated with the original "fine" mesh. Together with the $H(\mathrm{div})$- and L_2-conforming coarse spaces described in our previous work, the proposed spaces form an exact sequence (with respect to exterior derivatives) when the spatial domain is homeomorphic to a ball. In contrast to the previous work, all four coarse spaces locally contain (on each coarse element) the set of all polynomials of the same order as do the corresponding fine-grid spaces. For example, the coarse counterpart of the H^1-conforming space locally contains all four linear functions, and the coarse version of the Nédélec space locally contains all three vector constants. This property ensures that the constructed coarse spaces can exhibit approximation properties comparable to those of the original fine-grid spaces. Our numerical experiments do confirm this. As the original mesh is refined and the average agglomerate size is kept constant (in other words, the ratio H/h is kept constant, where H and h are coarse and fine mesh parameters, respectively), the corresponding norm (H^1, $H(\mathrm{curl})$, $H(\mathrm{div})$, or L_2) of the error of the coarse-grid approximation (based on a respective Galerkin projection) exhibits $\mathcal{O}(H) \simeq \mathcal{O}(h)$ behavior, that is, the same as of the fine-grid approximation. We additionally observe that the coarse H^1-conforming space approximates smooth scalar functions in L^2-norm with error of order $\mathcal{O}(h^2)$.

Related approaches are found in the mimetic/virtual finite element literature, see e.g., [2, 3]. In recent years, several mimetic/virtual methods have been developed for finite elements with rather general polygonal/polyhedral shape. We stress upon the fact that mimetic methods are discretization methods, i.e., they take a PDE as input and produce a discrete system of equations as output. Our approach differs in that we assume the existence of a fine-grid discretization of a PDE utilizing standard finite elements, and then we create, by local procedures, coarse discretization spaces associated with agglomerates. Note that obtaining the fine-grid finite element solution may be computationally infeasible, while the created coarse (upscaled) problem may be more tractable.

To actually create the coarse elements (agglomerates) we use mesh partitioners, such as (Par)METIS [5], with post-processing to ensure certain topological properties of the agglomerates. This is by itself a very challenging task, and the implementation details will be addressed elsewhere.

The main results of the paper that we prove are exactness of the sequence of coarse spaces, and also that the coarse spaces possess certain approximation properties in the sense that locally on each agglomerate the coarse spaces span the lowest order polynomials as the ones of the respective fine-grid space. The latter property is described in detail at the end of Section 2.

We finally mention, that the approach presented here is general and extends to de Rham sequences of arbitrary order finite element spaces. The details however will be presented in a follow-up paper.

The remainder of the paper is structured as follows. We begin with a general problem setup in Section 2, then introduce our assumptions on the coarse topology in Section 3, and in Section 4, we provide some auxiliary results that are used throughout the paper. Then in the following several sections we describe the construction of the four coarse counterparts of the originally given lowest order spaces. The coarse L_2-conforming space and the coarse counterpart of the Raviart–Thomas space are described in Section 5. The coarse version of the Nédélec space is described in Section 6, and the coarse H^1-conforming space is described in Section 7. We present some numerical results in Section 8 that illustrate the improved approximation properties of the new coarse spaces.

2. Problem Setup

We consider a bounded three-dimensional polyhedral domain Ω, exactly covered by an unstructured tetrahedral mesh $\mathcal{T}_h$. We assume that there exists a polyhedral domain $\widehat{\Omega}$, exactly covered by an unstructured tetrahedral mesh $\widehat{\mathcal{T}}_h$ such that $\Omega \subset \widehat{\Omega}$, $\mathcal{T}_h$ is a "sub-mesh" of $\widehat{\mathcal{T}}_h$, and $\widehat{\Omega}$ is homeomorphic to a 3D ball. This assumption is used later in Section 7.7.

We use the following definitions, which are customary in the finite element literature. The *element* is an individual tetrahedron from $\mathcal{T}_h$. The boundary of each element consists of four triangular *faces*. The boundary of each face consists of three straight segments, referred to as *edges*. Finally, the two endpoints of each edge are called *vertices*. We treat elements, faces and edges as closed sets of points in 3D. Then, for example, we can express the fact that an edge e is part of the boundary of a face f by writing $e \subset f$. We say that two mesh entities a and b (an entity is an element, a face, an edge, or a vertex) are incident iff either $a \subset b$ or $b \subset a$. In what follows we often call $\mathcal{T}_h$ the *fine* mesh, as opposite to the *coarse* mesh that we seek to construct. We also often call the vertices, edges, faces, and elements of $\mathcal{T}_h$ the *fine* vertices, edges, faces, and elements.

In this paper, we use letters in bold font to denote 3D vectors and 3D vector fields. All other objects (e.g., scalar fields and vectors in $\mathbb{R}^k$) are denoted by letters in normal font.

Throughout the paper we use the following notation for vector constant functions:

$$\mathbf{c}_1 = (1,0,0)^T, \quad \mathbf{c}_2 = (0,1,0)^T, \quad \mathbf{c}_3 = (0,0,1)^T, \quad C = \operatorname{span}(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3). \tag{2.1}$$

We assume that each fine face f is assigned a particular unit normal vector $\mathbf{n} = \mathbf{n}(f)$, once and for all. Similarly, we assign each edge a direction (we label one of the vertices as the "head" and another as the "tail"). Let a fine edge e have tail vertex A and head vertex B. Let $\overrightarrow{AB}$ denote the vector starting at point A and ending at point B. The length of vector $\overrightarrow{AB}$ will be denoted by $|\overrightarrow{AB}|$. We associate with e a unit tangential vector

$$\boldsymbol{\tau} = \boldsymbol{\tau}(e) = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}.$$

Let $\widetilde{S}_h$ denote the space of continuous functions which are linear on each tetrahedron $t \in \mathcal{T}_h$. Let $\widetilde{Q}_h$ denote the lowest order Nédélec space (Nédélec curl-conforming finite element space of the first type) associated with $\mathcal{T}_h$, and let $\widetilde{R}_h$ denote the lowest-order Raviart–Thomas space (also known as divergence-conforming Nédélec space of the first type), see, e.g., [9]. Finally, let $\widetilde{M}_h$ denote the space of functions which are constant on each tetrahedron (clearly, such functions are generally discontinuous).

It is well known (e.g., [9]) that

$$\mathbb{R} \hookrightarrow \widetilde{S}_h \xrightarrow{\nabla} \widetilde{Q}_h \xrightarrow{\nabla\times} \widetilde{R}_h \xrightarrow{\nabla\cdot} \widetilde{M}_h \to 0$$

is an "exact sequence", when Ω is homeomorphic to a ball. The term "exact sequence" in this context means

(1) $\nabla \widetilde{S}_h \subset \widetilde{Q}_h$, $\nabla\times\widetilde{Q}_h \subset \widetilde{R}_h$, $\nabla\cdot\widetilde{R}_h \subseteq \widetilde{M}_h$;

(2) $\ker(\nabla) = \mathbb{R}$, $\ker(\nabla\times) = \nabla\widetilde{S}_h$, $\ker(\nabla\cdot) = \nabla\times\widetilde{Q}_h$, $\widetilde{M}_h = \nabla\cdot\widetilde{R}_h$.

We also need the spaces "with zero boundary conditions": S_h, Q_h, R_h and M_h (note the absence of tildes). $S_h \subset \widetilde{S}_h$ is defined to consist of functions from $\widetilde{S}_h$ which vanish on $\partial\Omega$. In a similar way, $Q_h \subset \widetilde{Q}_h$ consists of functions which have zero *tangential* component on $\partial\Omega$. Next, $R_h \subset \widetilde{R}_h$ consists of functions with zero *normal* component on $\partial\Omega$. Finally, $M_h \subset \widetilde{M}_h$ is defined to consist of functions which have zero average over Ω. It is well known that the sequence

$$0 \hookrightarrow S_h \xrightarrow{\nabla} Q_h \xrightarrow{\nabla\times} R_h \xrightarrow{\nabla\cdot} M_h \to 0 \tag{2.2}$$

is also exact when Ω is homeomorphic to a ball. Note that this time the sequence starts with zero, which means that the zero function is the only function in S_h which has zero gradient.

Let $D \subset \Omega$ be an *open* subdomain such that its closure $\overline{D}$ is exactly covered by a union of elements from $\mathcal{T}_h$. We define $S_h(D) \subset \widetilde{S}_h$ to contain those functions $g \in \widetilde{S}_h$ which satisfy $\mathrm{supp}(g) \subset \overline{D}$ *and* $g = 0$ on ∂D. Let $Q_h(D) \subset \widetilde{Q}_h$ contain those functions $\mathbf{q} \in \widetilde{Q}_h$ which satisfy $\mathrm{supp}(\mathbf{q}) \subset \overline{D}$ *and* have vanishing tangential component on ∂D. Similarly, let $R_h(D) \subset \widetilde{R}_h$ consist of those functions $\mathbf{r} \in \widetilde{R}_h$ which satisfy $\mathrm{supp}(\mathbf{r}) \subset \overline{D}$ *and* have vanishing normal component on ∂D.

Finally, let $M_h(D) \subset \widetilde{M}_h$ contain those functions $u \in \widetilde{M}_h$ which satisfy $\mathrm{supp}(u) \subset \overline{D}$ *and* $\int_D u \, \mathrm{d}V = 0$.

When D is homeomorphic to a ball, the sequence

$$0 \hookrightarrow S_h(D) \xrightarrow{\nabla} Q_h(D) \xrightarrow{\nabla\times} R_h(D) \xrightarrow{\nabla\cdot} M_h(D) \to 0 \tag{2.3}$$

is exact, since (2.3) is just (2.2) written for a different domain $\Omega' = D$.

Suppose we are given a partitioning $\Omega = \bigcup T_i$ of the domain into subdomains such that each subdomain T_i is a union of fine-grid elements (tetrahedrons) from $\mathcal{T}_h$, and the interiors of the subdomains do not intersect. In other words, we partition the set of all fine-grid elements into non-intersecting groups. Throughout the rest of the paper we interchangeably call the subdomains T_i *agglomerates*, *coarse elements* or *agglomerated elements*. We seek to construct four spaces, $\widetilde{S}_H \subset \widetilde{S}_h$, $\widetilde{Q}_H \subset \widetilde{Q}_h$, $\widetilde{R}_H \subset \widetilde{R}_h$ and $\widetilde{M}_H \subset \widetilde{M}_h$, so that the sequence

$$\mathbb{R} \hookrightarrow \widetilde{S}_H \xrightarrow{\nabla} \widetilde{Q}_H \xrightarrow{\nabla\times} \widetilde{R}_H \xrightarrow{\nabla\cdot} \widetilde{M}_H \to 0 \tag{2.4}$$

is exact if Ω is homeomorphic to a ball, and the constructed spaces have the following approximation properties:

- There exists a projection $\Pi^S : \widetilde{S}_h \to \widetilde{S}_H$ such that for any $g \in \widetilde{S}_h$ and any agglomerate T, the functions $\Pi^S g$ and g coincide on T, as long as g is linear on T.

- There exists a projection $\Pi^Q : \widetilde{Q}_h \to \widetilde{Q}_H$ such that for any $\mathbf{q} \in \widetilde{Q}_h$ and any agglomerate T, the functions $\Pi^Q \mathbf{q}$ and $\mathbf{q}$ coincide on T, as long as $\mathbf{q}$ is a vector constant on T.

- There exists a projection $\Pi^R : \widetilde{R}_h \to \widetilde{R}_H$ such that for any $\mathbf{r} \in \widetilde{R}_h$ and any agglomerate T, the functions $\Pi^R \mathbf{r}$ and $\mathbf{r}$ coincide on T, as long as $\mathbf{r}$ is a vector constant on T.

- There exists a projection $\Pi^M : \widetilde{M}_h \to \widetilde{M}_H$ such that for any $u \in \widetilde{M}_h$ and any agglomerate T, the functions $\Pi^M u$ and u coincide on T, as long as u is constant on T.

Our construction guarantees that the coarse spaces have "locally" supported basis functions. "Locally" means that the support of each coarse basis function only includes agglomerates which are "neighbors" of the geometrical entity (coarse vertex, coarse edge, etc.) with which the coarse basis function is associated.

We note that the spaces introduced in [10] do form an exact sequence (under certain conditions), but do not satisfy the approximation properties outlined above, unless the "faces" of the agglomerates (coarse faces) are flat and the "edges" of the agglomerates (coarse edges) are straight. We rigorously define coarse faces and coarse edges in Section 3. The space $\widetilde{R}_H \subset \widetilde{R}_h$ locally containing all vector constants (and satisfying the exactness property together with the space $\widetilde{M}_H$) has been described in our previous work [8]. We include its description here for completeness.

3. Element Agglomeration and Coarse Topology

We assume that we are given the partitioning of all fine elements into non-intersecting subsets, called agglomerates. We call a fine face a *boundary* face of an agglomerate T iff that face is a subset of exactly one fine element from T. A fine edge (vertex) is called a boundary edge (boundary vertex) of T iff that edge (vertex) is a subset of any boundary face of T.

Assumption 3.1. We further assume that each agglomerate T has the following properties:

- Define the following *dual graph*. Its vertices are the fine-grid elements belonging to T. Any two such vertices are connected by an arc iff the corresponding fine-grid elements share a face. Our assumption is that the *dual graph* of T is connected.

- The graph made of boundary edges and boundary vertices of T is connected.

- Let $N^v_{\text{int}}(T)$, $N^e_{\text{int}}(T)$, $N^f_{\text{int}}(T)$ be, respectively, the number of interior (non-boundary) vertices, edges and faces of the agglomerate. Let $N^t(T)$ be the number of tetrahedrons (elements) in the agglomerate. We require that

$$E_0(T) := N^v_{\text{int}}(T) - N^e_{\text{int}}(T) + N^f_{\text{int}}(T) - N^t(T) = -1.$$

  The number E_0 is similar to the Euler characteristic, except that we are counting only interior entities.

3.1. Coarse Faces

Consider the set of all fine faces, which are boundary faces of at least one agglomerate. We assume that this set is partitioned into non-intersecting subsets, which we call *coarse faces*. We additionally assume that for any such coarse face F, each fine face f in F is a boundary face of the same set of agglomerated elements. That is, either there exist two agglomerates, T_1 and T_2, such that each fine face from F is a boundary face of both T_1 and T_2 (and, clearly, of no other agglomerate), or there exists an agglomerate T such that each fine face from F is a boundary face of T (and of no other agglomerate).

With each coarse face F, we associate a "consistent normal" $\mathbf{n}_F$ defined to be a vector function on F, equal on each fine face $f \in F$ to either $\mathbf{n}(f)$ or $-\mathbf{n}(f)$, so that $\mathbf{n}_F$ always points inside the same agglomerated element T. The latter means that on each fine face $f \in F$ we have $\mathbf{n}_F = \epsilon(f)\mathbf{n}(f)$ where $\epsilon(f)$ is either 1 or -1, chosen such that $\mathbf{n}_F$ points inside the fine-grid element that is contained in T. Generally, two such choices of $\mathbf{n}_F$ are possible (they differ by a sign). For each coarse face F, we fix one particular choice, once and for all.

We call a fine edge e a *boundary fine edge* of a coarse face F iff e is a subset of exactly one fine face from F. A vertex v is called a boundary vertex of F iff v is an endpoint of some boundary edge of F. We call all other edges and vertices of F *interior*.

Assumption 3.2. We assume that each coarse face F additionally meets the following requirements:

- Define a dual graph of F with its vertices being the fine faces in F, and any two such vertices are connected by an arc iff the corresponding fine faces share an edge. Our assumption is that the dual graph of F is connected.

- Each interior edge of F is a subset of exactly two fine faces from F.

- Interior edges and vertices of F do not belong to any other coarse face. In other words, coarse faces can only "touch" each other along their boundary fine edges (and their boundary vertices).

- F has at least one boundary edge (and thus, at least two boundary vertices).

- Let $N^v_{\text{int}}(F)$ and $N^e_{\text{int}}(F)$ be, respectively, the number of interior vertices and edges of F. Let $N^f(F)$ be the number of faces in F. We require that

$$E_0(F) := N^v_{\text{int}}(F) - N^e_{\text{int}}(F) + N^f(F) = 1.$$

- The following is a technical assumption which simplifies the construction to follow. It can be avoided as demonstrated in [8]. We assume that the *average normal* of F,

$$\mathbf{n}_F^{\text{avg}} = \begin{bmatrix} \int_F \mathbf{n}_F \cdot \mathbf{c}_1 \, \mathrm{d}A \\ \int_F \mathbf{n}_F \cdot \mathbf{c}_2 \, \mathrm{d}A \\ \int_F \mathbf{n}_F \cdot \mathbf{c}_3 \, \mathrm{d}A \end{bmatrix},$$

  is a non-zero vector. We recall that $\mathbf{c}_i$ are the vector constant functions introduced in (2.1).

The above assumptions are in essence a 2D analog of Assumption 3.1. Note that for any agglomerate, the set of its boundary fine faces can always be represented as a union of several coarse faces.

3.2. Coarse Edges and Coarse Vertices

Consider the set of all fine edges, which are the boundary edges (as defined above) of at least one coarse face. We assume that this set is partitioned into non-intersecting subsets, which we call *coarse edges*.

We call a fine vertex v an *endpoint* of a coarse edge E iff v is a subset of exactly one fine edge from E. We refer to the endpoints of the coarse edges also as to *coarse vertices*.

Assumption 3.3. We assume that each coarse edge E meets the following requirements:

- Each fine edge in E is a boundary edge of the same set of coarse faces.
- The fine edges in E together with their endpoints form a connected graph.
- Each fine vertex can be an endpoint of at most two fine edges from E.
- If a fine vertex v is an endpoint of two fine edges from E, then v is not an endpoint of any fine edge from any *other* coarse edge. In other words, coarse edges only "meet" each other at coarse vertices.

Let a coarse edge E contain n fine edges. From the assumptions itemized above it follows that E contains $n+1$ fine vertices and the edges and the vertices can be ordered in such way that the fine edge $e_i \in E$ connects the fine vertices v_i and v_{i+1}. This way the vertices v_1 and v_{n+1} are the endpoints of E. (There are two such possible orderings. We choose one, once and for all.) We call v_1 the tail of E, and v_{n+1} the head of E. We define $\boldsymbol{\tau}_E$ to be a vector function on the coarse edge E, equal on each fine edge e_i to $\pm\boldsymbol{\tau}(e_i)$, where the sign is chosen so that $\boldsymbol{\tau}_E|_{e_i}$ points from v_i to v_{i+1}. Since the endpoints of the coarse edge are two distinct vertices, one can easily see that at least one of the integrals $\int_E \mathbf{c}_i \cdot \boldsymbol{\tau}_E \,\mathrm{d}L$ must be non-zero (for $\mathbf{c}_i$ see (2.1)).

As we shall show in the course of the paper, the conditions in Assumptions 3.1–3.3 guarantee the exactness of the sequence

$$0 \hookrightarrow S_h(T) \xrightarrow{\nabla} Q_h(T) \xrightarrow{\nabla\times} R_h(T) \xrightarrow{\nabla\cdot} M_h(T) \to 0. \tag{3.1}$$

Note that this is a sequence of spaces with homogeneous boundary conditions.

3.3. Stokes Theorems for Coarse Entities

Let E be a coarse edge consisting of fine edges $e_1, \ldots, e_n$ and fine vertices $v_1, \ldots, v_{n+1}$, so that edge e_i connects vertices v_i and v_{i+1}, and $\boldsymbol{\tau}_E|_{e_i}$ points from v_i to v_{i+1}. The vertices v_1 and v_{n+1} are hence the tail and the head of E, respectively. Let $f \in \widetilde{S}_h$. By integrating $\nabla f \in \widetilde{Q}_h$ over the fine edge e_i, we obtain

$$\int_{e_i} \nabla f \cdot \boldsymbol{\tau}_E \,\mathrm{d}L = f(v_{i+1}) - f(v_i). \tag{3.2}$$

By summing (3.2) over i we obtain

$$\int_E \nabla f \cdot \boldsymbol{\tau}_E \,\mathrm{d}L = f(v_{n+1}) - f(v_n). \tag{3.3}$$

Lemma 3.4. *Let T be a coarse element. Then for each coarse face $F \subset \partial T$ there exists a number $\alpha(F) \in \{-1, 1\}$ such that for any sufficiently smooth $\mathbf{r} \in H(\mathrm{div})$*

$$\int_T \nabla\cdot\mathbf{r}\,\mathrm{d}V = \sum_{F\subset\partial T} \alpha(F) \int_F \mathbf{r}\cdot\mathbf{n}_F\,\mathrm{d}A. \tag{3.4}$$

Proof. By applying the divergence theorem to each fine element (tetrahedron) contained in T, summing up the resulting equalities and noting that integrals over interior faces (inside T) cancel out in the sum, we obtain

$$\int_T \nabla\cdot\mathbf{r}\,\mathrm{d}V = \int_{\partial T} \mathbf{r}\cdot\mathbf{n}_T\,\mathrm{d}A = \sum_{F\subset\partial T} \int_F \mathbf{r}\cdot\mathbf{n}_T\,\mathrm{d}A,$$

where $\mathbf{n}_T$ is outward normal to ∂T. Take any coarse face $F \subset \partial T$. Consider the "consistent normal" $\mathbf{n}_F$ (defined in Section 3.1). There are two possibilities. Either $\mathbf{n}_F$ points outside T and then for each fine face $f \subset F$ we have $\mathbf{n}_F|_f = \mathbf{n}_T|_f$. Or $\mathbf{n}_F$ points inside T and then for each fine face $f \subset F$ we have $\mathbf{n}_F|_f = -\mathbf{n}_T|_f$. In any case there exists $\alpha(F) \in \{-1, 1\}$ such that

$$\int_F \mathbf{r}\cdot\mathbf{n}_T\,\mathrm{d}A = \alpha(F) \int_F \mathbf{r}\cdot\mathbf{n}_F\,\mathrm{d}A. \qquad \square$$

Lemma 3.5. *Given a coarse edge E consisting of the fine edges $e_1, \ldots, e_n$, let the numbers $\alpha_i \in \{-1, 1\}$, $i = 1, \ldots, n$, satisfy*

$$\sum_{i=1}^n \alpha_i \int_{e_i} \nabla\theta\cdot\boldsymbol{\tau}\,\mathrm{d}L = 0$$

for any $\theta \in \widetilde{S}_h$ vanishing at the endpoints (coarse vertices) of E. Then either

$$\alpha_i\boldsymbol{\tau}(e_i) = \boldsymbol{\tau}_E|_{e_i}, \quad i = 1, \ldots, n, \tag{3.5}$$

or

$$\alpha_i\boldsymbol{\tau}(e_i) = -\boldsymbol{\tau}_E|_{e_i}, \quad i = 1, \ldots, n. \tag{3.6}$$

Proof. Denote the fine vertices of E as $v_1, \ldots, n_{n+1}$, so that the fine edge e_i connects the vertices v_i and v_{i+1}, and $\boldsymbol{\tau}_E|_{e_i}$ points from v_i to v_{i+1}. We shall assume

$$\alpha_1\boldsymbol{\tau}(e_1) = \boldsymbol{\tau}_E|_{e_1} \tag{3.7}$$

and prove (3.5). The case $\alpha_1\boldsymbol{\tau}(e_1) = -\boldsymbol{\tau}_E|_{e_1}$ is treated analogously, yielding (3.6).

Suppose (3.5) does not hold. Denote by k the first index i for which (3.5) is violated. Due to our assumption (3.7), $k \geqslant 2$. Let $\theta \in \widetilde{S}_h$ satisfy $\theta(v_k) = 1$ and vanish at all other fine vertices. We then have

$$\begin{aligned}\sum_{i=1}^n \alpha_i \int_{e_i} \nabla\theta\cdot\boldsymbol{\tau}\,\mathrm{d}L &= \alpha_{k-1} \int_{e_{k-1}} \nabla\theta\cdot\boldsymbol{\tau}\,\mathrm{d}L + \alpha_k \int_{e_k} \nabla\theta\cdot\boldsymbol{\tau}\,\mathrm{d}L \\ &= \int_{e_{k-1}} \nabla\theta\cdot\boldsymbol{\tau}_E\,\mathrm{d}L - \int_{e_k} \nabla\theta\cdot\boldsymbol{\tau}_E\,\mathrm{d}L \\ &= \theta(v_k) - \theta(v_{k-1}) - \theta(v_{k+1}) + \theta(v_k) = 2 \neq 0,\end{aligned}$$

which contradicts the assumption of the lemma. $\square$

Lemma 3.6. *Let F be a coarse face. Then for each coarse edge $E \subset \partial F$ there exists a number $\beta(E) \in \{-1, 1\}$ such that for any sufficiently smooth $\mathbf{q} \in H(\mathrm{curl})$*

$$\int_F \nabla \times \mathbf{q} \cdot \mathbf{n}_F \, \mathrm{d}A = \sum_{E \subset \partial F} \beta(E) \int_E \mathbf{q} \cdot \boldsymbol{\tau}_E \, \mathrm{d}L. \tag{3.8}$$

Proof. By applying the Stokes theorem to each fine face contained in F and summing up the resulting equalities, we can see that for each fine edge $e \subset F$ there exists a number $\alpha(e)$ such that

$$\int_F \nabla \times \mathbf{q} \cdot \mathbf{n}_F \, \mathrm{d}A = \sum_{e \in F} \alpha(e) \int_e \mathbf{q} \cdot \boldsymbol{\tau} \, \mathrm{d}L \tag{3.9}$$

for any sufficiently smooth $\mathbf{q} \in H(\mathrm{curl})$. We further observe that $\alpha(e) \in \{-1, 1\}$ for the boundary edges of F, and $\alpha(e) \in \{-2, 0, 2\}$ for the interior edges of F.

We shall prove that $\alpha(e) = 0$ for any interior fine edge $e \subset F$. Choose $\mathbf{q}$ to satisfy

$$\int_{e'} \mathbf{q} \cdot \boldsymbol{\tau} \, \mathrm{d}L = \begin{cases} 1 & \text{if } e' = e, \\ 0 & \text{if } e' \neq e \end{cases}$$

for each fine edge e' of the mesh. Then for any coarse face F' we have

$$\int_{F'} (\nabla \times \mathbf{q}) \cdot \mathbf{n}_{F'} \, \mathrm{d}A = \begin{cases} \alpha(e) & \text{if } F' = F, \\ 0 & \text{if } F' \neq F. \end{cases}$$

There exists at least one agglomerate T such that F is part of ∂T. We then have, applying Lemma 3.4 to T,

$$0 = \int_T \nabla \cdot \nabla \times \mathbf{q} \, \mathrm{d}V = \pm \alpha(e),$$

which is only possible when $\alpha(e) = 0$.

We can now re-write (3.9) as

$$\int_F \nabla \times \mathbf{q} \cdot \mathbf{n}_F \, \mathrm{d}A = \sum_{E \subset \partial F} \sum_{e \subset E} \alpha(e) \int_e \mathbf{q} \cdot \boldsymbol{\tau} \, \mathrm{d}L, \tag{3.10}$$

where $\alpha(e) = \pm 1$ for each e on ∂F. Consider any coarse edge $E \subset \partial F$. Let $\theta \in \widetilde{S}_h$ be an arbitrary function vanishing at all boundary vertices of F, except the interior vertices of E. Note that θ vanishes at the endpoints of E and $\int_e \nabla \theta \cdot \boldsymbol{\tau} \, \mathrm{d}L = 0$ for any fine edge $e \subset \partial F$, $e \not\subset E$. Substituting $\mathbf{q} = \nabla \theta$ into (3.10), we get

$$0 = \int_F (\nabla \times \nabla \theta) \cdot \mathbf{n}_F \, \mathrm{d}A = \sum_{e \subset E} \alpha(e) \int_e \nabla \theta \cdot \boldsymbol{\tau} \, \mathrm{d}L.$$

Then, by Lemma 3.5 there exists a $\beta(E) \in \{-1, 1\}$ such that

$$\alpha(e) \boldsymbol{\tau}(e) = \beta(E) \boldsymbol{\tau}_E|_e \quad \text{for all } e \subset E,$$

which substituted in (3.10) implies

$$\sum_{e \subset E} \alpha(e) \int_e \mathbf{q} \cdot \boldsymbol{\tau} \, \mathrm{d}L = \beta(E) \int_E \mathbf{q} \cdot \boldsymbol{\tau}_E \, \mathrm{d}L.$$

The desired result follows. □

4. Some Auxiliary Results

4.1. Solvability of Certain Discrete Mixed Systems

Our construction of the coarse spaces involves solving various "small" linear systems, associated with either coarse elements or coarse faces. The unique solvability of these linear systems follows from, e.g., [1, Lemma 3.10]. We restate the necessary corollary of the lemma below for completeness.

Let V_i, $i = 1, 2, 3$, be finite dimensional linear spaces with inner products $(\cdot, \cdot)_i$. In particular, we allow $V_i = \{0\}$. Note that in this case $(\cdot, \cdot)_i \equiv 0$ is still formally an inner product. Let $\mathrm{d}_i : V_i \to V_{i+1}$, $i = 1, 2$, be linear operators. Consider the problem of finding a pair $\sigma \in V_1$, $u \in V_2$ satisfying

$$\begin{aligned} (\sigma, \tau)_1 - (\mathrm{d}_1 \tau, u)_2 &= \varphi(\tau) \quad \text{for all } \tau \in V_1, \\ (\mathrm{d}_1 \sigma, v)_2 + (\mathrm{d}_2 u, \mathrm{d}_2 v)_3 &= \psi(v) \quad \text{for all } v \in V_2, \end{aligned} \tag{4.1}$$

where φ is a linear functional on the space V_1, and ψ is a linear functional on the space V_2.

Lemma 4.1. *If* $\ker(\mathrm{d}_2) = \mathrm{d}_1(V_1)$*, then the system* (4.1) *has a unique solution pair for any* φ *and* ψ*.*

Proof. Since the spaces V_i are finite-dimensional, it is sufficient to show that the homogeneous system

$$\begin{aligned} (\sigma, \tau)_1 - (\mathrm{d}_1 \tau, u)_2 &= 0 \quad \text{for all } \tau \in V_1, \\ (\mathrm{d}_1 \sigma, v)_2 + (\mathrm{d}_2 u, \mathrm{d}_2 v)_3 &= 0 \quad \text{for all } v \in V_2 \end{aligned}$$

has only zero solution, when $\ker(\mathrm{d}_2) = \mathrm{d}_1(V_1)$, which is readily checked (as shown in [1, Lemma 3.10]). $\square$

4.2. Some Properties of Projectors

We summarize here two properties of projection operators that we use throughout the paper. Let V be a linear space. An operator P is called a projector onto the subspace $L \subset V$ if $P(V) \subset L$ and $Px = x$ for any $x \in L$. The definition obviously implies $P^2 = P$, $P(V) = P(L) = L$.

Lemma 4.2. *Let* $P_1, \ldots, P_k$ *be projectors onto subspaces* $V_1, \ldots, V_k$ *of* V*, respectively. Let* $P_i P_j = P_j P_i = 0$ *for* $i \neq j$*. Then the sum* $\sum_i V_i$ *is direct and the operator* $P = \sum_i P_i$ *is a projector onto* $\sum_i V_i = \bigoplus_i V_i$*.*

The proof is straightforward and can be found, e.g., in [11, p. 131].

Lemma 4.3. *Let* $P : V \to V$ *be a projector onto a linear space* $L \subset V$*. Let* $A_i : V \to V$*,* $i = 1, \ldots, n$*, be linear operators satisfying* $PA_i = 0$ *for each* i*. Then the operator*

$$\Big(\mathbb{1} + \sum_i A_i\Big) P$$

is a projector onto the linear space

$$\Big(\mathbb{1} + \sum_i A_i\Big) L.$$

Proof. Let $A = \mathbb{1} + \sum_i A_i$. We need to prove that AP is a projector onto $A(L)$. Clearly $APx \in A(L)$ for any $x \in V$. Let $x = Ay$ for $y \in L$. From the condition $PA_i = 0$ it follows that $PA = P$. We then have $APx = APAy = APy = Ay = x$. $\square$

5. L_2- and H(div)-Conforming Coarse Spaces

We define the space $\widetilde{M}_H \subset \widetilde{M}_h \subset L_2(\Omega)$ to consist of functions which are constant in (the interior of) each agglomerated element. The functions in $\widetilde{M}_H$ are elements of $L_2(\Omega)$; they are discontinuous across the boundaries of the agglomerated elements. We define Π^M to be the orthogonal projection onto $\widetilde{M}_H$ (with respect to the L_2 inner product).

In the rest of this section we describe the construction of the coarse Raviart–Thomas space, $\widetilde{R}_H \subset \widetilde{R}_h$. This construction was introduced in [8], and we include it here for completeness. We define $\widetilde{R}_H$ in two steps. First, for each coarse face F we define a certain space $\widetilde{R}_H(F) \subset \widetilde{R}_h$. We then describe the "interior extension" mapping from $\bigoplus_F \widetilde{R}_H(F)$ to $\widetilde{R}_h$, i.e., from the traces on the set of coarse faces F into the interior of the agglomerated elements.

The standard degrees of freedom for the lowest order Raviart–Thomas space are the integrals of the normal component over the fine faces:

$$\mathbf{r} \to \int_f \mathbf{r}\cdot\mathbf{n}\,\mathrm{d}A, \quad \mathbf{r} \in \widetilde{R}_h, \quad f \text{ is a fine face.} \tag{5.1}$$

Specifying the values of all such degrees of freedom completely defines a function in $\widetilde{R}_h$.

Definition 5.1 (Restrictions of fine-grid RT functions)**.** Let $\mathbf{r} \in \widetilde{R}_h$. Let K be a coarse face or a coarse element. Define $\pi_K^R\mathbf{r} \in \widetilde{R}_h$ to have the following degrees of freedom:

$$\int_f (\pi_K^R\mathbf{r})\cdot\mathbf{n}\,\mathrm{d}A = \begin{cases} \int_f \mathbf{r}\cdot\mathbf{n}\,\mathrm{d}A & \text{if } f \in K, \\ 0 & \text{if } f \notin K. \end{cases}$$

Recall that we treat all coarse entities as closed sets, e.g., a coarse element contains all its boundary fine faces.

5.1. Defining Coarse Raviart–Thomas Basis Functions on Coarse Faces

We associate at least one and up to three coarse Raviart–Thomas basis functions with a given coarse face. Let F be any coarse face. Consider the space

$$\widetilde{R}_h(F) = \pi_F^R \widetilde{R}_h.$$

We define $\widetilde{R}_H(F) \subset \widetilde{R}_h(F)$ to be

$$\widetilde{R}_H(F) = \pi_F^R C.$$

Remark 5.2. In an actual computer implementation, we form a matrix with entries $W_{ij} = \int_{f_i} \mathbf{c}_j\cdot\mathbf{n}_F\,\mathrm{d}A$, where $f_i \in F$, and perform the singular value decomposition (SVD) of W. The singular vectors which correspond to singular values above certain threshold give rise to coarse basis functions. This is discussed in more detail in [8]. The dimension of $\widetilde{R}_H(F)$ can be one (the case for a planar coarse face), two, or three.

For $\mathbf{r}, \mathbf{s} \in \widetilde{R}_h$, we define

$$(\mathbf{r},\mathbf{s})_F = (\mathbf{r},\mathbf{s})_F^R = \sum_{f\in F} \Big(\int_f \mathbf{r}\cdot\mathbf{n}_F\,\mathrm{d}A\Big)\Big(\int_f \mathbf{s}\cdot\mathbf{n}_F\,\mathrm{d}A\Big). \tag{5.2}$$

Clearly, this bilinear form $(\mathbf{r},\mathbf{s})_F$ is an inner product on $\widetilde{R}_h(F)$. Note that we are using the "global" normal $\mathbf{n}_F(f)$ on each fine face $f \in F$. (It is easy to check that using local normal in each integral of (5.2) would lead to an equivalent definition. Using global normals in (5.2) is more convenient when proving (6.12).)

Remark 5.3. Alternatively, we could use the following bilinear form:

$$(\mathbf{r},\mathbf{s})_F = \sum_{f\in F}\Big(\int_f \mathbf{r}\cdot\mathbf{n}_F\ \mathbf{s}\cdot\mathbf{n}_F\,\mathrm{d}A\Big).$$

The mapping

$$\mathbf{r} \to \int_F \mathbf{r}\cdot\mathbf{n}_F\,\mathrm{d}A \tag{5.3}$$

can be viewed as a linear functional on the space $\widetilde{R}_H(F)$. Due to the Riesz representation theorem, there exists a $\boldsymbol{\nu}_F \in \widetilde{R}_H(F)$ such that

$$\int_F \mathbf{r}\cdot\mathbf{n}_F\,\mathrm{d}A = (\boldsymbol{\nu}_F,\mathbf{r})_F \quad \text{for all } \mathbf{r}\in\widetilde{R}_H(F). \tag{5.4}$$

We assume that the average normal of F is non-zero (last bullet in Assumption 3.2). Thus, the linear functional (5.3) is nonzero on $\widetilde{R}_H(F)$ and consequently $\boldsymbol{\nu}_F \neq 0$. Let the projector $\Pi^R_{F,1}:\widetilde{R}_h \to \operatorname{span}(\boldsymbol{\nu}_F)$ be defined as follows:

$$\Pi^R_{F,1}\mathbf{r} = \frac{\int_F \mathbf{r}\cdot\mathbf{n}_F\,\mathrm{d}A}{(\boldsymbol{\nu}_F,\boldsymbol{\nu}_F)_F}\boldsymbol{\nu}_F \quad \text{for all } \mathbf{r}\in\widetilde{R}_h. \tag{5.5}$$

From (5.4) it follows that $\Pi^R_{F,1}$ is indeed a projector onto $\operatorname{span}(\boldsymbol{\nu}_F)$. We now define the space

$$R_H(F) = \big\{\mathbf{r}\in\widetilde{R}_H(F) : (\mathbf{r},\boldsymbol{\nu}_F)_F = 0\big\}.$$

Note that the dimension of $R_H(F)$ can be zero (for a flat F), one, or two. For any $\mathbf{r}\in\widetilde{R}_h$, we define $\Pi^R_{F,0}\mathbf{r}$ to be the (unique) element of $R_H(F)$ satisfying

$$\big(\Pi^R_{F,0}\mathbf{r},\mathbf{s}\big)_F = (\mathbf{r},\mathbf{s})_F \quad \text{for all } \mathbf{s}\in R_H(F). \tag{5.6}$$

It is easy to check that $\Pi^R_{F,0}\Pi^R_{F,1} = \Pi^R_{F,1}\Pi^R_{F,0} = 0$. Consequently,

$$\Pi^R_F = \Pi^R_{F,1} + \Pi^R_{F,0}$$

is a projector onto $\operatorname{span}(\boldsymbol{\nu}_F)\oplus R_H(F) = \widetilde{R}_H(F)$. Note that

$$\int_F \mathbf{r}\cdot\mathbf{n}_F\,\mathrm{d}A = \int_F (\Pi^R_{F,1}\mathbf{r})\cdot\mathbf{n}_F\,\mathrm{d}A = \int_F (\Pi^R_F\mathbf{r})\cdot\mathbf{n}_F\,\mathrm{d}A. \tag{5.7}$$

Definition 5.4. We define

$$\Pi^R_2 = \sum_F \Pi^R_F.$$

Lemma 5.5. Π^R_2 *is a projector onto* $\bigoplus_F \widetilde{R}_H(F)$.

Proof. It is easy to check that $\Pi^R_F\Pi^R_{F'} = 0$ for $F\neq F'$. The desired result then follows from Lemma 4.2. □

Lemma 5.6. *Let F be a coarse face. Let $\mathbf{r}\in\widetilde{R}_h$. Suppose $\pi^R_F\mathbf{r} = \pi^R_F\mathbf{c}$, where $\mathbf{c}$ is a vector constant. Then $\pi^R_F\Pi^R_2\mathbf{r} = \pi^R_F\mathbf{c}$.*

Proof. It is easy to check that $\pi^R_F\Pi^R_2\mathbf{r} = \Pi^R_F\mathbf{r} = \Pi^R_F\pi^R_F\mathbf{r}$. Since $\pi^R_F\mathbf{r} = \pi^R_F\mathbf{c}\in\widetilde{R}_H(F)$ and Π^R_F is a projector onto $\widetilde{R}_H(F)$, we have $\Pi^R_F\pi^R_F\mathbf{r} = \pi^R_F\mathbf{c}$. □

5.2. Extending Coarse Raviart–Thomas Basis Functions into the Interior of Agglomerated Elements

After we have constructed the spaces $\widetilde{R}_H(F)$, it remains to describe the interior extension mapping $\mathcal{I}_T^R$ for each coarse element T. We first prove that the sequence 3.1 is exact at the term $M_h(T)$. We need this result to show that a certain linear system that we use for interior extension is non-degenerate.

Lemma 5.7. *We have*

$$\nabla\cdot R_h(T) = M_h(T).$$

Proof. We first prove $\nabla\cdot R_h(T) \subset M_h(T)$. Fix $\mathbf{r} \in R_h(T)$. On any tetrahedron $\nabla\cdot\mathbf{r}$ is constant. If tetrahedron t is outside T, then $\int_f \mathbf{r}\cdot\mathbf{n}\,\mathrm{d}A = 0$ for any triangular face f of t. Thus, due to the divergence theorem applied to t, we have that $\nabla\cdot\mathbf{r}$ vanishes on t. The condition $\int_T \nabla\cdot\mathbf{r}\,\mathrm{d}V = 0$ follows from Lemma 3.4.

We now show the equality $\nabla\cdot R_h(T) = M_h(T)$. Just for the sake of this proof, consider the following inner product on $M_h(T)$:

$$\langle u, v\rangle = \sum_{t\subset T}\Big(\int_t u\,\mathrm{d}V\Big)\Big(\int_t v\,\mathrm{d}V\Big).$$

This inner product is not the $L_2(T)$ inner product on $R_h(T)$, unless each tetrahedron in T has unit volume. Let $w \in M_h(T)$ satisfy

$$\langle w, \nabla\cdot\mathbf{r}\rangle = 0 \quad \text{for all } \mathbf{r} \in R_h(T).$$

We shall prove $w = 0$, which implies the desired equality. We first establish that

$$\int_{t_1} w\,\mathrm{d}V = \int_{t_2} w\,\mathrm{d}V \tag{5.8}$$

for any pair of fine elements $t_1, t_2 \subset T$ which share an (interior) face $f \subset T$. Let $\mathbf{r} \in R_h(T)$ satisfy

$$\int_{f'} \mathbf{r}\cdot\mathbf{n}\,\mathrm{d}A = \begin{cases} 1 & \text{if } f' = f,\\ 0 & \text{otherwise.}\end{cases}$$

By applying the divergence theorem to the fine elements t_1, t_2 we obtain

$$\Big|\int_{t_i} \nabla\cdot\mathbf{r}\,\mathrm{d}V\Big| = 1, \quad i = 1, 2. \tag{5.9}$$

Clearly, f can be a subset of at most two tetrahedrons from T. Consequently, by applying the divergence theorem to any fine element $t \subset T$ other than t_1 and t_2, we get

$$\int_t \nabla\cdot\mathbf{r}\,\mathrm{d}V = 0, \quad t \notin \{t_1, t_2\}.$$

As we already established, $\nabla\cdot R_h(T) \subset M_h(T)$, and thus

$$0 = \int_T \nabla\cdot\mathbf{r}\,\mathrm{d}V = \int_{t_1} \nabla\cdot\mathbf{r}\,\mathrm{d}V + \int_{t_2} \nabla\cdot\mathbf{r}\,\mathrm{d}V. \tag{5.10}$$

We can re-write (5.9) and (5.10) as

$$\int_{t_1} \nabla\cdot\mathbf{r}\,\mathrm{d}V = \alpha, \qquad \int_{t_2} \nabla\cdot\mathbf{r}\,\mathrm{d}V = -\alpha, \qquad |\alpha| = 1.$$

The condition $\langle w, \nabla\cdot\mathbf{r}\rangle = 0$ can then be re-written as

$$\alpha \int_{t_1} w\,\mathrm{d}V - \alpha \int_{t_2} w\,\mathrm{d}V = 0.$$

Since $\alpha \neq 0$, this implies (5.8).

By Assumption 3.1 the dual graph of T is connected. Because of this, (5.8) holds even if t_1 and t_2 do not share a face. Since $w \in R_h(T)$, we have

$$\sum_{t\subset T} \int_t w\,\mathrm{d}V = 0.$$

We have already established that all terms in the above sum are equal, consequently they are all zero. Since w is constant on each tetrahedron, we have $w = 0$. □

We now describe the actual extension procedure. Given a function $\mathbf{r} \in \widetilde{R}_h$, let $\mathbf{r}_T \in R_h(T)$ satisfy

$$\begin{aligned} (\mathbf{r}_T, \mathbf{s})_T + (u_T, \nabla\cdot\mathbf{s})_T &= -(\mathbf{r}, \mathbf{s})_T && \text{for all } \mathbf{s} \in R_h(T), \\ (\nabla\cdot\mathbf{r}_T, w)_T &= -(\nabla\cdot\mathbf{r}, w)_T && \text{for all } w \in M_h(T), \end{aligned} \tag{5.11}$$

for some $u_T \in M_h(T)$. Here $(\mathbf{r}, \mathbf{s})_T = \int_T \mathbf{r}\cdot\mathbf{s}\,\mathrm{d}V$ and $(u, w)_T = \int_T uw\,\mathrm{d}V$. The system (5.11) is uniquely solvable due to Lemma 4.1 (take $V_1 = R_h(T)$, $V_2 = M_h(T)$, $V_3 = \{0\}$, $\mathrm{d}_1 = \nabla\cdot$, $\mathrm{d}_2 = 0$, and use (5.7)).

Definition 5.8. Let $\mathbf{r} \in \widetilde{R}_h$. Let $\mathbf{r}_T \in R_h(T)$ be the corresponding solution of (5.11). We define

$$\mathcal{I}_T^R \mathbf{r} = \mathbf{r}_T.$$

Note that we have

$$\mathcal{I}_T^R \mathbf{r} = -\mathbf{r} \quad \text{for all } \mathbf{r} \in R_h(T), \tag{5.12}$$

since the pair $(\mathbf{r}_T = -\mathbf{r}, u_T = 0)$ solves (5.11).

Lemma 5.9. *Let $\mathbf{r} \in \widetilde{R}_h$. Let T be a coarse element. Suppose $\pi_T^R \mathbf{r} \in R_h(T)$. Then*

$$\pi_T^R (\mathbb{1} + \mathcal{I}_T^R)\mathbf{r} = 0.$$

Proof. Fix $\mathbf{r} \in \widetilde{R}_h$. It is easy to verify the identities $\pi_T^R \mathcal{I}_T^R \mathbf{r} = \mathcal{I}_T^R \pi_T^R \mathbf{r} = \mathcal{I}_T^R \mathbf{r}$. Using these identities we obtain

$$\pi_T^R (\mathbb{1} + \mathcal{I}_T^R)\mathbf{r} = (\mathbb{1} + \mathcal{I}_T^R)\pi_T^R \mathbf{r} \overset{(5.12)}{=} \pi_T^R \mathbf{r} - \pi_T^R \mathbf{r} = 0.$$ □

Definition 5.10. The coarse Raviart–Thomas space is defined to be

$$\widetilde{R}_H = \Big(\mathbb{1} + \sum_T \mathcal{I}_T^R\Big) \bigoplus_F \widetilde{R}_H(F).$$

The corresponding projector, $\Pi^R : \widetilde{R}_h \to \widetilde{R}_H$, is defined to be

$$\Pi^R = \Big(\mathbb{1} + \sum_T \mathcal{I}_T^R\Big)\Pi_2^R = \Big(\mathbb{1} + \sum_T \mathcal{I}_T^R\Big) \sum_F \Pi_F^R. \tag{5.13}$$

We show next that Π^R is indeed a projector onto $\widetilde{R}_H$. It is clear that equations (5.7) and (5.13) imply that for any coarse face F and any function $\mathbf{r} \in \widetilde{R}_h$, we have

$$\int_F \mathbf{r}\cdot\mathbf{n}_F \,\mathrm{d}A = \int_F (\Pi_H^R \mathbf{r})\cdot\mathbf{n}_F \,\mathrm{d}A. \tag{5.14}$$

Lemma 5.11. Π^R *is a projector onto* $\widetilde{R}_H$.

Proof. It is easy to check that for any coarse element T we have $\Pi_2^R \mathcal{I}_T^R = 0$. The desired result then follows from Lemma 4.3. □

5.3. The Commutativity Property

Theorem 5.12. *For all* $\mathbf{s} \in \widetilde{R}_h$, *we have*

$$\nabla\cdot\Pi^R\mathbf{s} = \Pi^M\nabla\cdot\mathbf{s}.$$

Proof. Since $\Pi^R\mathbf{s} \in \widetilde{R}_H$, we have $\Pi^R\mathbf{s} = \mathbf{r} + \sum_T \mathbf{r}_T$, where $\mathbf{r} \in \bigoplus_F \widetilde{R}_H(F)$ and each $\mathbf{r}_T$ satisfies (5.11). For any agglomerate T and any function $w \in M_h(T)$, due to the second equation of (5.11), we have

$$(\nabla\cdot\Pi^R\mathbf{s}, w)_T = \big(\nabla\cdot(\mathbf{r}+\mathbf{r}_T), w\big)_T = 0.$$

In other words, $\nabla\cdot\Pi^R\mathbf{s}$ is constant on T, just like $\Pi^M\nabla\cdot\mathbf{s}$. To prove that the constants are equal, it is obviously sufficient to show that

$$\int_T \nabla\cdot\Pi^R\mathbf{s}\,\mathrm{d}V = \int_T \Pi^M\nabla\cdot\mathbf{s}\,\mathrm{d}V.$$

Let $F_1,\ldots,F_n$ be the coarse faces incident to T. We have

$$\begin{aligned}\int_T \nabla\cdot\Pi^R\mathbf{s}\,\mathrm{d}V &\overset{(3.4)}{=} \sum_{i=1}^n \alpha_i \int_{F_i} (\Pi^R\mathbf{s})\cdot\mathbf{n}_{F_i}\,\mathrm{d}A \\ &\overset{(5.14)}{=} \sum_{i=1}^n \alpha_i \int_{F_i} \mathbf{s}\cdot\mathbf{n}_F\,\mathrm{d}A \overset{(3.4)}{=} \int_T \nabla\cdot\mathbf{s}\,\mathrm{d}V = \int_T \Pi^M\nabla\cdot\mathbf{s}\,\mathrm{d}V,\end{aligned}$$

where the last equality is true by the definition of Π^M. □

5.4. The “Exactness” Property

Corollary 5.13. *If* $\nabla\cdot\widetilde{R}_h = \widetilde{M}_h$, *then* $\nabla\cdot\widetilde{R}_H = \widetilde{M}_H$.

Proof. Let $u \in \widetilde{M}_H$. Since $\nabla\cdot\widetilde{R}_h = \widetilde{M}_h$, there exists $\hat{\mathbf{r}} \in \widetilde{M}_h$ such that $\nabla\cdot\hat{\mathbf{r}} = u$. Since $\nabla\cdot\hat{\mathbf{r}} \in \widetilde{M}_H$ and due to Theorem 5.12, we have

$$\nabla\cdot\hat{\mathbf{r}} = \Pi^M\nabla\cdot\hat{\mathbf{r}} = \nabla\cdot\Pi^R\hat{\mathbf{r}}.$$

That is, $u = \nabla\cdot\mathbf{r}$, where $\mathbf{r} = \Pi^R\hat{\mathbf{r}} \in \widetilde{R}_H$. □

5.5. The Local Approximation Property

Theorem 5.14. *If a function* $\mathbf{s} \in \widetilde{R}_h$ *coincides with a vector constant* $\mathbf{c}$ *on an agglomerate* T*, then* $\Pi^R \mathbf{s}$ *also coincides with* $\mathbf{c}$ *on* T*.*

Proof. We have

$$\Pi^R \mathbf{s} = \mathbf{r} + \sum_{T'} \mathbf{r}_{T'},$$

where each $\mathbf{r}_{T'}$ satisfies (5.11), and

$$\mathbf{r} = \sum_F \Pi_F^R \mathbf{s}.$$

On the coarse element T, $\Pi^R \mathbf{s}$ obviously equals $\mathbf{r} + \mathbf{r}_T$. In what follows we shall restrict all functions to T. Due to Lemma 5.6 we have for any fine face $f \in \partial T$

$$\int_f \mathbf{r} \cdot \mathbf{n}_f \, \mathrm{d}A = \int_f \mathbf{c} \cdot \mathbf{n}_f \, \mathrm{d}A,$$

and thus the function $\mathbf{c} - \mathbf{r}$ belongs to $R_h(T)$. It now suffices to prove that $\mathbf{r}_T = \mathbf{c} - \mathbf{r}$ solves (5.11), since (5.11) has only one solution. The second equation of (5.11) is satisfied, since $\nabla \cdot \mathbf{c} = 0$. Let $\hat{u}_T$ be any linear function satisfying $\nabla \hat{u}_T = \mathbf{c}$. Pick any $\mathbf{s} \in R_h(T)$. Using the "integration by parts" (applying (3.4) to $\nabla \cdot (\hat{u}_T \mathbf{s})$ and noting that $\mathbf{s} \cdot \mathbf{n}$ vanishes on ∂T), we obtain

$$(\mathbf{c}, \mathbf{s})_T + (\hat{u}_T, \nabla \cdot \mathbf{s})_T = 0.$$

Let u_T be an L_2 projection of $\hat{u}_T$ onto $M_h(T)$. By the properties of the orthogonal projection, $(\hat{u}_T, w)_T = (u_T, w)_T$ for any $w \in M_h(T)$. Since $\nabla \cdot \mathbf{s} \in M_h(T)$, we have

$$(\mathbf{c}, \mathbf{s})_T + (u_T, \nabla \cdot \mathbf{s})_T = 0,$$

or equivalently

$$(\mathbf{r}_T, \mathbf{s})_T + (u_T, \nabla \cdot \mathbf{s})_T = -(\mathbf{r}, \mathbf{s})_T,$$

which is the first equation of (5.11). □

6. Coarse Nédélec Space

As is well known, the standard degrees of freedom for the lowest-order Nédélec space are the integrals of tangential component over the edges:

$$\mathbf{q} \to \int_e \mathbf{q} \cdot \boldsymbol{\tau} \, \mathrm{d}L, \quad \mathbf{q} \in \widetilde{Q}_h, \ e \text{ is a fine edge.} \tag{6.1}$$

Specifying all such degrees of freedom completely specifies a function from $\widetilde{Q}_h$.

Definition 6.1. Let $\mathbf{q} \in \widetilde{Q}_h$. Let K be a coarse edge, a coarse face or a coarse element. We define $\pi_K^Q \mathbf{q} \in \widetilde{Q}_h$ to have the following degrees of freedom:

$$\int_e (\pi_K^Q \mathbf{q}) \cdot \boldsymbol{\tau} \, \mathrm{d}L = \begin{cases} \int_e \mathbf{q} \cdot \boldsymbol{\tau} \, \mathrm{d}L & \text{if } e \in K, \\ 0 & \text{if } e \notin K. \end{cases}$$

Note that we treat all coarse entities as closed sets, i.e., a coarse face F contains all its boundary fine edges and so forth.

6.1. Defining Coarse Nédélec (ND) Basis Functions on Coarse Edges

With each coarse edge we associate up to three coarse basis functions. Let E be any coarse edge. Consider the space

$$\widetilde{Q}_h(E) = \pi^Q_E \widetilde{Q}_h.$$

Recall that the space of vector constant functions is denoted by C. We define the space $\widetilde{Q}_H(E) \subset \widetilde{Q}_h(E)$ to be

$$\widetilde{Q}_H(E) = \pi^Q_E C. \tag{6.2}$$

Remark 6.2. In an actual computer implementation, a basis of $\widetilde{Q}_H(E)$ can be constructed by calculating the left singular vectors of the matrix with entries $W_{ij} = \int_{e_i} \mathbf{c}_j \cdot \boldsymbol{\tau} \,\mathrm{d}L$, where $e_i \subset E$. If E is a straight line, the dimension of $\widetilde{Q}_H(E)$ is one. If E is not a straight line, but still a planar polygonal line, the dimension of $\widetilde{Q}_H(E)$ is two. If E is not planar, the dimension of $\widetilde{Q}_H(E)$ is three.

Let $(\cdot,\cdot)_E = (\cdot,\cdot)_{E,Q}$ be the following bilinear form $\widetilde{Q}_h \times \widetilde{Q}_h \to \mathbb{R}$:

$$(\mathbf{p},\mathbf{q})_E = \sum_{e\subset E} \Big(\int_e \mathbf{p}\cdot\boldsymbol{\tau}_E \,\mathrm{d}L \Big) \Big(\int_e \mathbf{q}\cdot\boldsymbol{\tau}_E \,\mathrm{d}L \Big).$$

Note that $(\cdot,\cdot)_E$ is an inner product, when both arguments are restricted to $\widetilde{Q}_h(E)$.

Remark 6.3. We can alternatively use the following inner product:

$$(\mathbf{p},\mathbf{q})_E = \sum_{e\subset E} \int_e (\mathbf{p}\cdot\boldsymbol{\tau}_E)(\mathbf{q}\cdot\boldsymbol{\tau}_E) \,\mathrm{d}L.$$

Due to the Riesz representation theorem, there exists a $\boldsymbol{\nu}_E \in \widetilde{Q}_H(E)$ such that

$$\int_E \mathbf{q}\cdot\boldsymbol{\tau}_E \,\mathrm{d}L = (\boldsymbol{\nu}_E, \mathbf{q})_E \quad \text{for all } \mathbf{q} \in \widetilde{Q}_H(E). \tag{6.3}$$

At least one of the integrals $\int_E \mathbf{c}_i \cdot \boldsymbol{\tau}_E \,\mathrm{d}L$ is nonzero (see Section 3.2), so, in view of (6.2), $\boldsymbol{\nu}_E \neq 0$. Define the space $Q_H(E)$ to be

$$Q_H(E) = \big\{ \mathbf{q} \in \widetilde{Q}_H(E) : (\mathbf{q}, \boldsymbol{\nu}_E)_E = 0 \big\}. \tag{6.4}$$

The dimension of $Q_H(E)$ can be zero, one, or two.

Define the 1D projector $\Pi^Q_{E,1} : \widetilde{Q}_h \to \mathrm{span}(\boldsymbol{\nu}_E)$ as follows:

$$\Pi^Q_{E,1}\mathbf{q} = \frac{\int_E \mathbf{q}\cdot\boldsymbol{\tau}_E \,\mathrm{d}L}{(\boldsymbol{\nu}_E,\boldsymbol{\nu}_E)_E} \boldsymbol{\nu}_E \quad \text{for all } \mathbf{q} \in \widetilde{Q}_h. \tag{6.5}$$

From (6.3) it follows that $\Pi^Q_{E,1}$ is indeed a projector onto $\mathrm{span}(\boldsymbol{\nu}_E)$.

For any $\mathbf{q} \in \widetilde{Q}_h$, we define $\Pi^Q_{E,0}\mathbf{q}$ to satisfy

$$\big(\Pi^Q_{E,0}\mathbf{q}, \mathbf{p}\big)_E = (\mathbf{q},\mathbf{p})_E \quad \text{for all } \mathbf{p} \in Q_H(E). \tag{6.6}$$

It is easy to check that such $\Pi^Q_{E,0}\mathbf{q}$ exists, is unique and $\Pi^Q_{E,0}\mathbf{q} = \mathbf{q}$ when $\mathbf{q} \in Q_H(E)$. From the fact $\boldsymbol{\nu}_E \perp Q_H(E)$ and (6.6) it follows that

$$\Pi^Q_{E,0}\Pi^Q_{E,1} = \Pi^Q_{E,1}\Pi^Q_{E,0} = 0,$$

and consequently the operator

$$\Pi^Q_E = \Pi^Q_{E,0} + \Pi^Q_{E,1}$$

is a projector onto $\widetilde{Q}_H(E) = \mathrm{span}(\boldsymbol{\nu}_E) \oplus Q_H(E)$. Note that for any $\mathbf{q} \in \widetilde{Q}_h$

$$\int_E (\Pi^Q_E\mathbf{q})\cdot\boldsymbol{\tau}_E \,\mathrm{d}L = \int_E (\Pi^Q_{E,1}\mathbf{q})\cdot\boldsymbol{\tau}_E \,\mathrm{d}L = \int_E \mathbf{q}\cdot\boldsymbol{\tau}_E \,\mathrm{d}L. \tag{6.7}$$

6.2. Projector Π^Q_1 and its Properties

Definition 6.4. We define

$$\Pi^Q_1 = \sum_E \Pi^Q_E.$$

Then the following result holds:

Lemma 6.5. *Π^Q_1 is a projector onto $\bigoplus_E \widetilde{Q}_H(E)$.*

Proof. Since for each coarse edge E the operator Π^Q_E is a projector onto $\widetilde{Q}_H(E)$, it is sufficient to prove that $\Pi^Q_E\Pi^Q_{E'} = 0$ for distinct coarse edges E and E'. The latter is true since distinct coarse edges do not have common fine edges. □

The following result follows from (6.7) and the fact that $\int_E (\Pi^Q_{E'}\mathbf{q})\cdot\boldsymbol{\tau}_E \,\mathrm{d}L = 0$ for $E \neq E'$. For any coarse edge E, the following identity holds:

$$\int_E (\Pi^Q_1\mathbf{q})\cdot\boldsymbol{\tau}_E \,\mathrm{d}L = \int_E \mathbf{q}\cdot\boldsymbol{\tau}_E \,\mathrm{d}L. \tag{6.8}$$

Lemma 6.6. *Let E be any coarse edge. Let $\mathbf{c}$ be a vector constant. Let $\mathbf{q} \in \widetilde{Q}_h$ be such that its restriction to E is $\mathbf{c}$ restricted to E, i.e., $\pi^Q_E\mathbf{q} = \pi^Q_E\mathbf{c}$. Then the restriction of $\Pi^Q_1\mathbf{q}$ to E also coincides with $\mathbf{c}$ on E, i.e., $\pi^Q_E\Pi^Q_1\mathbf{q} = \Pi^Q_E\mathbf{q} = \pi^Q_E\mathbf{c}$.*

Proof. We have $\Pi^Q_E\mathbf{q} = \Pi^Q_E\pi^Q_E\mathbf{q} = \Pi^Q_E\pi^Q_E\mathbf{c} = \pi^Q_E\mathbf{c}$. □

Lemma 6.7. *For any coarse face F, we have*

$$\Pi^R_{F,1}\nabla\times\mathbf{q} = \Pi^R_{F,1}\nabla\times\Pi^Q_1\mathbf{q} \quad \textit{for all } \mathbf{q} \in \widetilde{Q}_h.$$

Proof. Let the boundary of the coarse face F consist of the coarse edges $E_1, \ldots, E_n$. We then have

$$\begin{aligned}
\Pi^R_{F,1}\nabla\times\mathbf{q} &\overset{(5.5)}{=} \frac{\int_F (\nabla\times\mathbf{q})\cdot\mathbf{n}_F \,\mathrm{d}A}{(\boldsymbol{\nu}_F, \boldsymbol{\nu}_F)_F}\boldsymbol{\nu}_F \overset{(3.8)}{=} \frac{\sum_{i=1}^n \alpha_i \int_{E_i} \mathbf{q}\cdot\boldsymbol{\tau}_{E_i} \,\mathrm{d}L}{(\boldsymbol{\nu}_F, \boldsymbol{\nu}_F)_F}\boldsymbol{\nu}_F \\
&\overset{(6.8)}{=} \frac{\sum_{i=1}^n \alpha_i \int_{E_i} (\Pi^Q_1\mathbf{q})\cdot\boldsymbol{\tau}_{E_i} \,\mathrm{d}L}{(\boldsymbol{\nu}_F, \boldsymbol{\nu}_F)_F}\boldsymbol{\nu}_F \overset{(3.8)}{=} \frac{\int_F (\nabla\times(\Pi^Q_1\mathbf{q}))\cdot\mathbf{n}_F \,\mathrm{d}A}{(\boldsymbol{\nu}_F, \boldsymbol{\nu}_F)_F}\boldsymbol{\nu}_F \\
&\overset{(5.5)}{=} \Pi^R_{F,1}\nabla\times(\Pi^Q_1\mathbf{q}).
\end{aligned}$$

□

6.3. Coarse ND Basis Functions: Edge-to-Face Extension and Face Bubbles

In this section we describe how we interpolate the coarse ND basis functions from coarse edges to coarse faces. We also describe the construction of additional coarse ND basis functions that we associate with coarse faces. Our approach is similar to that of [10] modified accordingly.

6.3.1. Preliminaries. We start with defining two subspaces of $\widetilde{Q}_h$. These subspaces are associated with a given coarse face F. The space $\widetilde{Q}_h(F) \subset \widetilde{Q}_h$ is defined to be

$$\widetilde{Q}_h(F) = \pi_F^Q(\widetilde{Q}_h).$$

We shall equip $\widetilde{Q}_h(F)$ with an inner product

$$(\mathbf{p}, \mathbf{q})_F^Q = \sum_{e \subset F} \Big(\int_e \mathbf{p} \cdot \boldsymbol{\tau} \, \mathrm{d}L \Big) \Big(\int_e \mathbf{q} \cdot \boldsymbol{\tau} \, \mathrm{d}L \Big).$$

We note that we have defined previously (see (5.2)) one more inner product on coarse faces, denoted $(\cdot, \cdot)_F = (\cdot, \cdot)_F^R$ acting on traces of functions from $\widetilde{R}_h$. If there is no ambiguity, we will omit the superscript R or Q, since which inner product is used will be clear from the arguments that it is applied to.

The space $Q_h(F) \subset \widetilde{Q}_h(F)$ is defined to be

$$Q_h(F) = \Big\{ \mathbf{q} \in \widetilde{Q}_h(F) : \int_e \mathbf{q} \cdot \boldsymbol{\tau} \, \mathrm{d}L = 0 \text{ for all } e \subset \partial F \Big\}.$$

Definition 6.8. We also need the space $R_h(F) \subset \widetilde{R}_h(F)$ defined as follows:

$$R_h(F) = \Big\{ \mathbf{r} \in \widetilde{R}_h(F) : \int_F \mathbf{r} \cdot \mathbf{n}_F \, \mathrm{d}A = 0 \Big\}.$$

Observe that $R_H(F) \subset R_h(F)$.

Definition 6.9. Let $\mathbf{q} \in \widetilde{Q}_h$, and let K be a coarse face or a coarse element. We define

$$\nabla_K \times \mathbf{q} = \pi_K^R \nabla \times \mathbf{q}.$$

Note that above $\nabla \times \mathbf{q}$ is viewed as an element of $\widetilde{R}_h$.

Remark 6.10. An important observation is that $\nabla_F \times \mathbf{q}$ depends only on degrees of freedom of $\mathbf{q}$ associated with edges $e \in F$, i.e., we have $\nabla_F \times \mathbf{q} = \nabla_F \times \pi_F^Q \mathbf{q}$. This is a consequence of the Stokes theorem applied to each fine face $f \in F$:

$$\int_f (\nabla \times \mathbf{q}) \cdot \mathbf{n}_f \, \mathrm{d}A = \sum_{e \subset \partial f} \epsilon_e \int_e \mathbf{q} \cdot \boldsymbol{\tau}_e \, \mathrm{d}L \quad \text{for some } \epsilon_e = \pm 1.$$

Also, note that

$$(\nabla_F \times \mathbf{q}, \mathbf{r})_F = (\nabla \times \mathbf{q}, \mathbf{r})_F \quad \text{for all } \mathbf{q} \in \widetilde{Q}_h, \ \mathbf{r} \in \widetilde{R}_h, \tag{6.9}$$

and

$$\int_F (\nabla_F \times \mathbf{q}) \cdot \mathbf{n}_F \, \mathrm{d}A = \int_F (\nabla \times \mathbf{q}) \cdot \mathbf{n}_F \, \mathrm{d}A \quad \text{for all } \mathbf{q} \in \widetilde{Q}_h. \tag{6.10}$$

Due to (3.8) we have

$$\nabla_F \times Q_h(F) \subset R_h(F). \tag{6.11}$$

The following lemma ensures the solvability of the linear systems that we utilize in the course of the edge-to-face interpolation.

Lemma 6.11. *The following "exactness" result on coarse faces holds:*

$$\nabla_F \times Q_h(F) = R_h(F). \tag{6.12}$$

Proof. The proof is analogous to that of Lemma 5.7. Let $\mathbf{r} \in R_h(F)$ satisfy

$$(\mathbf{r}, \nabla_F \times \mathbf{q})_F = 0 \quad \text{for all } \mathbf{q} \in Q_h(F).$$

The above equation actually reads

$$\sum_f \Big(\int_f \mathbf{r} \cdot \mathbf{n}_F \, \mathrm{d}A \Big) \Big(\int_f \nabla_F \times \mathbf{q} \cdot \mathbf{n}_F \, \mathrm{d}A \Big) = 0 \quad \text{for all } \mathbf{q} \in Q_h(F). \tag{6.13}$$

We shall prove $\mathbf{r} = 0$, which would imply (6.12). We first establish that

$$\int_{f_1} \mathbf{r} \cdot \mathbf{n}_F \, \mathrm{d}A = \int_{f_2} \mathbf{r} \cdot \mathbf{n}_F \, \mathrm{d}A \tag{6.14}$$

for any pair of fine faces $f_1, f_2 \subset F$ which share an (interior) edge $e \subset F$. Let $\mathbf{q} \in Q_h(F)$ satisfy

$$\int_{e'} \mathbf{q} \cdot \boldsymbol{\tau} \, \mathrm{d}A = \begin{cases} 1 & \text{if } e' = e, \\ 0 & \text{otherwise.} \end{cases}$$

By applying the Stokes theorem to the fine faces f_1, f_2 we obtain

$$\Big| \int_{f_i} (\nabla_F \times \mathbf{q}) \cdot \mathbf{n}_F \, \mathrm{d}A \Big| = 1, \quad i = 1, 2. \tag{6.15}$$

Due to our assumptions about coarse faces (Assumption 3.2), e can be a subset of at most two fine faces from F. Consequently, by applying the Stokes theorem to any fine face $f \subset F$ other than f_1 and f_2, we get

$$\int_f (\nabla_F \times \mathbf{q}) \cdot \mathbf{n}_F \, \mathrm{d}A = 0, \quad f \notin \{f_1, f_2\}. \tag{6.16}$$

Due to (6.11) we have

$$0 = \int_F (\nabla_F \times \mathbf{q}) \cdot \mathbf{n}_F \, \mathrm{d}A = \int_{f_1} (\nabla_F \times \mathbf{q}) \cdot \mathbf{n}_F \, \mathrm{d}A + \int_{f_2} (\nabla_F \times \mathbf{q}) \cdot \mathbf{n}_F \, \mathrm{d}A. \tag{6.17}$$

We can re-write (6.15) and (6.17) as

$$\int_{f_1} (\nabla_F \times \mathbf{q}) \cdot \mathbf{n}_F \, \mathrm{d}A = \alpha, \quad \int_{f_2} (\nabla_F \times \mathbf{q}) \cdot \mathbf{n}_F \, \mathrm{d}A = -\alpha, \quad |\alpha| = 1.$$

Due to (6.13) and (6.16) we have

$$\alpha \int_{f_1} \mathbf{r} \cdot \mathbf{n}_F \, \mathrm{d}A - \alpha \int_{f_2} \mathbf{r} \cdot \mathbf{n}_F \, \mathrm{d}A = 0.$$

Since $\alpha \neq 0$, this implies (6.14).

Due to the way we construct the coarse faces (Assumption 3.2), the dual graph of F is connected. (We recall that "vertices" of the dual graph are the fine faces of F, and any two such "vertices" are connected by an arc iff the corresponding fine faces share an edge.) Because of this, (6.14) holds even if f_1 and f_2 do not share an edge (since there exists a finite sequence $\mathcal{S}$ of fine faces starting with f_1 and ending with f_2 such that all faces in $\mathcal{S}$ belong to F and each face in $\mathcal{S}$ shares an edge with the next face in $\mathcal{S}$). Since $\mathbf{r} \in R_h(F)$, we have

$$\sum_{f \subset F} \int_f \mathbf{r} \cdot \mathbf{n}_F \,\mathrm{d}A = 0.$$

We have already established that all terms in the above sum are equal, consequently they are all zero. Thus, for any fine face $f \subset F$ we have

$$\int_f \mathbf{r} \cdot \mathbf{n} \,\mathrm{d}A = \pm \int_f \mathbf{r} \cdot \mathbf{n}_F \,\mathrm{d}A = 0,$$

i.e., $\mathbf{r} = 0$. □

6.3.2. The Bilinear Form for the Edge-to-Face Extension. To interpolate coarse ND basis functions from coarse edges to the interior of the coarse faces, we solve certain linear system, similar to the mixed system (5.5) of [10]. However, the construction there does not guarantee the exact interpolation of the vector constants, unless the coarse faces are flat. We modify one of the bilinear forms used in the mixed system to achieve the exact interpolation of constants. However, this modification (to be described) is not always sufficient. For example, if a coarse face F is not flat but has a flat boundary, those vector constant functions which are orthogonal to the plane containing ∂F will "vanish" on ∂F. "Vanish" means that the degrees of freedom associated with the edges in ∂F will vanish, when evaluated on those functions. Since we assume *linear* extension procedures, zero function can only be extended as zero function. Thus, we have to add the "problematic" vector constant as a "bubble" basis function. We also add a bubble basis function when the boundary of a non-flat face is *almost* flat, to avoid ill-conditioned interpolation operators.

Let the functions $\mathbf{c}^Q_{i,F} \in \widetilde{Q}_h(F)$ have degrees of freedom

$$\int_e \mathbf{c}^Q_{i,F} \cdot \boldsymbol{\tau} \,\mathrm{d}L = \int_e \mathbf{c}_i \cdot \boldsymbol{\tau} \,\mathrm{d}L \quad \text{for all } e \subset F.$$

Let $C_F \subset \widetilde{Q}_h(F)$ be the linear span of $\{\mathbf{c}^Q_{1,F}, \mathbf{c}^Q_{2,F}, \mathbf{c}^Q_{3,F}\}$.

Remark 6.12. In the computer implementation we perform SVD on a certain matrix with three columns in order to obtain an orthonormal basis of C_F.

Let $P_{C_F} : \widetilde{Q}_h(F) \to C_F$ be an orthogonal projection onto C_F, and let $P_0 : \widetilde{Q}_h(F) \to Q_h(F)$ be an orthogonal projection onto $Q_h(F)$. Consider the eigenvalue problem

$$P_{C_F} P_0 P_{C_F} \psi = \sigma^2 \psi, \quad 0 \neq \psi \in C_F.$$

It is easy to check that all eigenvalues σ^2 lie in the segment $[0, 1]$. The eigenvalues are in fact the squared cosines of the *principal angles* between C_F and $Q_h(F)$. The largest eigenvalue will be 1 if F is non-planar but has planar boundary (then the spaces $Q_h(F)$ and C_F will have non-trivial intersection). Or, if the boundary is "getting close" to being planar (but the whole coarse face "stays far" from being planar), the largest eigenvalue will approach one. Given some tolerance γ (in practice we use $\gamma = 0.01$), we select those eigenvalues $\{\sigma_i^2\}_{i=1}^m$, $m \leqslant 3$ which satisfy

$$\sigma_i^2 \leqslant 1 - \gamma. \tag{6.18}$$

Choose $D_F \subseteq C_F$ to be the linear span of the corresponding eigenvectors $\{\psi_i\}_{i=1}^m$. In our numerical experiments we have observed only two scenarios: either D_F coincided with C_F, or D_F had one dimension less.

To define the bilinear form of interest, we consider the orthogonal projection $P_{D_F} : \widetilde{Q}_h(F) \to D_F$ onto D_F. The following coercivity property holds.

Lemma 6.13. *For any* $\mathbf{q} \in Q_h(F)$*, we have*

$$\big(\mathbf{q}, (\mathbb{1} - P_{D_F})\mathbf{q}\big) \geqslant \gamma(\mathbf{q}, \mathbf{q}),$$

where γ *is the tolerance level used in* (6.18).

Proof. A detailed proof is given as an appendix. □

6.3.3. Face Extension of the Edge-Based Basis Functions. Let F be a coarse face. Here we describe the construction of the "edge-to-face" extension mapping $\mathcal{I}_F^Q : \widetilde{Q}_h \to Q_h(F)$. We assume that P_{D_F} is an orthogonal projection with respect to the $(\cdot,\cdot)_F^Q$ inner product and the form $(\cdot, (I - P_{D_F})\cdot)_F$ is coercive on $Q_h(F)$. Such a projector can be constructed as described in Section 6.3.2.

For any $\mathbf{q} \in \widetilde{Q}_h$, consider the pair $\mathbf{q}_F \in Q_h(F)$, $\mathbf{r}_F \in R_h(F)$ solving the saddle-point (mixed) system

$$\begin{aligned} &\big(\mathbf{q}_F, (\mathbb{1} - P_{D_F})\mathbf{p}\big)_F + (\nabla_F \times \mathbf{p}, \mathbf{r}_F)_F = -\big(\mathbf{q}, (\mathbb{1} - P_{D_F})\mathbf{p}\big)_F \quad \text{for all } \mathbf{p} \in Q_h(F), \\ &(\nabla_F \times \mathbf{q}_F, \mathbf{s})_F = \frac{\int_F (\nabla \times \mathbf{q}) \cdot \mathbf{n}_F \, \mathrm{d}A}{\int_F \boldsymbol{\nu}_F \cdot \mathbf{n}_F \, \mathrm{d}A} (\boldsymbol{\nu}_F, \mathbf{s})_F - (\nabla_F \times \mathbf{q}, \mathbf{s})_F \quad \text{for all } \mathbf{s} \in R_h(F). \end{aligned} \tag{6.19}$$

The system (6.19) is uniquely solvable due to Lemma 4.1. Indeed, take $V_1 = Q_h(F)$. Due to Lemma 6.13, $(\cdot, (\mathbb{1} - P_{D_F})\cdot)_F^Q$ is an inner product on V_1. Take $V_2 = R_h(F)$, $(\cdot,\cdot)_2 = (\cdot,\cdot)_F^R$, $\mathrm{d}_1 = \nabla_F \times$, $\mathrm{d}_2 = 0$, $V_3 = \{0\}$.

We define $\mathcal{I}_F^Q \mathbf{q} = \mathbf{q}_F$. Note that

$$\nabla_F \times \mathbf{q} = 0 \implies \nabla_F \times (\mathcal{I}_F^Q \mathbf{q}) = 0 \tag{6.20}$$

for all $\mathbf{q} \in \widetilde{Q}_h$, which follows from the second equation of (6.19) and from (6.11). Also note that

$$\mathcal{I}_F^Q \mathbf{q} = -\mathbf{q} \quad \text{for all } \mathbf{q} \in Q_h(F). \tag{6.21}$$

Identity (6.21) holds since the pair $(\mathbf{q}_F = -\mathbf{q}, \mathbf{r}_F = 0)$ solves (6.19), which can be easily checked, taking into account the fact that

$$\int_F (\nabla \times \mathbf{q}) \cdot \mathbf{n}_F \, \mathrm{d}A = 0 \quad \text{for all } \mathbf{q} \in Q_h(F)$$

due to (3.8).

Lemma 6.14. *Suppose the function* $\mathbf{q} \in \widetilde{Q}_h$ *satisfies* $\pi_F^Q \mathbf{q} \in Q_h(F)$. *Then*

$$\pi_F^Q(\mathbb{1} + \mathcal{I}_F^Q)\mathbf{q} = 0.$$

In other words, if $\mathbf{q}$ *vanishes on* ∂F, *then its extension* $\mathbf{q} + \mathbf{q}_F$ *is zero on* F.

Proof. For any $\mathbf{q} \in \widetilde{Q}_h$, we have

$$\mathcal{I}_F^Q \mathbf{q} = \mathcal{I}_F^Q \pi_F^Q \mathbf{q}, \tag{6.22}$$

seen from (6.19), i.e., if we replace $\mathbf{q}$ in the right-hand side of (6.19) with $\pi_F^Q \mathbf{q}$, the solution $\mathbf{q}_F$ does not change (due to (6.9) and (6.10)). Also, $\mathcal{I}_F^Q \mathbf{q} = \pi_F^Q \mathcal{I}_F^Q \mathbf{q}$ (since $\mathcal{I}_F^Q \mathbf{q} \in Q_h(F)$). Consequently, using also the assumption that $\pi_F^Q \mathbf{q} \in Q_h(F)$, we have

$$\pi_F^Q(\mathbb{1} + \mathcal{I}_F^Q)\mathbf{q} = (\mathbb{1} + \mathcal{I}_F^Q)\pi_F^Q \mathbf{q} \overset{(6.21)}{=} \pi_F^Q \mathbf{q} - \pi_F^Q \mathbf{q} = 0.$$

□

Lemma 6.15. *For any* $\mathbf{q} \in \widetilde{Q}_h$, *recalling the definition* (5.5) *of the 1D projection* $\Pi_{F,1}^R$, *we have*

$$\nabla_F \times \Big(\mathbb{1} + \sum_{F'} \mathcal{I}_{F'}^Q\Big)\mathbf{q} = \Pi_{F,1}^R \nabla \times \mathbf{q} = \Pi_{F,1}^R \nabla_F \times \mathbf{q}. \tag{6.23}$$

Proof. Let $\mathbf{q} \in \widetilde{Q}_h$. From (5.5) it follows that the function

$$\mathbf{z} = \frac{\int_F (\nabla \times \mathbf{q}) \cdot \mathbf{n}_F \, \mathrm{d}A}{\int_F \boldsymbol{\nu}_F \cdot \mathbf{n}_F \, \mathrm{d}A} \boldsymbol{\nu}_F - \nabla_F \times \mathbf{q} = \Pi_{F,1}^R(\nabla_F \times \mathbf{q}) - \nabla_F \times \mathbf{q}$$

lies in $R_h(F)$, i.e., $\int_F \mathbf{z} \cdot \mathbf{n}_F \, \mathrm{d}A = 0$. From (6.11) and the definition of $\mathcal{I}_F^Q$ it follows that $\nabla_F \times \mathcal{I}_F^Q \mathbf{q}$ also lies in $R_h(F)$. The second equation of (6.19) states that the difference $\nabla_F \times \mathcal{I}_F^Q \mathbf{q} - \mathbf{z} \in R_h(F)$ is orthogonal to all $R_h(F)$. Thus $\nabla_F \times \mathcal{I}_F^Q \mathbf{q} = \mathbf{z}$, i.e.,

$$\nabla_F \times (\mathbf{q} + \mathcal{I}_F^Q \mathbf{q}) = \Pi_{F,1}^R(\nabla_F \times \mathbf{q}). \tag{6.24}$$

From the definitions of the operators $\nabla_F \times$, $\mathcal{I}_F^Q$, and $\Pi_{F,1}^R$ it follows that

$$\Pi_{F,1}^R \nabla_F \times \; = \Pi_{F,1}^R \nabla \times$$

and

$$\nabla_F \times \mathcal{I}_{F'}^Q = 0, \quad F' \neq F.$$

Using these identities and (6.24), we obtain (6.23). □

Lemma 6.16. *The operator*

$$\Big(\mathbb{1} + \sum_F \mathcal{I}_F^Q\Big)\Pi_1^Q, \quad \textit{where } \Pi_1^Q = \sum_E \Pi_E^Q,$$

is a projector onto

$$\Big(\mathbb{1} + \sum_F \mathcal{I}_F^Q\Big) \bigoplus_E \widetilde{Q}_H(E).$$

Proof. For any coarse face F, any coarse edge E and any function $\mathbf{q} \in Q_h(F)$, we have $\Pi_E^Q \mathbf{q} = 0$, since a coarse edge E cannot contain interior fine edges of F. Consequently $\Pi_1^Q \mathcal{I}_F^Q = 0$, and the proof is finished by applying Lemma 4.3. □

6.3.4. Face Bubbles. To ensure the exact interpolation of the vector constants and the exactness of (2.4), we associate with some coarse faces up to three "face-bubble" basis functions, i.e., functions with vanishing degrees of freedom corresponding to the fine edges on the boundary of the coarse face. Define the space $Q^1_H(F)$ to be

$$Q^1_H(F) = \mathcal{I}^Q_F(C_F). \tag{6.25}$$

We begin with the following auxiliary result.

Lemma 6.17. *For any* $\mathbf{q} \in D_F$*, we have* $\mathcal{I}^Q_F \mathbf{q} = 0$.

Proof. We recall that $D_F \subset C_F$. Thus, there exists a vector constant $\mathbf{c}$ such that $\mathbf{q} = \pi^Q_F \mathbf{c}$. We then have $\nabla_F \times \mathbf{q} = \pi^R_F \nabla \times \mathbf{c} = 0$. To complete the proof we show next that the pair $(\mathbf{q}_F = 0, \mathbf{r}_F = 0)$ solves (6.19). The first equation is satisfied:

$$\big(\mathbf{q}, (\mathbb{1} - P_{D_F})\mathbf{p}\big)_F = \big((\mathbb{1} - P_{D_F})\mathbf{q}, \mathbf{p}\big)_F = 0,$$

since $\mathbf{q} \in D_F$. The second equation is satisfied because of $\nabla_F \times \mathbf{q} = 0$. $\square$

Lemma 6.17 implies that

$$\mathcal{I}^Q_F(C_F) = \mathcal{I}^Q_F(C_F \ominus D_F), \tag{6.26}$$

where $C_F \ominus D_F$ is the $(\cdot,\cdot)_F$-orthogonal complement of D_F in C_F, i.e.,

$$C_F \ominus D_F = \big\{\mathbf{q} \in C_F : (\mathbf{q}, \mathbf{p})_F = 0 \text{ for all } \mathbf{p} \in D_F\big\}. \tag{6.27}$$

Remark 6.18. In our computer implementation, to construct the basis of $Q^1_H(F)$, we compute a basis of $C_F \ominus D_F$, apply $\mathcal{I}^Q_F$ to each vector of the basis and perform SVD on the resulting vectors. As mentioned previously, in Section 6.3.2, we have only encountered cases when the dimension of $C_F \ominus D_F$ is either one or zero.

Definition 6.19. Consider the operator $\mathcal{J}_F : \widetilde{R}_h \to Q_h(F)$, defined as follows. For any $\mathbf{r} \in \widetilde{R}_h$, let $\mathbf{q} \in Q_h(F)$ be the solution of

$$\begin{aligned} &\big(\mathbf{q}, (\mathbb{1} - P_{D_F})\mathbf{p}\big)_F + (\nabla_F \times \mathbf{p}, \mathbf{z})_F = 0 && \text{for all } \mathbf{p} \in Q_h(F), \\ &(\nabla_F \times \mathbf{q}, \mathbf{s})_F = (\mathbf{r}, \mathbf{s})_F && \text{for all } \mathbf{s} \in R_h(F), \end{aligned} \tag{6.28}$$

for some $\mathbf{z} \in R_h(F)$. The system (6.28) is uniquely solvable for the same reasons as the system (6.19). We define $\mathbf{q} = \mathcal{J}_F \mathbf{r}$. We further define

$$Q^2_H(F) = \mathcal{J}_F(R_H(F)).$$

In a computer implementation, we construct a basis of $R_H(F)$ and solve (6.28) for $\mathbf{r}$ equal to each of the basis vectors.

We further define

$$Q_H(F) = Q^1_H(F) \oplus Q^2_H(F). \tag{6.29}$$

The sum above is indeed direct, since

$$\big(\mathbf{p}, (\mathbb{1} - P_{D_F})\mathbf{q}\big)_F = 0 \quad \text{for all } \mathbf{p} \in Q^1_H(F),\ \mathbf{q} \in Q^2_H(F). \tag{6.30}$$

Indeed, due to (6.20), $\nabla_F \times \mathbf{p} = 0$ for any $\mathbf{p} \in Q^1_H(F)$. Because of that, (6.30) follows from the first equation of (6.28).

We now define operator $\Pi^Q_F : \widetilde{Q}_h \to Q_H(F)$, which turns out to be a projection (see Lemma 6.21 below).

Definition 6.20. Fix any $\mathbf{q} \in \widetilde{Q}_h$. Let

$$\mathbf{p} = -\mathcal{I}_F^Q \mathbf{q} \in Q_h(F), \tag{6.31}$$
$$\mathbf{p}_1 = \mathcal{J}_F \Pi_{F,0}^R \nabla_F \times \mathbf{p},$$

and let $\mathbf{p}_0$ be the (unique) element of $Q_H^1(F)$ satisfying

$$\big(\mathbf{p}_0, (\mathbb{1} - P_{D_F})\mathbf{z}\big)_F = \big(\mathbf{p}, (\mathbb{1} - P_{D_F})\mathbf{z}\big) \quad \text{for all } \mathbf{z} \in Q_H^1(F). \tag{6.32}$$

We define

$$\Pi_F^Q \mathbf{q} = \mathbf{p}_1 + \mathbf{p}_0. \tag{6.33}$$

Note that the above definition of Π_F^Q implies

$$\Pi_F^Q \mathbf{c}_F = -\mathcal{I}_F^Q \mathbf{c}_F \quad \text{for all } \mathbf{c}_F \in C_F. \tag{6.34}$$

Indeed, $\mathbf{p} = -\mathcal{I}_F^Q \mathbf{c}_F \in Q_H^1(F)$ and thus $\mathbf{p}_0 = \mathbf{p}$. We also have $\nabla_F \times \mathbf{p} = 0$ due to (6.20), and thus $\mathbf{p}_1 = 0$.

Lemma 6.21. Π_F^Q *is a projector onto* $Q_H(F)$.

Proof. From (6.33) it follows that $\Pi_F^Q \widetilde{Q}_h \subseteq Q_H(F)$. Let $\mathbf{q} \in Q_H(F)$. Then, due to (6.21) and (6.31), we have $\mathbf{p} \equiv -\mathcal{I}_F^Q \mathbf{q} = \mathbf{q}$. From the definition (6.29) of $Q_H(F)$, we have

$$\mathbf{q} = \mathcal{J}_F \mathbf{r} + \mathbf{q}_0, \quad \mathbf{r} \in R_H(F),\ \mathbf{q}_0 \in Q_H^1(F).$$

Due to (6.30) and (6.32), we have $\mathbf{p}_0 = \mathbf{q}_0$ where $\mathbf{p}_0$ is from the decomposition (6.33), $\Pi_F^Q \mathbf{q} = \mathbf{p}_1 + \mathbf{p}_0$.

It remains to prove that $\mathbf{p}_1 = \mathcal{J}_F \mathbf{r}$. From the second equation of (6.28) and from (6.11) it follows that

$$\nabla_F \times \mathcal{J}_F \mathbf{r} = \mathbf{r} \quad \text{for all } \mathbf{r} \in R_h(F). \tag{6.35}$$

We then have

$$\nabla_F \times \mathbf{q} = \nabla_F \times \mathcal{J}_F \mathbf{r} + \nabla_F \times \mathbf{q}_0 = \nabla_F \times \mathcal{J}_F \mathbf{r},$$

where we have used (6.20). Thus we have

$$\mathbf{p}_1 = \mathcal{J}_F \Pi_{F,0}^R \nabla_F \times \mathbf{q} = \mathcal{J}_F \Pi_{F,0}^R \nabla_F \times \mathcal{J}_F \mathbf{r} \overset{\text{via (6.35)}}{=} \mathcal{J}_F \Pi_{F,0}^R \mathbf{r} \overset{\mathbf{r} \in R_H(F)}{=} \mathcal{J}_F \mathbf{r}. \qquad \square$$

From (6.21) and (6.31) we have the identity

$$\Pi_F^Q \mathcal{I}_F^Q \mathbf{q} = -\Pi_F^Q \mathbf{q} \quad \text{for all } \mathbf{q} \in \widetilde{Q}_h. \tag{6.36}$$

Lemma 6.22. *For any coarse faces* F, F', *and any* $\mathbf{q} \in \widetilde{Q}_h$, *we have*

$$\nabla_{F'} \times \Pi_F^Q \mathbf{q} = \begin{cases} \Pi_{F,0}^R \nabla \times \mathbf{q} & \text{if } F = F', \\ 0 & \text{if } F \neq F'. \end{cases}$$

Proof. If $F \neq F'$, then $\Pi_F^Q \mathbf{q} \in Q_h(F)$ has vanishing degrees of freedom for all fine edges in F'. Consequently, $\nabla_{F'} \times \Pi_F^Q \mathbf{q} = 0$. From now on we assume that $F = F'$. From (6.33) and (6.35) it follows that

$$\nabla_F \times \Pi_F^Q \mathbf{q} = \Pi_{F,0}^R \nabla_F \times (-\mathcal{I}_F^Q \mathbf{q}).$$

Since $\Pi_{F,0}^R$ is an orthogonal projector from $\widetilde{R}_h(F)$ onto $R_H(F)$ (see (5.6)), it is now sufficient to prove that

$$\big(\nabla_F \times (-\mathcal{I}_F^Q \mathbf{q}), \mathbf{s}\big)_F = (\nabla \times \mathbf{q}, \mathbf{s})_F \quad \text{for all } \mathbf{s} \in R_H(F).$$

Let $\mathbf{s} \in R_H(F)$. Using the second equation of (6.19) and noting that $\boldsymbol{\nu}_F \perp R_H(F)$, we obtain

$$-(\nabla_F \times \mathcal{I}_F^Q \mathbf{q}, \mathbf{s})_F = (\nabla_F \times \mathbf{q}, \mathbf{s})_F \overset{(6.9)}{=} (\nabla \times \mathbf{q}, \mathbf{s})_F. \qquad \square$$

6.4. Projector Π_2^Q and its Properties

Definition 6.23. We define

$$\Pi_2^Q = \sum_F \Pi_F^Q + \Big(\mathbb{1} + \sum_F \mathcal{I}_F^Q\Big)\Pi_1^Q.$$

(Recall that $\Pi_1^Q = \sum_E \Pi_E^Q$.)

Lemma 6.24. Π_2^Q *is a projector onto the following direct sum of spaces:*

$$\bigoplus_F Q_H(F) \oplus \Big(\mathbb{1} + \sum_F \mathcal{I}_F^Q\Big) \bigoplus_E \widetilde{Q}_H(E).$$

Proof. Let

$$A = \sum_F \Pi_F^Q, \quad B = \Big(\mathbb{1} + \sum_F \mathcal{I}_F^Q\Big)\Pi_1^Q.$$

It is easy to check that for distinct faces $F \neq F'$ we have $\Pi_F^Q \Pi_{F'}^Q = 0$. Each Π_F^Q is a projector onto $Q_H(F)$. Then, by Lemma 4.2, A is a projector onto $\bigoplus_F Q_H(F)$. By Lemma 6.16, B is a projector onto

$$\Big(\mathbb{1} + \sum_F \mathcal{I}_F^Q\Big) \bigoplus_E \widetilde{Q}_H(E).$$

It is therefore sufficient to prove that $AB = BA = 0$ and apply Lemma 4.2 to $A + B$. For any coarse face F and any coarse edge E, we have $\Pi_E^Q \Pi_F^Q = 0$, since fine edges of E cannot be interior edges of F. Consequently

$$\Pi_1^Q \sum_F \Pi_F^Q = \Pi_1^Q A = 0 \implies BA = 0.$$

Distinct coarse faces cannot have common interior fine edges, thus

$$\Pi_F^Q \mathcal{I}_{F'}^Q = 0, \quad F \neq F'.$$

Together with (6.36), this implies

$$\Pi_F^Q\Big(\mathbb{1} + \sum_{F'} \mathcal{I}_{F'}^Q\Big) = 0.$$

By summing the above equation over F we obtain $A(\mathbb{1} + \sum_{F'} \mathcal{I}_{F'}^Q) = 0$ and thus $AB = 0$ which completes the proof. □

6.4.1. Approximation Property on Coarse Faces.

Lemma 6.25. *Let F be a coarse face. Let $\mathbf{c}$ be a vector constant. If $\mathbf{q} \in \widetilde{Q}_h$ satisfies $\pi_F^Q \mathbf{q} = \pi_F^Q \mathbf{c}$, then we have $\pi_F^Q \Pi_2^Q \mathbf{q} = \pi_F^Q \mathbf{c}$. In other words, if $\mathbf{q}$ "coincides" with $\mathbf{c}$ on F, so does the projection $\Pi_2^Q \mathbf{q}$.*

Proof. For any coarse edge $E \subset \partial F$, we have $\pi_E^Q \Pi_1^Q \mathbf{q} = \pi_E^Q \mathbf{c}$. This follows from the identity

$$\pi_E^Q \mathbf{q} = \pi_E^Q \pi_F^Q \mathbf{q} = \pi_E^Q \pi_F^Q \mathbf{c} = \pi_E^Q \mathbf{c}$$

and Lemma 6.6. Since any fine edge $e \subset \partial F$ is a subset of some coarse edge $E \subset F$, we have

$$\pi_F^Q (\Pi_1^Q \mathbf{q} - \mathbf{c}) \in Q_h(F).$$

From Lemma 6.14 we have

$$\pi_F^Q (\mathbb{1} + \mathcal{I}_F^Q)(\Pi_1^Q \mathbf{q} - \mathbf{c}) = 0.$$

Hence, using (6.34), we obtain

$$\pi_F^Q (\mathbb{1} + \mathcal{I}_F^Q) \Pi_1^Q \mathbf{q} = \pi_F^Q (\mathbb{1} + \mathcal{I}_F^Q) \mathbf{c} = \pi_F^Q \mathbf{c} - \pi_F^Q \Pi_F^Q \mathbf{c}. \tag{6.37}$$

It is easy to check that $\Pi_F^Q = \Pi_F^Q \pi_F^Q$ (based on (6.22)). Consequently

$$\Pi_F^Q \mathbf{c} = \Pi_F^Q \pi_F^Q \mathbf{c} = \Pi_F^Q \pi_F^Q \mathbf{q} = \Pi_F^Q \mathbf{q}. \tag{6.38}$$

Combining (6.37) with (6.38) we obtain

$$\pi_F^Q \big(\Pi_F^Q \mathbf{q} + (\mathbb{1} + \mathcal{I}_F^Q) \Pi_1^Q \mathbf{q} \big) = \pi_F^Q \mathbf{c}.$$

Using the definition of Π_2^Q and noting that $\pi_F^Q \Pi_{F'}^Q = \pi_F^Q \mathcal{I}_{F'}^Q = 0$ for $F \neq F'$, we conclude that

$$\pi_F^Q \Pi_2^Q \mathbf{q} = \pi_F^Q \Big(\sum_{F'} \Pi_{F'}^Q \mathbf{q} + \Big(\mathbb{1} + \sum_{F'} \mathcal{I}_{F'}^Q \Big) \Pi_1^Q \mathbf{q} \Big) = \pi_F^Q \big(\Pi_F^Q \mathbf{q} + (\mathbb{1} + \mathcal{I}_F^Q) \Pi_1^Q \mathbf{q} \big) = \pi_F^Q \mathbf{c}. \qquad \square$$

6.4.2. Commutativity Property on Coarse Faces.

Definition 6.26. We define

$$\nabla_2 \times = \sum_F \nabla_F \times .$$

Lemma 6.27. *We have*

$$\nabla_2 \times \Pi_2^Q = \Pi_2^R \nabla \times . \tag{6.39}$$

Proof. Using Lemmas 6.22, 6.15, and 6.7, we obtain

$$\nabla_2 \times \Pi_2^Q = \sum_F \Pi_{F,0}^R \nabla \times + \sum_F \Pi_{F,1}^R \nabla \times \Pi_1^Q = \sum_F \Pi_{F,0}^R \nabla \times + \sum_F \Pi_{F,1}^R \nabla \times = \Pi_2^R \nabla \times . \qquad \square$$

6.5. Completing the Coarse Nédélec Space by Face-to-Interior Extension

Let T be a coarse element. We first prove that, under our assumptions on the agglomerates, Assumption 3.1, the sequence (3.1) is exact at the term $R_h(T)$. We need this result to establish the solvability of (6.43).

Lemma 6.28. *Assumption 3.1 implies that*

$$\ker(\nabla \cdot) \cap R_h(T) = \nabla \times Q_h(T).$$

Proof. We first prove $\ker(\nabla\cdot) \cap R_h(T) \supset \nabla\times Q_h(T)$. Since $\nabla\cdot\nabla\times = 0$, it is sufficient to prove $R_h(T) \supset \nabla\times Q_h(T)$. That is, we need to prove that for any fine face f which is not an interior face of T and for any function $\mathbf{q} \in Q_h(T)$ we have $\int_f (\nabla\times\mathbf{q})\cdot\mathbf{n}\,\mathrm{d}A = 0$. This however follows from the Stokes theorem applied to f, since by definition of $Q_h(T)$, we have $\int_e \mathbf{q}\cdot\boldsymbol{\tau}\,\mathrm{d}L = 0$ for any edge e, except interior edges of T.

It now remains to prove that the two spaces in question have the same dimension, i.e.,

$$\dim(\nabla\times Q_h(T)) = \dim\big(\ker(\nabla\cdot) \cap R_h(T)\big).$$

We start by observing $\dim Q_h(T) = N^e_{\text{int}}(T)$, $\dim S_h(T) = N^v_{\text{int}}(T)$, and $\dim \widetilde{M}_h(T) = N^t(T)$ (see Assumption 3.1 for notation). Applying the rank-nullity theorem to operator $\nabla\times :$ $Q_h(T) \to R_h(T)$ yields

$$\begin{aligned}\dim(\nabla\times Q_h(T)) &= \dim Q_h(T) - \dim\big(\ker(\nabla\times) \cap Q_h(T)\big)\\ &= N^e_{\text{int}}(T) - \dim\big(\ker(\nabla\times) \cap Q_h(T)\big).\end{aligned} \tag{6.40}$$

Now, we use Lemmas 7.27 and 7.32 which are proven, independently of this result, in Section 7.6 and 7.7, respectively. By Lemma 7.32, we have

$$\dim\big(\ker(\nabla\times) \cap Q_h(T)\big) = \dim \nabla S_h(T). \tag{6.41}$$

Due to Lemma 7.27, the gradient operator has trivial kernel on $S_h(T)$, and consequently

$$\dim \nabla S_h(T) = \dim S_h(T) = N^v_{\text{int}}(T). \tag{6.42}$$

Combining (6.40), (6.41), and (6.42), we obtain

$$\dim(\nabla\times Q_h(T)) = N^e_{\text{int}}(T) - N^v_{\text{int}}(T).$$

By applying the rank-nullity theorem to the nonzero linear functional $\widetilde{M}_h(T) \ni w \to \int_T w\,\mathrm{d}V$ we observe that $\dim M_h(T) = N^t(T) - 1$. Now, applying the rank-nullity theorem to the operator $\nabla\cdot : R_h(T) \to M_h(T)$ and using Lemma 5.7, we obtain

$$\begin{aligned}\dim\big(\ker(\nabla\cdot) \cap R_h(T)\big) &= \dim R_h(T) - \dim\big(\nabla\cdot(R_h(T))\big)\\ &= N^f_{\text{int}}(T) - \dim(M_h(T)) = N^f_{\text{int}}(T) - (N^t(T) - 1).\end{aligned}$$

Due to Assumption 3.1,

$$E_0(T) = N^v_{\text{int}}(T) - N^e_{\text{int}}(T) + N^f_{\text{int}}(T) - N^t(T) = -1,$$

that is

$$\begin{aligned}\dim(\nabla\times Q_h(T)) &= N^e_{\text{int}}(T) - N^v_{\text{int}}(T)\\ &= N^f_{\text{int}}(T) - (N^t(T) - 1) = \dim\big(\ker(\nabla\cdot) \cap R_h(T)\big),\end{aligned}$$

which completes the proof. □

Consider the volume extension mapping $\mathcal{I}^Q_T : \widetilde{Q}_h \to Q_h(T)$, defined as follows. For any $\mathbf{q} \in \widetilde{Q}_h$, let the pair $\mathbf{q}_T \in Q_h(T), \mathbf{r}_T \in R_h(T)$ solve the local mixed system

$$\begin{aligned}(\mathbf{q}_T, \mathbf{p})_T - (\nabla\times\mathbf{p}, \mathbf{r}_T)_T &= -(\mathbf{q}, \mathbf{p})_T && \text{for all } \mathbf{p} \in Q_h(T),\\ (\nabla\times\mathbf{q}_T, \mathbf{s})_T + (\nabla\cdot\mathbf{r}_T, \nabla\cdot\mathbf{s})_T &= -(\nabla\times\mathbf{q}, \mathbf{s})_T && \text{for all } \mathbf{s} \in R_h(T).\end{aligned} \tag{6.43}$$

The system (6.43) is uniquely solvable due to Lemma 4.1. Indeed, take $V_1 = Q_h(T)$, $V_2 = R_h(T)$, $V_3 = M_h(T)$ with the standard $L_2(T)$ inner products, and use Lemma 6.28. We define $\mathcal{I}^Q_T \mathbf{q}$ to be $\mathbf{q}_T$. Note that

$$\mathcal{I}^Q_T \mathbf{q} = -\mathbf{q} \quad \text{for all } \mathbf{q} \in Q_h(T), \tag{6.44}$$

since for such $\mathbf{q}$ the pair ($\mathbf{q}_T = -\mathbf{q}, \mathbf{r}_T = 0$) solves (6.43).

Definition 6.29. The coarse Nédélec (ND) space $\widetilde{Q}_H$ is defined by

$$\widetilde{Q}_H = \Big(\mathbb{1} + \sum_T \mathcal{I}^Q_T\Big)\Big(\bigoplus_F Q_H(F) \oplus \Big(\mathbb{1} + \sum_F \mathcal{I}^Q_F\Big) \bigoplus_E \widetilde{Q}_H(E)\Big).$$

The operator Π^Q of our main interest (proven later on to be a projection) is defined to be

$$\Pi^Q = \Big(\mathbb{1} + \sum_T \mathcal{I}^Q_T\Big)\Pi^Q_2 = \Big(\mathbb{1} + \sum_T \mathcal{I}^Q_T\Big)\Big(\sum_F \Pi^Q_F + \Big(\mathbb{1} + \sum_F \mathcal{I}^Q_F\Big) \sum_E \Pi^Q_E\Big).$$

Lemma 6.30. Π^Q *is indeed a projector onto* $\widetilde{Q}_H$.

Proof. For any coarse element T, we have

$$\Pi^Q_2 \mathcal{I}^Q_T = 0,$$

since any function from $Q_h(T)$ has vanishing degree of freedom corresponding to any fine edge which is part of some coarse edge or coarse face. The desired result then follows from Lemma 4.3. □

Lemma 6.31. *For any agglomerate* T*, we have*

$$\nabla \times \mathcal{I}^Q_T = \mathcal{I}^R_T \nabla \times . \tag{6.45}$$

Proof. Fix $\mathbf{q} \in \widetilde{Q}_h$. Let $\mathbf{q}_T = \mathcal{I}^Q_T \mathbf{q}$. From the second equation of (6.43) and the identity $\nabla \cdot \nabla \times = 0$ it follows that $\mathbf{q}_T$ satisfies

$$\begin{aligned} (\nabla \times \mathbf{q}_T, \mathbf{s})_T + (\nabla \cdot \mathbf{z}_T, \nabla \cdot \mathbf{s})_T &= -(\nabla \times \mathbf{q}, \mathbf{s})_T && \text{for all } \mathbf{s} \in R_h(T), \\ (\nabla \cdot \nabla \times \mathbf{q}_T, w)_T &= 0 && \text{for all } w \in M_h(T), \end{aligned}$$

for some $\mathbf{z}_T \in R_h(T)$. Note that $\nabla \cdot \mathbf{z}_T \in M_h(T)$, and the pair $(\nabla \times \mathbf{q}_T, \nabla \cdot \mathbf{z}_T)$ solves the mixed system (5.11) for $\mathbf{r} = \nabla \times \mathbf{q}$. Let $\mathbf{r}_T = \mathcal{I}^R_T \nabla \times \mathbf{q}$. By definition of $\mathcal{I}^R_T$, there exists $u_T \in M_h(T)$ such that the pair $(\mathbf{r}_T, u_T)$ solves (5.11) for $\mathbf{r} = \nabla \times \mathbf{q}$. Since (5.11) has a unique solution, we conclude $\nabla \times \mathbf{q}_T = \mathbf{r}_T$, which implies (6.45). □

Lemma 6.32. *Let* $\mathbf{q} \in \widetilde{Q}_h$. *Suppose* $\pi^Q_T \mathbf{q} \in Q_h(T)$. *Then* $\pi^Q_T(\mathbb{1} + \mathcal{I}^Q_T)\mathbf{q} = 0$.

Proof. It is straightforward to verify the identities

$$\mathcal{I}^Q_T \mathbf{q} = \pi^Q_T \mathcal{I}^Q_T \mathbf{q} = \mathcal{I}^Q_T \pi^Q_T \mathbf{q}.$$

Using these identities, we obtain

$$\pi^Q_T(\mathbb{1} + \mathcal{I}^Q_T)\widetilde{Q}_h = (\mathbb{1} + \mathcal{I}^Q_T)\pi^Q_T \mathbf{q} \overset{(6.44)}{=} \pi^Q_T \mathbf{q} - \pi^Q_T \mathbf{q} = 0.$$ □

Lemma 6.33. *Let operator $\nabla_2\times$ be as in Definition 6.26. Then*

$$\Big(\mathbb{1}+\sum_T \mathcal{I}_T^R\Big)\nabla_2\times\mathbf{q}=\Big(\mathbb{1}+\sum_T \mathcal{I}_T^R\Big)\nabla\times\mathbf{q}\quad \textit{for all } \mathbf{q}\in\widetilde{Q}_h. \tag{6.46}$$

Proof. Fix $\mathbf{q}\in\widetilde{Q}_h$. Both sides of (6.46) belong to $\widetilde{R}_h$, thus it is sufficient to prove that for any fine face the corresponding degree of freedom (5.1) gives the same number when evaluated on both sides of (6.46). Each fine face is a part of (at least one) coarse element, thus it is sufficient to prove that for each agglomerate T we have

$$\pi_T^R(\mathbb{1}+\mathcal{I}_T^R)\nabla_2\times\mathbf{q}=\pi_T^R(\mathbb{1}+\mathcal{I}_T^R)\nabla\times\mathbf{q}$$

(note that $\pi_T^R\mathcal{I}_{T'}^R=0$ when $T\neq T'$). This however follows from Lemma 5.9, since

$$\pi_T^R(\nabla_2\times\mathbf{q}-\nabla\times\mathbf{q})\in R_h(T),$$

i.e., since the degrees of freedom of both $\nabla_2\times\mathbf{q}$ and $\nabla\times\mathbf{q}$ coincide on each coarse face. □

Lemma 6.34. *The following commutativity property holds:*

$$\nabla\times\Big(\mathbb{1}+\sum_T \mathcal{I}_T^Q\Big)\mathbf{q}=\Big(\mathbb{1}+\sum_T \mathcal{I}_T^R\Big)\nabla_2\times\mathbf{q}\quad \textit{for all } \mathbf{q}\in\widetilde{Q}_h. \tag{6.47}$$

Proof. We have

$$\nabla\times\Big(\mathbb{1}+\sum_T \mathcal{I}_T^Q\Big)\mathbf{q}\overset{(6.45)}{=}\Big(\mathbb{1}+\sum_T \mathcal{I}_T^R\Big)\nabla\times\mathbf{q}\overset{(6.46)}{=}\Big(\mathbb{1}+\sum_T \mathcal{I}_T^R\Big)\nabla_2\times\mathbf{q}. \qquad \square$$

6.6. Main Commutativity and Exactness Properties

We are ready to prove the following main commutativity result.

Theorem 6.35. *Consider the projection operators $\Pi^Q:\widetilde{Q}_h\mapsto\widetilde{Q}_H$ and $\Pi^R:\widetilde{R}_h\mapsto\widetilde{R}_H$. Then*

$$\nabla\times\Pi^Q\mathbf{q}=\Pi^R\nabla\times\mathbf{q}\quad \textit{for all } \mathbf{q}\in\widetilde{Q}_h. \tag{6.48}$$

Proof. Fix any $\mathbf{q}\in\widetilde{Q}_h$. We have

$$\begin{aligned}\nabla\times\Pi^Q\mathbf{q}&=\nabla\times\Big(\mathbb{1}+\sum_T \mathcal{I}_T^Q\Big)\Pi_2^Q\mathbf{q}\\ &\overset{(6.47)}{=}\Big(\mathbb{1}+\sum_T \mathcal{I}_T^R\Big)\nabla_2\times\Pi_2^Q\mathbf{q}\overset{(6.39)}{=}\Big(\mathbb{1}+\sum_T \mathcal{I}_T^R\Big)\Pi_2^R\nabla\times\mathbf{q}=\Pi^R\nabla\times\mathbf{q}.\end{aligned} \qquad \square$$

Corollary 6.36 (Exactness property). *If $\nabla\times\widetilde{Q}_h=\ker(\nabla\cdot)\cap\widetilde{R}_h$, then*

$$\nabla\times\widetilde{Q}_H=\ker(\nabla\cdot)\cap\widetilde{R}_H.$$

Proof. Let $\mathbf{r}\in\ker(\nabla\cdot)\cap\widetilde{R}_H$. Since $\mathbf{r}\in\widetilde{R}_H$, we have $\mathbf{r}=\Pi^R\mathbf{r}$. Since $\widetilde{R}_H\subset\widetilde{R}_h$, we have $\mathbf{r}\in\ker(\nabla\cdot)\cap\widetilde{R}_h$. Then, by assumption of the corollary, $\mathbf{r}=\nabla\times\mathbf{q}$ for some $\mathbf{q}\in\widetilde{Q}_h$. We conclude

$$\mathbf{r}=\Pi^R\mathbf{r}=\Pi^R\nabla\times\mathbf{q}\overset{(6.48)}{=}\nabla\times\Pi^Q\mathbf{q}.$$

Denoting $\mathbf{p}=\Pi^Q\mathbf{q}$, we have proved that $\mathbf{r}=\nabla\times\mathbf{p}$ for $\mathbf{p}\in\widetilde{Q}_H$. □

6.7. Local Approximation Property of the Coarse ND Space

We begin with the following auxiliary result.

Lemma 6.37. *For each constant vector* $\mathbf{c}$*, we have*

$$\mathcal{I}_T^Q \mathbf{c} = 0 \quad \text{for all } T.$$

Proof. We need to prove that there exists an $\mathbf{r}_T \in R_h(T)$ such that the pair $(\mathbf{q}_T = 0, \mathbf{r}_T)$ solves (6.43) for $\mathbf{q} = \mathbf{c}$. We use the representation

$$\mathbf{c} = \nabla \times \mathbf{r}, \quad \text{where } \mathbf{r} = \frac{1}{2}\mathbf{c}\times\mathbf{x},\ \mathbf{x} \text{ is the position vector.} \tag{6.49}$$

Let $\mathbf{p} \in Q_h(T)$. Since $\mathbf{p}$ has zero tangential component on ∂T, the function $\mathbf{r}\times\mathbf{p}$ has zero normal component on ∂T. By applying (3.4) to $\mathbf{r}\times\mathbf{p}$, and using the identity

$$\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B),$$

we obtain

$$0 = \int_T \nabla\cdot(\mathbf{r}\times\mathbf{p})\,\mathrm{d}V = \int_T \mathbf{p}\cdot(\nabla\times\mathbf{r})\,\mathrm{d}V - \int_T \mathbf{r}\cdot(\nabla\times\mathbf{p})\,\mathrm{d}V. \tag{6.50}$$

Let $\mathbf{r}_T$ be the $(L_2(T))^3$-orthogonal projection of $\mathbf{r}$ onto $\ker(\nabla\cdot)\cap R_h(T)$. For any $\mathbf{p}\in Q_h(T)$, we have $\nabla\times\mathbf{p}\in\ker(\nabla\cdot)\cap R_h(T)$, and thus

$$(\nabla\times\mathbf{p}, \mathbf{r}_T)_T = (\nabla\times\mathbf{p}, \mathbf{r})_T \overset{(6.50)}{=} (\mathbf{p}, \nabla\times\mathbf{r})_T \overset{(6.49)}{=} (\mathbf{p}, \mathbf{c}).$$

That is, the pair $(\mathbf{q}_T = 0, \mathbf{r}_T)$ satisfies the first equation of (6.43) for $\mathbf{q} = \mathbf{c}$. The second equation is also satisfied by the pair, since $\nabla\times\mathbf{c} = 0$ and by construction $\nabla\cdot\mathbf{r}_T = 0$. □

Theorem 6.38 (Local approximation property)**.** *If a function* $\mathbf{q}\in\widetilde{Q}_h$ *equals a vector constant* $\mathbf{c}$ *on a coarse element* T*, so does the projection* $\Pi^Q\mathbf{q}$*. Equivalently, if* $\pi_T^Q\mathbf{q} = \pi_T^Q\mathbf{c}$*, then* $\pi_T^Q\Pi^Q\mathbf{q} = \pi_T^Q\mathbf{c}$*.*

Proof. For any coarse face $F\subset\partial T$, we have $\pi_F^Q\Pi_2^Q\mathbf{q} = \pi_F^Q\mathbf{c}$. This follows from the identities

$$\pi_F^Q\mathbf{q} = \pi_F^Q\pi_T^Q\mathbf{q} = \pi_F^Q\pi_T^Q\mathbf{c} = \pi_F^Q\mathbf{c}$$

and Lemma 6.25. Since every fine edge $e\subset\partial T$ is part of some coarse face $F\subset\partial T$, we have

$$\pi_T^Q(\Pi_2^Q\mathbf{q} - \mathbf{c}) \in Q_h(T).$$

By Lemmas 6.32 and 6.37 we have

$$\pi_T^Q(\mathbb{1} + \mathcal{I}_T^Q)\Pi_2^Q\mathbf{q} = \pi_T^Q(\mathbb{1} + \mathcal{I}_T^Q)\mathbf{c} = \pi_T^Q\mathbf{c}.$$

Using the definition of Π^Q and noting that $\pi_T^Q\mathcal{I}_{T'}^Q = 0$ for $T\neq T'$, we obtain

$$\pi_T^Q\Pi^Q\mathbf{q} = \pi_T^Q\Big(\mathbb{1} + \sum_{T'}\mathcal{I}_{T'}^Q\Big)\Pi_2^Q\mathbf{q} = \pi_T^Q(\mathbb{1} + \mathcal{I}_T^Q)\Pi_2^Q\mathbf{q} = \pi_T^Q\mathbf{c}.$$ □

7. The Coarse H^1-Conforming Space

Since $\widetilde{S}_h$ is the space of continuous piecewise linear functions, an element of $\widetilde{S}_h$ is completely defined by values at all fine vertices.

Definition 7.1. For any $\theta \in \widetilde{S}_h$ and any coarse mesh "entity" K (coarse edge, coarse face or coarse element), we define $\pi_K^S\theta \in \widetilde{S}_h$ to have the following values at fine vertices:

$$(\pi_K^S\theta)(v) = \begin{cases} \theta(v) & \text{if } v \in K, \\ 0 & \text{if } v \notin K. \end{cases}$$

Recall that we treat all coarse entities as closed sets, i.e., a coarse edge E contains its endpoints and so forth.

Definition 7.2. For any $\theta \in \widetilde{S}_h$ and any coarse mesh "entity" K, we define

$$\nabla_K\theta = \pi_K^Q\nabla\theta.$$

That is, $\nabla_K\theta \in \widetilde{Q}_h$ with only possibly non-zero dofs $\int_e \nabla\theta \cdot \boldsymbol{\tau}_e \,\mathrm{d}L$ that are associated with the fine edges $e \in K$.

7.1. Basis Functions Associated with Coarse Vertices

Let V be a coarse vertex. Consider the function $\eta_V \in \widetilde{S}_h$ with fine vertex values prescribed as follows. Let E be any coarse edge incident to V. Assume that E consists of fine edges $e_1, \ldots, e_n$ and fine vertices $v_1, \ldots, v_{n+1}$, where $v_1 = V$ and e_i connects v_i with v_{i+1}. We define $\eta_V(v_1) = 1$, and for $i > 1$, we let

$$\eta_V(v_i) = 1 - \frac{\sum_{k=1}^{i-1} \int_{e_k} \boldsymbol{\nu}_E \cdot \boldsymbol{\tau}_E \,\mathrm{d}L}{\int_E \boldsymbol{\nu}_E \cdot \boldsymbol{\tau}_E \,\mathrm{d}L}. \tag{7.1}$$

Note that $\int_E \boldsymbol{\nu}_E \cdot \boldsymbol{\tau}_E \,\mathrm{d}L \neq 0$ (see the argument below (6.3)). For any fine vertex v which is not part of any coarse edge incident to V, we define $\eta_V(v) = 0$. This concludes the definition of η_V. Note $\eta_V(V) = 1$ and $\eta_V(V') = 0$ for any coarse vertex $V' \neq V$. Also, for any coarse edge E, we have $\nabla_E\eta_V = \alpha\boldsymbol{\nu}_E$ ($\alpha = 0$ if E is not incident to V).

Definition 7.3. For any function $\theta \in \widetilde{S}_h$, define the coarse vertex interpolant

$$\Pi_V^S\theta = \theta(V)\eta_V.$$

Note that $\Pi_V^S\eta_V = \eta_V$, i.e., Π_V^S is a projector onto $\operatorname{span}(\eta_V)$.

Lemma 7.4. *For any $\theta \in \widetilde{S}_h$ and for any coarse edge E, we have*

$$\nabla_E \sum_V \Pi_V^S\theta = \Pi_{E,1}^Q\nabla\theta. \tag{7.2}$$

Proof. From the definition of ∇_E, Π_V^S, and $\Pi_{E,1}^Q$ it follows that both sides of (7.2) are multiples of $\boldsymbol{\nu}_E$. Thus, it is sufficient to prove that

$$\int_E \Big(\nabla_E \sum_V \Pi_V^S\theta\Big) \cdot \boldsymbol{\tau}_E \,\mathrm{d}L = \int_E (\Pi_{E,1}^Q\nabla\theta) \cdot \boldsymbol{\tau}_E \,\mathrm{d}L.$$

Let coarse vertices V_1 and V_2 be the "tail" and the "head" of E, respectively. We have

$$\int_E \Big(\nabla_E \sum_V \Pi_V^S \theta\Big) \cdot \boldsymbol{\tau}_E \, \mathrm{d}L \overset{(3.3)}{=} \Big(\sum_V \Pi_V^S \theta\Big)(V_2) - \Big(\sum_V \Pi_V^S \theta\Big)(V_1) = \theta(V_2) - \theta(V_1).$$

On the other hand we have (see (6.5))

$$\int_E (\Pi_{E,1}^Q \nabla \theta) \cdot \boldsymbol{\tau}_E \, \mathrm{d}L = \int_E \nabla \theta \cdot \boldsymbol{\tau}_E \, \mathrm{d}L = \theta(V_2) - \theta(V_1). \qquad \square$$

7.2. Edge Bubbles

Let E be a coarse edge with endpoint coarse vertices V_1 and V_2. Define the space

$$S_H(E) = \big\{\theta \in \pi_E^S(\widetilde{S}_h) : \theta(V_1) = 0 \text{ and } \nabla_E \theta \in Q_H(E)\big\}.$$

Note that due to the definition (6.4) of $Q_H(E)$ and formula (3.3), we have $\theta(V_2) = \theta(V_1) = 0$ for any $\theta \in S_H(E)$.

Definition 7.5. For any $\eta \in \widetilde{S}_h$ and any coarse edge E, let

$$\Pi_E^S \eta = \theta \in S_H(E), \quad \text{where } \nabla_E \theta = \Pi_{E,0}^Q \nabla \eta.$$

Note that for any $\mathbf{q} \in Q_H(E)$ there exists a unique $\theta \in S_H(E)$ such that $\nabla_E \theta = \mathbf{q}$. Existence can be shown, e.g., by explicitly constructing θ in a manner similar to (7.1) (using the fact that gradients of linear functions span all vector constants). Uniqueness follows from the (assumed) connectivity of the coarse edge: if $\nabla_E \theta_1 = \nabla_E \theta_2$ then $\theta_1 - \theta_2$ is constant on E, but then it has to be zero, since $\theta_1(V_1) = \theta_2(V_1) = 0$.

Π_E^S is a projection onto $S_H(E)$, since $\Pi_{E,0}^Q$ is a projector onto $Q_H(E)$. By definition of Π_E^S we have

$$\nabla_E \Pi_E^S \eta = \Pi_{E,0}^Q \nabla \eta = \Pi_{E,0}^Q \nabla_E \eta \quad \text{for all } \eta \in \widetilde{S}_h. \tag{7.3}$$

7.3. Projector $\boldsymbol{\Pi_1^S}$ and its Properties

Definition 7.6. We define

$$\Pi_1^S = \sum_E \Pi_E^S + \sum_V \Pi_V^S.$$

Lemma 7.7. Π_1^S *is a projector onto*

$$\bigoplus_E S_H(E) \oplus \bigoplus_V \operatorname{span}(\eta_V)$$

Proof. The result follows from Lemma 4.2 and the identities

$$\begin{aligned} \Pi_E^S \Pi_{E'}^S &= 0 \quad \text{for } E \neq E', \\ \Pi_V^S \Pi_{V'}^S &= 0 \quad \text{for } V \neq V', \\ \Pi_E^S \Pi_V^S = \Pi_V^S \Pi_E^S &= 0 \quad \text{for all } E, V. \end{aligned}$$

The first two identities and the identity $\Pi_V^S \Pi_E^S = 0$ are straightforward. Let $\eta \in \widetilde{S}_h$. Denote $\theta = \Pi_E^S \Pi_V^S \eta$. We claim that $\theta = 0$. Since $\theta \in S_H(E)$, it is sufficient to prove $\nabla_E \theta = 0$. By (7.3) we have $\nabla_E \theta = \Pi_{E,0}^Q \nabla_E \Pi_V^S \eta$. By the definition of Π_V^S we have that $\nabla_E \Pi_V^S \eta$ is a multiple of $\boldsymbol{\nu}_E$. However, by (6.4) and (6.6), we have $\Pi_{E,0}^Q \boldsymbol{\nu}_E = 0$. $\square$

Definition 7.8. We define

$$\nabla_1 = \sum_E \nabla_E.$$

Lemma 7.9. *We have*

$$\nabla_1 \Pi_1^S \eta = \Pi_1^Q \nabla \eta. \tag{7.4}$$

Proof. Combining (7.2) with (7.3), we obtain for any $\eta \in \widetilde{S}_h$

$$\nabla_E \Pi_1^S \eta = \nabla_E \Big(\sum_{E'} \Pi_{E'}^S \eta + \sum_V \Pi_V^S \eta \Big) = \Pi_{E,0}^Q \nabla \eta + \Pi_{E,1}^Q \nabla \eta = \Pi_E^Q \nabla \eta. \tag{7.5}$$

We used that $\nabla_E \Pi_{E'}^S = 0$ when $E \neq E'$. By summing (7.5) over all coarse edges we obtain (7.4). □

7.4. Edge-to-Face Extension of Coarse H^1 Basis Functions

Let F be a coarse face. Define

$$\widetilde{S}_h(F) = \pi_F^S(\widetilde{S}_h).$$

We also need the space $S_h(F) \subset \widetilde{S}_h(F)$, defined as follows:

$$S_h(F) := \big\{ \eta \in \widetilde{S}_h(F) : \eta(v) = 0 \text{ for any fine vertex } v \in \partial F \big\}.$$

To aid our analysis we introduce an inner product on $\widetilde{S}_h(F)$:

$$(\theta, \eta)_F = \sum_{v \in F} \theta(v)\eta(v) \quad \text{for all } \theta, \eta \in \widetilde{S}_h.$$

This inner product is only introduced for theoretical purposes (we do not use it in a computer implementation). We stress that many other choices for this inner product would do just as well.

Lemma 7.10. *Let $\eta \in S_h(F)$. Then the condition $\nabla_F \eta = 0$ implies $\eta = 0$.*

Proof. The dual graph of F is connected (see Section 3.1). Since the boundary of a triangular face is connected, the graph made of vertices and edges of F is also connected. The condition $\nabla_F \eta = 0$ then implies that η has the same value at any vertex of F. Since η vanishes at the boundary vertices of F (and F does have boundary vertices, as required in Section 3.1), $\eta = 0$. □

Let $\theta \in \widetilde{S}_h$. Define $\theta_F \in S_h(F)$ as the solution of

$$\big(\nabla_F \theta_F, (\mathbb{1} - P_{D_F}) \nabla_F \eta \big)_F = -\big(\nabla_F \theta, (\mathbb{1} - P_{D_F}) \nabla_F \eta \big)_F \quad \text{for all } \eta \in S_h(F), \tag{7.6}$$

where the inner product $(\cdot,\cdot)_F = (\cdot,\cdot)_F^Q$ and the projector P_{D_F} are those defined in Section 6.3. The equation (7.6) has a unique solution due to Lemma 7.10. We define

$$\mathcal{I}_F^S \theta = \theta_F.$$

Lemma 7.11. *The operator $(\mathbb{1} + \sum_F \mathcal{I}_F^S)\Pi_1^S$ is a projector onto*

$$\Big(\mathbb{1} + \sum_F \mathcal{I}_F^S \Big) \Big(\bigoplus_E S_H(E) \oplus \bigoplus_V \operatorname{span}(\eta_V) \Big).$$

Proof. For any coarse face F, a function from $S_h(F)$ has vanishing degrees of freedom corresponding to all fine edges which are part of some coarse edge. Because of that

$$\Pi_1^S \mathcal{I}_F^Q = 0.$$

The desired result now follows from Lemmas 7.7 and 4.3. □

Lemma 7.12 (Exactness on coarse faces)**.** *For any coarse face F, we have*

$$\ker(\nabla_F \times) \cap Q_h(F) = \nabla_F S_h(F).$$

Proof. The proof is analogous to that of Lemma 6.28. Since for any fine face f

$$\int_f \big(\nabla_F \times (\nabla_F \theta)\big) \cdot \mathbf{n}_F \, \mathrm{d}A = \pm \int_{\partial f} \nabla\theta \cdot \boldsymbol{\tau} \, \mathrm{d}L = 0,$$

we see that $\nabla_F \times \nabla_F \theta = 0$ for any $\theta \in S_h(F)$. The endpoints of each fine edge from ∂F also lie on ∂F. Thus from (3.2) it follows that $\nabla_F \theta \in Q_h(F)$ for any $\theta \in S_h(F)$. Overall, we have

$$\nabla_F S_h(F) \subseteq \ker(\nabla_F \times) \cap Q_h(F),$$

and it is now sufficient to prove that the dimension of $\nabla_F S_h(F)$ equals that of $\ker(\nabla_F \times) \cap Q_h(F)$. The dimension of $S_h(F)$ is $N^v_{\text{int}}(F)$, the number of interior vertices of F. Applying the rank-nullity theorem to the operator $\nabla_F : S_h(F) \to Q_h(F)$ and taking into account Lemma 7.10, we obtain

$$\dim \nabla_F S_h(F) = N^v_{\text{int}}(F) - \dim\big(\ker(\nabla_F) \cap S_h(F)\big) = N^v_{\text{int}}(F).$$

The space $R_h(F)$ is the kernel of a non-zero linear functional $l(\mathbf{r}) = \int_F \mathbf{r} \cdot \mathbf{n}_F \, \mathrm{d}A$ defined on $\widetilde{R}_h(F)$. The dimension of $\widetilde{R}_h(F)$ is $N^f(F)$, the number of fine faces in F. By rank-nullity theorem applied to l, the dimension of $R_h(F)$ is $N^f(F) - 1$. The dimension of $Q_h(F)$ equals $N^e_{\text{int}}(F)$, the number of interior edges of F. Applying the rank-nullity theorem to the operator $\nabla_F \times : Q_h(F) \to R_h(F)$ and taking into account (6.12), we obtain

$$\begin{aligned} \dim\big(\ker(\nabla_F \times) \cap Q_h(F)\big) &= \dim Q_h(F) - \dim \nabla_F \times Q_h(F) \\ &= N^e_{\text{int}}(F) - (N^f(F) - 1) = N^e_{\text{int}}(F) - N^f(F) + 1. \end{aligned}$$

By Assumption 3.2, we have

$$N^v_{\text{int}}(F) - N^e_{\text{int}}(F) + N^f(F) = 1 \iff N^v_{\text{int}}(F) = N^e_{\text{int}}(F) - N^f(F) + 1.$$

Therefore,

$$\dim\big(\ker(\nabla_F \times) \cap Q_h(F)\big) = N^v_{\text{int}}(F),$$

which completes the proof. □

Lemma 7.13 (Commutativity on coarse faces)**.** *For any $\theta \in \widetilde{S}_h$ and any coarse face F, we have*

$$\nabla_F \mathcal{I}_F^S \theta = \mathcal{I}_F^Q \nabla_F \theta. \tag{7.7}$$

Proof. Consider the mixed system for the unknowns $\sigma \in S_h(F)$ and $\mathbf{z} \in Q_h(F)$,

$$\begin{aligned} (\sigma,\tau)_F - \big(\nabla_F \tau, (\mathbb{1} - P_{D_F})\mathbf{z}\big)_F &= -(\theta,\tau)_F && \text{for all } \tau \in S_h(F),\\ \big(\nabla_F \sigma, (\mathbb{1} - P_{D_F})\mathbf{p}\big)_F + (\nabla_F \times \mathbf{z}, \nabla_F \times \mathbf{p})_F &= -\big(\nabla_F \theta, (\mathbb{1} - P_{D_F})\mathbf{p}\big)_F && \text{for all } \mathbf{p} \in Q_h(F). \end{aligned} \tag{7.8}$$

Due to Lemmas 7.12 and 4.1, the system is solvable. To apply Lemma 4.1 we take $V_1 = S_h(F)$, $V_2 = Q_h(F)$, $V_3 = R_h(F)$, $\mathrm{d}_1 = \nabla_F$, and $\mathrm{d}_2 = \nabla_F \times$.

Since for any $\eta \in S_h(F)$ we have $\mathbf{p} = \nabla_F \eta \in Q_h(F)$ and $\nabla_F \times \mathbf{p} = 0$, σ also solves (7.6). The latter equation has the unique solution $\theta_F = \mathcal{I}_F^S \theta$, thus $\sigma = \theta_F$. We now see that the pair ($\mathbf{q}_F = \nabla_F \theta_F, \mathbf{r}_F = \nabla_F \times \mathbf{z}$) solves the first equation of (6.19) for $\mathbf{q} = \nabla_F \theta$. The second equation of (6.19) is trivially satisfied since $\nabla_F \times \mathbf{q}_F = \nabla_F \times \mathbf{q} = 0$. Thus, $\mathbf{q}_F = \mathcal{I}_F^Q \mathbf{q}$, which is (7.7). □

Note that in a computer implementation we can solve system (7.6), and not necessarily (7.8).

We recall that a boundary fine edge of a coarse entity (face or element) is considered to be a part of that coarse entity. Similarly to coarse edge gradient ∇_1, we introduce the coarse face gradient ∇_2.

Definition 7.14. For any $\theta \in \widetilde{S}_h$, define $\nabla_2 \theta \in \widetilde{Q}_h$ to have the following degrees of freedom:

$$\int_e (\nabla_2 \theta) \cdot \boldsymbol{\tau} \,\mathrm{d}L = \begin{cases} \int_e \nabla\theta \cdot \boldsymbol{\tau} \,\mathrm{d}L & \text{if } e \text{ is a part of some coarse face,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\nabla_2 \neq \sum_F \nabla_F$, since distinct coarse faces can have common fine edges (of course, each such common fine edge has to be part of some coarse edge).

Lemma 7.15. *For any function $\theta \in \widetilde{S}_h$, the following commutativity property holds:*

$$\nabla_2 \Big(\mathbb{1} + \sum_{F'} \mathcal{I}_{F'}^S\Big)\theta = \Big(\mathbb{1} + \sum_{F'} \mathcal{I}_{F'}^Q\Big)\nabla_1 \theta. \tag{7.9}$$

In other words, one could say that edge-to-face extension commutes with applying the respective gradient operator.

Proof. Suppose that a fine edge is not part of any coarse face. Then the corresponding fine degree of freedom (6.1) vanishes on both sides of (7.9). Consequently, it is sufficient to check that for each coarse face F

$$\pi_F^Q \nabla_2 \Big(\mathbb{1} + \sum_{F'} \mathcal{I}_{F'}^S\Big)\theta \stackrel{?}{=} \pi_F^Q \Big(\mathbb{1} + \sum_{F'} \mathcal{I}_{F'}^Q\Big)\nabla_1 \theta. \tag{7.10}$$

We have $\pi_F^Q \nabla_2 = \pi_F^Q \nabla_F$, as well as $\nabla_F \mathcal{I}_{F'}^S = 0$ and $\pi_F^Q \mathcal{I}_{F'}^Q = 0$ for $F' \neq F$. Therefore, (7.10) is equivalent to

$$\pi_F^Q \nabla_F (\mathbb{1} + \mathcal{I}_F^S)\theta \stackrel{?}{=} \pi_F^Q (\mathbb{1} + \mathcal{I}_F^Q)\nabla_1 \theta. \tag{7.11}$$

From Lemma 7.13, we see that

$$\pi_F^Q \nabla_F (\mathbb{1} + \mathcal{I}_F^S)\theta = \pi_F^Q (\nabla_F \theta + \mathcal{I}_F^Q \nabla_F \theta) = \pi_F^Q (\mathbb{1} + \mathcal{I}_F^Q)\nabla_F \theta.$$

Since $\pi_F^Q (\nabla_1 \theta - \nabla_F \theta) \in Q_h(F)$, by Lemma 6.14 we have

$$\pi_F^Q (\mathbb{1} + \mathcal{I}_F^Q)\nabla_F \theta = \pi_F^Q (\mathbb{1} + \mathcal{I}_F^Q)\nabla_1 \theta,$$

hence, (7.11) holds, which completes the proof. □

7.4.1. Face Bubbles.

Definition 7.16. Define the space

$$L_F = \Big\{\eta \in \widetilde{S}_h(F) : \nabla_F \eta \in C_F \ominus D_F \text{ and } \sum_{v\in F} \eta(v) = 0\Big\},$$

where the sum is taken over all fine vertices in F. Refer to (6.27) for the definition of $C_F \ominus D_F$.

Clearly, the dimension of L_F equals that of $C_F \ominus D_F$. As we have mentioned in Section 6.3.2, in our numerical experiments the dimension of $C_F \ominus D_F$ never exceeded one (for most coarse faces we had $L_F = \{0\}$, i.e., no face bubbles were added).

Definition 7.17. The face-bubble space is defined to be

$$S_H(F) = \mathcal{I}_F^S(L_F).$$

In a computer implementation, we build a basis of L_F, apply $\mathcal{I}_F^S$ to each basis vector by solving the system (7.6), and perform SVD on the resulting vectors to extract a linearly independent set.

Lemma 7.18. *Let $Q_H^1(F) = \mathcal{I}_F^Q(C_F)$ be as in* (6.25). *Then*

$$\nabla_F S_H(F) = Q_H^1(F).$$

Proof. We have

$$\nabla_F S_H(F) = \nabla_F \mathcal{I}_F^S(L_F) \overset{(7.7)}{=} \mathcal{I}_F^Q \nabla_F L_F = \mathcal{I}_F^Q(C_F \ominus D_F) \overset{(6.26)}{=} \mathcal{I}_F^Q C_F = Q_H^1(F). \qquad \square$$

Lemma 7.19. *We have*

$$S_H(F) = \big\{\theta \in S_h(F) : \nabla_F \theta \in Q_H^1(F)\big\}.$$

Proof. If $\theta \in S_H(F)$ then clearly $\theta \in S_h(F)$, and due to Lemma 7.18 we have $\nabla_F \theta \in Q_H^1(F)$. Conversely, if $\nabla_F \theta \in Q_H^1(F)$, then due to Lemma 7.18 there exists a $\theta' \in S_H(F)$ such that $\nabla_F \theta' = \nabla_F \theta$. Then, since both θ and θ' lie in $S_h(F)$, we have $\theta = \theta'$ by Lemma 7.10. $\square$

Lemma 7.20. *For any $\theta \in \widetilde{S}_h$, we have $\Pi_F^Q \nabla\theta \in Q_H^1(F)$.*

Proof. Let $\mathbf{q} = \nabla_F \theta$. Define $\mathbf{p} = -\mathcal{I}_F^Q \mathbf{q}$. Since $\nabla_F \times \mathbf{q} = 0$, we have (from (6.20)) that $\nabla_F \times \mathbf{p} = -\nabla_F \times (\mathcal{I}_F^Q \mathbf{q}) = 0$. From Definition 6.20, we have $\Pi_F^Q \mathbf{q} = \mathbf{p}_1 + \mathbf{p}_0$ where $\mathbf{p}_1 = 0$ (since $\nabla_F \times \mathbf{p} = -\nabla_F \times (\mathcal{I}_F^Q \mathbf{q}) = 0$) and $\mathbf{p}_0 \in Q_H^1(F)$. $\square$

Definition 7.21. For any $\theta \in \widetilde{S}_h$, let $\Pi_F^S \theta$ be the element of $S_h(F)$ satisfying

$$\nabla_F \Pi_F^S \theta = \Pi_F^Q \nabla_F \theta. \tag{7.12}$$

The existence of $\Pi_F^S \theta$ follows from Lemmas 7.20 and 7.18. The uniqueness follows from Lemma 7.10.

Lemma 7.22. Π_F^S *is a projector onto $S_H(F)$.*

Proof. By Lemma 7.20, we have $\nabla_F \Pi_F^S \theta = \Pi_F^Q \nabla_F \theta \in Q_H^1(F)$. Then, due to Lemma 7.19, $\Pi_F^S \theta \in S_H(F)$. If $\theta \in S_H(F)$, then $\nabla_F \theta \in Q_H^1(F) \subset Q_H(F)$, and thus $\nabla_F \Pi_F^S \theta = \Pi_F^Q \nabla_F \theta = \nabla_F \theta$. Then, due to Lemma 7.10, $\Pi_F^S \theta = \theta$. □

The last fact we need is stated next.

Lemma 7.23. *We have*
$$\Pi_F^S \mathcal{I}_F^S = -\Pi_F^S.$$

Proof. Due to Lemma 7.10, it is sufficient to prove that
$$\nabla_F \Pi_F^S \mathcal{I}_F^S = -\nabla_F \Pi_F^S.$$

We have
$$\nabla_F \Pi_F^S \mathcal{I}_F^S \overset{(7.12)}{=} \Pi_F^Q \nabla_F \mathcal{I}_F^S \overset{(7.7)}{=} \Pi_F^Q \mathcal{I}_F^Q \nabla_F \overset{(6.36)}{=} -\Pi_F^Q \nabla_F \overset{(7.12)}{=} -\nabla_F \Pi_F^S.$$ □

7.5. Projector Π_2^S and its Properties

Definition 7.24. We define
$$\Pi_2^S = \sum_F \Pi_F^S + \Big(\mathbb{1} + \sum_F \mathcal{I}_F^S\Big)\Pi_1^S.$$

Lemma 7.25. Π_2^S *is a projector onto*
$$\bigoplus_F S_H(F) \oplus \Big(\mathbb{1} + \sum_F \mathcal{I}_F^S\Big)\Big(\bigoplus_E S_H(E) \oplus \bigoplus_V \operatorname{span}(\eta_V)\Big).$$

Proof. Let
$$B = \Big(\mathbb{1} + \sum_F \mathcal{I}_F^S\Big)\Pi_1^S.$$

It is straightforward to check that for distinct faces $F \neq F'$ we have $\Pi_F^S \Pi_{F'}^S = 0$. For any coarse face F, a function from $S_h(F)$ has vanishing degrees of freedom corresponding to all fine vertices which are part of some coarse edge (including the coarse vertices). We thus have
$$\Pi_1^S \Pi_F^S = 0 \implies B\Pi_F^S = 0.$$

For distinct coarse faces F and F', the spaces $\widetilde{S}_h(F)$ and $S_h(F')$ have a trivial intersection, thus $\Pi_F^S \mathcal{I}_{F'}^S = 0$. Together with Lemma 7.23 this implies
$$\Pi_F^S\Big(\mathbb{1} + \sum_{F'} \mathcal{I}_{F'}^S\Big) = 0 \implies \Pi_F^S B = 0.$$

The desired result now follows from Lemmas 4.2, 7.11, and 7.22. □

Lemma 7.26. *We have*
$$\nabla_2 \Pi_2^S = \Pi_2^Q \nabla. \tag{7.13}$$
Recall that $\Pi_2^Q = \sum_F \Pi_F^Q + (\mathbb{1} + \sum_F \mathcal{I}_F^Q)\Pi_1^Q$ *was defined in Definition 6.23.*

Proof. Let F be any coarse face. For any $\theta \in S_h(F)$, we have $\nabla_2\theta = \nabla_F\theta$, and thus $\nabla_2\Pi_F^S = \nabla_F\Pi_F^S$. On the other hand, $\Pi_F^S = \Pi_F^S\pi_F^S$ and thus $\Pi_F^S\nabla_F = \Pi_F^S\nabla$. Combining this with (7.12), we obtain

$$\nabla_2\Pi_F^S = \Pi_F^Q\nabla. \tag{7.14}$$

From (7.9) and (7.4) it follows that

$$\nabla_2\Big(\mathbb{1} + \sum_F \mathcal{I}_F^S\Big)\Pi_1^S = \Big(\mathbb{1} + \sum_F \mathcal{I}_F^Q\Big)\nabla_1\Pi_1^S = \Big(\mathbb{1} + \sum_F \mathcal{I}_F^Q\Big)\Pi_1^Q\nabla. \tag{7.15}$$

By summing (7.14) over all coarse faces and adding the resulting equality to (7.15) we obtain (7.13). □

7.6. Volume Extension

Lemma 7.27. *Let $\eta \in S_h(T)$. Then the condition $\nabla_F\eta = 0$ implies $\eta = 0$.*

Proof. The dual graph of T is connected by Assumption 3.1. Since the boundary of a tetrahedral element is connected, the graph made of all (interior and boundary) fine vertices and edges of T is also connected. The condition $\nabla_F\eta = 0$ then implies that η has the same value at any vertex of F. Since η vanishes at the boundary vertices of T, we have $\eta = 0$. □

For any $\theta \in \widetilde{S}_h$, define $\theta_T \in S_h(T)$ as the solution of

$$(\nabla\theta_T, \nabla\eta)_T = -(\nabla\theta, \nabla\eta)_T, \tag{7.16}$$

where $(\cdot,\cdot)_T$ is the $L_2(T)$ inner product. Due to Lemma 7.27 this problem has a unique solution.

Definition 7.28. For a given $\theta \in \widetilde{S}_h$, we define $\mathcal{I}_T^S\theta$ to be the corresponding θ_T from (7.16).

Definition 7.29. The coarse H^1-conforming space is defined to be

$$\widetilde{S}_H = \Big(\mathbb{1} + \sum_T \mathcal{I}_T^S\Big) \times \Big(\sum_F S_H(F) + \Big(\mathbb{1} + \sum_F \mathcal{I}_F^S\Big)\Big(\sum_E S_H(E) + \sum_V \operatorname{span}(\eta_V)\Big)\Big).$$

Definition 7.30. The main fine-to-coarse projector is defined to be

$$\Pi^S = \Big(\mathbb{1} + \sum_T \mathcal{I}_T^S\Big)\Pi_2^S.$$

7.7. Properties of Π^S

Lemma 7.31. Π^S *is indeed a projector onto* $\widetilde{S}_H$.

Proof. For any agglomerate T, any function from $S_h(T)$ vanishes at all boundary fine vertices of T. Consequently, $\Pi_2^S\mathcal{I}_T^S = 0$. The desired result then follows from Lemma 4.3. □

With the next result we complete the proof that the sequence (3.1) is exact.

Lemma 7.32 (Local exactness)**.** *For any agglomerate T, we have*

$$\ker(\nabla\times) \cap Q_h(T) = \nabla S_h(T).$$

Proof. Let $\mathbf{q} = \nabla\theta$ for some $\theta \in S_h(T)$. The endpoints of each fine edge from ∂T also lie on ∂T. Thus from (3.2) it follows that $\mathbf{q} \in Q_h(T)$. Since $\nabla\times\nabla = 0$, we also have $\mathbf{q} \in \ker(\nabla\times)$.

Now, let $\mathbf{q} \in \ker(\nabla\times) \cap Q_h(T)$. Let $\widehat{Q}_h$ be the lowest order Nédélec space corresponding to the mesh $\widehat{\mathcal{T}}_h$ that covers a domain $\widehat{\Omega}$. We recall that $\widehat{\mathcal{T}}_h$ is introduced in the beginning of Section 2. Let $\widehat{\mathbf{q}} \in \widehat{Q}_h$ have the following degrees of freedom:

$$\int_e \widehat{\mathbf{q}}\cdot\boldsymbol{\tau}\,\mathrm{d}L = \begin{cases} \int_e \mathbf{q}\cdot\boldsymbol{\tau}\,\mathrm{d}L & \text{if fine edge } e \in T, \\ 0 & \text{if } e \notin T. \end{cases}$$

Note that $\nabla\times\widehat{\mathbf{q}} = 0$ for any point in $\widehat{\Omega}$ (since tangential component of $\mathbf{q}$ vanishes on ∂T). Let $\widehat{S}_h$ be the space of scalar continuous piecewise linear functions associated with the mesh $\widehat{\mathcal{T}}_h$. Since by assumption $\widehat{\Omega}$ is homeomorphic to a ball, there exists a function $\widehat{\theta} \in \widehat{S}_h$ such that $\nabla\widehat{\theta} = \widehat{\mathbf{q}}$. The graph made of boundary fine edges and boundary fine vertices of T is connected (by Assumption 3.1), and for each fine edge $e \in \partial T$ we have

$$\int_e \nabla\widehat{\theta}\cdot\boldsymbol{\tau}\,\mathrm{d}L = \int_e \widehat{\mathbf{q}}\cdot\boldsymbol{\tau}\,\mathrm{d}L = \int_e \mathbf{q}\cdot\boldsymbol{\tau}\,\mathrm{d}L = 0.$$

Because of this and formula (3.2), the function $\widehat{\theta}$ assumes the same constant value c at all fine vertices on ∂T. Now, consider the function $\theta \in \widetilde{S}_h$ defined as follows:

$$\theta(v) = \begin{cases} \widehat{\theta}(v) - c & \text{if the fine vertex } v \in T, \\ 0 & \text{if } v \notin T. \end{cases}$$

We have $\theta \in S_h(T)$ and $\nabla\theta = \mathbf{q}$. $\square$

Lemma 7.33. *For any agglomerate T and any $\theta \in \widetilde{S}_h$, we have*

$$\nabla \mathcal{I}_T^S \theta = \mathcal{I}_T^Q \nabla\theta. \tag{7.17}$$

Proof. The proof is analogous to that of Lemma 7.13. Consider the linear system for the unknowns $\sigma \in S_h(T)$ and $\mathbf{z} \in Q_h(T)$,

$$\begin{aligned} (\sigma,\tau)_T - (\nabla\tau,\mathbf{z})_T &= -(\theta,\tau)_T && \text{for all } \tau \in S_h(T), \\ (\nabla\sigma,\mathbf{p})_T + (\nabla\times\mathbf{z},\nabla\times\mathbf{p})_T &= -(\nabla\theta,\mathbf{p})_T && \text{for all } \mathbf{p} \in Q_h(T). \end{aligned} \tag{7.18}$$

Due to Lemmas 7.32 and 4.1, the system is solvable. To apply Lemma 4.1, take $V_1 = S_h(T)$, $V_2 = Q_h(T)$, $V_3 = R_h(T)$, $\mathrm{d}_1 = \nabla$, and $\mathrm{d}_2 = \nabla\times$.

Since for any $\eta \in S_h(T)$ we have $\mathbf{p} = \nabla\eta \in Q_h(T)$ and $\nabla\times\mathbf{p} = 0$, σ also solves (7.16). The latter equation has the unique solution $\theta_T = \mathcal{I}_T^S\theta$, thus $\sigma = \theta_T$. We now see that the pair ($\mathbf{q}_T = \nabla\theta_T, \mathbf{r}_T = -\nabla\times\mathbf{z}$) solves the first equation of (6.43) for $\mathbf{q} = \nabla\theta$. The second equation of (6.43) is trivially satisfied since $\nabla\times\mathbf{q}_T = \nabla\times\mathbf{q} = 0$. Thus, $\mathbf{q}_T = \mathcal{I}_T^Q\mathbf{q}$, which is (7.17). $\square$

Note that in a computer implementation we can solve the system (7.16), and not necessarily (7.18).

Lemma 7.34. *For any function $\theta \in \widetilde{S}_h$, we have*

$$\nabla\Big(\mathbb{1} + \sum_T \mathcal{I}_T^S\Big)\theta = \Big(\mathbb{1} + \sum_T \mathcal{I}_T^Q\Big)\nabla_2\theta. \tag{7.19}$$

Proof. The proof is essentially the same as that of Lemma 7.15. $\square$

Theorem 7.35 (Main commutativity property)**.** *For any $\theta \in \widetilde{S}_h$, we have*

$$\nabla \Pi^S \theta = \Pi^Q \nabla \theta.$$

Proof. We have

$$\begin{aligned}\nabla \Pi^S \theta &= \nabla \Big(\mathbb{1} + \sum_T \mathcal{I}_T^S\Big) \Pi_2^S \theta \overset{(7.19)}{=} \Big(\mathbb{1} + \sum_T \mathcal{I}_T^Q\Big) \nabla_2 \Pi_2^S \theta \\ &\overset{(7.13)}{=} \Big(\mathbb{1} + \sum_T \mathcal{I}_T^Q\Big) \Pi_2^Q \nabla \theta = \Pi^Q \nabla \theta. \qquad \square\end{aligned}$$

Corollary 7.36 (Exactness of coarse spaces)**.** *If $\nabla \widetilde{S}_h = \ker(\nabla \times) \cap \widetilde{Q}_h$, then*

$$\nabla \widetilde{S}_H = \ker(\nabla \times) \cap \widetilde{Q}_H.$$

Proof. Let $\mathbf{q} \in \ker(\nabla \times) \cap \widetilde{Q}_H$. Since $\widetilde{Q}_H \subset \widetilde{Q}_h$, there exists an $\eta \in \widetilde{S}_h$ such that $\mathbf{q} = \nabla \eta$. Since $\mathbf{q} \in \widetilde{Q}_H$,

$$\mathbf{q} = \Pi^Q \mathbf{q} = \Pi^Q \nabla \eta = \nabla \Pi^S \eta \in \nabla \widetilde{S}_H.$$

Conversely, let $\mathbf{q} \in \nabla \widetilde{S}_H$, i.e., $\mathbf{q} = \nabla \eta$, $\eta \in \widetilde{S}_H$. Clearly, $\mathbf{q} \in \ker(\nabla \times)$. We also have

$$\mathbf{q} = \nabla \eta = \nabla \Pi^S \eta = \Pi^Q \nabla \eta \in \widetilde{Q}_H. \qquad \square$$

Corollary 7.37 (Local approximation property)**.** *If on a coarse element T a function $\eta \in \widetilde{S}_h$ equals an affine (linear) function $\xi = \mathbf{c} \cdot \mathbf{x} + d \in P_1(T)$, so does the projection $\Pi^S \eta$.*

Proof. Let $\nabla \xi = \mathbf{c}$, where $\mathbf{c}$ is a vector constant. We have $\pi_T^Q \nabla \eta = \pi_T^Q \mathbf{c}$. Using Theorems 7.35 and 6.38 we can see that

$$\pi_T^Q \nabla \Pi^S \eta = \pi_T^Q \Pi^Q \nabla \eta = \pi_T^Q \mathbf{c}.$$

In other words, gradients of $\Pi^S \eta$ and ξ coincide on T. By construction, $\Pi^S \eta$ and ξ assume the same value at each coarse vertex of T, and T has at least one coarse vertex (each agglomerate has at least one coarse face, each coarse face is required to have boundary fine edges, each boundary fine edge is part of some coarse edge and each coarse edge has two coarse vertices as endpoints). If we consider the graph made of fine vertices and fine edges of T, this graph is connected (since the dual graph of T is required to be connected, and each tetrahedron has connected boundary). Consequently, ξ and $\Pi^S \eta$ have to agree on all of T. $\square$

8. Numerical Results

In this section we provide a number of tests that illustrate the approximation properties of the coarse spaces proposed in the paper. The PDEs we solve are the Poisson equation in mixed form (i.e., Darcy flow), two discretizations of the vector Laplace operator, namely, the $H(\mathrm{curl}) - H(\mathrm{div})$ and $H^1 - H(\mathrm{curl})$ formulations, and also the second order scalar elliptic equation $-\Delta u + u = f$ (which we discretize using H^1-conforming FE space).

All these PDEs can be written in the form $Lu = f$. We compute a corresponding right-hand side $f = Lu$ for a given exact solution u. For a given tetrahedral mesh $\mathcal{T}_h$, using

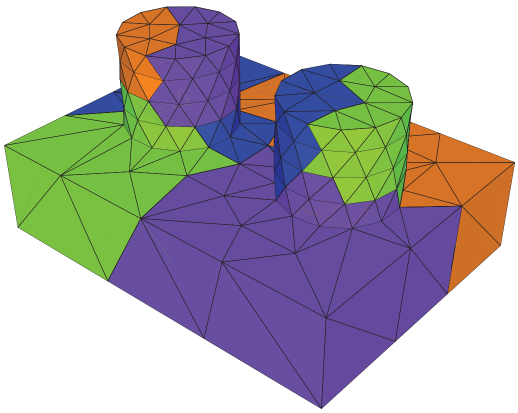

Figure 1. Agglomerates for box–cylinder domain.

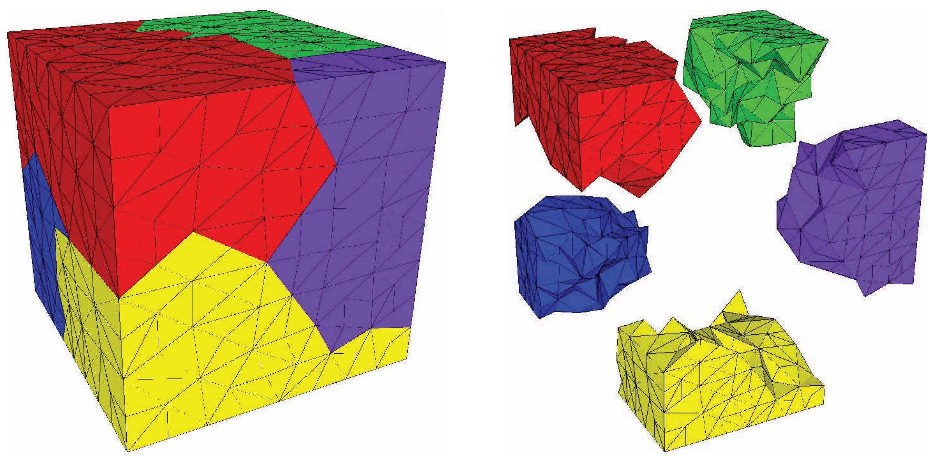

Figure 2. Agglomerates for unit cube with unstructured mesh.

element agglomeration, we create a coarse mesh $\mathcal{T}_H$ (consisting of agglomerates). We then compute the solution u_H of the Galerkin discretization of $Lu = f$ using the proposed coarse spaces associated with the agglomerates in $\mathcal{T}_H$.

We study the behavior of the error $e_H = u - u_H$, as the original mesh $\mathcal{T}_h$ is refined. That is, we refine $\mathcal{T}_h$ to obtain another tetrahedral mesh $\mathcal{T}_{h_2}$ ($h_2 = h/2$), build the corresponding coarse mesh $\mathcal{T}_{H_2}$, compute the Galerkin approximation u_{H_2} and the error $e_{H_2} = u - u_{H_2}$. In a similar way, we refine $\mathcal{T}_{h_2}$ to obtain $\mathcal{T}_{h_3}$, build the corresponding coarse mesh $\mathcal{T}_{H_3}$, compute $e_{H_3} = u - u_{H_3}$ and so forth. As we proceed, we keep the average number of tetrahedrons per agglomerate fixed, i.e., in a sense the ratio H_k/h_k stays approximately the same. We then measure the convergence rate $\log_2(\|e_{H_k}\|/\|e_{H_{k+1}}\|)$, $k = 0, 1, 2, \ldots$

In all experiments involving mixed systems, we use the initial unstructured mesh on the box–cylinder domain (domain scaled to fit into unit cube) seen in Figure 1. For the last H^1-tests, we also use the unit cube with unstructured mesh as shown in Figure 2.

8.1. Numerical Results for the Coarse Raviart–Thomas Space

The test PDE here is Darcy flow $u = \nabla p$, $\nabla \cdot u = f$ with exact solution

$$p = \cos(\pi x)\cos(\pi y)\cos(\pi z)$$

and variational boundary conditions.

# times refined	u (flux): $\|\cdot\|_{L_2}$ conv. rate	u: $\|\cdot\|_{H(\text{div})}$ conv. rate	p (pressure): $\|\cdot\|_{L_2}$ conv. rate
0	—	—	—
1	1.02	0.59	0.58
2	1.05	0.90	0.90
3	1.04	0.82	0.82
4	1.02	0.97	0.97

Table 1. Results for coarse Raviart–Thomas spaces.

We solve the *coarse* mixed system, interpolate the flux and pressure to the fine grid and measure the errors with respect to the exact solution. We observe the error behavior as we refine the original mesh (this increases the number of agglomerates, since we keep the agglomerate size constant). Some details for this set of experiments read:

- The finest mesh has 5,715,200 faces and 2,834,432 elements.
- We have approximately 77 fine-grid elements per agglomerate. The size of the coarse-grid mixed system is approximately 5% of that of the fine-grid system. The number of non-zeros in the coarse-grid matrix is at most 28% of that of the fine-grid one.

As it is clearly seen from Table 1, the expected first order of approximation is confirmed.

8.2. Numerical Results for Coarse Nédélec and Raviart–Thomas Spaces

Here, we consider $H(\text{curl})$-$H(\text{div})$ mixed formulation of the vector Laplacian

$$\begin{gathered}\sigma = \nabla\times u \text{ in } \Omega,\\ -\nabla\times\sigma + \nabla(\nabla\cdot u) = f \text{ in } \Omega,\\ u\times\mathbf{n} \text{ and } \nabla\cdot u \text{ are given on } \partial\Omega,\end{gathered}$$

where Ω is the same box-cylinder domain as in the previous tests. The exact solution is

$$u = \begin{bmatrix}\cos(\pi x)\sin(\pi y)\sin(\pi z)\\ \cos(\pi x)\cos(\pi y)\sin(\pi z)\\ \cos(\pi x)\cos(\pi y)\cos(\pi z)\end{bmatrix}.$$

As before, using Galerkin coarsening with our AMGe-constructed coarse Nédélec space to approximate σ and the respective coarse Raviart–Thomas space to approximate u, we form the coarse-grid mixed system.

We have the following characteristics of the coarse discretization:

- About 77 elements per agglomerate.
- After the last refinement, the fine mesh has 430,892 edges, 720,192 faces, 354,304 elements.
- The size of the coarse-grid mixed system is at most 10% of that of the fine-grid one and the number of non-zeros in coarse-grid matrix is at most 60% of the fine-grid one.

As expected, Table 2 demonstrates the first order of approximation.

# times refined	σ: $\|\cdot\|_{L_2}$ conv. rate	σ: $|\cdot|_{H(\mathrm{curl})}$ conv. rate	u: $\|\cdot\|_{L_2}$ conv. rate	u: $|\cdot|_{H(\mathrm{div})}$ conv. rate
0	—	—	—	—
1	0.87	0.74	0.78	0.58
2	0.84	0.83	0.86	0.80
3	0.98	0.92	0.97	0.91

Table 2. Tests for coarse Nédélec and Raviart–Thomas spaces.

8.3. Numerical Results for the Pair Coarse H^1-Conforming and Nédélec Spaces

We consider the H^1-$H(\mathrm{curl})$ mixed formulation of the vector Laplacian

$$\begin{gathered}\sigma = -\nabla\cdot u,\\ -\nabla\sigma - \nabla\times\nabla\times u = f \text{ in } \Omega,\\ u\cdot\mathbf{n} \text{ and } (\nabla\times u)\times\mathbf{n} \text{ are given on } \partial\Omega,\end{gathered}$$

again posed on the box-cylinder domain. The exact solution is

$$u = \begin{bmatrix}\cos(\pi x)\sin(\pi y)\sin(\pi z)\\ \cos(\pi x)\cos(\pi y)\sin(\pi z)\\ \cos(\pi x)\cos(\pi y)\cos(\pi z)\end{bmatrix}.$$

We use the AMGe-constructed coarse H^1-conforming space to approximate σ and the coarse Nédélec space to approximate u. Here, we keep each agglomerate to contain about 450 fine-grid elements.

The characteristics of the discretization are:

- After the last refinement, the fine-grid mesh has 495,897 vertices, 3,376,664 edges, 5,715,200 faces and 2,834,432 elements.
- The size of the coarse-grid mixed system is at most 10% of that of the fine-grid one.
- The number of non-zeros in the coarse-grid matrix is at most 52% of the fine-grid one.

As seen from Table 3, the expected first order of approximation for the coarse Nédélec space is clearly demonstrated. The same holds for the H^1-norm of the error and close to second order in the L_2-norm (note that the domain we use is not convex). In the next subsection, we use, in addition to the box-cylinder domain, a convex domain, for which we have full-regularity of the solution of the Laplace equation.

8.4. Numerical Results for Coarse H^1-Conforming Spaces

We solve the scalar equation $-\Delta p + p = f$, first on the box-cylinder domain with exact solution $p = \cos(\pi x)\cos(\pi y)\cos(\pi z)$.

The AMGe-constructed coarse H^1-conforming space is used in standard Galerkin approximation procedure to generate the coarse discrete system. In the agglomeration procedure, we keep about 679 fine-grid elements per agglomerate.

As before, Table 4 demonstrates first order of approximation in energy and less than second order in L_2 of the AMGe-constructed coarse H^1-conforming spaces.

# times refined	σ: $\|\cdot\|_{L_2}$ conv. rate	σ: $|\cdot|_{H^1}$ conv. rate	u: $\|\cdot\|_{L_2}$ conv. rate	u: $|\cdot|_{H(\mathrm{curl})}$ conv. rate
0	—	—	—	—
1	0.65	0.40	0.57	0.43
2	1.59	0.77	1.17	1.02
3	1.63	0.86	0.91	0.93
4	1.87	0.97	1.02	1.00

Table 3. Tests with pairs of coarse H^1 and coarse Nédélec spaces.

# times refined	p: $\|\cdot\|_{L_2}$ conv. rate	p: $|\cdot|_{H^1}$ conv. rate
0	—	—
1	0.47	0.24
2	1.12	0.49
3	1.66	0.92
4	1.78	0.93

Table 4. Tests with coarse H^1-conforming spaces for box-cylinder domain.

# times refined	p: $\|\cdot\|_{L_2}$ conv. rate	p: $|\cdot|_{H^1}$ conv. rate
0	—	—
1	0.17	0.12
2	1.46	0.74
3	2.02	1.02
4	1.98	0.99

Table 5. Tests with coarse H^1-conforming spaces for unit cube with unstructured fine-grid mesh.

Next, we solve the same problem, $-\Delta p + p = f$, now on the unit cube with unstructured mesh. The difference from the previous test is in the better behavior of the L_2-error, as seen in Table 5.

In the last test, the number of non-zeros in the coarse-grid matrix is about 80% of that in fine-grid matrix, and the size of the coarse space is about 10 times smaller than the fine-grid one.

9. Conclusions

In the present paper, we have extended the approach from [8], to complete the de Rham sequence of finite element spaces, so that the resulting coarse spaces contain locally (on each agglomerated element) either the vector constants (in the case of Nédélec and Raviart–Thomas spaces) or the affine functions in the case of H^1-conforming spaces. The results hold if the agglomerates satisfy certain topological restrictions. We proved that the coarse de Rham sequence is exact. Moreover, the coarse projection operators we constructed satisfy important commutativity relations with the respective differential operators. Our numerical results confirm the expected first order of approximation of the coarse spaces, a result that makes them suitable for numerical upscaling, i.e., to be used as an accurate discretization tool (in addition to being used in multigrid solvers) for general unstructured meshes.

The methodology we developed is naturally extendable to higher order elements and also to adaptively incorporate into the coarse spaces any given set of fine-grid functions, details of which will be given in a follow-up paper.

A. Appendix

Here, we provide a proof (in a more general setting) of Lemma 6.13.

Lemma A.1. *Let U, V, and $\widetilde{U}$ be three finite dimensional spaces such that $V \subset \widetilde{U}$ and $U \subset \widetilde{U}$. Also, consider a given symmetric and positive definite bilinear form $a(\cdot,\cdot) : \widetilde{U} \times \widetilde{U}$. Let $P_V : \widetilde{U} \mapsto V$ and $P_U : \widetilde{U} \mapsto U$ be the respective Galerkin ($a(\cdot,\cdot)$-orthogonal) projections. Consider the eigenproblem*

$$P_V P_U P_V q = \sigma^2 q, \quad q \in V. \tag{A.1}$$

For a given tolerance $\gamma \in (0,1)$, select all eigenvalues σ_i^2, $i = 1,\dots,m$, such that

$$\sigma^2 \leqslant 1 - \gamma,$$

and define the space $D \subset V$ to be spanned by the respective eigenvectors q_i, $i = 1,\dots,m$. Finally, let $P_D : \widetilde{U} \mapsto D$ be the Galerkin ($a(\cdot,\cdot)$-orthogonal) projection onto D. Then the bilinear form $a((I - P_D)\mathbf{u}, \mathbf{u})$ is coercive on U and the following coercivity estimate holds:

$$a((I - P_D)\mathbf{u}, \mathbf{u}) \geqslant \gamma a(\mathbf{u}, \mathbf{u}) \quad \textit{for all } \mathbf{u} \in U.$$

Proof. We need to show that

$$\frac{a(P_D\mathbf{u}, \mathbf{u})}{a(\mathbf{u}, \mathbf{u})} \leqslant 1 - \gamma \quad \text{for all } \mathbf{u} \in U.$$

Introduce bases in U and V represented by the columns of the respective matrices $\mathcal{P}_U$ and $\mathcal{P}_V$. Then if $a(\cdot,\cdot)$ gives rise to a symmetric positive definite matrix A (in some basis of $\widetilde{U}$), we have the subspace matrices $A_U = \mathcal{P}_U^T A \mathcal{P}_U$, $A_V = \mathcal{P}_V^T A \mathcal{P}_V$, and $A_D = \mathcal{P}_D^T A \mathcal{P}_D$, where $\mathcal{P}_D = [q_1, \dots, q_m]$. Using the above matrix notation, we have to show that

$$\mathbf{u}^T P_U^T A P_D A_D^{-1} P_D^T A P_U \mathbf{u} \leqslant (1 - \gamma)\ \mathbf{u}^T A_U \mathbf{u}.$$

Equivalently, using the fact that C^T and $C = A_D^{-\frac{1}{2}} \mathcal{P}_D^T A \mathcal{P}_U A_U^{-\frac{1}{2}}$ have the same norms, we need to show that

$$\mathbf{d}^T \mathcal{P}_D^T A \mathcal{P}_U A_U^{-1} \mathcal{P}_U^T A \mathcal{P}_D \mathbf{d} \leqslant (1 - \gamma)\ \mathbf{d}^T A_D \mathbf{d}. \tag{A.2}$$

Due to our choice of $\mathcal{P}_D = [q_1, \dots, q_m]$, we have, rewriting (A.1) in matrix form,

$$\mathbf{d}^T \mathcal{P}_D^T A \mathcal{P}_V A_V^{-1} \mathcal{P}_V^T A \mathcal{P}_U A_U^{-1} \mathcal{P}_U^T A \mathcal{P}_V A_V^{-1} \mathcal{P}_V^T A \mathcal{P}_D \mathbf{d} \leqslant (1 - \gamma)\ \mathbf{d}^T A_D \mathbf{d}. \tag{A.3}$$

We can choose $\mathcal{P}_V = [\mathcal{P}_D, \mathcal{P}_{D^\perp}]$, that is, the second block represents $D^\perp$, the A-orthogonal complement of D in V. Then

$$A_V = \mathcal{P}_V^T A \mathcal{P}_V = \begin{bmatrix} A_D & 0 \\ 0 & A_{D^\perp} \end{bmatrix}, \quad \text{where } A_{D^\perp} = (\mathcal{P}_{D^\perp})^T A \mathcal{P}_{D^\perp},$$

and

$$\mathcal{P}_V^T A \mathcal{P}_D = \begin{bmatrix} A_D \\ 0 \end{bmatrix}.$$

Hence

$$A \mathcal{P}_V A_V^{-1} \mathcal{P}_V^T A \mathcal{P}_D = A[\mathcal{P}_D, \mathcal{P}_{D^\perp}] \begin{bmatrix} A_D^{-1} & 0 \\ 0 & A_{D^\perp}^{-1} \end{bmatrix} \begin{bmatrix} A_D \\ 0 \end{bmatrix} = A[\mathcal{P}_D, \mathcal{P}_{D^\perp}] \begin{bmatrix} I \\ 0 \end{bmatrix} = A \mathcal{P}_D.$$

That is, estimate (A.3) simplifies to

$$\mathbf{d}^T P_D^T A \mathcal{P}_U A_U^{-1} \mathcal{P}_U^T A \mathcal{P}_D \mathbf{d} \leqslant (1 - \gamma)\ \mathbf{d}^T A_D \mathbf{d},$$

which is exactly the desired estimate (A.2). $\square$

Acknowledgments

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