Research Article

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A Posteriori Error Bounds for Approximations of the Oseen Problem and Applications to the Uzawa Iteration Algorithm

Abstract: We derive computable bounds of deviations from the exact solution of the stationary Oseen problem. They are applied to approximations generated by the Uzawa iteration method. Also, we derive an advanced form of the estimate, which takes into account approximation errors arising due to discretization of the boundary value problem, generated by the main step of the Uzawa method. Numerical tests confirm our theoretical results and show practical applicability of the estimates.

Keywords: Oseen Problem, Estimates of Deviations from Exact Solutions, Uzawa Iteration Method

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1 Introduction

We consider the stationary Oseen problem in a bounded connected domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz continuous boundary $\partial\Omega$. Throughout the paper, we use the following notation: n denotes the outward unit normal vector to the boundary $\partial\Omega$; the space of scalar valued square summable functions with zero mean is denoted by $\tilde{L}^2(\Omega)$; $S_0(\Omega)$ denotes the closure of smooth solenoidal functions with compact supports in Ω with respect to the norm of $V(\Omega) := H^1(\Omega, \mathbb{R}^d)$; and $V_0(\Omega)$ denotes the subspace of $V(\Omega)$ that consists of the functions with zero traces on $\partial\Omega$.

Also, we use spaces of tensor valued functions $\Sigma(\Omega) := L^2(\Omega, \mathbb{M}^{d\times d})$, where $\mathbb{M}^{d\times d}$ is the space of $d \times d$-matrices (tensors). $\mathbb{I}$ denotes the unit element of $\mathbb{M}^{d\times d}$. The L^2-norms of scalar and vector valued functions are denoted by $\|\cdot\|$ and the corresponding inner products are denoted by $(\,,\,)$. The scalar product of tensors is denoted by two dots (:), and the norm of Σ is denoted by $\|\cdot\|_\Sigma$. By div and Div, we denote the divergence of vector and tensor fields, respectively. Finally, we introduce the Hilbert space

$$\Sigma(\mathrm{Div}, \Omega) := \{w \in \Sigma(\Omega) \mid \mathrm{Div}\, w \in L^2(\Omega, \mathbb{R}^d)\},$$

which can be viewed as a tensor analogous to the vector space $H(\Omega, \mathrm{div})$ containing L^2 vector functions with square summable divergence.

The classical formulation of the stationary Oseen problem is to find the velocity field $u \in S_0(\Omega) + u_D$ and the pressure function $p \in \tilde{L}^2(\Omega)$, which satisfy the relations

$$-\mathrm{Div}(\nu\nabla u) + \mathrm{Div}(a \otimes u) = f - \nabla p \quad \text{in } \Omega, \tag{1.1}$$

$$\mathrm{div}\, u = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$u = u_D \quad \text{on } \partial\Omega, \tag{1.3}$$

where a, u_D, and f are given vector valued functions. It is assumed that

$$\int_{\partial\Omega} u_D \cdot n \, \mathrm{d}x = 0, \tag{1.4}$$

that the viscosity ν is a positive bounded function, i.e., $0 < \underline{\nu} \le \nu(x) \le \bar{\nu}$ for all $x \in \overline{\Omega}$, and that $a \in S_0(\Omega)$ is a bounded vector function.

The generalized solution of (1.1)–(1.4) is a function $u \in S_0(\Omega) + u_D$ such that

$$\int_\Omega (\nu \nabla u : \nabla w - (a \otimes u) : \nabla w)\, \mathrm{d}x = \int_\Omega f \cdot w \, \mathrm{d}x \quad \text{for all } w \in S_0(\Omega). \tag{1.5}$$

Existence and uniqueness of generalized solutions to the Stokes and Oseen problems are well established (see, e.g., [9]). In essence, the corresponding results are based on the following lemma.

Lemma 1.1. *For any function $g \in \tilde{L}^2(\Omega)$, there exists a function $v \in V_0(\Omega)$ satisfying the condition* $\operatorname{div} v = g$ *such that*

$$\|\nabla v\|_\Sigma \le \kappa_\Omega \|g\|.$$

Here κ_Ω is a positive constant depending only on the domain Ω.

We note that the constant inverse to κ_Ω arises in the so-called Ladyzhenskaya–Babuška–Brezzi (LBB) condition (see, e.g., [2, 4]), which can be viewed as a different form of Lemma 1.1. Also, these results guarantee boundedness of the energy norm of the exact solution, namely, $|||u||| := \|\nu^{1/2}\nabla u\|_\Sigma \le c$, where the constant c depends on the problem data and on the constant $C_{F\Omega}$ in the Friedrichs type inequality

$$\|w\| \le C_{F\Omega} \|\nabla w\|_\Sigma \quad \text{for all } w \in V_0.$$

The constants $C_{F\Omega}$ and κ_Ω play an important role in our analysis because they control distances between a vector valued function and the set of solenoidal fields evaluated in different norms (see [12, 15–17]). In particular, Lemma 1.1 implies an important corollary: for any $v \in V_0(\Omega)$ there exists $v_0 \in S_0(\Omega)$ such that

$$\|\nabla(v - v_0)\|_\Sigma \le \kappa_\Omega \|\operatorname{div} v\|. \tag{1.6}$$

A similar estimate holds for $v \in V_0(\Omega) + u_D$ with some $v_0 \in S_0(\Omega) + u_D$.

If functions vanish on the whole boundary, then a guaranteed upper bound of $C_{F\Omega}$ is easy to find. For some domains the constant $C_{\mathrm{LBB}} = \kappa_\Omega^{-1}$ or computable bounds for it can be found if the field satisfies some additional requirements (see, e.g., [5, 8, 13, 18, 20]).

In [16, 17], guaranteed and fully computable bounds of the distance between the exact solution of the stationary Stokes problem and any function in $V_0(\Omega)+u_D$ were derived by transformations of integral relations similar to (1.5). If the function compared with u is an approximation, then these estimates yield robust and efficient a posteriori error bounds (for the Stokes problem, they were numerically tested in [6, 7], see also [11]). In [20], analogous estimates were derived for the generalized Stokes problem. In Section 2 of the present paper, we use the same ideas in order to derive estimates of the distance to the exact solution of (1.1)–(1.4). We obtain estimates for the velocity, pressure, and stress fields. In Section 3, similar estimates are derived for the combined error norm, which encompasses errors of approximations related to all fields. In Section 4, the estimates are applied to approximations generated by the Uzawa algorithm. Section 5 contains results of numerical tests, which confirm practical applicability and efficiency of the estimates.

2 Estimates of Deviations from the Exact Velocity Field

Theorem 2.1. *Let $v \in V_0(\Omega) + u_D$. Then for all $q \in \tilde{L}^2(\Omega)$ and $\tau \in \Sigma(\Omega)$ we have*

$$|||u - v||| \le \underline{\nu}^{-1/2} \|r(\tau)\|_{-1,\Omega} + \|\nu^{-1/2} d(v, \tau, q)\|_\Sigma + (2\bar{\nu}^{1/2} + C_\Omega)\kappa_\Omega \|\operatorname{div} v\| := M_\oplus(v, \tau, q), \tag{2.1}$$

where

$$r(\tau) := f + \operatorname{Div} \tau, \quad C_\Omega = C_{F\Omega} \|\nu^{-1/2} a\|_{\infty,\Omega}, \quad d(v, \tau, q) := \tau - \nu \nabla v + a \otimes v + \mathbb{I} q,$$

and

$$\|r(\tau)\|_{-1,\Omega} := \sup_{w \in V_0(\Omega)} \frac{\int_\Omega (f \cdot w - \tau : \nabla w)\, \mathrm{d}x}{\|\nabla w\|_\Sigma}.$$

Proof. For any $v_0 \in S_0(\Omega) + u_D$, we have

$$|||u - v||| \le |||u - v_0||| + |||v_0 - v|||. \tag{2.2}$$

First, we estimate from above the first term of the right-hand side of (2.2). Let $w \in S_0(\Omega)$. By subtracting the integral

$$\int_\Omega \big(\nu\nabla v_0 : \nabla w - (a \otimes v_0) : \nabla w\big)\,\mathrm{d}x$$

from both sides of (1.5), we obtain

$$\int_\Omega \big(\nu\nabla(u - v_0) : \nabla w - (a \otimes (u - v_0)) : \nabla w\big)\,\mathrm{d}x = \int_\Omega \big(f \cdot w - \nu\nabla v_0 : \nabla w + (a \otimes v_0) : \nabla w\big)\,\mathrm{d}x. \tag{2.3}$$

For any $\tau \in \Sigma(\Omega)$ and $q \in \tilde{L}^2(\Omega)$ we rewrite the right-hand side and estimate it as follows:

$$\begin{aligned}
&\int_\Omega \big(f \cdot w - \tau : \nabla w + (\tau - \nu\nabla v_0 + a \otimes v_0 + \mathbb{I}q) : \nabla w\big)\,\mathrm{d}x \\
&\quad = \int_\Omega \big(r(\tau) \cdot w + d(v_0, \tau, q) : \nabla w\big)\,\mathrm{d}x \\
&\quad \le \|r(\tau)\|_{-1,\Omega}\|\nabla w\|_\Sigma + \|\nu^{-1/2} d(v_0, \tau, q)\|_\Sigma \|\nu^{1/2}\nabla w\|_\Sigma \\
&\quad \le \big(\underline{\nu}^{-1/2}\|r(\tau)\|_{-1,\Omega} + \|\nu^{-1/2} d(v_0, \tau, q)\|_\Sigma\big)\,\|\nu^{1/2}\nabla w\|_\Sigma.
\end{aligned} \tag{2.4}$$

Set $w = u - v_0$. Since

$$\int_\Omega (a \otimes (u - v_0)) : \nabla(u - v_0)\,\mathrm{d}x = 0,$$

the estimates (2.4) and (2.3) yield

$$|||u - v_0||| \le \underline{\nu}^{-1/2}\|r(\tau)\|_{-1,\Omega} + \|\nu^{-1/2} d(v_0, \tau, q)\|_\Sigma. \tag{2.5}$$

Now, we estimate the second term in the right-hand side of (2.5). We have

$$\|\nu^{-1/2} d(v_0, \tau, q)\|_\Sigma \le \|\nu^{1/2}\nabla(v_0 - v)\|_\Sigma + \|\nu^{-1/2} d(v, \tau, q)\|_\Sigma + \|\nu^{-1/2} a \otimes (v_0 - v)\|_\Sigma.$$

Note that

$$\|\nu^{-1/2} a \otimes (v_0 - v)\|_\Sigma \le \|\nu^{-1/2} a\|_{\infty,\Omega}\|v_0 - v\|,$$

where

$$\|\nu^{-1/2} a\|_{\infty,\Omega} := \max_{i=1,\dots,d} \sup_{x\in\Omega}\{\nu^{-1/2} a_i\}.$$

We find that

$$\|\nu^{-1/2} d(v_0, \tau, q)\|_\Sigma \le (\bar{\nu}^{1/2} + C_\Omega)\|\nabla(v_0 - v)\|_\Sigma + \|\nu^{-1/2} d(v, \tau, q)\|_\Sigma.$$

Hence,

$$|||u - v||| \le (2\bar{\nu}^{1/2} + C_\Omega)\|\nabla(v_0 - v)\|_\Sigma + \|\nu^{-1/2} d(v, \tau, q)\|_\Sigma + \underline{\nu}^{-1/2}\|r(\tau)\|_{-1,\Omega}.$$

In view of (1.6), we finally obtain (2.1). □

Remark 2.2. If $\tau \in \Sigma(\mathrm{Div}, \Omega)$, then it is easy to show that

$$\|r(\tau)\|_{-1,\Omega} \le C_{F\Omega}\|r(\tau)\|.$$

In this case, (2.1) is reduced to the majorant derived for the Oseen problem in [17].

3 Estimate of Deviations from the Exact Pressure and Stress Fields

Let $q \in \tilde{L}^2(\Omega)$ be a function considered as an approximation of the exact pressure p. Then $(p-q) \in \tilde{L}^2(\Omega)$ and due to Lemma 1.1 there exists a function $\bar{w} \in V_0(\Omega)$ such that

$$\operatorname{div}(\bar{w}) = p - q \tag{3.1}$$

and

$$\|\nabla \bar{w}\|_\Sigma \le \kappa_\Omega \|p - q\|. \tag{3.2}$$

As in the case of the Stokes problem (see [16, 17]), this fact allows us to deduce computable majorants of $\|p-q\|$.

Theorem 3.1. *Let $q \in \tilde{L}^2(\Omega)$. Then for all $\tau \in \Sigma(\Omega)$*

$$\frac{1}{\kappa_\Omega}\|p-q\| \le C_{a,\nu}|||u-v||| + \|d(v,\tau,q)\|_\Sigma + \|r(\tau)\|_{-1,\Omega}, \tag{3.3}$$

where $C_{a,\nu} := (\bar{\nu}^{1/2} + \underline{\nu}^{-1/2} C_{F\Omega}\|a\|_{\infty,\Omega})$ and $|||u-v|||$ is estimated by (2.1).

Proof. From (3.1) we have

$$\|p-q\|^2 = \int_\Omega \operatorname{div}\bar{w}(p-q)\,\mathrm{d}x = \int_\Omega (\operatorname{div}\bar{w}\, p - q\mathbb{I} : \nabla\bar{w})\,\mathrm{d}x. \tag{3.4}$$

Multiplying (1.1) by $\bar{w}$ and integrating over Ω, we obtain

$$\int_\Omega \operatorname{div}\bar{w}\, p\,\mathrm{d}x = \int_\Omega \big(\nu\nabla u : \nabla\bar{w} - (a\otimes u) : \nabla\bar{w} - f\cdot\bar{w}\big)\,\mathrm{d}x. \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$\|p-q\|^2 \le \int_\Omega \big((\nu\nabla v - \tau - a\otimes v - \mathbb{I}q) : \nabla\bar{w}\big)\,\mathrm{d}x + \|r(\tau)\|_{-1,\Omega}\|\nabla\bar{w}\| + \int_\Omega \big(\nu\nabla(u-v) : \nabla\bar{w} - a\otimes(u-v) : \nabla\bar{w}\big)\,\mathrm{d}x.$$

Here

$$\begin{aligned}\int_\Omega \big(\nu\nabla(u-v) : \nabla\bar{w} - a\otimes(u-v) : \nabla\bar{w}\big)\,\mathrm{d}x &\le \bar{\nu}^{1/2}|||u-v|||\|\nabla\bar{w}\|_\Sigma + \|a\|_{\infty,\Omega}\|u-v\|\|\nabla\bar{w}\|_\Sigma \\ &\le \kappa_\Omega\big(\bar{\nu}^{1/2} + \underline{\nu}^{-1/2}C_{F\Omega}\|a\|_{\infty,\Omega}\big)|||u-v|||\|p-q\|\end{aligned}$$

and in view of (3.2) we have

$$\int_\Omega d(v,\tau,q) : \nabla\bar{w}\,\mathrm{d}x \le \|d(v,\tau,q)\|_\Sigma\|\nabla\bar{w}\|_\Sigma \le \kappa_\Omega\|d(v,\tau,q)\|_\Sigma\|p-q\|.$$

Thus, we arrive at the estimate (3.3). □

The exact solution generates the tensor

$$\sigma := \nu\nabla u - a\otimes u - p\mathbb{I}.$$

Assume that $v \in V_0(\Omega) + u_D$, $\eta \in \Sigma(\Omega)$, and $q \in \tilde{L}^2(\Omega)$ approximate u, σ, and p, respectively. Then,

$$\begin{aligned}\|\eta-\sigma\|_\Sigma = \|\eta - \nu\nabla u + a\otimes u + p\mathbb{I}\|_\Sigma &\le \|\eta - \nu\nabla v + a\otimes v + q\mathbb{I}\|_\Sigma + \|\nu\nabla(u-v)\|_\Sigma + \|a\|_{\infty,\Omega}\|u-v\| + \sqrt{d}\|p-q\| \\ &\le \|d(v,\eta,q)\|_\Sigma + C_{a,\nu}|||u-v||| + \sqrt{d}\|p-q\|.\end{aligned}$$

By (3.3) we obtain

$$\|\eta-\sigma\|_\Sigma \le \sqrt{d}\kappa_\Omega\|d(v,\tau,q)\|_\Sigma + \|d(v,\eta,q)\|_\Sigma + \sqrt{d}\kappa_\Omega\|r(\tau)\|_{-1,\Omega} + (1+\sqrt{d}\kappa_\Omega)C_{a,\nu}M_\oplus(v,\tau,q), \tag{3.6}$$

where $C_{a,\nu}$ is defined in Theorem 3.1.

Remark 3.2. If we choose $\eta = \nu\nabla v - a \otimes v - q\mathbb{I}$, then $\|d(v, \eta, q)\|_\Sigma = 0$.

Also, we can measure the error in terms of the norm of the product space

$$W := (V_0(\Omega) + u_D) \times \tilde{L}^2(\Omega) \times \Sigma(\Omega),$$

which is

$$\|(v, q, \eta)\|_W := |||v||| + \|q\| + \|\eta\|_\Sigma.$$

Combining the estimates (2.1), (3.3) and (3.6) we find that

$$\|(u - v, p - q, \eta - \sigma)\|_W \le c_\oplus M_\oplus(v, \tau, q) + \|d(v, \eta, q)\|_\Sigma,$$

where

$$c_\oplus := 1 + (\kappa_\Omega + \sqrt{d}\kappa_\Omega)(C_{a,\nu} + \max\{1, C_{F\Omega}\}).$$

4 Error Estimates for Approximate Solutions Generated by the Uzawa Algorithm

Uzawa type algorithms are commonly used for solving various saddle point problems (see, e.g., the survey article [3]). They are widely used in numerical analysis of incompressible media. In our case, the algorithm can be used in the following form:

(1) Set $k = 0$ and $p^0 \in \tilde{L}^2(\Omega)$.

(2) Find $u^k \in V_0(\Omega) + u_D$ such that

$$\int_\Omega (\nu\nabla u^k : \nabla w - (a \otimes u^k) : \nabla w)\,\mathrm{d}x = \int_\Omega (f \cdot w + p^k \operatorname{div} w)\,\mathrm{d}x \quad \text{for all } w \in V_0. \tag{4.1}$$

(3) Find

$$p^{k+1} = p^k - \rho \operatorname{div} u^k, \quad \text{where } \rho \in (0, \bar{\rho}). \tag{4.2}$$

(4) Set $k = k + 1$ and go to step (2).

It is well known (see, e.g., [21]) that approximations generated by the Uzawa algorithm converge (as $k \to \infty$) in the sense that

$$u^k \to u \quad \text{in } V(\Omega, \mathbb{R}^d), \quad p^k \to p \quad \text{weakly in } L^2(\Omega)$$

provided that

$$0 < \rho < \bar{\rho} := 2\underline{\nu}.$$

Our first goal is to deduce computable and realistic estimates of $u^k - u$ and $p^k - p$ in terms of the respective norms.

For this purpose, we use results of previous sections. We set

$$v = u^k, \quad q = p^k, \quad \tau = \nu\nabla u^k - a \otimes u^k - \mathbb{I}p^k.$$

In this case,

$$d(v, \tau, q) := \tau - \nu\nabla u^k + a \otimes u^k + \mathbb{I}p^k = 0$$

and in view of (4.1),

$$\|r(\tau)\|_{-1,\Omega} := \sup_{w \in V_0(\Omega)} \frac{\int_\Omega (f \cdot w - (\nu\nabla u^k - a \otimes u^k - \mathbb{I}p^k) : \nabla w)\,\mathrm{d}x}{\|\nabla w\|_\Sigma} = 0.$$

We use the estimate (2.1) and arrive at the following result.

Theorem 4.1. *Let u^k be the exact solution computed in the k-th step of the Uzawa algorithm. Then*

$$|||u - u^k||| \le (2\bar{\nu}^{1/2} + C_\Omega)\kappa_\Omega \|\operatorname{div} u^k\| := M_\oplus^{Uz}(u^k) \tag{4.3}$$

and

$$\|p - p^k\| \le \kappa_\Omega C_{a,\nu} M_\oplus^{Uz}(u^k). \tag{4.4}$$

Remark 4.2. Since

$$|||u^k - u|||^2 \ge \frac{1}{d}\,\underline{\nu}\,\|\operatorname{div} u^k\|^2 =: M_\ominus^{Uz}(u^k),$$

we find that

$$M_\ominus^{Uz}(u^k) \le |||u - u^k||| \le M_\oplus^{Uz}(u^k). \tag{4.5}$$

This means that the efficiency index of the majorant is bounded by an explicitly computable constant, namely

$$I^{\text{eff}}\big(M_\oplus^{Uz}(u^k)\big) := \frac{M_\oplus^{Uz}(u^k)}{|||u - u^k|||} \le I_\oplus^{\text{eff}}, \tag{4.6}$$

where

$$I_\oplus^{\text{eff}} := \frac{M_\oplus^{Uz}(u^k)}{M_\ominus^{Uz}(u^k)} \le \frac{2\bar{\nu}^{1/2} + C_{F\Omega}\|\nu^{-1/2}a\|_{\infty,\Omega}}{\underline{\nu}^{1/2}}\sqrt{d}\kappa_\Omega.$$

We note that $\kappa_\Omega \ge \frac{1}{\sqrt{d}}$, so that $I_\oplus^{\text{eff}} \ge 1$ (which of cause also follows directly from (4.5) and (4.6)).

In particular, for the Stokes problem with constant ν the ratio is smaller than $2\sqrt{d}\kappa_\Omega$. The estimate (4.5) shows that the quantity $\|\operatorname{div} u^k\|$ reliably controls convergence of u^k to u in V.

The estimates (4.3) and (4.4) are of theoretical relevance. In practice, the problem (4.1) is solved numerically on a certain mesh $\mathcal{T}_h$, whose cells have the characteristic size h. For this case, we need an advanced form of the error majorant, which is derived below.

Let $V_{0h}(\Omega, \mathbb{R}^d)$ and $\tilde{L}_h^2(\Omega)$ be finite dimensional subspaces of $V_0(\Omega)$ and $\tilde{L}^2(\Omega)$, respectively. We also assume that the spaces are constructed so that the corresponding numerical problem is stable and satisfies the discrete LBB-condition.

Let $u_h^k \in V_{0h} + u_D$ be an approximation of u^k calculated in the k-th step (4.1) of the Uzawa algorithm and $p_h^k, p_h^{k+1} \in \tilde{L}_h^2(\Omega)$ be approximations of the pressure related to step (4.2). Our goal is to derive a fully computable error majorant for the pair (u_h^k, p_h^k) generated in step k.

Theorem 4.3. *For any $\eta \in \Sigma(\Omega, \operatorname{Div})$,*

$$|||u - u_h^k||| \le \mathcal{E}^h(u_h^k, p_h^k, \eta) + M_\oplus^{Uz}(u_h^k) := M_\oplus^{Uz}(u_h^k, p_h^k, \eta),$$

where the first term

$$\mathcal{E}(u_h^k, p_h^k, \eta) = \underline{\nu}^{-1/2}\big(C_{F\Omega}\|r(\eta)\|_\Omega + \|d(u_h^k, \eta, p_h^k)\|_\Sigma\big)$$

is related to the approximation error and the second term presents the error associated with the Uzawa method.

Analogously,

$$\frac{1}{\kappa_\Omega}\|p - p_h^k\| \le (1 + C_{a,\nu})\mathcal{E}(u_h^k, p_h^k, \eta) + C_{a,\nu}M_\oplus^{Uz}(u_h^k).$$

Proof. We set

$$v = u_h^k, \quad q = p_h^k, \quad \tau = \nu\nabla u_h^k - a \otimes u_h^k - \mathbb{I}p_h^k$$

and use the estimate (2.1). In this case, $d(u_h^k, \tau, p_h^k) = 0$ and

$$\|r(\tau)\|_{-1,\Omega} := \sup_{w \in V_0(\Omega)} \frac{\int_\Omega (f \cdot w - (\nu\nabla u_h^k - a \otimes u_h^k - \mathbb{I}p_h^k) : \nabla w)\,\mathrm{d}x}{\|\nabla w\|_\Sigma}.$$

Let $\eta \in \Sigma(\operatorname{Div}, \Omega)$. Then,

$$\|r(\eta)\|_{-1,\Omega} = \sup_{w \in V_0(\Omega)} \frac{\int_\Omega ((f + \operatorname{Div}\eta) \cdot w - (d(u_h^k, \eta, p_h^k)) : \nabla w)\,\mathrm{d}x}{\|\nabla w\|_\Sigma} \le C_{F\Omega}\|f + \operatorname{Div}\eta\|_\Omega + \|d(u_h^k, \eta, p_h^k)\|_\Sigma.$$

Hence, we arrive at estimate (4.3). Now, we use (3.3) and find that

$$\frac{1}{\kappa_\Omega}\|p - p_h\| \le C_{a,\nu}|||u - u_h^k||| + C_{F\Omega}\|f + \operatorname{Div}\eta\|_\Omega + \|d(u_h^k, \eta, p_h^k)\|_\Sigma. \qquad \square$$

Remark 4.4. Analogously to (4.5), we find that

$$M_\ominus^{Uz}(u_h^k) \le |||u - u_h^k||| \le M_\oplus^{Uz}(u_h^k) + \mathcal{E}(u_h^k, p_h^k, \eta), \tag{4.7}$$

which means that the guaranteed efficiency index of the error majorant is subject to similar estimates, namely

$$I^{\text{eff}}\big(M_\oplus^{Uz}(u_h^k)\big) \le I_\oplus^{\text{eff}} + I_k^{\text{eff}}, \tag{4.8}$$

where the second term

$$I_k^{\text{eff}} := \frac{\mathcal{E}(u_h^k, p_h^k, \eta)}{|||u - u_h^k|||}$$

represents the efficiency index associated with the approximation error.

We end up this section with a short comment on practical applications of the estimate (4.7). First we note that it has the form which is natural to expect. It is clear that the quality of error estimation related to solving the boundary value problem (4.1) by means of a certain numerical method should enter the estimate and increase the overall value of the majorant (cf. (4.8)). In the numerical tests presented below, we indeed observed this effect. In these examples, the function η was defined by means of very simple (and very cheap) reconstructions of the numerical stress (based on local averaging) and, therefore, the term $\mathcal{E}(u_h^k, p_h^k, \eta)$ made a considerable contribution to the overall error bound. Nevertheless, the majorant correctly reflects the decreasing of the error in the process of the Uzawa iterations. Certainly more sophisticated stress reconstruction procedures (e.g., global minimization) would lead to much better results (see a consequent discussion of the corresponding methods in [11]). However, even if the approximation error would be defined sharply, for sufficiently large k the right-hand side of (4.7) will not decrease because the mesh $\mathcal{T}_h$ is too coarse for getting approximations with a required accuracy. In practice, this "saturation" phenomenon is easily detected by comparing the values of two terms forming the majorant (in our tests this phenomenon was observed). This means that fully reliable computations based on the Uzawa type methods require "modeling-discretization" adaptive algorithms in the spirit of, e.g., [19].

5 Numerical Experiments

Below we present results from numerical computations performed to test the majorants and minorants. Approximations for model problem were calculated with MINI-elements [1] for the velocity field, linear triangular elements for the pressure field, and linear Raviart–Thomas elements [14] for the stress field.

We consider the Oseen problem with $\nu = 1$ in $\Omega = [0,1] \times [0,1]$ and homogeneous Dirichlet boundary conditions. The exact velocity

$$u(x,y) = \begin{pmatrix} 20x^2y(2y-1)(x-1)^2(y-1) \\ -20xy^2(2x-1)(x-1)(y-1)^2 \end{pmatrix}$$

and the pressure $p(x,y) = 2x - 1$ generate the right-hand side of the equation. The iterations were started with $p^0 = 0$ in Ω. Computations were performed with the help of FEniCS Project open source software [10]. Uniform refinements of the mesh were performed if the majorant for the velocity field shows that practically the error does not decrease (if the absolute value of difference between the values computed for two consecutive iterations was less than 10%). At the very beginning we had 512 elements. At every refinement one triangle element was divided into four similarly shaped triangles (so that we had 2048 degrees of freedom after the first refinement, and then 8192 after the second refinement). The algorithm was stopped after the third mesh refinement.

k	$\frac{M_{\oplus}(v,\tau,q)}{\|v\|}$	$\frac{\|\operatorname{div} v\|}{\|v\|}$	$\frac{\|d(v,\tau,q)\|}{\|v\|}$	$\frac{\|r(\tau)\|}{\|v\|}$	$\frac{\|(u-v,p-q,\eta-\sigma)\|_W}{\|(v,q,\eta)\|_W}$	$\frac{c_{\oplus}M_{\oplus}(v,\tau,q)}{\|(v,q,\eta)\|_W}$
6	0.814861	0.108033	0.186701	6.14041×10^{-5}	0.18877	3.3947
9	0.394441	0.0533815	0.0840590	3.76671×10^{-6}	0.0921263	1.61059
12	0.197608	0.0266837	0.0424575	2.34323×10^{-7}	0.0509854	0.798602

Table 1. Components of the majorant, $a = (1, 0)$.

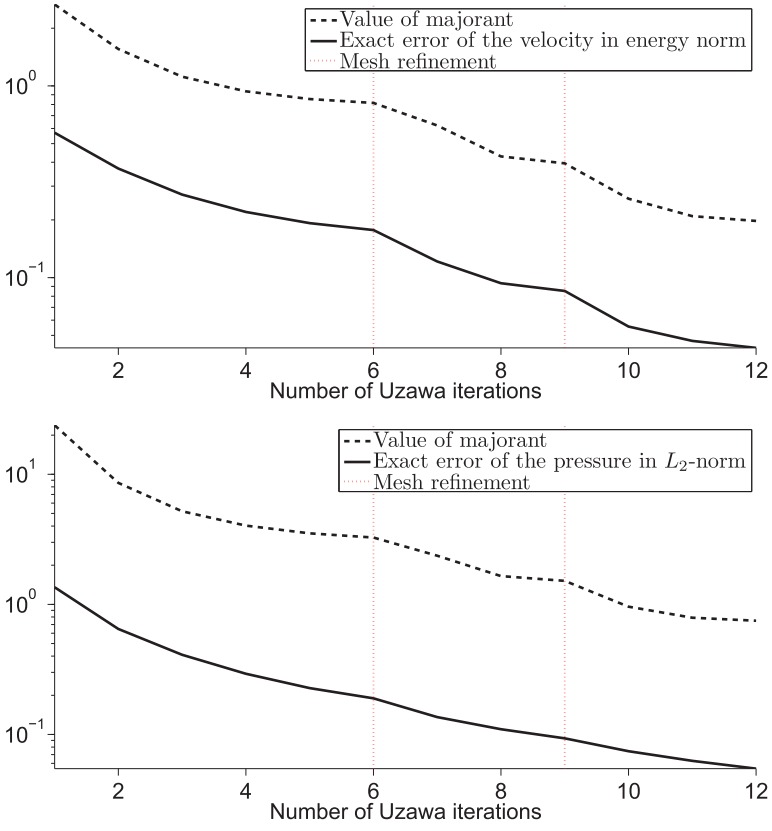

Figure 1. Behavior of the majorants, $a = (1, 0)$.

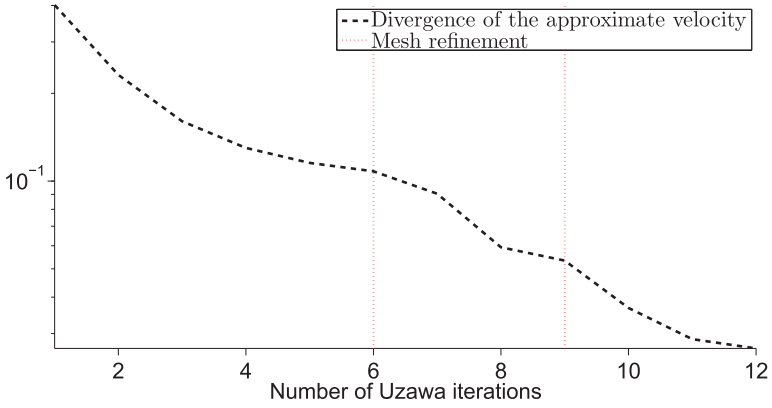

Figure 2. Divergence of the approximate velocity, $a = (1, 0)$.

k	$\frac{M_\oplus(v,\tau,q)}{\|v\|}$	$\frac{\|\operatorname{div} v\|}{\|v\|}$	$\frac{\|d(v,\tau,q)\|}{\|v\|}$	$\frac{\|r(\tau)\|}{\|v\|}$	$\frac{\|(u-v,p-q,\eta-\sigma)\|_W}{\|(v,q,\eta)\|_W}$	$\frac{c_\oplus M_\oplus(v,\tau,q)}{\|(v,q,\eta)\|_W}$
6	0.816497	0.108253	0.187053	6.13097×10^{-5}	0.187327	3.36203
9	0.394957	0.0534651	0.0840883	3.76496×10^{-6}	0.0913371	1.59401
12	0.197734	0.0267046	0.0424624	2.34283×10^{-7}	0.0505014	0.78988

Table 2. Components of the majorant, $a = (1, 1)$.

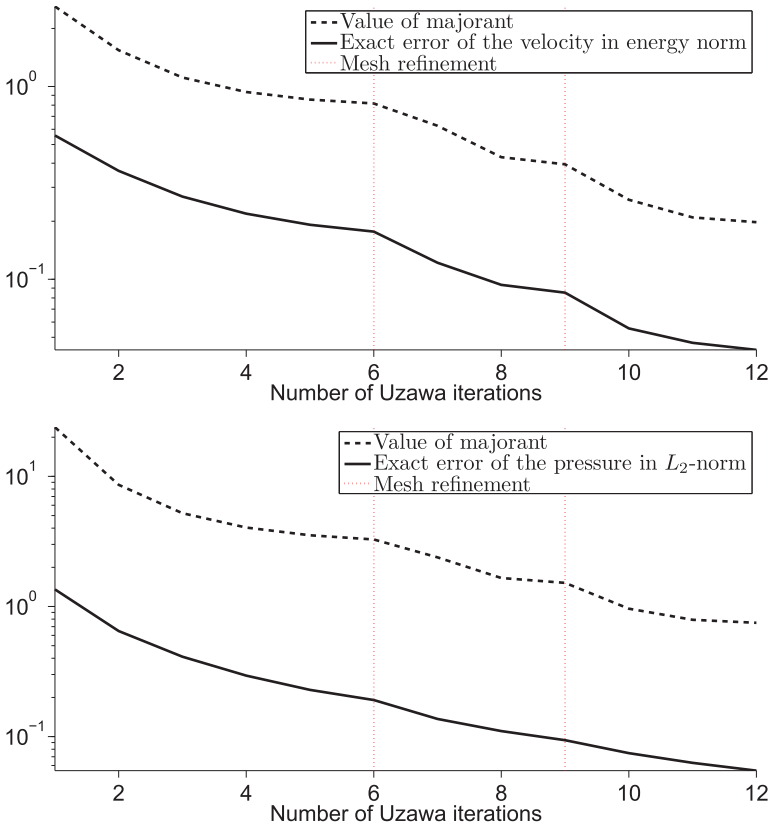

Figure 3. Behavior of the majorants, $a = (1, 1)$.

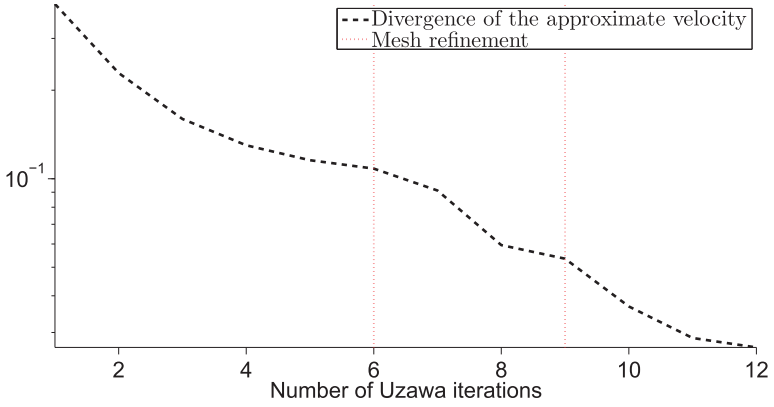

Figure 4. Divergence of the approximate velocity, $a = (1, 1)$.

k	$\frac{M_{\oplus}(v,\tau,q)}{\|v\|}$	$\frac{\|\operatorname{div} v\|}{\|v\|}$	$\frac{\|d(v,\tau,q)\|}{\|v\|}$	$\frac{\|r(\tau)\|}{\|v\|}$	$\frac{\|(u-v,p-q,\eta-\sigma)\|_W}{\|(v,q,\eta)\|_W}$	$\frac{c_{\oplus}M_{\oplus}(v,\tau,q)}{\|(v,q,\eta)\|_W}$
5	0.78383	0.114972	0.182948	6.12371×10^{-5}	0.219877	3.03241
8	0.372967	0.0553391	0.0837532	3.76496×10^{-6}	0.102283	1.39201
11	0.185204	0.0273769	0.0421268	2.34320×10^{-7}	0.0542261	0.680833

Table 3. Components of the majorant, $a = (0, 0)$.

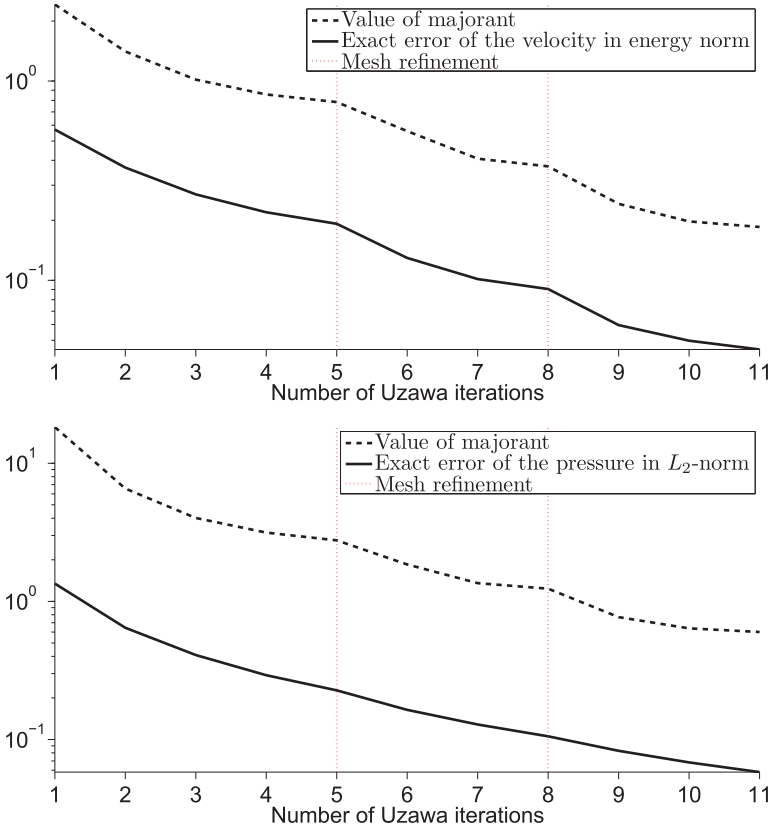

Figure 5. Behavior of the majorants, $a = (0, 0)$.

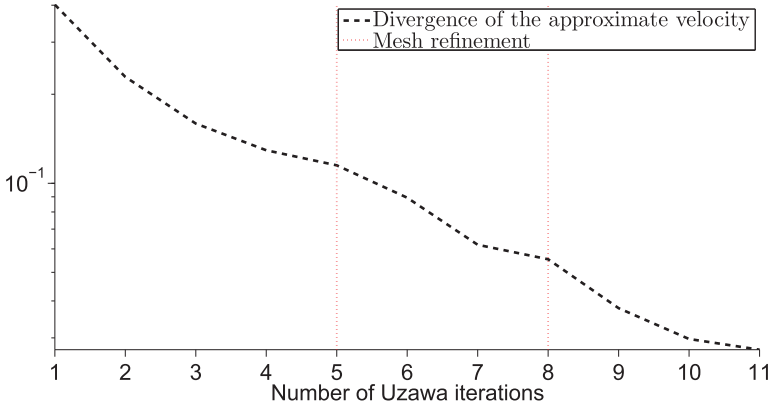

Figure 6. Divergence of the approximate velocity, $a = (0, 0)$.

We tested the algorithm for different a. Below, we focus attention on three examples, which present typical results. We set $a = (1, 0)$, $a = (1, 1)$ and $a = (0, 0)$ (this case corresponds to the Stokes problem). Values for majorants and exact errors are shown in Figure 1 for $a = (1, 0)$, Figure 3 for $a = (1, 1)$ and Figure 5 for $a = (0, 0)$. In Figures 2, 4 and 6 we show how the norm of the divergence decreases in the process of Uzawa iterations. For the velocity field, errors are calculated in the energy norm and for the pressure in the L^2-norm. Values of the majorants and exact errors for the velocity and pressure are normalized with the norms $|||v|||$ and $\|q\|$, respectively. Dotted vertical lines mark the iterations after which mesh refinements were done. In the examples, the "free" function τ was computed by minimization of the majorant on the same mesh that was used for the velocity field. Also, we can compute guaranteed bounds on the errors in terms of stresses and the combined primal-dual norm (see Table 1 for $a = (1, 0)$, Table 2 for $a = (1, 1)$ and Table 3 for $a = (0, 0)$). We see that the estimates indeed provide guaranteed upper bounds of errors in the functions computed by means of the Uzawa iterations. These bounds correctly reflect decrease of the corresponding errors and indicate the moment when adaptation of the mesh is required.

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