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Review Article

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The Tailored Finite Point Method

Abstract: In this paper, a brief review of tailored finite point methods (TFPM) is given. The TFPM is a new approach to construct the numerical solutions of partial differential equations. The TFPM has been tailored based on the local properties of the solution for each given problem. Especially, the TFPM is very efficient for solutions which are not smooth enough, e.g., for solutions possessing boundary/interior layers or solutions being highly oscillated. Recently, the TFPM has been applied to singular perturbation problems, the Helmholtz equation with high wave numbers, the first-order wave equation in high frequency cases, transport equations with interface, second-order elliptic equations with rough or highly oscillatory coefficients, etc.

Keywords: Tailored Finite Point Method, Singular Perturbation Problem, Boundary/Interior Layer, Discrete Maximum Principle, High Frequency Waves, Discrete-Ordinate Transport Equation, Multiscale Elliptic Problem

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1 Introduction

The tailored finite point method (TFPM) is a new approach to constructing discrete numerical schemes for the solutions of the differential equations. Especially, TFPM is very efficient for the solutions which are not smooth enough, for example, when the solutions possess boundary/interior layers or the solutions are highly oscillatory.

The TFPM provides a new point of view for designing the discrete numerical schemes for the solution of a given differential equation with suitable boundary and/or initial conditions. At each given interior point, the TFPM scheme is constructed based on the properties of the solution of the given problem. Therefore, the main properties of the solution can be preserved in the numerical scheme of the TFPM in some sense.

At first, the tailored finite point method was proposed by H. Han, Z. Huang and R. B. Kellogg [20, 21] for solving the Hemker problem numerically. This is an open problem proposed by P. Hemker [27]. The international conference BAIL2008 awarded the Pieter Hemker Prize to Han, Huang and Kellogg for their contribution to the goal of designing the best computational algorithm for the Hemker problem [21].

The Hemker problem is a typical singular perturbation problem of a second-order elliptic equation with constant coefficients in two dimensions. The solution of the Hemker problem possesses boundary and interior layers. The computational algorithm given by the TFPM for the Hemker problem can achieve good accuracy with a very coarse mesh whenever the perturbed parameter ε is very small or large, and the numerical solution can capture the boundary/interior layers, even though the mesh is very coarse. To solve numerically singular perturbation problems of second-order elliptic equations, the TFPM was applied systematically by Han and Huang [15–17, 19, 31], Shih, Kellogg and Chang [55], Shih, Kellogg and Tsai [56]. Han and Huang proposed the TFPM scheme for the numerical solution of a singular perturbation problem of fourth-order elliptic equation [18], and an iterative TFPM scheme of which was given by Han, Huang and Zhang [23]. Hsieh, Shih and Yang [29] proposed a TFPM scheme for solving the steady magnetohydrodynamic (MHD) Duct flow problem with boundary layers, which is a singularly perturbed system of second-order elliptic equations. A uniformly convergent semi-discrete TFPM for a class of anisotropic diffusion problems was proposed by Han, Huang and Ying [22]. A parameter-uniform TFPM for a singularly perturbed linear ODE system with multiple perturbation parameters was obtained by Han, Miller and Tang [24]. Two parameter-uniform tailored finite point schemes for two-dimensional discrete ordinary differential equations with boundary layers and interfaces were given by Han, Tang and Ying [25], in which the problem can be treated as a singularly perturbed problem of a system of ordinary differential equations.

In addition to singularly perturbed problems, TFPM has been applied to various other fields as a new approach for the numerical solution of differential equations. For example, a uniformly convergent TFPM for the one-dimensional Helmholtz equation with high wave numbers in heterogenous medium was obtained by Han and Huang [15], and a tailored finite cell method for solving the Helmholtz equation in layered heterogenous medium appears in Huang and Yang [32]. The TFPM was applied to interface problems by Huang [30] and the first-order wave equation by Huang and Yang [33]. The multi-scale TFPM for second-order elliptic equations with rough or highly oscillatory coefficients was given by Han and Zhang [26], in which a class of multi-scale problems was studied.

The TFPM provides new ideas and a new perspective when constructing numerical schemes for differential equations. The numerical scheme at each point is tailored/constructed based on some properties of the solution of the given problem at that point. In many cases, the schemes given by TFPM preserve important properties of the solution of the given problem. Therefore, even on a very coarse mesh, the numerical solutions given by TFPM can still capture the important properties of the given problem. The TFPM is now at the development stage. Further applications of the method are expected in the future.

The rest of this paper is organized as follows. First, we describe the principle of the TFPM in Section 2. Then we review the TFPM for singular perturbation problems in Section 3, for wave problems in Section 4, for transport equations in Section 5, and for multiscale elliptic problems in Section 6. Finally, we give a short summary in Section 7.

2 The Principle of the Tailored Finite Point Method

In this section, a couple of examples are given to explain the principle of the tailored finite point method.

At first, we start from the five-point and nine-point difference schemes for the Laplace equation

$$-\Delta u = 0, \tag{2.1}$$

to explain the basic idea of the tailored finite point method.

Take a point $\mathbf{x}^0$ and four points around it (cf. Figure 1):

$$\mathbf{x}^1 = (h_1, 0), \quad \mathbf{x}^2 = (0, h_2), \quad \mathbf{x}^3 = (-h_3, 0), \quad \mathbf{x}^4 = (0, -h_4),$$

where $h_i > 0$ $(i = 1, 2, 3, 4)$ are given and $0 < h_i \ll 1$.

We try to find a five-point scheme for the Laplace equation (2.1) at the point $\mathbf{x}^0$, namely, find five numbers α_i $(i = 0, 1, \dots, 4)$ such that

$$\alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \alpha_4 u_4 = 0$$

approximates the Laplace equation at the point $\mathbf{x}^0$, where u_i $(i = 0, 1, \dots, 4)$ denote the approximate values of $u(\mathbf{x})$, the solution of Laplace equation (2.1), at the point $\mathbf{x}^i$. How do we find the constants α_i $(i = 0, 1, \dots, 4)$?

The solution $u(\mathbf{x})$ of the equation (2.1) can be expanded into a sum of harmonic polynomials around the point $\mathbf{x}^0 = (0, 0)$:

$$u(\mathbf{x}) = c_0 + c_1 x_1 + c_2 x_2 + c_3(x_1^2 - x_2^2) + c_4 x_1 x_2 + c_5(x_1^3 - 3x_1 x_2^2) + c_6(x_2^3 - 3x_1^2 x_2) + \cdots .$$

At the cell $\Omega_0 = \{(x_1, x_2) \mid -h_3 \le x_1 \le h_1,\ -h_4 \le x_2 \le h_2\}$, the solution $u(\mathbf{x})$ can be approximated by the first few terms, for example,

$$u(\mathbf{x}) - \{c_0 + c_1 x_1 + c_2 x_2 + c_3(x_1^2 - x_2^2) + c_4 x_1 x_2\} = \mathcal{O}(h^3) \quad \text{for all } \mathbf{x} \in \Omega_0$$

with $h = \max_{1 \le i \le 4} h_i$.

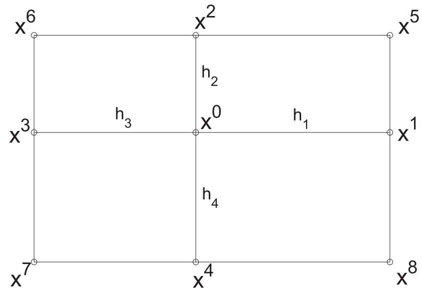

Figure 1. The location of the points $\mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^8$.

To find the constants α_i $(i = 0, 1, \dots, 4)$, we take the subspace of the harmonic polynomials

$$W_I^4 = \{v(\mathbf{x}) \mid v(\mathbf{x}) = c_0 + c_1 x_1 + c_2 x_2 + c_3(x_1^2 - x_2^2) \text{ for all } c_i \in \mathbb{R},\ i = 0, 1, 2, 3\}.$$

We then choose the constants α_i $(i = 0, 1, \dots, 4)$ such that

$$\alpha_0 v(\mathbf{x}^0) + \alpha_1 v(\mathbf{x}^1) + \alpha_2 v(\mathbf{x}^2) + \alpha_3 v(\mathbf{x}^3) + \alpha_4 v(\mathbf{x}^4) = 0 \quad \text{for all } v(\mathbf{x}) \in W_I^4.$$

Taking $v(\mathbf{x}) = 1$, x_1, x_2 and $x_1^2 - x_2^2$, respectively, we arrive at

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -\alpha_0, \quad h_1\alpha_1 - h_3\alpha_3 = 0, \quad h_2\alpha_2 - h_4\alpha_4 = 0, \quad h_1^2\alpha_1 - h_2^2\alpha_2 + h_3^2\alpha_3 - h_4^2\alpha_4 = 0. \tag{2.2}$$

Solving system (2.2), we obtain

$$\alpha_1 = -\frac{h_2h_3h_4}{(h_1+h_3)(h_1h_3+h_2h_4)}\alpha_0, \qquad \alpha_2 = -\frac{h_1h_3h_4}{(h_2+h_4)(h_1h_3+h_2h_4)}\alpha_0,$$
$$\alpha_3 = -\frac{h_1h_2h_4}{(h_1+h_3)(h_1h_3+h_2h_4)}\alpha_0, \qquad \alpha_4 = -\frac{h_1h_2h_3}{(h_2+h_4)(h_1h_3+h_2h_4)}\alpha_0.$$

In practice, if we take $h_1 = h_2 = h_3 = h_4 = h$, we have

$$\alpha_j = -\frac{1}{4}\alpha_0, \quad j = 1, 2, 3, 4.$$

Letting $\alpha_0 = 4/h^2$, from a new point of view, we reconstruct the famous five-point difference scheme for the Laplace equation (2.1),

$$-\frac{u_1 + u_2 + u_3 + u_4 - 4u_0}{h^2} = 0.$$

Second, we design a five-point scheme (with $h_1 = h_2 = h_3 = h_4 = h$) for the equation

$$-\varepsilon\Delta u + \partial_{x_1} u = 0 \tag{2.3}$$

with a small perturbation parameter $0 < \varepsilon \ll 1$, which was considered in the Hemker problem [27]. For any solution $u(x_1, x_2, \varepsilon)$ of the equation (2.3), let

$$v(x_1, x_2, \varepsilon) = e^{-\frac{x_1}{2\varepsilon}} u(x_1, x_2, \varepsilon).$$

Then $v(x_1, x_2, \varepsilon)$ satisfies

$$-\Delta v + \frac{1}{\mu^2} v = 0 \tag{2.4}$$

with $\mu = \frac{1}{2\varepsilon}$. The solution $v(x_1, x_2, \varepsilon)$ of equation (2.4) can be expanded at $\mathbf{x} = \mathbf{x}^0$ as

$$v(x_1, x_2, \varepsilon) = a_0 I_0(\mu r) + \sum_{n=1}^{\infty} I_n(\mu r)\big(a_n \cos n\theta + b_n \sin n\theta\big),$$

where (r, θ) is the polar coordinate of $\mathbf{x} = (x_1, x_2)$ based at $\mathbf{x}^0$, and I_n is the n-th order modified Bessel function of first kind.

Since the solution $u(x_1, x_2, \varepsilon)$ of (2.3) satisfies

$$u(x_1, x_2, \varepsilon) = e^{\mu x_1} v(x_1, x_2, \varepsilon),$$

u can be expanded around $\mathbf{x} = \mathbf{x}^0$ as

$$u(\mathbf{x}) = b_0 + e^{\mu r \cos\theta}\Big\{a_0 I_0(\mu r) + \sum_{n=1}^{\infty} I_n(\mu r)(a_n \cos n\theta + b_n \sin n\theta)\Big\}. \tag{2.5}$$

We take the first few terms in (2.5) to construct a four-dimensional function space

$$W_{II}^4 = \{u \mid u = b_0 + e^{\mu r \cos\theta}[a_0 I_0(\mu r) + a_1 I_1(\mu r)\cos\theta + b_1 I_1(\mu r)\sin\theta] \text{ for all } a_0, a_1, b_0, b_1 \in \mathbb{R}\}.$$

We seek some constants α_i ($i = 0, 1, \dots, 4$) to construct a five-point scheme for the equation (2.3) at the five points $\mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^4$, namely

$$\alpha_1 u(x^1) + \alpha_2 u(x^2) + \alpha_3 u(x^3) + \alpha_4 u(x^4) + \alpha_0 u(x^0) = 0.$$

The constants α_i ($i = 0, 1, \dots, 4$) are determined by

$$\alpha_1 u(x^1) + \alpha_2 u(x^2) + \alpha_3 u(x^3) + \alpha_4 u(x^4) + \alpha_0 u(x^0) = 0 \quad \text{for all } u \in W_{II}^4. \tag{2.6}$$

In equation (2.6), taking $u = 1$, $e^{\mu r\cos\theta} I_0(\mu r)$, $e^{\mu r\cos\theta} I_1(\mu r)\cos\theta$, $e^{\mu r\cos\theta} I_1(\mu r)\sin\theta$, respectively, yields

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -\alpha_0, \quad I_0(\mu h)(e^{\mu h}\alpha_1 + \alpha_2 + e^{-\mu h}\alpha_3 + \alpha_4) = -\alpha_0, \quad \alpha_2 - \alpha_4 = 0, \quad e^{\mu h}\alpha_1 - e^{-\mu h}\alpha_3 = 0. \tag{2.7}$$

Setting

$$\beta(\mu h) = \frac{\cosh(\mu h) - I_0(\mu h)}{I_0(\mu h) - 1},$$

then from (2.7) we have

$$\alpha_1 = -\frac{e^{-\mu h}}{2(\cosh(\mu h) + \beta(\mu h))}\alpha_0 \equiv -\alpha_1(\mu h)\alpha_0, \qquad \alpha_2 = -\frac{\beta(\mu h)}{2(\cosh(\mu h) + \beta(\mu h))}\alpha_0 \equiv -\alpha_2(\mu h)\alpha_0, \tag{2.8}$$

$$\alpha_3 = -\frac{e^{\mu h}}{2(\cosh(\mu h) + \beta(\mu h))}\alpha_0 \equiv -\alpha_3(\mu h)\alpha_0, \qquad \alpha_4 = \alpha_2 \equiv -\alpha_4(\mu h)\alpha_0, \quad \text{with } \alpha_2(\mu h) = \alpha_4(\mu h). \tag{2.9}$$

In this manner we obtain a new five-point scheme for equation (2.3), which has been called the five-point tailored finite point scheme for this equation:

$$-\alpha_1(\mu h)u_1 - \alpha_2(\mu h)u_2 - \alpha_3(\mu h)u_3 - \alpha_4(\mu h)u_4 + u_0 = 0. \tag{2.10}$$

The coefficients $\alpha_j(\mu h)$ ($j = 1, 2, 3, 4$) given by (2.8) and (2.9) contain the Bessel function $I_0(\mu h)$ and exponential functions $e^{\mu h}, e^{-\mu h}$. By an integral representation of $I_0(x)$ (see [14]), we have

$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} \cosh(x\cos\theta)d\theta \quad \text{and} \quad I_0(x) - 1 > 0 \quad \text{for all } x > 0.$$

Hence, $\beta(\mu h) > 0$ for $\mu h > 0$, and

$$\alpha_j(\mu h) > 0 \quad \text{for } \mu h > 0,\ j = 1, 2, 3, 4 \quad \text{and} \quad \sum_{j=1}^{4}\alpha_j(\mu h) = 1.$$

Furthermore, a nine-point scheme of equation (2.3) can be constructed at the nine points $\mathbf{x}^j$ ($j = 0, 1, \dots, 8$) (see Figure 1). In this case we take the 8-dimensional function space

$$W_{II}^8 = \Big\{u \mid u = b_0 + e^{\mu r\cos\theta}\Big[a_0 I_0(\mu r) + \sum_{n=1}^{3} I_n(\mu r)(a_n \cos n\theta + b_n \sin n\theta)\Big] \text{ for all } a_n, b_n \in \mathbb{R}\Big\}.$$

The nine-point scheme we seek is of the form

$$\sum_{j=0}^{8} \beta_j u_j = 0,$$

where the constants β_j ($j = 0, 1, \dots, 8$) are determined by

$$\sum_{j=0}^{8} \beta_j v(\mathbf{x}^j) = 0 \quad \text{for all } v \in W_{II}^8. \tag{2.11}$$

Set

$$\begin{aligned} A(\mu h) &= I_0(\sqrt{2}\mu h) - \cosh(\mu h), \quad B(\mu h) = \cosh^2(\mu h/2) - I_0(\mu h), \\ C(\mu h) &= 2I_0(\sqrt{2}\mu h)\big(1 + \cosh(\mu h)\big) - 4I_0(\mu h)\cosh(\mu h). \end{aligned}$$

Taking $\beta_0 = -1$ in the linear equations (2.11), we obtain

$$\begin{aligned} \beta_1 &= \frac{e^{-\mu h} A(\mu h)}{C(\mu h)} \equiv \beta_1(\mu h), & \beta_3 &= \frac{e^{\mu h} A(\mu h)}{C(\mu h)} \equiv \beta_3(\mu h), \\ \beta_2 = \beta_4 &= \frac{A(\mu h)}{C(\mu h)} \equiv \beta_2(\mu h) = \beta_4(\mu h), & \beta_5 = \beta_8 &= \frac{e^{-\mu h} B(\mu h)}{C(\mu h)} \equiv \beta_5(\mu h) = \beta_8(\mu h), \\ \beta_6 = \beta_7 &= \frac{e^{\mu h} B(\mu h)}{C(\mu h)} \equiv \beta_6(\mu h) = \beta_7(\mu h). \end{aligned}$$

Thus, we arrive at the following tailored nine-point scheme for equation (2.3):

$$u_0 - \sum_{j=1}^{8} \beta_j(\mu h) u_j = 0. \tag{2.12}$$

Remark 2.1. For $\xi = \mu h > 0$, we have $A(\xi) > 0$, $B(\xi) > 0$ and $C(\xi) > 0$. That means $\beta_j(\xi) > 0$ ($j = 1, 2, \dots, 8$) for $\xi > 0$. Furthermore, we have $\sum_{j=1}^{8} \beta_j(\xi) = 1$ for $\xi > 0$.

For the second-order singularly perturbed elliptic equation (2.3), we obtain a five-point scheme (2.10) and a nine-point scheme (2.12), which possess the following properties:

(i) Schemes (2.10) and (2.12) satisfy the maximum principle.

(ii) When $0 < \varepsilon \ll h$, the schemes (2.10) and (2.12) naturally reduce to

$$\frac{u_0 - u_3}{h} = \mathcal{O}\Big(\sqrt{\frac{\mu}{h}} \exp(-\mu h)\Big),$$

which is a good approximation of the upwind scheme of equation (2.3):

$$-\varepsilon \frac{u_1 + u_2 + u_3 + u_4 - 4u_0}{h^2} + \frac{u_0 - u_3}{h} = 0,$$

since the first term of the above equation is a higher-order term in the case $0 < \varepsilon \ll h$.

Principles of TFPM

From the above examples, we can see the procedure for constructing a discrete scheme for a given differential equation by TFPM.

(i) For a given point $\mathbf{x}^0$, choose several points $\mathbf{x}^j$ ($j = 1, \dots, k$) around $\mathbf{x}^0$.

(ii) Construct a finite-dimensional function space W^k, where the basis functions of the function space W^k are tailored based on some properties of the solution of the given problem at point $\mathbf{x}^0$. For example, the solutions of the approximate problem at point $\mathbf{x}^0$ could be used to construct the space W^k.

(iii) To construct a TFPM at $\mathbf{x}^0$ (with $\mathbf{x}^j$, $j = 1, \dots, k$) is equivalent to finding constants α_j ($j = 1, \dots, k$) such that

$$u_0 = \sum_{j=1}^{k} \alpha_j u_j,$$

where u_j denotes the approximation of $u(\mathbf{x}^j)$.

(iv) The constants α_j ($j = 1, \dots, k$) are determined by requiring that

$$v(\mathbf{x}^0) = \sum_{j=1}^{k} \alpha_j v(\mathbf{x}^j) \quad \text{for all } v(\mathbf{x}) \in W^k.$$

3 TFPM for Singular Perturbation Problems

Let us consider the following model problems:

$$\begin{cases} \mathbb{L}_{\varepsilon,1} u \equiv -\varepsilon^2 \Delta u + p(\mathbf{x}) u_x + q(\mathbf{x}) u_y + b(\mathbf{x}) u = f(\mathbf{x}) \quad \text{for all } \mathbf{x} = (x, y) \in \Omega, \\ u|_\Gamma = 0, \\ b(\mathbf{x}) \geq b^0 > 0 \quad \text{for all } \mathbf{x} \in \overline{\Omega}, \end{cases} \tag{3.1}$$

$$\begin{cases} \mathbb{L}_{\varepsilon,2} u \equiv \varepsilon^2 \Delta^2 u - \Delta u = f(\mathbf{x}) \quad \text{for all } \mathbf{x} = (x, y) \in \Omega, \\ u|_\Gamma = 0, \\ \left.\dfrac{\partial u}{\partial n}\right|_\Gamma = 0, \end{cases} \tag{3.2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary Γ, while $p(\mathbf{x})$, $q(\mathbf{x})$, $b(\mathbf{x})$ and $f(\mathbf{x})$ are four given smooth functions on $\overline{\Omega}$.

When $0 < \varepsilon \ll 1$, problems (3.1) and (3.2) are two typical *singular perturbation problems*. Problem (3.1) is a singularly perturbed second-order elliptic equation. Problem (3.2) is a singularly perturbed fourth-order elliptic equation. The solutions of these problems possess some *boundary layers* on a portion of Γ, and maybe also have some *interior layers* in Ω. These layers are characterized by *rapid transitions* in the solutions. The existence of these layers in the solution is a major difficulty for the numerical simulation of singular perturbation problems.

The numerical solutions of similar singular perturbation problems have been studied by many mathematicians, for example, one could refer to the books by Doolan, Miller and Schilders [9], Morton [46], Roos, Stynes and Tobiska [51], and the review paper by Stynes [58]. Generally speaking, for capturing the boundary layers and interior layers in the numerical solutions of those given problems, one usually need the finest mesh size $h \sim \mathcal{O}(\varepsilon^2)$ in problem (3.1). One of the main goals in the study of numerical solutions of singular perturbation problems is to construct "uniformly convergent methods", which means that the numerical solution converges to the true solution, uniformly in ε, in some norm. For problem (3.1) in the one-dimensional case, exponentially fitted schemes on a uniform mesh are given, e.g., by Il'in [37], Miller [45], Berger, Han and Kellogg [5]. The uniform convergence of the numerical solutions was proved in those papers. On the other hand, mesh refinement is also used frequently for capturing the boundary layers in the numerical solutions; see, for example, Shishkin meshes [57]. Shishkin meshes can give a uniformly convergent method, but they need a prior knowledge of the position and behavior of the boundary layers (see [6, 42–44, 48, 57]).

For problem (3.2) in the one-dimensional case, there are many results. For example, uniformly convergent conforming finite element methods (FEM) are given by Roos and Stynes [50] and Semper [52]. Shishkin meshes are used in the FEM by Sun and Stynes [60, 61]. An exponentially fitted finite difference scheme is proposed by Shanthi and Ramanujam [53] and a high-order finite volume method is given by Chen, He and Wu [7]. For the two-dimensional case of problem (3.2), a nonconforming H^2-element FEM is obtained by Nilssen, Tai and Winther [47] and a modified Morley element is developed by Wang, Xu and Hu [66], in which they obtained a half order uniform convergence rate in an energy norm. The modified Morley element for three-dimensional problems with half order uniform convergence rate is given by Wang and Meng [65].

Recently, Han, Huang et al. applied the TFPM to a wide variety of such problems [16–18, 20, 21, 23, 30]. At each point, they choose the exact solutions of the local approximation problems as the basis functions for constructing the discrete schemes. Therefore, good approximate solutions can be obtained by these new schemes even on a uniform coarse mesh ($h \gg \varepsilon$) without any prior knowledge of the boundary/interior layers.

3.1 The Tailored Finite Point Scheme for Problem (3.1)

For the sake of simplicity, we assume that $\Omega = [0,1] \times [0,1]$. Let $h = N^{-1}$ be the mesh size and

$$x_i = ih, \quad y_j = jh, \quad 0 \le i, j \le N.$$

Then $P_{i,j} = (x_i, y_j)$ $(0 \le i, j \le N)$ are the mesh points, i.e., one has a uniform mesh.

We now construct the tailored finite point scheme for the first equation of problem (3.1) at mesh point $\mathbf{x}^0$. Around $\mathbf{x}^0$, there are eight mesh points $\mathbf{x}^i$ $(i = 1, 2, \dots, 8)$. Then the cell Ω_0 contains $\mathbf{x}^i$ $(i = 0, 1, \dots, 8)$; see Figure 1.

First we approximate the first equation of problem (3.1) on the cell Ω_0 by

$$-\varepsilon^2 \Delta u + p_0 u_x + q_0 u_y + b_0 u = f_0$$

with $p_0 = p(\mathbf{x}^0)$, $q_0 = q(\mathbf{x}^0)$, $b_0 = b(\mathbf{x}^0)$, $f_0 = f(\mathbf{x}^0)$. Introduce a new function

$$v(x, y) = \left(u(x, y) - \frac{f_0}{b_0}\right) \exp\left(-\frac{p_0 x + q_0 y}{2\varepsilon^2}\right).$$

Then v satisfies $-\varepsilon^2 \Delta v + d_0^2 v = 0$ with $d_0^2 = b_0 + \frac{p_0^2 + q_0^2}{4\varepsilon^2}$.

Let $\mu_0 = d_0/\varepsilon$ and

$$H_4 = \{v(x, y) \mid v = c_1 e^{-\mu_0 x} + c_2 e^{\mu_0 x} + c_3 e^{-\mu_0 y} + c_4 e^{\mu_0 y} \text{ for all } c_i \in \mathbb{R}\}.$$

Then we take the discrete scheme as

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \alpha_4 V_4 + \alpha_0 V_0 = 0 \tag{3.3}$$

with $V_j = v(\mathbf{x}^j)$, such that (3.3) holds for all $v \in H_4$. By the procedure given in Section 2, we have

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{-\alpha_0}{e^{\mu_0 h} + e^{-\mu_0 h} + 2} \equiv \frac{-\alpha_0}{4\cosh^2(\frac{\mu_0 h}{2})}.$$

Choosing

$$\alpha_0 = \frac{e^{\mu_0 h} + e^{-\mu_0 h} + 2}{e^{\mu_0 h} + e^{-\mu_0 h} - 2} \equiv \frac{\cosh^2(\frac{\mu_0 h}{2})}{\sinh^2(\frac{\mu_0 h}{2})},$$

we get

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -\frac{1}{e^{\mu_0 h} + e^{-\mu_0 h} - 2} \equiv -\frac{1}{4\sinh^2(\frac{\mu_0 h}{2})}.$$

Then we have the following five-point scheme for problem (3.1):

$$U_0 - \frac{e^{-\frac{p_0 h}{2\varepsilon^2}} U_1 + e^{-\frac{q_0 h}{2\varepsilon^2}} U_2 + e^{\frac{p_0 h}{2\varepsilon^2}} U_3 + e^{\frac{q_0 h}{2\varepsilon^2}} U_4}{4\cosh^2(\frac{\mu_0 h}{2})} = \frac{f_0}{b_0}\left(1 - \frac{e^{-\frac{p_0 h}{2\varepsilon^2}} + e^{-\frac{q_0 h}{2\varepsilon^2}} + e^{\frac{p_0 h}{2\varepsilon^2}} + e^{\frac{q_0 h}{2\varepsilon^2}}}{4\cosh^2(\frac{\mu_0 h}{2})}\right) \tag{3.4}$$

with

$$U_j = u(\mathbf{x}^j) = \frac{f_0}{b_0} + V_j \exp\left(\frac{p_0 x_j + q_0 y_j}{2\varepsilon^2}\right).$$

Finally, we get the following discrete problem of problem (3.1):

$$\begin{cases} \mathbb{L}_h U_{i,j} \equiv \dfrac{b_{ij}}{1-\eta_{ij}} \left\{ U_{ij} - \dfrac{e^{-\frac{p_{ij}h}{2\varepsilon^2}} U_{i+1,j} + e^{-\frac{q_{ij}h}{2\varepsilon^2}} U_{i,j+1} + e^{\frac{p_{ij}h}{2\varepsilon^2}} U_{i-1,j} + e^{\frac{q_{ij}h}{2\varepsilon^2}} U_{i,j-1}}{4\cosh^2(\frac{\mu_{ij}h}{2})} \right\} = f_{i,j}, \quad 1 \le i,j \le N-1, \\ U_{i,0} = U_{i,N} = 0, \quad i = 0,1,\dots,N, \\ U_{0,j} = U_{N,j} = 0, \quad j = 1,\dots,N-1, \end{cases} \tag{3.5}$$

with

$$\eta_{ij} = \frac{e^{-\frac{p_{ij}h}{2\varepsilon^2}} + e^{-\frac{q_{ij}h}{2\varepsilon^2}} + e^{\frac{p_{ij}h}{2\varepsilon^2}} + e^{\frac{q_{ij}h}{2\varepsilon^2}}}{4\cosh^2(\frac{\mu_{ij}h}{2})}.$$

Lemma 3.1 (cf. [19]). *It is easy to check that the discrete problem* (3.5) *satisfies the discrete maximum principle.*

Remark 3.2 (cf. [19]). If $h \ll \varepsilon^2$, i.e., $\mu_0 h \ll 1$ and $\nu_0 h \ll 1$, we have

$$\cosh^2\left(\frac{\mu_0 h}{2}\right) = 1 + \frac{(\mu_0 h)^2}{4} + \mathcal{O}((\mu_0 h)^4), \quad \cosh^2\left(\frac{\nu_0 h}{2}\right) = 1 + \frac{(\nu_0 h)^2}{4} + \mathcal{O}((\nu_0 h)^4)$$

and

$$e^{\pm\frac{p_0 h}{2\varepsilon^2}} = 1 \pm \frac{p_0 h}{2\varepsilon^2} + \frac{(\frac{p_0 h}{2\varepsilon^2})^2}{2} \pm \frac{(\frac{p_0 h}{2\varepsilon^2})^3}{6} + \mathcal{O}\left(\left(\frac{p_0 h}{2\varepsilon^2}\right)^4\right), \quad e^{\pm\frac{q_0 h}{2\varepsilon^2}} = 1 \pm \frac{q_0 h}{2\varepsilon^2} + \frac{(\frac{q_0 h}{2\varepsilon^2})^2}{2} \pm \frac{(\frac{q_0 h}{2\varepsilon^2})^3}{6} + \mathcal{O}\left(\left(\frac{q_0 h}{2\varepsilon^2}\right)^4\right).$$

If we omit the high-order terms, the tailored finite point scheme reduces to the *standard second-order finite difference scheme* of the first equation of problem (3.1) at $\mathbf{x}^0$, i.e.,

$$-\varepsilon^2 \frac{U_1 + U_2 + U_3 + U_4 - 4U_0}{h^2} + p_0 \frac{U_1 - U_3}{2h} + q_0 \frac{U_2 - U_4}{2h} + b_0 U_0 = f_0.$$

Remark 3.3. For $0 < \varepsilon \ll h$, the scheme (3.4) reduces to the upwind scheme.

3.2 Error Analysis of the Discrete Problem (3.5)

We only consider the simple case

$$\Omega = (0,1)^2, \quad p(\mathbf{x}) \equiv p_0 > 0, \quad q(\mathbf{x}) \equiv 0, \quad b(\mathbf{x}) \equiv b_0 > 0, \quad f(\mathbf{x}) \in \mathbf{C}^{2l,\alpha}(\overline{\Omega}) \tag{3.6}$$

for some integers $3 \le l \in \mathbb{N}$ and some real numbers $\alpha \in (0,1)$.

Theorem 3.4 (cf. [19]). *Suppose that* (3.6) *holds and* $\{U_{i,j}, 0 \le i,j \le N\}$ *is a solution of problem* (3.5). *Then the following estimate holds:*

$$|U_{i,j}| \le \max_{0\le i,j\le N} \frac{|f_{i,j}|}{b_0}, \quad 0 \le i,j \le N.$$

Let

$$E_{i,j} = U_{i,j} - u(P_{i,j}), \quad 0 \le i,j \le N,$$

then

$$\mathbb{L}_h E = \mathbb{T}_h u = \mathcal{O}\{h + \varepsilon^{-2} e^{-\frac{p_0 h}{\varepsilon^2}} + e^{-\beta\frac{h}{2\varepsilon}}\}.$$

By Theorem 3.4 and Lemma 3.1, we have the following result.

Theorem 3.5 (cf. [19]). *Suppose that* (3.6) *and* $0 < \varepsilon \ll h$ *hold. Then we have the error estimate*

$$|E_{i,j}| \le C\{h + \varepsilon^{-2} e^{-\frac{p_0 h}{\varepsilon^2}} + e^{-\beta\frac{h}{2\varepsilon}}\}, \quad 0 \le i,j \le N.$$

Remark 3.6. Many numerical examples are given in [16, 17, 20, 21, 30], which show that the tailored finite point method works efficiently and displaces the uniform convergence in ε. But the uniformly convergent error analysis is still open for the general case.

Remark 3.7. We also proposed TFPM for the anisotropic diffusion problems [22] and the ODEs with different parameters [24]. We proved the uniform convergence of our TFPM for those problems.

3.3 TFPM for the Fourth-Order Singular Perturbation Problem

First, we discuss the decomposition of the fourth-order partial differential equation of problem (3.2). Let

$$v = \Delta u \quad \text{and} \quad w = u - \varepsilon^2 v. \tag{3.7}$$

It is straightforward to check that the functions v, w satisfy the following system of two second-order elliptic boundary value problems:

$$\begin{cases} -\varepsilon^2 \Delta v + v = -f(\mathbf{x}) & \text{for all } \mathbf{x} = (x, y) \in \Omega, \\ -\Delta w = f(\mathbf{x}) & \text{for all } \mathbf{x} = (x, y) \in \Omega, \\ (w + \varepsilon^2 v)|_\Gamma = 0, \\ \left.\dfrac{\partial(w + \varepsilon^2 v)}{\partial n}\right|_\Gamma = 0. \end{cases} \tag{3.8}$$

For problem (3.8), we have the following stability estimate:

Theorem 3.8 (cf. [18]). *Suppose that* (v, w) *is a solution of problem* (3.8) *and* $v, w \in H^1(\Omega)$*. Then the following stability estimate holds:*

$$\int_\Omega \nabla(w + \varepsilon^2 v) \cdot \nabla(w + \varepsilon^2 v)\, dxdy + \varepsilon^2 \int_\Omega v^2\, dxdy \le C \int_\Omega f^2\, dxdy,$$

where C *is a constant independent of* ε*.*

On the other hand, we know that problem (3.2) has a solution $u_\varepsilon \in H_0^2(\Omega)$ for all $\varepsilon > 0$; see [12]. By the definition (3.7), we know (v, w) is a solution of problem (3.8). Therefore, from the above stability estimate, we obtain the uniqueness of problem (3.8) directly:

Theorem 3.9. *Problem* (3.8) *has a unique solution and is equivalent to problem* (3.2).

From the view of finding the numerical solution, using problem (3.8) is more convenient than problem (3.2). We now construct the tailored finite point scheme for problem (3.8).

3.3.1 TFPM in Interior Domain

For the sake of simplicity, we assume that $\Omega = [0, 1] \times [0, 1]$ and we have a uniform mesh, i.e., let $h = N^{-1}$ be the mesh size and

$$x_i = ih, \quad y_j = jh, \quad 0 \le i, j \le N.$$

Then $P_{i,j} = (x_i, y_j)$ $(0 \le i, j \le N)$ are the mesh points.

We now construct our tailored finite point scheme for (3.8) on a cell Q_0 (see Figure 1). First, we approximate the function $f(x, y)$ by piecewise constants, i.e., we approximate f by $f_0 = \frac{1}{|Q_0|}\int_{Q_0} f(x, y)\, dxdy$ in cell Q_0.

For the second equation of problem (3.8), we can just use the standard five-point difference scheme:

$$-\frac{W_1 + W_2 + W_3 + W_4 - 4W_0}{h^2} = f_0 \tag{3.9}$$

with $W_j = w(P_j)$.

For the first equation of problem (3.8), let $\lambda = \varepsilon^2 v$. Then λ satisfies

$$-\varepsilon^2 \Delta\lambda + \lambda = \varepsilon^2 f. \tag{3.10}$$

In cell Q_0, let $\psi = \lambda + \varepsilon^2 f_0$. Then ψ satisfies

$$-\varepsilon^2 \Delta\psi + \psi = 0. \tag{3.11}$$

Then we construct a scheme for (3.11) as

$$\alpha_1\Psi_1 + \alpha_2\Psi_2 + \alpha_3\Psi_3 + \alpha_4\Psi_4 + \alpha_0\Psi_0 = 0$$

with $\Psi_j = \psi(P_j)$, by the procedure given in Section 2, where we take

$$\alpha_0 = \frac{e^{\mu h} + e^{-\mu h} + 2}{e^{\mu h} + e^{-\mu h} - 2} \equiv \frac{\cosh^2(\frac{\mu h}{2})}{\sinh^2(\frac{\mu h}{2})}.$$

Then

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -\frac{1}{e^{\mu h} + e^{-\mu h} - 2} \equiv -\frac{1}{4\sinh^2(\frac{\mu h}{2})}. \tag{3.12}$$

We finally have the following five-point scheme for (3.10):

$$\Lambda_0 - \frac{\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4}{4\cosh^2(\frac{\mu h}{2})} = \varepsilon^2 f_0\Big(\frac{1}{\cosh^2(\frac{\mu h}{2})} - 1\Big) \tag{3.13}$$

with $\Lambda_j = \lambda(P_j) = \Psi_j - \varepsilon^2 f_0$.

3.3.2 TFPM on the Boundary

It is straightforward to implement the first boundary condition of problem (3.8). For example, if $P_3 \in \partial\Omega$, we have

$$W_3 + \Lambda_3 = 0. \tag{3.14}$$

Furthermore, in terms of the function λ, we replace the second boundary condition of problem (3.8) by

$$\frac{\partial(w + \lambda)}{\partial n} = 0. \tag{3.15}$$

The boundary condition (3.15) can be discretized similarly to our procedure above. For example (cf. Figure 1), suppose P_3 is on the boundary $\partial\Omega$. We want to approximate $\frac{\partial w}{\partial n}(P_3)$ by

$$\frac{\partial w}{\partial n}(P_3) \approx \alpha_3 W_3 + \alpha_6 W_6 + \alpha_7 W_7 + \alpha_0 W_0. \tag{3.16}$$

We expect that the approximation (3.16) has no error for all the functions in the space

$$\mathcal{W}_4 = \{\phi(x,y) = c_1 + c_2 x + c_3 y + c_4(x^2 - y^2) \text{ for all } c_j \in \mathbb{R},\ j = 1,2,3,4\}.$$

Then we obtain the coefficients

$$\alpha_0 = -\frac{1}{h}, \quad \alpha_3 = \frac{2}{h}, \quad \alpha_6 = \alpha_7 = -\frac{1}{2h}.$$

Next, we want to approximate $\frac{\partial\lambda}{\partial n}(P_3)$ by

$$\frac{\partial\lambda}{\partial n}(P_3) \approx \beta_3\Lambda_3 + \beta_6\Lambda_6 + \beta_7\Lambda_7 + \beta_0\Lambda_0.$$

We arrive at

$$\beta_0 = -\frac{1}{\varepsilon\sinh\frac{h}{\varepsilon}}, \quad \beta_3 = \frac{1 + \cosh\frac{h}{\varepsilon}}{\varepsilon\sinh\frac{h}{\varepsilon}}, \quad \beta_6 = \beta_7 = \frac{1}{2}\beta_0. \tag{3.17}$$

Then we get the discretization of (3.15) as

$$\beta_3\Lambda_3 + \beta_6\Lambda_6 + \beta_7\Lambda_7 + \beta_0\Lambda_0 + \alpha_3 W_3 + \alpha_6 W_6 + \alpha_7 W_7 + \alpha_0 W_0 = 0. \tag{3.18}$$

When $h \gg \varepsilon$ (for example, $h > 5\varepsilon$), it is easy to check that

$$\beta_3 > |\beta_6| + |\beta_7| + |\beta_0| + |\alpha_3| + |\alpha_6| + |\alpha_7| + |\alpha_0|. \tag{3.19}$$

The discretization of the boundary condition (3.15) at other boundary points is similar.

Remark 3.10. It is clear that the system of equations (3.9), (3.12) and (3.14) is *diagonally dominant*. When $h \gg \varepsilon$, from (3.19), we know that (3.18) is also *diagonally dominant*. That means, in this case, our linear system for W_j and Λ_j has a unique solution.

After solving this linear system, we get the approximation of $u(P_j) = U_j$ by $U_j = W_j + \Lambda_j$.

Remark 3.11. If $h \ll \varepsilon$, i.e., $\mu h \ll 1$, we obtain

$$\cosh^2\left(\frac{\mu h}{2}\right) = 1 + \frac{(\mu h)^2}{4} + \mathcal{O}((\mu h)^4).$$

Omitting high order terms, the tailored finite point scheme (3.13) reduces to the *standard second-order finite difference scheme* of the first equation of problem (3.8) at P_0, i.e.,

$$-\varepsilon^2 \frac{V_1 + V_2 + V_3 + V_4 - 4V_0}{h^2} + V_0 = f_0.$$

Remark 3.12. When $\varepsilon \to 0$, from (3.17), we obtain

$$\beta_0, \beta_6, \beta_7 \to 0, \quad \beta_3 \to +\infty,$$

i.e., we should have $\lambda(P_3) \to 0$. That means the second boundary condition of problem (3.8) reduces to $\lambda|_\Gamma = 0$ as $\varepsilon \to 0$. Therefore, by our discretization of the boundary conditions, when $\varepsilon \to 0$, problem (3.8) is really decomposed into two decoupled problems:

$$\begin{cases} -\Delta w = f(x, y) \quad \text{for all } (x, y) \in \Omega, \\ w|_\Gamma = 0, \end{cases} \tag{3.20}$$

$$\begin{cases} -\varepsilon^2 \Delta \lambda + \lambda = -\varepsilon^2 f(x, y) \quad \text{for all } (x, y) \in \Omega, \\ \lambda|_\Gamma = 0. \end{cases} \tag{3.21}$$

It is straightforward to check that λ, the solution of (3.21), goes to zero when $\varepsilon \to 0$. And the solution of (3.20) is really the leading order approximation of the original problem (3.2) as $\varepsilon \to 0$. That means, we can get a good approximation of u_ε as $\varepsilon \to 0$, i.e., our method is an *asymptotic-preserving method*.

Remark 3.13. We also propose an iterative method [23] to solve problem (3.8). At each step we only need to solve two boundary value problems of second-order elliptic equation which are only coupled on the boundary. We prove the convergence of our method on a disc. The convergence theory in the general case is still open.

4 TFPM for Wave Equation

Here we consider the inhomogeneous Helmholtz equation in the one-dimensional case:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x}\left(c^2(x)\dfrac{\mathrm{d}u}{\mathrm{d}x}\right) + k^2 n^2(x) u = f(x) \quad \text{for all } x \in \Omega = (a, b) \subset \mathbb{R}, \\ u(a) = 0, \quad (cu' - \mathrm{i}knu)(b) = 0, \\ u(x) \text{ and } c^2(x)u'(x) \text{ are continuous on } \Omega, \end{cases} \tag{4.1}$$

where i is the imaginary unit, $k > 0$, $f \in L^2(\Omega)$, and $c(x)$ and $n(x)$ are two piecewise smooth functions, namely the local speed of sound and the index of refraction, respectively, such that

$$0 < c_0 \le c(x) \le C_0 < \infty, \quad 0 < n_0 \le n(x) \le N_0 < \infty.$$

The boundary value problem of the Helmholtz equation arises in many physical fields, such as the acoustic/electromagnetic/seismic wave propagation. It is well known that the numerical simulation of the Helmholtz equation with high wave numbers in inhomogeneous medium is extremely difficult [4, 35, 36, 38].

In recent decades, many scientists presented efficient methods for this class of problems with constant coefficients, including the discrete singular convolution method [2], the hybrid numerical asymptotic method [13], the spectral approximation method [54], the element-free Galerkin method [59, 63], the so-called ultra weak variational formulation [34], and the hybrid numerical-asymptotic boundary integral method [8]. In general, these methods need the restriction $kh = \mathcal{O}(1)$ for the mesh size h in the simulation.

For problem (4.1), if we let

$$y(x) = \int_a^x \frac{1}{c^2(\xi)}\,d\xi, \quad \tau = y(b), \quad s(y) \equiv c(x(y))n(x(y)), \quad F(y) \equiv f(x(y)),$$

the function $U(y) \equiv u(x(y))$ shall satisfy the following problem:

$$\begin{cases} U''(y) + k^2 s^2(y)U(y) = F(y) & \text{for all } y \in I = (0,\tau), \\ U(0) = 0, \quad U'(\tau) - \mathrm{i}ks(\tau)U(\tau) = 0, \\ U(y) \text{ and } U'(y) \text{ are continuous on } I. \end{cases} \tag{4.2}$$

4.1 Stability Analysis for Analytical Solution

Without loss of generality, we assume that $I = (0,\tau) \equiv (0,1)$. Let

$$L^2(I) = \Big\{v \mid \int_I |v(y)|^2\,dy < +\infty\Big\}$$

denote the space of all square-integrable complex-valued functions equipped with the inner product

$$(v,w) := \int_I v(y)\bar{w}(y)\,dy$$

and the induced norm

$$\|v\|_{0,I} := \sqrt{(v,v)}.$$

Furthermore, for $m \in \mathbb{N}$, let

$$H^m(I) = \{v \mid v \in L^2(I),\ v^{(j)} \in L^2(I),\ j = 1,\dots,m\},$$

where $v^{(j)}$ are the derivatives of order j in the distribution sense. By the semi-norm $|v|_{l,I} := \|v^{(l)}\|_{0,I}$ given in $H^l(I)$, one norm of the space $H^m(I)$ is defined as

$$\|v\|_{m,I} = \Big(\sum_{j=0}^m |v|^2_{j,I}\Big)^{1/2}.$$

In addition, assume that the piecewise smooth function $s(y)$ is also piecewise monotone, i.e., there are some points y_j $(j = 0,1,\dots,J)$ such that

$$0 = y_0 < y_1 < \cdots < y_J = 1, \quad I_j = (y_{j-1}, y_j), \quad s|_{I_j} \text{ is monotone}, \quad m \in C^1(\bar{I}_j), \quad j = 1,\dots,J.$$

Setting

$$\mathcal{M}_j^0 \equiv \|s\|_{\infty,I_j}, \quad \mathcal{M}_j^1 \equiv \|s'\|_{\infty,I_j}, \quad j = 1,\dots,J,$$

we obtain

$$n_0 \le \max_{1\le j\le J} \mathcal{M}_j^0 \le N_0, \quad \mathcal{M}_1 \equiv \max_{1\le j\le J} \mathcal{M}_j^1 < +\infty.$$

Then the following estimates for problem (4.2) hold (cf. [35, 36, 54]).

Lemma 4.1 (Stability analysis, cf. [15]). *Suppose $F \in L^2(I)$ in the first equation of problem* (4.2), *$s(y)$ is piecewise smooth and piecewise monotone, U is the solution of* (4.2). *Then $U \in H^2(I) \cap C^1(\bar{I})$ and the estimates*

$$|U|_{1,I} + k\|U\|_{0,I} \le C\|F\|_{0,I}, \quad |U|_{2,I} \le C(1+k)\|F\|_{0,I}$$

hold for a positive constant C which is independent of F and k.

4.2 TFPM for the Helmholtz Equation

First, we take a partition as

$$0 = \xi_0 < \xi_1 < \cdots < \xi_N = 1$$

with

$$h_j = \xi_j - \xi_{j-1}, \quad j = 1, 2, \dots, N, \quad \text{and} \quad h = \max_{1 \le j \le N} h_j,$$

such that $s(y)$ is smooth and monotone on each subdomain $D_j = (\xi_{j-1}, \xi_j)$ (see Figure 2).

Figure 2. The local mesh around points ξ_{j-1}, ξ_j, and ξ_{j+1}.

Then we approximate the coefficient $s(y)$ by a piecewise constant function, i.e., we introduce a function $s_h(y)$ defined by

$$s_h(y) = s_j \equiv s(\xi_j^-) \quad \text{for } y \in D_j, \ j = 1, \dots, N. \tag{4.3}$$

Now we obtain an approximate problem of (4.2) for U_h:

$$\begin{cases} U_h''(y) + k^2 s_h^2 U_h(y) = F & \text{for all } y \in D_j, \ j = 1, \dots, N, \\ U_h(0) = 0, \quad U_h'(1) - \mathrm{i}ks_N U_h(1) = 0, \\ U_h(\xi_j^-) = U_h(\xi_j^+), \quad U_h'(\xi_j^-) = U_h'(\xi_j^+), & j = 1, \dots, N-1. \end{cases} \tag{4.4}$$

For $y, z \in D_j$, $j = 1, \dots, N$, let

$$G_j(y,z) = \frac{1}{ks_j} \begin{cases} e^{\mathrm{i}ks_j(z-\xi_{j-1})} \sin\big(ks_j(y - \xi_{j-1})\big), & y \ge z, \\ e^{\mathrm{i}ks_j(y-\xi_{j-1})} \sin\big(ks_j(z - \xi_{j-1})\big), & z > y. \end{cases}$$

Then the solution of the first equation of problem (4.4) can be expressed as

$$U_h(y) = A_j e^{\mathrm{i}ks_j(y-\xi_{j-1})} + B_j e^{-\mathrm{i}ks_j(y-\xi_{j-1})} + \int_{\xi_{j-1}}^{\xi_j} f(z) G_j(y,z)\, dz \quad \text{for } y \in D_j \tag{4.5}$$

with some constants $A_j, B_j \in \mathbb{C}$, $j = 1, \dots, N$. From the boundary conditions and the interface conditions of problem (4.4), we have

$$A_1 + B_1 + f_1^s = 0, \quad -2\mathrm{i}B_N + f_N^e = 0, \tag{4.6}$$

$$A_j e^{\mathrm{i}ks_jh_j} + B_j e^{-\mathrm{i}ks_jh_j} + f_j^e \sin(ks_jh_j) = A_{j+1} + B_{j+1} + f_{j+1}^s, \quad 1 \le j \le N-1, \tag{4.7}$$

$$A_j e^{\mathrm{i}ks_jh_j} - B_j e^{-\mathrm{i}ks_jh_j} - \mathrm{i} f_j^e \cos(ks_jh_j) = \frac{s_{j+1}}{s_j}(A_{j+1} - B_{j+1} + f_{j+1}^s), \quad 1 \le j \le N-1, \tag{4.8}$$

with

$$f_j^s = \int_{\xi_{j-1}}^{\xi_j} \frac{f(y)}{ks_j} \sin\big(ks_j(y - \xi_{j-1})\big)\, dy, \quad f_j^c = \int_{\xi_{j-1}}^{\xi_j} \frac{f(y)}{ks_j} \cos\big(ks_j(y - \xi_{j-1})\big)\, dy, \quad f_j^e = f_j^c + \mathrm{i} f_j^s. \tag{4.9}$$

Now we obtain a linear system of $2N$ equations for all $2N$ unknowns A_j, B_j ($j = 1, \dots, N$). Solving this linear system (4.6)–(4.8), we can get our approximate solution $U_h(y)$.

Remark 4.2. Certainly, in practice, we need some quadrature rules to get the integrals in (4.9). When the wave number k is large, it is not easy to get these integrals by standard quadrature rules. However, if we expand the function F by a series of piecewise trigonometric functions or if we approximate F by piecewise polynomials, we can get the approximation of these integrals explicitly with high accuracy (cf. [30]).

4.3 Well-Posedness of TFPM

The following well-posedness theorem holds.

Theorem 4.3 (Uniqueness theorem, [15]). *The linear system* (4.6)–(4.8) *has a unique solution.*

Remark 4.4. From the proof of Theorem 4.3 (cf. [15]), we also give a procedure to solve our linear system (4.6)–(4.8) very easily. That means, from (4.6), we have

$$\begin{pmatrix} A_j \\ B_j \end{pmatrix} = \begin{pmatrix} \alpha_j & \beta_j \\ \bar{\beta}_j & \bar{\alpha}_j \end{pmatrix} \begin{pmatrix} A_{j+1} \\ B_{j+1} \end{pmatrix} + \begin{pmatrix} \mu_j \\ \nu_j \end{pmatrix}, \quad j = 1, \dots, N-1,$$

with

$$\mu_j = \alpha_j f_{j+1}^s + \frac{\mathrm{i}}{2} f_j^e, \quad \nu_j = \bar{\beta}_j f_{j+1}^s - \frac{\mathrm{i}}{2} f_{j+1}^e.$$

Then we can get

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} A_N \\ B_N \end{pmatrix} + \begin{pmatrix} \mu \\ \nu \end{pmatrix}$$

by iteration. Combining with (4.6), we can get A_1, B_1, A_N, B_N immediately. Finally, we can get A_j, B_j $(j = 2, \dots, N-1)$ by recursion. The total cost of solving the linear system (4.6)–(4.8) is $\mathcal{O}(N)$.

4.4 Convergence Analysis of TFPM

From the definition (4.3) of s_h, we have

$$|s^2(y) - s_h^2(y)| \le 2N_0\mathcal{M}_1 h \quad \text{for all } y \in I.$$

Suppose that U is the solution of problem (4.2), and U_h is the solution of problem (4.4). Set

$$E(y) \equiv U(y) - U_h(y), \quad R_h \equiv k^2(s_h^2 - s^2)U.$$

Then we arrive at

$$\begin{cases} E''(y) + k^2 s_h^2 E(y) = R_h(y) & \text{for all } y \in I, \\ E(0) = 0, \quad E'(1) - \mathrm{i}ks_N E(1) = 0, \\ E \text{ and } E' \text{ are continuous on } I. \end{cases}$$

Lemma 4.5 (Stability analysis for TFPM, [15]). *Suppose $F \in L^2(I)$ and that $s(y)$ is piecewise smooth and piecewise monotone on I. Then we have $E \in H^2(I) \cap C^1(\bar{I})$ and the estimates*

$$|E|_{1,I} + k\|E\|_{0,I} \le C\|R_h\|_{0,I}, \quad |E|_{2,I} \le C(1+k)\|R_h\|_{0,I}$$

with a constant C independent of R_h and k.

Theorem 4.6 (Error estimate for TFPM, [15]). *The error estimates*

$$|E|_{1,I} + k\|E\|_{0,I} \le C\mathcal{M}_1 kh\|F\|_{0,I}, \quad |E|_{2,I} \le C\mathcal{M}_1 k^2 h\|F\|_{0,I}$$

hold with a constant C independent of h and k.

Remark 4.7. From Theorem 4.6, we know that $E(y) \equiv 0$ if $s(y) = c(y)n(y)$ is a piecewise constant function, which implies that our method can get the exact solution in this case.

Remark 4.8. We extended the ideas of this section to higher-dimensional problems [33]. The algorithms and the stability analysis are more complicated.

Remark 4.9. We also apply our TFPM to the first-order wave equation [32]. We get very good approximations for the first-order wave equation and conservation laws, especially for the cases with high frequency waves and discontinuities.

5 TFPM for Discrete-Ordinate Transport Equations

5.1 The Discrete-Ordinate Transport Equations

The neutron transport equation is widely used in nuclear engineering, thermal radiation transport, charged-particle transport and oil-well logging tool design, etc. Numerical methods for the neutron transport equation have been developed for decades [39–41] as an active area.

The discrete-ordinate transport equations are given by [41]

$$\varepsilon\Big(c_m\frac{\partial}{\partial x}\psi_m + s_m\frac{\partial}{\partial y}\psi_m\Big) + \sigma_T\psi_m = (\sigma_T - \varepsilon^2\,\sigma_a)\sum_{n\in V}\psi_n w_n + \varepsilon^2 q, \quad m \in V, \tag{5.1}$$

where V represents the index set $V = \{1, 2, \ldots, 4M\}$ with positive integer M,

$$c_m = (1-\zeta_m^2)^{\frac{1}{2}}\cos\theta_m \quad \text{and} \quad s_m = (1-\zeta_m^2)^{\frac{1}{2}}\sin\theta_m \quad \text{for } |\zeta_m| \le 1,\ m \in V.$$

Let $\psi_m = \psi_m(x, y)$ be an approximation of the density function $\tilde{\psi}(x, y, \zeta_m, \theta_m)$ for $m \in V$. For simplicity, we assume that the spatial variables x, y satisfy $x \in (0, a)$ and $y \in (0, b)$ for two positive real numbers a and b, that is,

$$\mathbf{D} = \{(x, y) \mid x \in (0, a),\ y \in (0, b)\}$$

is the computational domain. At the boundary $\partial\mathbf{D}$, the approximate particle density functions $\{\psi_m(x, y)\}_{m\in V}$ satisfy the boundary conditions

$$\begin{cases} \psi_m(0, y) = \psi_{Lm}(y),\ c_m > 0; & \psi_m(a, y) = \psi_{Rm}(y),\ c_m < 0; \quad y \in [0, b]; \\ \psi_m(x, 0) = \psi_{Bm}(x),\ s_m > 0; & \psi_m(x, b) = \psi_{Tm}(x),\ s_m < 0, \quad x \in [0, a]. \end{cases} \tag{5.2}$$

Here $\psi_{Bm}(x)$, $\psi_{Tm}(x)$, $\psi_{Lm}(y)$ and $\psi_{Rm}(y)$ $(m \in V)$ are known functions. For any interface line α, we should have the interface conditions

$$\psi_m^+|_\alpha = \psi_m^-|_\alpha, \quad m \in V. \tag{5.3}$$

In order to have the discrete-ordinate equations (5.1) converge to the same diffusion limit equation, as ε tends to zero (when the boundary conditions are independent of m), the quadrature set $\{c_m, s_m, w_m\}_{m\in V}$ is required to satisfy the conditions [62]

$$\sum_{n\in V} w_n = 1, \quad \sum_{n\in V} w_n c_n = 0, \quad \sum_{n\in V} w_n s_n = 0, \quad \sum_{n\in V} w_n c_n s_n = 0, \quad \sum_{n\in V} w_n(c_n^2 + s_n^2) = \frac{2}{3}. \tag{5.4}$$

We choose a symmetric quadrature set $\{c_m, s_m, w_m\}$ by assuming

$$\begin{cases} w_m = w_{m+M} = w_{m+2M} = w_{m+3M} > 0, \quad m = 1, \ldots, M, \\ \theta_m = \theta_{m+M} - \dfrac{\pi}{2} = \theta_{m+2M} - \pi = \theta_{m+3M} - \dfrac{3}{2}\pi \in (0, \pi/2), \quad m = 1, \ldots, M, \\ \zeta_m = \zeta_{m+M} = \zeta_{m+2M} = \zeta_{m+3M} \in [0, 1], \quad m = 1, \ldots, M, \\ c_m = (1-\zeta_m^2)^{\frac{1}{2}}\cos\theta_m, \quad s_m = (1-\zeta_m^2)^{\frac{1}{2}}\sin\theta_m, \quad m \in V. \end{cases} \tag{5.5}$$

The requirement (5.4) indicates $\sum_{n=1}^{M} w_n(1-\zeta_n^2) = \frac{1}{6}$ and further

$$\sum_{n=1}^{M} w_n\,\zeta_n^2 = \frac{1}{12}. \tag{5.6}$$

We can check that when the set $\{c_m, s_m, w_m\}$ is chosen by (5.5)–(5.6), the requirement (5.4) is satisfied, so that the discrete-ordinate system possesses the same diffusion limit as the original integral equation.

In this section, we consider the most commonly used Gaussian quadratures set

$$S_N = \{c_m, s_m, w_m\}_{m\in V}$$

with N a positive integer parameter [41]. In a quadrature set S_N, each quadrant has $M = N(N+1)/2$ ordinates and N distinct $\zeta_m \in (0,1)$, which are the positive roots of the standard Legendre polynomial of degree $2N$ on the interval $[-1,1]$. The corresponding c_m, s_m, w_m for S_1, S_2 are given in the following; for S_N with $N \geq 3$, they can be found in [25]. It is easy to check that the Gaussian quadratures satisfy (5.5)–(5.6).

- *Quadrature set S_1:* When $N = 1$ and $M = N(N+1)/2 = 1$, $\zeta_1^2 = 1/3$, $\theta = \pi/4$, then

$$(c_1, s_1) = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right), \quad (c_2, s_2) = \left(\frac{-\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right), \quad (c_3, s_3) = \left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right), \quad (c_4, s_4) = \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right),$$

  and

$$w_1 = w_2 = w_3 = w_4 = \frac{1}{4}.$$

- *Quadrature set S_2:* When $N = 2$ and $M = N(N+1)/2 = 3$, the quadrature nodes and weights of the quadrature set S_2 are presented in Table 1.

ζ_m	θ_m	c_m	s_m	$4w_m$
0.3399810	$\pi/8$	0.8688461	0.3598879	0.3260726
0.3399810	$3\pi/8$	0.3598879	0.8688461	0.3260726
0.8611363	$\pi/4$	0.3594748	0.3594748	0.3478548

Table 1. The nodes and weights of the quadrature set S_2.

From the view of mathematics, the discrete ordinate transport problem (5.1)–(5.3) is a singularly perturbed first-order partial differential system. We now discuss the numerical solution of the discrete ordinate transport problem (5.1)–(5.3) using TFPM.

5.2 Special Solutions of the Homogeneous Discrete-Ordinate Transport Equations with Constant Coefficients

At first, we find the special solutions of the homogeneous discrete-ordinate equations

$$\varepsilon\left(c_m \frac{\partial}{\partial x} + s_m \frac{\partial}{\partial y}\right)\psi_m + \sigma_T \psi_m = (\sigma_T - \varepsilon^2 \sigma_a) \sum_{n \in V} \omega_n \psi_n, \quad m \in V, \tag{5.7}$$

with constant coefficients σ_T and σ_a. Let

$$\boldsymbol{\Psi}(\mathbf{x}) = \left(\psi_1(\mathbf{x}), \psi_2(\mathbf{x}), \dots, \psi_{4M}(\mathbf{x})\right)^T \in \mathbb{R}^{4M}$$

with $\mathbf{x} = (x, y) \in \mathbb{R}^2$. Now we introduce an auxiliary function

$$C(\mathbf{x}) = \sum_{n \in V} \omega_n \psi_n(\mathbf{x})$$

and rewrite system (5.7) in the following form:

$$\begin{pmatrix} \mathbf{L} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\Psi}(\mathbf{x}) \\ C(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{e} \\ \mathbf{w}^T & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\Psi}(\mathbf{x}) \\ C(\mathbf{x}) \end{pmatrix}. \tag{5.8}$$

Here, $\mathbf{L}$ is a $4M \times 4M$ diagonal matrix, whose m-th diagonal entry reads $\varepsilon(c_m \partial_x + s_m \partial_y) + \sigma_T$ for $m \in V$. The vectors $\mathbf{e}, \mathbf{w} \in \mathbb{R}^{4M \times 1}$ are given by

$$\mathbf{e} = (\sigma_T - \varepsilon^2 \sigma_a)(1, 1, \dots, 1)^T, \quad \mathbf{w} = (\omega_1, \omega_2, \dots, \omega_{4M})^T.$$

System (5.8) contains $(4M+1)$ unknown functions $\boldsymbol{\Psi}(\mathbf{z})$ and $C(\mathbf{z})$.

In the following, we are going to find special solutions of system (5.8) in the form

$$\begin{pmatrix} \boldsymbol{\Psi}(\mathbf{z}) \\ C(\mathbf{z}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\xi} \\ \eta \end{pmatrix} \exp\left\{\frac{\lambda x + \mu y}{\varepsilon}\right\}. \tag{5.9}$$

In order to determine the nonzero vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \dots, \xi_{4M})^T$ and scalar constant η as well as λ and μ, we substitute (5.9) into (5.8). This yields a matrix eigenvalue problem: find $\lambda, \mu \in \mathbb{C}$ and nonzero vector $(\boldsymbol{\xi}, \eta) \in \mathbb{C}^{4M} \times \mathbb{C}$ such that

$$\begin{pmatrix} \mathbf{A} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi} \\ \eta \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{e} \\ \mathbf{w}^T & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi} \\ \eta \end{pmatrix}. \tag{5.10}$$

Here, $\mathbf{A} = \mathbf{A}(\lambda, \mu)$ is a $4M \times 4M$ diagonal matrix, whose m-th diagonal entry reads $c_m\lambda + s_m\mu + \sigma_T$ for $m \in V$. We define (λ, μ) to be an eigenvalue pair of the problem, if there exists a nonzero solution $\binom{\xi}{\eta}$ to system (5.10).

The eigenvalue pairs. Note that the eigenvalue pair (λ, μ) is a zero point of the characteristic polynomial:

$$\begin{aligned} p_{4M}(\lambda, \mu) &\equiv \det \begin{pmatrix} \mathbf{A}(\lambda, \mu) & -\mathbf{e} \\ -\mathbf{w}^T & 1 \end{pmatrix} \\ &= \prod_{m \in V} (c_m\lambda + s_m\mu + \sigma_T) - (\sigma_T - \varepsilon^2\sigma_a) \sum_{n \in V} \Big[\omega_n \prod_{m \neq n} (c_m\lambda + s_m\mu + \sigma_T)\Big]. \end{aligned} \tag{5.11}$$

We have either

$$\pi_{4M}(\lambda, \mu) \equiv \prod_{m \in V} (c_m\lambda + s_m\mu + \sigma_T) = 0$$

or

$$q_{4M}(\lambda, \mu) \equiv 1 - \sum_{n \in V} \frac{\omega_n(\sigma_T - \varepsilon^2\sigma_a)}{c_n\lambda + s_n\mu + \sigma_T} = 0.$$

A few characteristic curves when $\varepsilon = 0.1$ for $p_{4M}(\lambda, \mu)$ with $M = 1, 3, 6, 10$ can be found in [25].

The eigenvectors corresponding to (λ, μ). After (λ, μ) is obtained, the corresponding eigenvector $(\boldsymbol{\xi}, \eta)$ is given by the following two cases [24]:

(1) Suppose that $c_m\lambda + s_m\mu + \sigma_T \neq 0$ for all $m \in V$. Then we get

$$\xi_m = \frac{\sigma_T - \varepsilon^2\sigma_a}{c_m\lambda + s_m\mu + \sigma_T} \quad \text{for all } m \in V, \quad \text{and} \quad \eta = 1.$$

(2) Suppose there is at least one $m_1 \in V$ with $c_{m_1}\lambda + s_{m_1}\mu + \sigma_T = 0$. There exists another $m_2 \in V$ such that $\xi_{m_2} \neq 0$. Then the components of the eigenvector $(\boldsymbol{\xi}, \eta)$ are given by

$$\xi_m = \begin{cases} 0 & \text{for } m \neq m_1, m_2, \\ w_{m_2} & \text{for } m = m_1, \\ -w_{m_1} & \text{for } m = m_2 \end{cases} \quad \text{and} \quad \eta = 0.$$

Now it is clear that after an eigenvalue pair (a zero point (λ, μ) of (5.11)) is found, the corresponding eigenvector $(\boldsymbol{\xi}, \eta)$ can be obtained directly.

5.3 Tailored Finite Point Scheme

We now construct a five-point node-centered tailored finite point scheme for the boundary value problem (5.1)–(5.2) on the rectangular domain $\mathbf{D}$.

On the rectangular domain $\Omega = [0, a] \times [0, b]$, we have the grid nodes

$$\mathbf{z}_{i,j} = (x_i, y_j), \quad i = 0, 1, \dots, I, \; j = 0, 1, \dots, J.$$

Here I and J are two positive integers. Let $h_1 = a/I$ and $h_2 = b/J$ be two mesh parameters, $x_i = ih_1$ with $i = 0, 1, \dots, I$ and $y_j = jh_2$ with $j = 0, 1, \dots, J$.

For each interior grid node $\mathbf{z}_{i,j}$, which is not on the domain boundary, let

$$E_{i,j} = \{(x, y) \mid |x - x_i| \leq h_1, \; |y - y_j| \leq h_2\}$$

be the rectangular patch centered at $\mathbf{z}_{i,j}$. The four adjacent grid nodes $\{\mathbf{z}_{i+1,j}, \mathbf{z}_{i,j+1}, \mathbf{z}_{i-1,j}, \mathbf{z}_{i,j-1}\}$ are on the boundary of patch $E_{i,j}$.

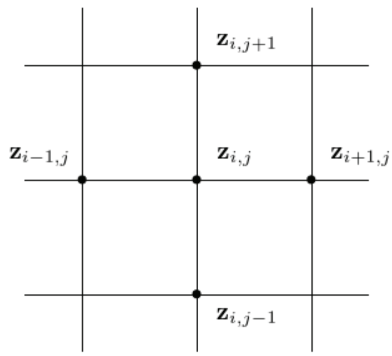

Figure 3. TFPM stencil for the five-point node-centered scheme.

On each patch $E_{i,j}$, we choose the values for σ_T, σ_a and q by their local averages on $E_{i,j}$.

That is, the discrete ordinates equations (5.1) on $E_{i,j}$ are approximated by the following first-order partial differential equations with constant coefficients:

$$\varepsilon\Big(c_m \frac{\partial}{\partial x}\tilde{\psi}_m + s_m \frac{\partial}{\partial y}\tilde{\psi}_m\Big) + \sigma_T\,\tilde{\psi}_m = \varepsilon^2 q + (\sigma_T - \varepsilon^2\sigma_a)\sum_{n\in V}\omega_n\tilde{\psi}_n \quad \text{for } m \in V. \tag{5.12}$$

Let $\mathbf{\Psi}^{(0)} = (\psi_1^{(0)}, \dots, \psi_{4M}^{(0)})^T = \frac{q}{\sigma_a}(1, 1, \dots, 1)^T \in \mathbb{R}^{4M}$. Then $\mathbf{\Psi}^{(0)}$ is a particular solution of the equations (5.12), and the difference $\mathbf{\Psi}(\mathbf{z}) = \tilde{\mathbf{\Psi}}(\mathbf{z}) - \mathbf{\Psi}^{(0)}$ satisfies the homogeneous equations

$$\varepsilon\Big(c_m \frac{\partial}{\partial x}\psi_m + s_m \frac{\partial}{\partial y}\psi_m\Big) + \sigma_T\psi_m = (\sigma_T - \varepsilon^2\sigma_a)\sum_{n\in V}\omega_n\psi_n \quad \text{for } m \in V. \tag{5.13}$$

Let K be a positive integer. We take K linearly independent special solutions of system (5.13) that have the form

$$\mathbf{\Psi}^{(k)}(\mathbf{z}) = \boldsymbol{\xi}^{(k)} \exp\Big\{\frac{\lambda_k(x - x_i) + \mu_k(y - y_j) - \max\{h_1|\lambda_k|, h_2|\mu_k|\}}{\varepsilon}\Big\} \quad \text{for } k = 1, 2, \dots, K.$$

Here, as discussed in Section 5.2, (λ_k, μ_k) is a real eigenvalue pair of system (5.13) and $\boldsymbol{\xi}^{(k)}$ is the eigenvector associated with (λ_k, μ_k) for $k = 1, 2, \dots, K$.

For any constants α_k ($k = 1, 2, \dots, K$), the vector-valued function

$$\tilde{\mathbf{\Psi}}(\mathbf{z}) = \mathbf{\Psi}^{(0)} + \sum_{k=1}^{K}\alpha_k\mathbf{\Psi}^{(k)}(\mathbf{z}) \tag{5.14}$$

is a solution to the nonhomogeneous system (5.12).

For each patch $E_{i,j}$, we choose four points $\mathbf{z}_{i+1,j}, \mathbf{z}_{i,j+1}, \mathbf{z}_{i-1,j}, \mathbf{z}_{i,j-1}$ on the boundary together with the center $\mathbf{z}_{i,j}$ to construct the five-point node-centered scheme for the local problem around $\mathbf{z}_{i,j}$ (see Figure 3).

The discrete in-flow boundary conditions for (5.12) are given by the $K = 4\cdot(4M/2) = 8M$ values at the four grid points $\mathbf{z}_{i+1,j}, \mathbf{z}_{i,j+1}, \mathbf{z}_{i-1,j}, \mathbf{z}_{i,j-1}$:

$$\tilde{\psi}_m(\mathbf{z}_{i+1,j}) \text{ with } c_m < 0, \quad \tilde{\psi}_m(\mathbf{z}_{i,j+1}) \text{ with } s_m < 0, \quad \tilde{\psi}_m(\mathbf{z}_{i-1,j}) \text{ with } c_m > 0, \quad \tilde{\psi}_m(\mathbf{z}_{i,j-1}) \text{ with } s_m > 0 \tag{5.15}$$

for $m \in V$. Then the constants α_k ($k = 1, 2, \dots, K$) in (5.14) are determined by the boundary conditions (5.15), namely, for $m \in V$,

$$\begin{cases} \psi_m^{(0)} + \sum_{k=1}^{K}\alpha_k\psi_m^{(k)}(\mathbf{z}_{i+1,j}) = \tilde{\psi}_m(\mathbf{z}_{i+1,j}) & \text{with } c_m < 0, \\ \psi_m^{(0)} + \sum_{k=1}^{K}\alpha_k\psi_m^{(k)}(\mathbf{z}_{i,j+1}) = \tilde{\psi}_m(\mathbf{z}_{i,j+1}) & \text{with } s_m < 0, \\ \psi_m^{(0)} + \sum_{k=1}^{K}\alpha_k\psi_m^{(k)}(\mathbf{z}_{i-1,j}) = \tilde{\psi}_m(\mathbf{z}_{i-1,j}) & \text{with } c_m > 0, \\ \psi_m^{(0)} + \sum_{k=1}^{K}\alpha_k\psi_m^{(k)}(\mathbf{z}_{i,j-1}) = \tilde{\psi}_m(\mathbf{z}_{i,j-1}) & \text{with } s_m > 0. \end{cases} \tag{5.16}$$

This is a system of $8M$ linear algebraic equations and the coefficients α_k ($k = 1, 2, \dots, K$) can be determined by (5.15). Moreover, from (5.14), at the point $\mathbf{z}_{i,j}$,

$$\tilde{\mathbf{\Psi}}(\mathbf{z}_{i,j}) = \mathbf{\Psi}^{(0)} + \sum_{k=1}^{K} \alpha_k \mathbf{\Psi}^{(k)}(\mathbf{z}_{i,j}). \tag{5.17}$$

If we express the constants α_k ($k = 1, 2, \dots, K$) in terms of the unknowns in (5.15) through solving (5.16), then (5.17) becomes a finite difference scheme that connects the unknowns at the grid node $\mathbf{z}_{i,j}$ with those at the four adjacent grid nodes. This is the five-point node-centered TFPM for the discrete ordinates equation.

Remark 5.1. If one of the grid nodes $\mathbf{z}_{i+1,j}$, $\mathbf{z}_{i,j+1}$, $\mathbf{z}_{i-1,j}$, $\mathbf{z}_{i,j-1}$ is on the physical boundary $\partial\mathbf{D}$ of the computational domain, we simply replace the corresponding component values $\tilde{\psi}_m$ with the physical boundary conditions (5.2).

Remark 5.2. The numerical examples given in [25] show that the two TFPM schemes discussed in this section are very effective, especially when the parameter ε is very small, when they can capture the boundary and interior layers of the solutions on coarse meshes.

6 Multiscale TFPM for Multiscale Elliptic Problems

6.1 Introduction

Second-order elliptic boundary value problems with rough or highly oscillatory coefficients arise in many applied fields, such as porous media and composite materials [1, 11, 64]. In this section, we discuss the numerical solution of the multiscale elliptic boundary value problem given by

$$\begin{cases} -\nabla \cdot (A^\varepsilon(\mathbf{x})\nabla u^\varepsilon(\mathbf{x})) + b^\varepsilon(\mathbf{x})u^\varepsilon(\mathbf{x}) = f^\varepsilon(\mathbf{x}) & \text{for all } \mathbf{x} = (x, y) \in \Omega, \\ u^\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega. \end{cases} \tag{6.1}$$

where $A^\varepsilon(\mathbf{x})$ is the given matrix function, the functions $A^\varepsilon(\mathbf{x})$, $b^\varepsilon(\mathbf{x})$, $f^\varepsilon(\mathbf{x})$ are highly oscillatory when the parameter ε is small, and $\Omega \subset \mathbb{R}^2$ is a bounded domain.

In practical applications, equation (6.1) describes the steady state heat conduction through a composite material, with $u^\varepsilon(\mathbf{x})$ and $A^\varepsilon(\mathbf{x})$ interpreted as the temperature and the thermal conductivity. Equation (6.1) is also the pressure equation in modeling two-phase flow in porous media, with $u^\varepsilon(\mathbf{x})$ and $A^\varepsilon(\mathbf{x})$ interpreted as the pressure and the relative permeability tensor [28].

In fact, $A^\varepsilon(\mathbf{x})$ is a 2×2 matrix function given by

$$A^\varepsilon(\mathbf{x}) = \left(a_{ij}^\varepsilon(\mathbf{x})\right)_{2\times 2}, \quad \mathbf{x} = (x, y) \in \Omega.$$

We assume that the matrix $A^\varepsilon(\mathbf{x})$ is positive definite with upper and lower bounds, namely, there exist positive constants m and M such that

$$m\|\xi\|^2 \le \sum_{i,j=1}^{2} a_{ij}^\varepsilon(\mathbf{x})\xi_i\xi_j \le M\|\xi\|^2, \quad \xi = (\xi_1, \xi_2)^T \in \mathbb{R}^2, \quad \mathbf{x} = (x, y) \in \overline{\Omega}.$$

$A^\varepsilon(\mathbf{x}) = (a_{ij}(\mathbf{x}, \varepsilon))_{2\times 2}$, $b^\varepsilon(\mathbf{x}) = b(\mathbf{x}, \varepsilon)$ and $f^\varepsilon(\mathbf{x}) = f(\mathbf{x}, \varepsilon)$ are oscillatory functions involving a small scale parameter ε. The main difficulty in finding the numerical solution of the given problem (6.1) is that the solution oscillates rapidly and requires a very fine mesh $h = \mathcal{O}(\varepsilon)$. This problem has attracted many researchers, for instance, the multiscale finite element method was given by Babuška and Osborn [3] for the one-dimensional case, and by Hou and Wu [28] for multidimensional cases. The heterogeneous multiscale method (HMM) was proposed by E and Engquist [10]. Wang, Guzman and Shu developed a multiscale discontinuous Galerkin (DG) method [67].

6.2 MsTFPM in Two-Dimensional Domain

We now discuss the numerical solution of problem (6.1) by the multiscale tailored finite point method (MsTFPM) in the two-dimensional case. Suppose that the domain Ω is $[0,1]\times[0,1]$.

Let $H = 1/N$ denote the coarse mesh size, where N is a positive integer. We divide the domain $\overline{\Omega}$ by a set of lines parallel to the x,y-axis to form a coarse mesh grid. The crossing points set Ω_H is called the coarse grid:

$$\Omega_H = \{(x_i, y_j) \mid x_i = iH,\ y_j = jH,\ i = 0,\dots,N,\ j = 0,\dots,N\}.$$

Suppose $U = \{u_{ij} \mid 0 \le i \le N,\ 0 \le j \le N\}$ is a grid function defined on the coarse gird Ω_H. We present two multiscale tailored finite point schemes to obtain the numerical solution of problem (6.1).

6.2.1 Numerical Scheme I

For each interior grid point (x_i, y_j) $(1 \le i \le N-1,\ 1 \le j \le N-1)$, we consider the local cell

$$\Omega_{ij} = \{(x,y) \mid (x-x_i)^2 + (y-y_j)^2 \le H^2\},$$

which is a disc with center $\mathbf{x}_0 = (x_i, y_j)$ and radius H in the domain $\overline{\Omega}$. On the boundary of the local cell Ω_{ij}, we take four points

$$\mathbf{x}_1 = (x_{i+1}, y_j),\quad \mathbf{x}_2 = (x_i, y_{j+1}),\quad \mathbf{x}_3 = (x_{i-1}, y_j),\quad \mathbf{x}_4 = (x_i, y_{j-1}).$$

We try to find a numerical scheme for the numerical solution u_{ij} at the points $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$. From the point of view of the TFPM, we only need to find four solutions of the homogeneous equation (6.1) and one solution of equation (6.1) on each local cell Ω_{ij}, but in this problem, we need a new idea to find these five solutions. A numerical method is used for this purpose.

First, on the circle $\partial\Omega_{ij}$, assume that the solution of equation (6.1) can be expanded as a Fourier series:

$$u^\varepsilon(\mathbf{x})|_{\partial\Omega_{ij}} = u^\varepsilon(H,\theta) = \frac{a_0^\varepsilon(H)}{2} + \sum_{n=1}^{\infty}\big(a_n^\varepsilon(H)\cos(n\theta) + b_n^\varepsilon(H)\sin(n\theta)\big),$$

where (H,θ) is the ordered pair of polar coordinates of $\mathbf{x} = (x,y) \in \partial\Omega_{ij}$ with the pole at $\mathbf{x}_0$. On each local cell Ω_{ij}, we consider the following cell problems:

$$\begin{cases} -\nabla\cdot(A^\varepsilon(\mathbf{x})\nabla U_0^\varepsilon(\mathbf{x})) + b^\varepsilon(\mathbf{x})U_0^\varepsilon(\mathbf{x}) = 0 & \text{for all } \mathbf{x}\in\Omega_{ij},\\ U_0^\varepsilon(\mathbf{x}) = 1, \quad \mathbf{x}\in\partial\Omega_{ij}, \end{cases} \tag{6.2}$$

$$\begin{cases} -\nabla\cdot(A^\varepsilon(\mathbf{x})\nabla U_n^\varepsilon(\mathbf{x})) + b^\varepsilon(\mathbf{x})U_n^\varepsilon(\mathbf{x}) = 0 & \text{for all } \mathbf{x}\in\Omega_{ij},\\ U_n^\varepsilon(\mathbf{x}) = \cos(n\theta), \quad \mathbf{x}\in\partial\Omega_{ij},\ n = 1,2,3,\dots, \end{cases} \tag{6.3}$$

$$\begin{cases} -\nabla\cdot(A^\varepsilon(\mathbf{x})\nabla V_n^\varepsilon(\mathbf{x})) + b^\varepsilon(\mathbf{x})V_n^\varepsilon(\mathbf{x}) = 0 & \text{for all } \mathbf{x}\in\Omega_{ij},\\ V_n^\varepsilon(\mathbf{x}) = \sin(n\theta), \quad \mathbf{x}\in\partial\Omega_{ij},\ n = 1,2,3,\dots, \end{cases} \tag{6.4}$$

$$\begin{cases} -\nabla\cdot(A^\varepsilon(\mathbf{x})\nabla U_f^\varepsilon(\mathbf{x})) + b^\varepsilon(\mathbf{x})U_f^\varepsilon(\mathbf{x}) = f^\varepsilon(\mathbf{x}) & \text{for all } \mathbf{x}\in\Omega_{ij},\\ U_f^\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x}\in\partial\Omega_{ij}. \end{cases} \tag{6.5}$$

The local cell problems (6.2)–(6.5) each have unique solutions. The solutions $U_0^\varepsilon(\mathbf{x})$, $U_n^\varepsilon(\mathbf{x})$, $V_n^\varepsilon(\mathbf{x})$, $n = 1,2,3,\dots$ of the local cell problems (6.2)–(6.4) form a complete basis for the homogeneous equation of problem (6.1) on the local cell Ω_{ij}. The solution $U_f^\varepsilon(\mathbf{x})$ comes from the inhomogeneous part, namely the forcing term $f^\varepsilon(\mathbf{x})$.

Furthermore we can see that on the local cell Ω_{ij}, the solution $u^\varepsilon(\mathbf{x})$ of problem (6.1) is given by

$$u^\varepsilon(\mathbf{x})|_{\partial\Omega_{ij}} = \frac{1}{2}a_0^\varepsilon(H)U_0^\varepsilon(\mathbf{x}) + \sum_{n=1}^{\infty}\big(a_n^\varepsilon(H)U_n^\varepsilon(\mathbf{x}) + b_n^\varepsilon(H)V_n^\varepsilon(\mathbf{x})\big) + U_f^\varepsilon(\mathbf{x}).$$

In practical computations, for instance in the five-point numerical scheme, we only have the solution values at the boundary grid points $\mathbf{x} = \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$. Therefore we take the solution in the space spanned by $U_0^\varepsilon(\mathbf{x})$, $U_1^\varepsilon(\mathbf{x})$, $V_1^\varepsilon(\mathbf{x})$, $U_2^\varepsilon(\mathbf{x})$ and $U_f^\varepsilon(\mathbf{x})$ to approximate the solution $u^\varepsilon(\mathbf{x})$. Recall that

$$U_0^\varepsilon(\mathbf{x})|_{\partial\Omega_{ij}} = 1, \quad U_1^\varepsilon(\mathbf{x})|_{\partial\Omega_{ij}} = \cos(\theta), \quad V_1^\varepsilon(\mathbf{x})|_{\partial\Omega_{ij}} = \sin(\theta), \quad U_2^\varepsilon(\mathbf{x})|_{\partial\Omega_{ij}} = \cos(2\theta).$$

We define the basis function as follows. Let

$$\begin{aligned} P_1^\varepsilon(\mathbf{x}) &= \frac{1}{4}U_0^\varepsilon(\mathbf{x}) + \frac{1}{2}U_1^\varepsilon(\mathbf{x}) + \frac{1}{4}U_2^\varepsilon(\mathbf{x}), \qquad & P_2^\varepsilon(\mathbf{x}) &= \frac{1}{4}U_0^\varepsilon(\mathbf{x}) + \frac{1}{2}V_1^\varepsilon(\mathbf{x}) - \frac{1}{4}U_2^\varepsilon(\mathbf{x}), \\ P_3^\varepsilon(\mathbf{x}) &= \frac{1}{4}U_0^\varepsilon(\mathbf{x}) - \frac{1}{2}U_1^\varepsilon(\mathbf{x}) + \frac{1}{4}U_2^\varepsilon(\mathbf{x}), \qquad & P_4^\varepsilon(\mathbf{x}) &= \frac{1}{4}U_0^\varepsilon(\mathbf{x}) - \frac{1}{2}V_1^\varepsilon(\mathbf{x}) - \frac{1}{4}U_2^\varepsilon(\mathbf{x}). \end{aligned}$$

Then at the boundary grid points $\mathbf{x} = \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$, we have $P_i^\varepsilon(\mathbf{x})|_{\mathbf{x}=\mathbf{x}_j} = \delta_{ij}$. On the local cell Ω_{ij}, let

$$u^{\varepsilon,F}(\mathbf{x}) = u^\varepsilon(\mathbf{x}_1)P_1^\varepsilon(\mathbf{x}) + u^\varepsilon(\mathbf{x}_2)P_2^\varepsilon(\mathbf{x}) + u^\varepsilon(\mathbf{x}_3)P_3^\varepsilon(\mathbf{x}) + u^\varepsilon(\mathbf{x}_4)P_4^\varepsilon(\mathbf{x}) + U_f^\varepsilon(\mathbf{x}) \tag{6.6}$$

denote the approximate solution based on the *Fourier approximation*. Moreover, let $E^\varepsilon(\mathbf{x}) = u^\varepsilon(\mathbf{x}) - u^{\varepsilon,F}(\mathbf{x})$ denote the error function. It is easy to see that $E^\varepsilon(\mathbf{x})$ satisfies

$$-\nabla\cdot\left(A^\varepsilon(\mathbf{x})\nabla E^\varepsilon(\mathbf{x})\right) + b^\varepsilon(\mathbf{x})E^\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x}\in\Omega_{ij}, \tag{6.7}$$

and

$$E^\varepsilon(\mathbf{x}_k) = 0, \quad \mathbf{x}_k \in \partial\Omega_{ij}, \; k = 1,2,3,4. \tag{6.8}$$

Therefore we use $u^{\varepsilon,F}(\mathbf{x})$ to approximate the solution of problem (6.1) on the cell Ω_{ij}. Plugging $\mathbf{x} = \mathbf{x}_0$ into the equation (6.6), we arrive at

$$u^\varepsilon(\mathbf{x}_0) \approx u^{\varepsilon,F}(\mathbf{x}_0) = u^\varepsilon(\mathbf{x}_1)P_1^\varepsilon(\mathbf{x}_0) + u^\varepsilon(\mathbf{x}_2)P_2^\varepsilon(\mathbf{x}_0) + u^\varepsilon(\mathbf{x}_3)P_3^\varepsilon(\mathbf{x}_0) + u^\varepsilon(\mathbf{x}_4)P_4^\varepsilon(\mathbf{x}_0) + U_f^\varepsilon(\mathbf{x}_0). \tag{6.9}$$

If we get all the solutions of the local cell problems (6.2)–(6.5), then from (6.9) we immediately obtain the following discrete scheme for problem (6.1) on the coarse grid Ω_H:

$$\begin{cases} u_{ij} = u_{i+1,j}p_{ij}^1 + u_{i,j+1}p_{ij}^2 + u_{i-1,j}p_{ij}^3 + u_{i,j-1}p_{ij}^4 + U_{ij}^f, & i,j = 1,\dots,N-1, \\ u_{ij} = 0, \quad j = 0 \text{ or } j = N, & \\ u_{ij} = 0, \quad i = 0 \text{ or } i = N, & \end{cases} \tag{6.10}$$

with $p_{ij}^k = P_k^\varepsilon(\mathbf{x}_0) = P_k^\varepsilon(x_i, y_j)$ $(k = 1,2,3,4)$ and $U_{ij}^f = U_f^\varepsilon(\mathbf{x}_0) = U_f^\varepsilon(x_i, y_j)$.

In general, we cannot obtain the exact solutions of the local problems (6.2)–(6.5). Therefore the coefficients p_{ij}^k $(k = 1,2,3,4)$ and U_{ij}^f in the scheme (6.10) are unknown and we cannot directly use the discrete scheme (6.10) to obtain the numerical solution of problem (6.1) on the coarse grid Ω_H. Therefore, we need to solve the local problems (6.2)–(6.5) numerically to obtain the numerical approximations of $\{P_k^\epsilon(\mathbf{x}),\ 1\le k\le 4;\ U_f^\epsilon(\mathbf{x})\}$. For example, if a finite difference scheme is used, the approximate solutions are obtained by $\{P_k^{\epsilon,h}(\mathbf{x}),\ 1\le k\le 4;\ U_f^{\epsilon,h}(\mathbf{x})\}$ with small mesh size $h \ll H$. Let $p_{ij}^{k,h} = P_k^{\varepsilon,h}(\mathbf{x}_0)$ $(k = 1,2,3,4)$ and $U_{ij}^{f,h} = U_f^{\varepsilon,h}(\mathbf{x}_0)$. Using $\{p_{ij}^{k,h},\ 1\le k\le 4;\ U_{ij}^{f,h}\}$ instead of $\{p_{ij}^k,\ 1\le k\le 4;\ U_{ij}^f\}$ in (6.10), we obtain a five-point multiscale tailored finite point scheme for problem (6.1).

6.2.2 Numerical Scheme II

We propose another multiscale tailored finite point scheme based on the *Lagrange interpolation approximation*. On the circle $\partial\Omega_{ij}$, we assume that the solution of equation (6.1) can be approximated by

$$u^\varepsilon(\mathbf{x})|_{\partial\Omega_{ij}} = u^\varepsilon(H,\theta) \approx u^\varepsilon(H,0)L_1(\theta) + u^\varepsilon\left(H,\frac{\pi}{2}\right)L_2(\theta) + u^\varepsilon(H,\pi)L_3(\theta) + u^\varepsilon\left(H,\frac{3\pi}{2}\right)L_4(\theta),$$

where $L_j(\theta)$ $(j = 1, 2, 3, 4)$ are four piecewise linear Lagrange interpolation basis functions on $\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$, $\theta_3 = \pi$, $\theta_4 = \frac{3\pi}{2}$. That is, for $j = 1, 2, 3, 4$, $L_j(\theta)$ are piecewise linear continuous functions of θ, $L_j(0) = L_j(2\pi)$ and

$$L_j(\theta_i) = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \tag{6.11}$$

On each local cell Ω_{ij}, we consider the following cell problems:

$$\begin{cases} -\nabla \cdot (A^\varepsilon(\mathbf{x}) \nabla W_k^\varepsilon(\mathbf{x})) + b^\varepsilon(\mathbf{x}) W_k^\varepsilon(\mathbf{x}) = 0 \quad \text{for all } \mathbf{x} \in \Omega_{ij}, \\ W_k^\varepsilon(\mathbf{x}) = L_k(\theta), \quad \mathbf{x} \in \partial\Omega_{ij}, \ k = 1, 2, 3, 4, \end{cases} \tag{6.12}$$

$$\begin{cases} -\nabla \cdot (A^\varepsilon(\mathbf{x}) \nabla W_f^\varepsilon(\mathbf{x})) + b^\varepsilon(\mathbf{x}) W_f^\varepsilon(\mathbf{x}) = f^\varepsilon(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \Omega_{ij}, \\ W_f^\varepsilon(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega_{ij}. \end{cases} \tag{6.13}$$

The local cell problems (6.12) and (6.13) each have unique solutions. Let

$$u^{\varepsilon,L}(\mathbf{x}) = u^\varepsilon(\mathbf{x}_1) W_1^\varepsilon(\mathbf{x}) + u^\varepsilon(\mathbf{x}_2) W_2^\varepsilon(\mathbf{x}) + u^\varepsilon(\mathbf{x}_3) W_3^\varepsilon(\mathbf{x}) + u^\varepsilon(\mathbf{x}_4) W_4^\varepsilon(\mathbf{x}) + W_f^\varepsilon(\mathbf{x}) \tag{6.14}$$

denote the approximate solution based on the Lagrange approximation. Moveover, let $E^\varepsilon(\mathbf{x}) = u^\varepsilon(\mathbf{x}) - u^{\varepsilon,L}(\mathbf{x})$ denote the error function. It is easy to see that $E^\varepsilon(\mathbf{x})$ also satisfies the equations (6.7)–(6.8). We use $u^{\varepsilon,L}(\mathbf{x})$ to approximate the solution of problem (6.1) on Ω_{ij}. Plugging $\mathbf{x} = \mathbf{x}_0$ into the equation (6.14), we arrive at

$$u^\varepsilon(\mathbf{x}_0) \approx u^{\varepsilon,L}(\mathbf{x}_0) = u^\varepsilon(\mathbf{x}_1) W_1^\varepsilon(\mathbf{x}_0) + u^\varepsilon(\mathbf{x}_2) W_2^\varepsilon(\mathbf{x}_0) + u^\varepsilon(\mathbf{x}_3) W_3^\varepsilon(\mathbf{x}_0) + u^\varepsilon(\mathbf{x}_4) W_4^\varepsilon(\mathbf{x}_0) + W_f^\varepsilon(\mathbf{x}_0). \tag{6.15}$$

If we get all the solutions of the local cell problems (6.12) and (6.13), then from (6.15) we immediately obtain the following discrete scheme for problem (6.1) on the coarse grid Ω_H:

$$\begin{cases} u_{ij} = u_{i+1,j} w_{ij}^1 + u_{i,j+1} w_{ij}^2 + u_{i-1,j} w_{ij}^3 + u_{i,j-1} w_{ij}^4 + w_{ij}^f, \quad i, j = 1, \dots, N-1, \\ u_{ij} = 0, \quad j = 0 \text{ or } j = N, \\ u_{ij} = 0, \quad i = 0 \text{ or } i = N, \end{cases} \tag{6.16}$$

with $w_{ij}^k = W_k^\varepsilon(\mathbf{x}_0) = W_k^\varepsilon(x_i, y_j)$ $(k = 1, 2, 3, 4)$ and $w_{ij}^f = W_f^\varepsilon(\mathbf{x}_0) = W_f^\varepsilon(x_i, y_j)$. From the properties of the Lagrange interpolation basis functions (6.11) and the maximum principle [49], we obtain the following lemma.

Lemma 6.1. *In the discrete scheme* (6.16) *for problem* (6.1), *we have the estimates*

$$0 < w_{ij}^k < 1, \quad 1 \le k \le 4, \qquad 0 < \sum_{k=1}^{4} w_{ij}^k \le 1$$

for $i, j = 1, \dots, N-1$.

From Lemma 6.1, we immediately find that the matrix of the linear equation system (6.16) is diagonal dominant, thus it is invertible. Therefore the discrete scheme (6.16) for problem (6.1) has a unique solution $U = \{u_{ij} \mid 0 \le i \le N, \ 0 \le j \le N\}$.

Before the scheme (6.16) is used, we need to obtain the approximate values of $\{w_{ij}^k, \ 1 \le k \le 4; \ w_{ij}^f\}$ by solving the local problems (6.12)–(6.13) as mentioned in Section 6.2.2.

In the multiscale tailored finite point method (MsTFPM), the construction of the base functions and load functions is fully decoupled from cell to cell; thus, this method is perfectly parallel and is naturally adapted to massively parallel computers. At each cell, after we obtain the numerical solutions of the five local problems, the MsTFPM scheme will be obtained directly.

7 Conclusion

In this paper, a review of the tailored finite point method (TFPM) is given. The TFPM is a new approach for the numerical solution of partial differential equations (PDEs). Essentially, the construction of a discrete scheme of a given PDE can be understood as a function approximation problem, in which the function is a solution of the given PDE. The TFPM implement the approximation of the solution to the given PDEs at each point from a different point of view, using the solutions of the locally reduced problems to construct the finite dimensional function space W^k (see Section 2). The tailored finite point schemes preserve the important properties of the original problems, such as *discrete maximum principle*, automatically. For singular perturbation problems, the TFPM can also achieve good accuracy on all mesh points, even when the mesh size $h \gg \varepsilon$, in some cases without any prior knowledge of the boundary layers. We also prove the uniform convergence of the TFPM for the boundary value problem of the Helmholtz equation with high wave numbers in some cases. Applications to the transport equation with interfaces and the multiscale elliptic problems are also reviewed. All of the numerical results support our mathematical theory.

The development of TFPM is at its beginning. There are still many interesting possibilities that remain to be explored, for example, the uniformly convergent analysis for singular perturbation problems or the construction of schemes for singularly perturbed eigenvalue problems. Although many questions are still open, we have seen that the framework of TFPM provides a basic idea for systematically designing discrete schemes in a wide variety of applications. Thus the TFPM may apply to more problems successfully in the future.

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