

Research Article

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Functional A Posteriori Error Estimates for Parabolic Time-Periodic Boundary Value Problems

Abstract: The paper is concerned with parabolic time-periodic boundary value problems which are of theoretical interest and arise in different practical applications. The multiharmonic finite element method is well adapted to this class of parabolic problems. We study properties of multiharmonic approximations and derive guaranteed and fully computable bounds of approximation errors. For this purpose, we use the functional *a posteriori* error estimation techniques earlier introduced by S. Repin. Numerical tests confirm the efficiency of the *a posteriori* error bounds derived.

Keywords: Parabolic Time-Periodic Boundary Value Problems, Multiharmonic Finite Element Methods, Functional A Posteriori Error Estimates

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1 Introduction

Initial-boundary value problems for parabolic equations describe various physical phenomena such as heat conduction, diffusion, chemical reactions, biological processes, and transient electromagnetic fields. The numerical simulation of these phenomena is usually based on time-integration methods together with a suitable space discretization, see, e.g., the well-known monograph [33] and references therein. In many practically interesting cases, for instance, in electromagnetics and chemistry, the processes are time-periodic, see, e.g., [1]. In this case, the initial condition must be replaced by the time-periodicity condition. Standard time-integration methods may be less efficient than methods based on approximations in terms of Fourier series. This paper deals with this type of approximations. In fact, it is devoted to the *a posteriori* error analysis of parabolic time-periodic boundary value problems in connection with their multiharmonic finite element discretization. More precisely, all functions are expanded into Fourier series, approximations are presented by truncated series and the Fourier coefficients are approximated by the finite element method (FEM). This so-called multiharmonic FEM (MhFEM) or harmonic-balanced FEM is a promising alternative to the existing temporal discretization schemes combined with the FEM, and is more on the lines of combining Laplace transformation tools with the FEM as used by Thomée and his collaborators for linear parabolic problems in $(0, \infty)$, see, e.g., [29, 30].

The MhFEM was successfully used for the simulation of electromagnetic devices described by nonlinear eddy current problems with harmonic excitations, see, e.g., [2, 3, 7, 35] and references therein. Later, this discretization technique has been applied to linear time-periodic parabolic boundary value and optimal con-

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trol problems [12, 13, 19, 22, 34] and to linear time-periodic eddy current problems and the corresponding optimal control problems [14–16]. In this framework, we deduce *a posteriori* error estimates which provide guaranteed and fully computable upper bounds (majorants) of the respective errors. To the best of our knowledge these estimates are new. Our approach is based on the works of Repin, see, e.g., the papers on parabolic problems [10, 25] as well as on optimal control problems [8, 9], the books [23, 26], and the references therein. In particular, our *a posteriori* error analysis uses the techniques close to the one suggested in [25], but the analysis contains essential changes. In the MhFEM setting, we are able to establish inf-sup and sup-sup conditions from which we deduce existence and uniqueness of solutions to parabolic time-periodic problems by applying the Babuška–Aziz theorem. Then, we deduce the *a posteriori* estimates, which are very well adapted to the nature of the multiharmonic solution because they can judge on the quality of approximation for any particular harmonic. This is important for linear time-periodic parabolic problems, where the computations of the Fourier coefficients corresponding to every single mode $k = 0, 1, \dots$ are decoupled. Hence, we can use different meshes independently generated by adaptive finite element approximations to the Fourier coefficients for different modes.

Let us mention here that the majorant suggests a clear way how to improve the reconstruction of the flux, which is important for parabolic time-periodic problems as well. First, we can detect the situation where the improvement is indeed necessary because there the norm of the error in the equilibrium equation dominates the majorant, and second, we have different efficient minimization procedures able to efficiently minimize this term and improve the balance with the other terms. Procedures of such a type are well studied for elliptic partial differential equations, see, e.g., [23], and can be used for multiharmonic approximations with minimal modifications. Moreover, it is well verified that under certain (non-restrictive) assumptions for elliptic problems the term of the majorant corresponding to the error in the constitutive relations is a good and robust indicator of local errors, see [27], and that this term tends to the exact error if a minimizing sequence of the majorant is provided, see [26]. All the same holds for the spatial parts of the multiharmonic approximations of the flux. This is the reason why we indeed have good error indicators for possible mesh adaptations. In order to finally realize the adaptivity in time, we then filter out the Fourier coefficients, which are most important for the numerical solution of the problem, by prescribing certain bounds. Altogether, such an adaptive multiharmonic finite element method (AMhFEM) yields complete adaptivity in space and time. This work is a starting point for the construction of AMhFEM, which utilizes the above principles. However, in this work we are not focused on mesh adaptation issues. This will be the subject of a separate paper. Our goal is to provide a detailed *a posteriori* error analysis of a parabolic time-periodic boundary value problem in the context of the MhFEM leading to guaranteed, computable upper bounds with efficiency indices close to one.

The paper is organized as follows. In Section 2, we discuss a space-time variational formulation for parabolic time-periodic boundary value problems that forms the basis of the MhFEM considered in Section 3. Section 4 is devoted to the derivation of functional type *a posteriori* error estimates adapted to problems in question. Finally, in Section 5, we discuss some implementation issues and present first numerical results.

2 A Parabolic Time-Periodic Boundary Value Problem

Let $Q_T := \Omega \times (0, T)$ denote the space-time cylinder and $\Sigma_T := \Gamma \times (0, T)$ its mantle boundary, where $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$ is a bounded Lipschitz domain with the boundary Γ , and $(0, T)$ is a given time interval. The following parabolic time-periodic boundary value problem is considered: Find u such that

$$\sigma(\mathbf{x}) \partial_t u(\mathbf{x}, t) - \operatorname{div}(v(\mathbf{x}) \nabla u(\mathbf{x}, t)) = f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in Q_T, \quad (2.1)$$

$$u(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \Sigma_T, \quad (2.2)$$

$$u(\mathbf{x}, 0) = u(\mathbf{x}, T), \quad \mathbf{x} \in \overline{\Omega}, \quad (2.3)$$

where $f(\mathbf{x}, t)$ is a given function in $L^2(Q_T)$, and $\sigma(\cdot)$ and $v(\cdot)$ satisfy the assumptions

$$0 < \underline{\sigma} \leq \sigma(\mathbf{x}) \leq \overline{\sigma}, \quad 0 < \underline{v} \leq v(\mathbf{x}) \leq \overline{v}, \quad \mathbf{x} \in \Omega. \quad (2.4)$$

In order to study the parabolic time-periodic boundary value problem (2.1)–(2.3), we will derive a space-time variational formulation in spaces of functions defined in the space-time cylinder Q_T using the approach similar to the one used by Ladyzhenskaya et al. [20, 21]. Let the Sobolev spaces

$$\begin{aligned} H^{1,0}(Q_T) &= \{u \in L^2(Q_T) : \nabla u \in [L^2(Q_T)]^d\}, \\ H^{1,1}(Q_T) &= \{u \in L^2(Q_T) : \nabla u \in [L^2(Q_T)]^d, \partial_t u \in L^2(Q_T)\} \end{aligned}$$

be equipped with the norms

$$\begin{aligned} \|u\|_{H^{1,0}(Q_T)} &:= \left(\int_{Q_T} (u(\mathbf{x}, t)^2 + |\nabla u(\mathbf{x}, t)|^2) d\mathbf{x} dt \right)^{1/2}, \\ \|u\|_{H^{1,1}(Q_T)} &:= \left(\int_{Q_T} (u(\mathbf{x}, t)^2 + |\nabla u(\mathbf{x}, t)|^2 + |\partial_t u(\mathbf{x}, t)|^2) d\mathbf{x} dt \right)^{1/2}, \end{aligned}$$

respectively, where $\nabla = \nabla_{\mathbf{x}}$ and ∂_t denote the generalized derivatives with respect to $\mathbf{x}$ and t . The space

$$H^{0,1}(Q_T) = \{u \in L^2(Q_T) : \partial_t u \in L^2(Q_T)\}$$

is defined analogously. Furthermore, the boundary and time-periodicity conditions are included by defining the spaces

$$\begin{aligned} H_0^{1,0}(Q_T) &= \{u \in H^{1,0}(Q_T) : u = 0 \text{ on } \Sigma_T\}, \\ H_0^{1,1}(Q_T) &= \{u \in H^{1,1}(Q_T) : u = 0 \text{ on } \Sigma_T\}, \\ H_{\text{per}}^{0,1}(Q_T) &= \{u \in H^{0,1}(Q_T) : u(\mathbf{x}, 0) = u(\mathbf{x}, T) \text{ for almost all } \mathbf{x} \in \Omega\}, \\ H_{\text{per}}^{1,1}(Q_T) &= \{u \in H^{1,1}(Q_T) : u(\mathbf{x}, 0) = u(\mathbf{x}, T) \text{ for almost all } \mathbf{x} \in \Omega\}, \\ H_{0,\text{per}}^{1,1}(Q_T) &= \{u \in H_0^{1,1}(Q_T) : u(\mathbf{x}, 0) = u(\mathbf{x}, T) \text{ for almost all } \mathbf{x} \in \Omega\}. \end{aligned}$$

For ease of notation, all inner products and norms in L^2 are denoted by $(\cdot, \cdot)$ and $\|\cdot\|$, if they are related to the whole space-time domain Q_T . If they are associated with the spatial domain Ω , then we write $(\cdot, \cdot)_{\Omega}$ and $\|\cdot\|_{\Omega}$, which denote the standard inner products and norms of the space $L^2(\Omega)$. The symbols $(\cdot, \cdot)_{1,\Omega}$ and $\|\cdot\|_{1,\Omega}$ denote the standard inner products and norms of $H^1(\Omega)$.

The functions used in our analysis will typically be presented as Fourier series, i.e.,

$$v(\mathbf{x}, t) = v_0^c(\mathbf{x}) + \sum_{k=1}^{\infty} (v_k^c(\mathbf{x}) \cos(k\omega t) + v_k^s(\mathbf{x}) \sin(k\omega t)) \quad (2.5)$$

with the Fourier coefficients

$$v_0^c(\mathbf{x}) = \frac{1}{T} \int_0^T v(\mathbf{x}, t) dt, \quad v_k^c(\mathbf{x}) = \frac{2}{T} \int_0^T v(\mathbf{x}, t) \cos(k\omega t) dt, \quad v_k^s(\mathbf{x}) = \frac{2}{T} \int_0^T v(\mathbf{x}, t) \sin(k\omega t) dt,$$

where T and $\omega = 2\pi/T$ denote the periodicity and the frequency, respectively. Moreover, we define additional spaces of functions, see [22], in order to derive a symmetric variational formulation of problem (2.1)–(2.3). The function spaces $H_{\text{per}}^{0,1/2}(Q_T)$, $H_{\text{per}}^{1,1/2}(Q_T)$ and $H_{0,\text{per}}^{1,1/2}(Q_T)$ are defined by

$$\begin{aligned} H_{\text{per}}^{0,1/2}(Q_T) &= \{u \in L^2(Q_T) : \|\partial_t^{1/2} u\| < \infty\}, \\ H_{\text{per}}^{1,1/2}(Q_T) &= \{u \in H^{1,0}(Q_T) : \|\partial_t^{1/2} u\| < \infty\}, \\ H_{0,\text{per}}^{1,1/2}(Q_T) &= \{u \in H_{\text{per}}^{1,1/2}(Q_T) : u = 0 \text{ on } \Sigma_T\}, \end{aligned}$$

respectively, where $\|\partial_t^{1/2} u\|$ is defined in the Fourier space by the relation

$$\|\partial_t^{1/2} u\|^2 := |u|_{H_{0,\text{per}}^{0,1/2}(Q_T)}^2 := \frac{T}{2} \sum_{k=1}^{\infty} k\omega \|u_k\|_{\Omega}^2, \quad (2.6)$$

where $\mathbf{u}_k := (u_k^c, u_k^s)^T$ for all $k \in \mathbb{N}$. These spaces are equipped with the scalar products

$$(\partial_t^{1/2} u, \partial_t^{1/2} v) := \frac{T}{2} \sum_{k=1}^{\infty} k\omega(\mathbf{u}_k, \mathbf{v}_k)_{\Omega}, \quad (\sigma \partial_t^{1/2} u, \partial_t^{1/2} v) := \frac{T}{2} \sum_{k=1}^{\infty} k\omega(\sigma \mathbf{u}_k, \mathbf{v}_k)_{\Omega}. \quad (2.7)$$

The seminorm and the norm of the space $H_{\text{per}}^{1,1/2}(Q_T)$ are defined by the relations

$$\begin{aligned} \|u\|_{H^{1,1/2}(Q_T)}^2 &:= \|\nabla u\|^2 + \|\partial_t^{1/2} u\|^2 = T \|\nabla u_0^c\|_{\Omega}^2 + \frac{T}{2} \sum_{k=1}^{\infty} (k\omega \|\mathbf{u}_k\|_{\Omega}^2 + \|\nabla \mathbf{u}_k\|_{\Omega}^2), \\ \|u\|_{H^{1,1/2}(Q_T)}^2 &:= \|u\|^2 + \|u\|_{H^{1,1/2}(Q_T)}^2 = T (\|u_0^c\|_{\Omega}^2 + \|\nabla u_0^c\|_{\Omega}^2) + \frac{T}{2} \sum_{k=1}^{\infty} ((1+k\omega) \|\mathbf{u}_k\|_{\Omega}^2 + \|\nabla \mathbf{u}_k\|_{\Omega}^2), \end{aligned}$$

respectively. Furthermore, we define

$$v^{\perp}(\mathbf{x}, t) := \sum_{k=1}^{\infty} (-v_k^c(\mathbf{x}) \sin(k\omega t) + v_k^s(\mathbf{x}) \cos(k\omega t)) = \sum_{k=1}^{\infty} \underbrace{(v_k^s(\mathbf{x}), -v_k^c(\mathbf{x}))}_{=: (-\mathbf{v}_k^{\perp})^T} \cdot \begin{pmatrix} \cos(k\omega t) \\ \sin(k\omega t) \end{pmatrix}. \quad (2.8)$$

Note that the relation $\|\mathbf{u}_k^{\perp}\|_{\Omega}^2 = \|\mathbf{u}_k\|_{\Omega}^2$ is valid.

Lemma 2.1. *The identities*

$$(\sigma \partial_t^{1/2} u, \partial_t^{1/2} v) = (\sigma \partial_t u, v^{\perp}) \quad \text{and} \quad (\sigma \partial_t^{1/2} u, \partial_t^{1/2} v^{\perp}) = (\sigma \partial_t u, v) \quad (2.9)$$

are valid for all $u \in H_{\text{per}}^{0,1}(Q_T)$ and $v \in H_{\text{per}}^{0,1/2}(Q_T)$.

Proof. Using the definition of the σ -weighted scalar product in (2.7) and inserting the Fourier expansions of

$$\partial_t u(\mathbf{x}, t) := \sum_{k=1}^{\infty} (k\omega u_k^s(\mathbf{x}) \cos(k\omega t) - k\omega u_k^c(\mathbf{x}) \sin(k\omega t))$$

as well as (2.8) into the inner products, we obtain

$$(\sigma \partial_t^{1/2} u, \partial_t^{1/2} v) = \frac{T}{2} \sum_{k=1}^{\infty} k\omega(\sigma \mathbf{u}_k, \mathbf{v}_k)_{\Omega} = \frac{T}{2} \sum_{k=1}^{\infty} k\omega(\sigma \mathbf{u}_k^{\perp}, \mathbf{v}_k^{\perp})_{\Omega} = \frac{T}{2} \sum_{k=1}^{\infty} k\omega(\sigma(-\mathbf{u}_k^{\perp}), (-\mathbf{v}_k^{\perp}))_{\Omega} = (\sigma \partial_t u, v^{\perp})$$

with $\mathbf{u}_k^{\perp} = (-u_k^s, u_k^c)^T$ for all $k \in \mathbb{N}$, and

$$(\sigma \partial_t^{1/2} u, \partial_t^{1/2} v^{\perp}) = \frac{T}{2} \sum_{k=1}^{\infty} k\omega(\sigma \mathbf{u}_k, \mathbf{v}_k^{\perp})_{\Omega} = \frac{T}{2} \sum_{k=1}^{\infty} k\omega(\sigma(-\mathbf{u}_k^{\perp}), \mathbf{v}_k)_{\Omega} = (\sigma \partial_t u, v). \quad \square$$

Hence, the following orthogonality relations hold:

$$\begin{cases} (\sigma \partial_t u, u) = 0 & \text{and} & (\sigma u^{\perp}, u) = 0 & \text{for all } u \in H_{\text{per}}^{0,1}(Q_T), \\ (\sigma \partial_t^{1/2} u, \partial_t^{1/2} u^{\perp}) = 0 & \text{and} & (v \nabla u, \nabla u^{\perp}) = 0 & \text{for all } u \in H_{\text{per}}^{1,1/2}(Q_T), \end{cases} \quad (2.10)$$

where, e.g.,

$$(v \nabla u, \nabla u^{\perp}) = \sum_{k=1}^{\infty} (v \nabla \mathbf{u}_k, \nabla \mathbf{u}_k^{\perp})_{\Omega} = 0 \quad \text{for all } u \in H_{\text{per}}^{1,1/2}(Q_T)$$

with $\nabla \mathbf{u}_k := ((\nabla u_k^c)^T, (\nabla u_k^s)^T)^T$ and $\nabla \mathbf{u}_k^{\perp} := (-\nabla u_k^s)^T, (\nabla u_k^c)^T)^T$ for all $k \in \mathbb{N}$. The identity

$$\int_0^T \xi \partial_t^{1/2} v^{\perp} dt = - \int_0^T \partial_t^{1/2} \xi^{\perp} v dt \quad \text{for all } \xi, v \in H_{\text{per}}^{0,1/2}(Q_T)$$

is also defined in the Fourier space yielding the definitions

$$(\xi, \partial_t^{1/2} v) := \frac{T}{2} \sum_{k=1}^{\infty} (k\omega)^{1/2} (\xi_k, \mathbf{v}_k)_{\Omega}$$

as well as

$$\partial_t^{1/2} \xi(\mathbf{x}, t) := \sum_{k=1}^{\infty} (k\omega)^{1/2} (\xi_k^c(\mathbf{x}) \cos(k\omega t) + \xi_k^s(\mathbf{x}) \sin(k\omega t))$$

and

$$\partial_t^{1/2} \xi^\perp(\mathbf{x}, t) := \sum_{k=1}^{\infty} (k\omega)^{1/2} (-\xi_k^s(\mathbf{x}) \cos(k\omega t) + \xi_k^c(\mathbf{x}) \sin(k\omega t)).$$

Hence,

$$\begin{aligned} (\xi, \partial_t^{1/2} v^\perp) &= \frac{T}{2} \sum_{k=1}^{\infty} (k\omega)^{1/2} (\xi_k, \mathbf{v}_k^\perp)_\Omega = -(\partial_t^{1/2} \xi, v^\perp), \\ (\xi, \partial_t^{1/2} v) &= \frac{T}{2} \sum_{k=1}^{\infty} (k\omega)^{1/2} (\xi_k, \mathbf{v}_k)_\Omega = \frac{T}{2} \sum_{k=1}^{\infty} (k\omega)^{1/2} (-\xi_k^\perp, \mathbf{v}_k)_\Omega \\ &= -\frac{T}{2} \sum_{k=1}^{\infty} (k\omega)^{1/2} (\xi_k^\perp, \mathbf{v}_k)_\Omega = -(\partial_t^{1/2} \xi^\perp, v) \end{aligned}$$

and all these identities coincide with the identities (2.9) in Lemma 2.1.

We note that for functions presented in terms of Fourier series the standard Friedrichs inequality holds in the form

$$\|\nabla u\|^2 = \int_{Q_T} |\nabla u|^2 d\mathbf{x} dt = T \|\nabla u_0^c\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^{\infty} \|\nabla \mathbf{u}_k\|_\Omega^2 \geq \frac{1}{C_F^2} \left(T \|u_0^c\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^{\infty} \|\mathbf{u}_k\|_\Omega^2 \right) = \frac{1}{C_F^2} \|u\|^2. \quad (2.11)$$

In order to derive the space-time variational formulation of the parabolic time-periodic problem (2.1)–(2.3), the parabolic partial differential equation (2.1) is multiplied by a test function $v \in H_{0,\text{per}}^{1,1/2}(Q_T)$, integrated over the space-time cylinder Q_T , and after integration by parts with respect to the space and time variables, the following “symmetric” space-time variational formulation of the parabolic time-periodic boundary value problem (2.1)–(2.3) is obtained: Given $f \in L^2(Q_T)$, find $u \in H_{0,\text{per}}^{1,1/2}(Q_T)$ such that

$$a(u, v) = \int_{Q_T} f(\mathbf{x}, t) v(\mathbf{x}, t) d\mathbf{x} dt \quad \text{for all } v \in H_{0,\text{per}}^{1,1/2}(Q_T) \quad (2.12)$$

with the space-time bilinear form

$$a(u, v) = \int_{Q_T} (\sigma(\mathbf{x}) \partial_t^{1/2} u(\mathbf{x}, t) \partial_t^{1/2} v^\perp(\mathbf{x}, t) + v(\mathbf{x}) \nabla u(\mathbf{x}, t) \cdot \nabla v(\mathbf{x}, t)) d\mathbf{x} dt, \quad (2.13)$$

where all functions are given in their Fourier series expansion in time, i.e., everything has to be understood in the sense of (2.6) and (2.7). In particular, this Fourier series approach makes sense due to the time-periodicity condition (for u and v).

3 Multiharmonic Finite Element Approximation

Inserting the Fourier series ansatz (2.5) into (2.12) and exploiting the orthogonality of the functions $\cos(k\omega t)$ and $\sin(k\omega t)$ with respect to the inner product $(\cdot, \cdot)_{L^2(0,T)}$, we arrive at the following variational formulation corresponding to every single mode $k \in \mathbb{N}$: Given $\mathbf{f}_k \in (L^2(\Omega))^2$, find $\mathbf{u}_k \in \mathbb{V} := V \times V = (H_0^1(\Omega))^2$ such that

$$\int_{\Omega} (v(\mathbf{x}) \nabla \mathbf{u}_k(\mathbf{x}) \cdot \nabla \mathbf{v}_k(\mathbf{x}) + k\omega \sigma(\mathbf{x}) \mathbf{u}_k(\mathbf{x}) \cdot \mathbf{v}_k^\perp(\mathbf{x})) d\mathbf{x} = \int_{\Omega} \mathbf{f}_k(\mathbf{x}) \cdot \mathbf{v}_k(\mathbf{x}) d\mathbf{x} \quad (3.1)$$

for all $\mathbf{v}_k \in \mathbb{V}$. In the case $k = 0$, we obtain the following variational formulation: Given $f_0^c \in L^2(\Omega)$, find $u_0^c \in V = H_0^1(\Omega)$ such that

$$\int_{\Omega} v(\mathbf{x}) \nabla u_0^c(\mathbf{x}) \cdot \nabla v_0^c(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f_0^c(\mathbf{x}) v_0^c(\mathbf{x}) d\mathbf{x} \quad (3.2)$$

for all $v_0^c \in V$. The variational problems (3.1) and (3.2) have a unique solution due to the Babuška–Aziz theorem, see [34]. In order to solve these problems numerically, the Fourier series are truncated at a finite index N and the unknown Fourier coefficients $\mathbf{u}_k = (u_k^c, u_k^s)^T \in \mathbb{V}$ are approximated by finite element functions $\mathbf{u}_{kh} = (u_{kh}^c, u_{kh}^s)^T \in \mathbb{V}_h = V_h \times V_h \subset \mathbb{V}$. Here, $V_h = \text{span}\{\varphi_1, \dots, \varphi_n\}$ with the standard nodal basis $\{\varphi_i(\mathbf{x}) = \varphi_{ih}(\mathbf{x}) : i = 1, 2, \dots, n_h\}$, and h denotes the usual discretization parameter such that $n = n_h = \dim V_h = O(h^{-d})$. We use continuous, piecewise linear functions on the finite elements on a regular triangulation $\mathcal{T}_h$ to construct the finite element subspace V_h and its basis, see, e.g., [4, 6, 11, 32]. However, the analysis can also be applied to more general higher-order finite elements, see, for instance, [36] and the monographs [6, 17, 31]. Under the assumptions (2.4), we then obtain the following saddle point system by using continuous, piecewise linear finite elements:

$$\begin{pmatrix} k\omega M_{h,\sigma} & -K_{h,v} \\ -K_{h,v} & -k\omega M_{h,\sigma} \end{pmatrix} \begin{pmatrix} \underline{u}_k^s \\ \underline{u}_k^c \end{pmatrix} = \begin{pmatrix} -\underline{f}_k^c \\ -\underline{f}_k^s \end{pmatrix}. \quad (3.3)$$

Equation (3.3) must be solved with respect to the nodal parameter vectors $\underline{u}_k^s = (u_{k,i}^s)_{i=1,\dots,n} \in \mathbb{R}^n$ and $\underline{u}_k^c = (u_{k,i}^c)_{i=1,\dots,n} \in \mathbb{R}^n$ of the corresponding finite element approximations. The matrices $K_{h,v}$ and $M_{h,\sigma}$ correspond to the weighted stiffness and mass matrices, respectively, whereas $\underline{f}_k^c$ and $\underline{f}_k^s$ are the load vectors. In the case $k = 0$, the system of linear simultaneous equations

$$K_{h,v} \underline{u}_0^c = \underline{f}_0^c \quad (3.4)$$

follows from the variational problem (3.2). Fast and robust solvers well adapted to systems (3.3) and (3.4) can be found in [13, 18, 22, 34]. We use these solvers in order to obtain the multiharmonic finite element approximation

$$u_{Nh}(\mathbf{x}, t) = u_{0h}^c(\mathbf{x}) + \sum_{k=1}^N (u_{kh}^c(\mathbf{x}) \cos(k\omega t) + u_{kh}^s(\mathbf{x}) \sin(k\omega t))$$

of the exact solution $u(\mathbf{x}, t)$. Computable *a posteriori* estimates of the difference between u_{Nh} and u are obtained in the next section.

4 Functional A Posteriori Error Estimates

First, we present inf-sup and sup-sup conditions for the bilinear form (2.13).

Lemma 4.1. *The space-time bilinear form $a(\cdot, \cdot)$ defined by (2.13) satisfies the following inf-sup and sup-sup conditions:*

$$\mu_1 \|u\|_{H^{1,1/2}(Q_T)} \leq \sup_{0 \neq v \in H_{0,\text{per}}^{1,1/2}(Q_T)} \frac{a(u, v)}{\|v\|_{H^{1,1/2}(Q_T)}} \leq \mu_2 \|u\|_{H^{1,1/2}(Q_T)} \quad (4.1)$$

for all $u \in H_{0,\text{per}}^{1,1/2}(Q_T)$ with positive constants

$$\mu_1 = \frac{1}{\sqrt{2}} \min \left\{ \frac{\underline{v}}{C_F^2 + 1}, \underline{\sigma} \right\} \quad \text{and} \quad \mu_2 = \max \{ \overline{\sigma}, \overline{v} \},$$

where C_F is the constant coming from the Friedrichs inequality.

Proof. Using the triangle and Cauchy–Schwarz inequalities, we obtain the estimate

$$\begin{aligned} |a(u, v)| &= \left| \int_{Q_T} (\sigma(\mathbf{x}) \partial_t^{1/2} u(\mathbf{x}, t) \partial_t^{1/2} v^\perp(\mathbf{x}, t) + v(\mathbf{x}) \nabla u(\mathbf{x}, t) \cdot \nabla v(\mathbf{x}, t)) \, d\mathbf{x} \, dt \right| \\ &\leq \overline{\sigma} \|\partial_t^{1/2} u\| \|\partial_t^{1/2} v\| + \overline{v} \|\nabla u\| \|\nabla v\| \\ &\leq \max \{ \overline{\sigma}, \overline{v} \} \|u\|_{H^{1,1/2}(Q_T)} \|v\|_{H^{1,1/2}(Q_T)} \\ &\leq \mu_2 \|u\|_{H^{1,1/2}(Q_T)} \|v\|_{H^{1,1/2}(Q_T)} \end{aligned}$$

with the constant $\mu_2 = \max \{ \overline{\sigma}, \overline{v} \}$, which justifies the second inequality in (4.1).

In order to prove the first inequality in (4.1), we select the test function $v = u - u^\perp$ and estimate the supremum from below. Using the σ - and v -weighted orthogonality relations (2.10) and the Friedrichs inequality (2.11), we find that

$$\begin{aligned} a(u, u) &= \int_{Q_T} (\sigma(\mathbf{x}) \partial_t^{1/2} u(\mathbf{x}, t) \partial_t^{1/2} u^\perp(\mathbf{x}, t) + v(\mathbf{x}) \nabla u(\mathbf{x}, t) \cdot \nabla u(\mathbf{x}, t)) \, d\mathbf{x} \, dt \\ &= \int_{Q_T} v(\mathbf{x}) \nabla u(\mathbf{x}, t) \cdot \nabla u(\mathbf{x}, t) \, d\mathbf{x} \, dt \geq \underline{v} \int_{Q_T} |\nabla u|^2 \, d\mathbf{x} \, dt \geq \frac{\underline{v}}{c_F^2 + 1} \|u\|_{H^{1,0}(Q_T)}^2 \end{aligned}$$

and

$$\begin{aligned} a(u, -u^\perp) &= \int_{Q_T} (\sigma(\mathbf{x}) \partial_t^{1/2} u(\mathbf{x}, t) \partial_t^{1/2} u^\perp(\mathbf{x}, t) - v(\mathbf{x}) \nabla u(\mathbf{x}, t) \cdot \nabla u^\perp(\mathbf{x}, t)) \, d\mathbf{x} \, dt \\ &= \int_{Q_T} \sigma(\mathbf{x}) \partial_t^{1/2} u(\mathbf{x}, t) \partial_t^{1/2} u^\perp(\mathbf{x}, t) \, d\mathbf{x} \, dt \geq \underline{\sigma} \|\partial_t^{1/2} u\|^2. \end{aligned}$$

Combining these estimates, we have

$$\begin{aligned} \sup_{0 \neq v \in H_{0,\text{per}}^{1,1/2}(Q_T)} \frac{a(u, v)}{\|v\|_{H^{1,1/2}(Q_T)}} &\geq \frac{a(u, u - u^\perp)}{\|u - u^\perp\|_{H^{1,1/2}(Q_T)}} \\ &\geq \frac{\frac{\underline{v}}{c_F^2 + 1} \|u\|_{H^{1,0}(Q_T)}^2 + \underline{\sigma} \|\partial_t^{1/2} u\|^2}{\|u - u^\perp\|_{H^{1,1/2}(Q_T)}} \\ &\geq \frac{\min\{\frac{\underline{v}}{c_F^2 + 1}, \underline{\sigma}\} \|u\|_{H^{1,1/2}(Q_T)}^2}{\sqrt{2} \|u\|_{H^{1,1/2}(Q_T)}} = \mu_1 \|u\|_{H^{1,1/2}(Q_T)}, \end{aligned}$$

with the constant $\mu_1 = \frac{1}{\sqrt{2}} \min\{\frac{\underline{v}}{c_F^2 + 1}, \underline{\sigma}\}$. □

Remark 4.2. Since the condition $u = 0$ is imposed on the whole boundary, we can easily find an upper bound of C_F . Indeed, $C_F(\Omega) \leq C_F(\hat{\Omega})$ if $\hat{\Omega} \supset \Omega$. Since for such domains as rectangles or balls the Friedrichs constants are known, we can easily obtain an upper bound of C_F for any Lipschitz domain.

Corollary 4.3. Since the norm $|\cdot|_{H^{1,1/2}(Q_T)}$ is equivalent to the norm $\|\cdot\|_{H^{1,1/2}(Q_T)}$ due to the Friedrichs inequality, the estimate (4.1) implies

$$\tilde{\mu}_1 |u|_{H^{1,1/2}(Q_T)} \leq \sup_{0 \neq v \in H_{0,\text{per}}^{1,1/2}(Q_T)} \frac{a(u, v)}{|v|_{H^{1,1/2}(Q_T)}} \leq \tilde{\mu}_2 |u|_{H^{1,1/2}(Q_T)} \quad (4.2)$$

for all $u \in H_{0,\text{per}}^{1,1/2}(Q_T)$ with positive constants $\tilde{\mu}_1 = \frac{1}{\sqrt{2}} \min\{\underline{v}, \underline{\sigma}\}$ and $\tilde{\mu}_2 = \mu_2 = \max\{\bar{\sigma}, \bar{v}\}$.

We now move on to the main part of this section related to a *a posteriori* error estimation. Let a function η be an approximation of u . First, we assume that η is a bit more regular than u . More precisely, we set $\eta \in H_{0,\text{per}}^{1,1}(Q_T)$. This is of course true for the multiharmonic finite element approximation u_{Nh} , which will later play the role of η . Now, the ultimate goal is to deduce a computable upper bound of the error $e := u - \eta$ in $H_{0,\text{per}}^{1,1/2}(Q_T)$. First, we notice that (2.12) implies the integral identity

$$\begin{aligned} &\int_{Q_T} (\sigma(\mathbf{x}) \partial_t^{1/2} (u - \eta) \partial_t^{1/2} v^\perp + v(\mathbf{x}) \nabla (u - \eta) \cdot \nabla v) \, d\mathbf{x} \, dt \\ &= \int_{Q_T} (f v - \sigma(\mathbf{x}) \partial_t^{1/2} \eta \partial_t^{1/2} v^\perp - v(\mathbf{x}) \nabla \eta \cdot \nabla v) \, d\mathbf{x} \, dt, \end{aligned} \quad (4.3)$$

which is valid for all $v \in H_{0,\text{per}}^{1,1/2}(Q_T)$. Here, the linear functional

$$\mathcal{F}_\eta(v) := \int_{Q_T} (f v - \sigma(\mathbf{x}) \partial_t^{1/2} \eta \partial_t^{1/2} v^\perp - v(\mathbf{x}) \nabla \eta \cdot \nabla v) \, d\mathbf{x} \, dt$$

is defined on $v \in H_{0,\text{per}}^{1,1/2}(Q_T)$. Now, identity (4.3) can be rewritten in the form

$$a(e, v) = \mathcal{F}_\eta(v).$$

Hence, getting an upper bound of the error is reduced to finding the quantities

$$\sup_{0 \neq v \in H_{0,\text{per}}^{1,1/2}(Q_T)} \frac{\mathcal{F}_\eta(v)}{\|v\|_{H^{1,1/2}(Q_T)}} \quad \text{or} \quad \sup_{0 \neq v \in H_{0,\text{per}}^{1,1/2}(Q_T)} \frac{\mathcal{F}_\eta(v)}{|v|_{H^{1,1/2}(Q_T)}}.$$

In order to find them, we reconstruct the functional $\mathcal{F}_\eta(v)$ using the identity

$$(\sigma \partial_t^{1/2} \eta, \partial_t^{1/2} v^\perp) = (\sigma \partial_t \eta, v) \quad \text{for all } \eta \in H_{0,\text{per}}^{1,1}(Q_T), v \in H_{0,\text{per}}^{1,1/2}(Q_T),$$

which follows from (2.9) and the identity

$$\int_{\Omega} \operatorname{div} \tau v \, d\mathbf{x} = - \int_{\Omega} \tau \cdot \nabla v \, d\mathbf{x},$$

which is valid for any $v \in H_0^1(\Omega)$ and any

$$\tau \in H(\operatorname{div}_x, Q_T) := \{\tau \in [L^2(Q_T)]^d : \operatorname{div}_x \tau(\cdot, t) \in L^2(\Omega) \text{ for a.e. } t \in (0, T)\}.$$

For ease of notation, the index $\mathbf{x}$ in div_x will be henceforth omitted, i.e., $\operatorname{div} = \operatorname{div}_x$ denotes the generalized spatial divergence. Using the Cauchy–Schwarz inequality leads to

$$\begin{aligned} \mathcal{F}_\eta(v) &= \int_{Q_T} (f v - \sigma(\mathbf{x}) \partial_t \eta v + \operatorname{div} \tau v + (\tau - v(\mathbf{x}) \nabla \eta) \cdot \nabla v) \, d\mathbf{x} \, dt \\ &\leq \|\mathcal{R}_1(\eta, \tau)\| \|v\| + \|\mathcal{R}_2(\eta, \tau)\| \|\nabla v\|, \end{aligned}$$

where

$$\mathcal{R}_1(\eta, \tau) := \sigma \partial_t \eta - \operatorname{div} \tau - f \quad \text{and} \quad \mathcal{R}_2(\eta, \tau) := \tau - v \nabla \eta. \quad (4.4)$$

In view of (2.11), we have

$$\begin{aligned} \mathcal{F}_\eta(v) &\leq \|\mathcal{R}_1(\eta, \tau)\| \|v\| + \|\mathcal{R}_2(\eta, \tau)\| \|\nabla v\| \\ &\leq \|\mathcal{R}_1(\eta, \tau)\| C_F \|\nabla v\| + \|\mathcal{R}_2(\eta, \tau)\| \|\nabla v\| = (C_F \|\mathcal{R}_1(\eta, \tau)\| + \|\mathcal{R}_2(\eta, \tau)\|) \|\nabla v\|. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \sup_{0 \neq v \in H_{0,\text{per}}^{1,1/2}(Q_T)} \frac{\mathcal{F}_\eta(v)}{|v|_{H^{1,1/2}(Q_T)}} &\leq \sup_{0 \neq v \in H_{0,\text{per}}^{1,1/2}(Q_T)} \frac{(C_F \|\mathcal{R}_1(\eta, \tau)\| + \|\mathcal{R}_2(\eta, \tau)\|) \|\nabla v\|}{|v|_{H^{1,1/2}(Q_T)}} \\ &= \sup_{0 \neq v \in H_{0,\text{per}}^{1,1/2}(Q_T)} \frac{(C_F \|\mathcal{R}_1(\eta, \tau)\| + \|\mathcal{R}_2(\eta, \tau)\|) \|\nabla v\|}{(\|\nabla v\|^2 + \|\partial_t^{1/2} v\|^2)^{1/2}} \\ &\leq C_F \|\mathcal{R}_1(\eta, \tau)\| + \|\mathcal{R}_2(\eta, \tau)\|. \end{aligned}$$

We use (4.2), i.e.,

$$|u - \eta|_{H^{1,1/2}(Q_T)} \leq \frac{1}{\tilde{\mu}_1} \sup_{0 \neq v \in H_{0,\text{per}}^{1,1/2}(Q_T)} \frac{a(u - \eta, v)}{|v|_{H^{1,1/2}(Q_T)}} = \frac{1}{\tilde{\mu}_1} \sup_{0 \neq v \in H_{0,\text{per}}^{1,1/2}(Q_T)} \frac{\mathcal{F}_\eta(v)}{|v|_{H^{1,1/2}(Q_T)}},$$

and arrive at the following result:

Theorem 4.4. *Let $\eta \in H_{0,\text{per}}^{1,1}(Q_T)$ and the bilinear form $a(\cdot, \cdot)$ satisfy (4.2). Then,*

$$|u - \eta|_{H^{1,1/2}(Q_T)} \leq \frac{1}{\tilde{\mu}_1} (C_F \|\mathcal{R}_1(\eta, \tau)\| + \|\mathcal{R}_2(\eta, \tau)\|) =: \mathcal{M}_{|\cdot|}^\oplus(\eta, \tau),$$

where $\tilde{\mu}_1 = \frac{1}{\sqrt{2}} \min\{\underline{\nu}, \underline{\sigma}\}$ and $\tau \in H(\operatorname{div}, Q_T)$.

We can also deduce an upper bound of the full $H^{1,1/2}$ -norm. Indeed,

$$\begin{aligned}\mathcal{F}_\eta(v) &\leq \|\mathcal{R}_1(\eta, \boldsymbol{\tau})\| \|v\| + \|\mathcal{R}_2(\eta, \boldsymbol{\tau})\| \|\nabla v\| \\ &\leq (\|\mathcal{R}_1(\eta, \boldsymbol{\tau})\|^2 + \|\mathcal{R}_2(\eta, \boldsymbol{\tau})\|^2)^{1/2} (\|v\|^2 + \|\nabla v\|^2)^{1/2}.\end{aligned}$$

In view of (4.1), we obtain

$$\begin{aligned}\sup_{0 \neq v \in H_{0,\text{per}}^{1,1/2}(Q_T)} \frac{\mathcal{F}_\eta(v)}{\|v\|_{H^{1,1/2}(Q_T)}} &\leq \sup_{0 \neq v \in H_{0,\text{per}}^{1,1/2}(Q_T)} \frac{(\|\mathcal{R}_1(\eta, \boldsymbol{\tau})\|^2 + \|\mathcal{R}_2(\eta, \boldsymbol{\tau})\|^2)^{1/2} (\|v\|^2 + \|\nabla v\|^2)^{1/2}}{\|v\|_{H^{1,1/2}(Q_T)}} \\ &\leq (\|\mathcal{R}_1(\eta, \boldsymbol{\tau})\|^2 + \|\mathcal{R}_2(\eta, \boldsymbol{\tau})\|^2)^{1/2}.\end{aligned}$$

Altogether, we deduce a similar estimate for $\|e\|_{H^{1,1/2}(Q_T)}$.

Theorem 4.5. Let $\eta \in H_{0,\text{per}}^{1,1}(Q_T)$ and the bilinear form $a(\cdot, \cdot)$ satisfy (4.1). Then,

$$\|u - \eta\|_{H^{1,1/2}(Q_T)} \leq \frac{1}{\mu_1} (\|\mathcal{R}_1(\eta, \boldsymbol{\tau})\|^2 + \|\mathcal{R}_2(\eta, \boldsymbol{\tau})\|^2)^{1/2} =: \mathcal{M}_{\|\cdot\|}^\oplus(\eta, \boldsymbol{\tau}),$$

where $\boldsymbol{\tau} \in H(\text{div}, Q_T)$ and now $\mu_1 = \frac{1}{\sqrt{2}} \min\{\frac{\nu}{C_F^2+1}, \underline{\sigma}\}$.

The functionals $\mathcal{M}_{\|\cdot\|}^\oplus(\eta, \boldsymbol{\tau})$ and $\mathcal{M}_{\|\cdot\|}^\oplus(\eta, \boldsymbol{\tau})$ present guaranteed and computable upper bounds (majorants) of the error with respect to the $H^{1,1/2}$ -norm.

Remark 4.6. It is easy to see that the majorants are nonnegative functionals vanishing if and only if $\eta = u$ and $\boldsymbol{\tau} = \nu \nabla u$. Indeed, if $\mathcal{R}_1(\eta, \boldsymbol{\tau}) = 0$ and $\mathcal{R}_2(\eta, \boldsymbol{\tau}) = 0$, then $\sigma \partial_t \eta - \text{div } \boldsymbol{\tau} = f$ and $\boldsymbol{\tau} = \nu \nabla \eta$. Since $\eta \in H_{0,\text{per}}^{1,1}(Q_T)$ is a periodic function and satisfies the Dirichlet condition on Σ_T , it is the solution. On the other hand, $\mathcal{R}_i(u, \nu \nabla u) = 0$, $i = 1, 2$.

The Multiharmonic Approximation

Since $f \in L^2(Q_T)$, it can be expanded into a Fourier series. Moreover, we choose our approximation η of the solution u as well as the vector-valued function $\boldsymbol{\tau}$ to be truncated Fourier series, i.e.,

$$\begin{cases} \eta(\mathbf{x}, t) = \eta_0^c(\mathbf{x}) + \sum_{k=1}^N (\eta_k^c(\mathbf{x}) \cos(k\omega t) + \eta_k^s(\mathbf{x}) \sin(k\omega t)), \\ \boldsymbol{\tau}(\mathbf{x}, t) = \boldsymbol{\tau}_0^c(\mathbf{x}) + \sum_{k=1}^N (\boldsymbol{\tau}_k^c(\mathbf{x}) \cos(k\omega t) + \boldsymbol{\tau}_k^s(\mathbf{x}) \sin(k\omega t)), \end{cases} \quad (4.5)$$

where all Fourier coefficients are from the space $L^2(\Omega)$ and are defined by the relations

$$\begin{aligned}\eta_0^c(\mathbf{x}) &= \frac{1}{T} \int_0^T \eta(\mathbf{x}, t) dt, & \boldsymbol{\tau}_0^c(\mathbf{x}) &= \frac{1}{T} \int_0^T \boldsymbol{\tau}(\mathbf{x}, t) dt, \\ \eta_k^c(\mathbf{x}) &= \frac{2}{T} \int_0^T \eta(\mathbf{x}, t) \cos(k\omega t) dt, & \boldsymbol{\tau}_k^c(\mathbf{x}) &= \frac{2}{T} \int_0^T \boldsymbol{\tau}(\mathbf{x}, t) \cos(k\omega t) dt, \\ \eta_k^s(\mathbf{x}) &= \frac{2}{T} \int_0^T \eta(\mathbf{x}, t) \sin(k\omega t) dt, & \boldsymbol{\tau}_k^s(\mathbf{x}) &= \frac{2}{T} \int_0^T \boldsymbol{\tau}(\mathbf{x}, t) \sin(k\omega t) dt.\end{aligned}$$

Hence, we get

$$\begin{aligned}\partial_t \eta(\mathbf{x}, t) &= \sum_{k=1}^N (k\omega \eta_k^s(\mathbf{x}) \cos(k\omega t) - k\omega \eta_k^c(\mathbf{x}) \sin(k\omega t)), \\ \nabla \eta(\mathbf{x}, t) &= \nabla \eta_0^c(\mathbf{x}) + \sum_{k=1}^N (\nabla \eta_k^c(\mathbf{x}) \cos(k\omega t) + \nabla \eta_k^s(\mathbf{x}) \sin(k\omega t)),\end{aligned}$$

and

$$\operatorname{div} \boldsymbol{\tau}(\mathbf{x}, t) = \operatorname{div} \boldsymbol{\tau}_0^c(\mathbf{x}) + \sum_{k=1}^N (\operatorname{div} \boldsymbol{\tau}_k^c(\mathbf{x}) \cos(k\omega t) + \operatorname{div} \boldsymbol{\tau}_k^s(\mathbf{x}) \sin(k\omega t)).$$

The $L^2(Q_T)$ -norms of the functions $\mathcal{R}_1(\boldsymbol{\eta}, \boldsymbol{\tau})$ and $\mathcal{R}_2(\boldsymbol{\eta}, \boldsymbol{\tau})$ presented in (4.4) can easily be computed. Thus, we arrive at

$$\begin{aligned} \|\mathcal{R}_1(\boldsymbol{\eta}, \boldsymbol{\tau})\|^2 &= T \|\operatorname{div} \boldsymbol{\tau}_0^c + f_0^c\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N (\| -k\omega \sigma \boldsymbol{\eta}_k^s + \operatorname{div} \boldsymbol{\tau}_k^c + f_k^c \|_\Omega^2 + \| k\omega \sigma \boldsymbol{\eta}_k^c + \operatorname{div} \boldsymbol{\tau}_k^s + f_k^s \|_\Omega^2) \\ &\quad + \frac{T}{2} \sum_{k=N+1}^\infty (\| f_k^c \|_\Omega^2 + \| f_k^s \|_\Omega^2) \\ &= T \|\operatorname{div} \boldsymbol{\tau}_0^c + f_0^c\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N \| k\omega \sigma \boldsymbol{\eta}_k^\perp + \operatorname{div} \boldsymbol{\tau}_k + \mathbf{f}_k \|_\Omega^2 + \frac{T}{2} \sum_{k=N+1}^\infty \| \mathbf{f}_k \|_\Omega^2, \end{aligned}$$

where $\boldsymbol{\eta}_k^\perp = (-\boldsymbol{\eta}_k^s, \boldsymbol{\eta}_k^c)^T$, $\operatorname{div} \boldsymbol{\tau}_k = (\operatorname{div} \boldsymbol{\tau}_k^c, \operatorname{div} \boldsymbol{\tau}_k^s)^T$, and

$$\begin{aligned} \|\mathcal{R}_2(\boldsymbol{\eta}, \boldsymbol{\tau})\|^2 &= T \|\boldsymbol{\tau}_0^c - \nu \nabla \boldsymbol{\eta}_0^c\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N (\| \boldsymbol{\tau}_k^c - \nu \nabla \boldsymbol{\eta}_k^c \|_\Omega^2 + \| \boldsymbol{\tau}_k^s - \nu \nabla \boldsymbol{\eta}_k^s \|_\Omega^2) \\ &= T \|\boldsymbol{\tau}_0^c - \nu \nabla \boldsymbol{\eta}_0^c\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N \| \boldsymbol{\tau}_k - \nu \nabla \boldsymbol{\eta}_k \|_\Omega^2 \end{aligned}$$

with $\boldsymbol{\tau}_k = ((\boldsymbol{\tau}_k^c)^T, (\boldsymbol{\tau}_k^s)^T)^T$.

Remark 4.7. We note that the remainder term

$$\mathcal{E}_N := \frac{T}{2} \sum_{k=N+1}^\infty \| \mathbf{f}_k \|_\Omega^2 = \frac{T}{2} \sum_{k=N+1}^\infty (\| f_k^c \|_\Omega^2 + \| f_k^s \|_\Omega^2) \quad (4.6)$$

is always computable, due to the knowledge on the given data f . In some cases, the computation of $\mathcal{E}_N$ is very easy, for example, if f is multiharmonic. However, even in the most complicated cases, in which $f = f(\mathbf{x}, t)$ and we do not refer to special (e.g., extra regularity) properties, the term $\mathcal{E}_N$ can be precomputed as $\|f - f_N\|^2$, where f_N is the truncated Fourier series of f .

In fact, the L^2 -norms of $\mathcal{R}_1$ and $\mathcal{R}_2$ corresponding to every single mode k are decoupled. Altogether, it follows that

$$\begin{aligned} \|\mathcal{R}_1(\boldsymbol{\eta}, \boldsymbol{\tau})\|^2 &= T \|\mathcal{R}_{10}^c(\boldsymbol{\tau}_0^c)\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N (\|\mathcal{R}_{1k}^c(\boldsymbol{\eta}_k^s, \boldsymbol{\tau}_k^c)\|_\Omega^2 + \|\mathcal{R}_{1k}^s(\boldsymbol{\eta}_k^c, \boldsymbol{\tau}_k^s)\|_\Omega^2) + \mathcal{E}_N, \\ \|\mathcal{R}_2(\boldsymbol{\eta}, \boldsymbol{\tau})\|^2 &= T \|\mathcal{R}_{20}^c(\boldsymbol{\eta}_0^c, \boldsymbol{\tau}_0^c)\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N (\|\mathcal{R}_{2k}^c(\boldsymbol{\eta}_k^c, \boldsymbol{\tau}_k^c)\|_\Omega^2 + \|\mathcal{R}_{2k}^s(\boldsymbol{\eta}_k^s, \boldsymbol{\tau}_k^s)\|_\Omega^2), \end{aligned}$$

where

$$\mathcal{R}_{10}^c(\boldsymbol{\tau}_0^c) := \operatorname{div} \boldsymbol{\tau}_0^c + f_0^c, \quad \mathcal{R}_{20}^c(\boldsymbol{\eta}_0^c, \boldsymbol{\tau}_0^c) := \boldsymbol{\tau}_0^c - \nu \nabla \boldsymbol{\eta}_0^c,$$

and, for $k = 1, \dots, N$, we have

$$\begin{cases} \mathcal{R}_{1k}^c(\boldsymbol{\eta}_k^s, \boldsymbol{\tau}_k^c) := -k\omega \sigma \boldsymbol{\eta}_k^s + \operatorname{div} \boldsymbol{\tau}_k^c + f_k^c, & \mathcal{R}_{2k}^c(\boldsymbol{\eta}_k^c, \boldsymbol{\tau}_k^c) := \boldsymbol{\tau}_k^c - \nu \nabla \boldsymbol{\eta}_k^c, \\ \mathcal{R}_{1k}^s(\boldsymbol{\eta}_k^c, \boldsymbol{\tau}_k^s) := k\omega \sigma \boldsymbol{\eta}_k^c + \operatorname{div} \boldsymbol{\tau}_k^s + f_k^s, & \mathcal{R}_{2k}^s(\boldsymbol{\eta}_k^s, \boldsymbol{\tau}_k^s) := \boldsymbol{\tau}_k^s - \nu \nabla \boldsymbol{\eta}_k^s. \end{cases} \quad (4.7)$$

Corollary 4.8. The error majorants $\mathcal{M}_{|\cdot|}^\oplus(\boldsymbol{\eta}, \boldsymbol{\tau})$ and $\mathcal{M}_{\|\cdot\|}^\oplus(\boldsymbol{\eta}, \boldsymbol{\tau})$ can be presented in the forms

$$\begin{aligned} \mathcal{M}_{|\cdot|}^\oplus(\boldsymbol{\eta}, \boldsymbol{\tau}) &= \frac{1}{\tilde{\mu}_1} (C_F \|\mathcal{R}_1(\boldsymbol{\eta}, \boldsymbol{\tau})\| + \|\mathcal{R}_2(\boldsymbol{\eta}, \boldsymbol{\tau})\|) \\ &= \frac{1}{\tilde{\mu}_1} \left(C_F \left(T \|\mathcal{R}_{10}^c(\boldsymbol{\tau}_0^c)\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N (\|\mathcal{R}_{1k}^c(\boldsymbol{\eta}_k^s, \boldsymbol{\tau}_k^c)\|_\Omega^2 + \|\mathcal{R}_{1k}^s(\boldsymbol{\eta}_k^c, \boldsymbol{\tau}_k^s)\|_\Omega^2) + \mathcal{E}_N \right)^{1/2} \right. \\ &\quad \left. + \left(T \|\mathcal{R}_{20}^c(\boldsymbol{\eta}_0^c, \boldsymbol{\tau}_0^c)\|_\Omega^2 + \frac{T}{2} \sum_{k=1}^N (\|\mathcal{R}_{2k}^c(\boldsymbol{\eta}_k^c, \boldsymbol{\tau}_k^c)\|_\Omega^2 + \|\mathcal{R}_{2k}^s(\boldsymbol{\eta}_k^s, \boldsymbol{\tau}_k^s)\|_\Omega^2) \right)^{1/2} \right) \quad (4.8) \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_{\|\cdot\|}^{\oplus}(\eta, \boldsymbol{\tau}) &= \frac{1}{\mu_1} (\|\mathcal{R}_1(\eta, \boldsymbol{\tau})\|^2 + \|\mathcal{R}_2(\eta, \boldsymbol{\tau})\|^2)^{1/2} \\ &= \frac{1}{\mu_1} \left(T(\|\mathcal{R}_{10}^c(\boldsymbol{\tau}_0^c)\|_{\Omega}^2 + \|\mathcal{R}_{20}^c(\eta_0^c, \boldsymbol{\tau}_0^c)\|_{\Omega}^2) \right. \\ &\quad \left. + \frac{T}{2} \sum_{k=1}^N (\|\mathcal{R}_{1k}^c(\eta_k^s, \boldsymbol{\tau}_k^c)\|_{\Omega}^2 + \|\mathcal{R}_{1k}^s(\eta_k^c, \boldsymbol{\tau}_k^s)\|_{\Omega}^2 + \|\mathcal{R}_{2k}^c(\eta_k^c, \boldsymbol{\tau}_k^c)\|_{\Omega}^2 + \|\mathcal{R}_{2k}^s(\eta_k^s, \boldsymbol{\tau}_k^s)\|_{\Omega}^2) + \varepsilon_N \right)^{1/2}, \end{aligned}$$

where $\tilde{\mu}_1 = \frac{1}{\sqrt{2}} \min\{\underline{\nu}, \underline{\sigma}\}$ and $\mu_1 = \frac{1}{\sqrt{2}} \min\{\frac{\underline{\nu}}{C_F^2+1}, \underline{\sigma}\}$.

Remark 4.9. Since the error (with respect to the truncation index N) between the exact solution u and its multiharmonic approximation η decreases with $\mathcal{O}(N^{-1})$, see [22, 34], the contributions in the majorants coming from the functionals $\mathcal{R}_{1k}^c$ and $\mathcal{R}_{1k}^s$ cannot blow up.

We see that the majorants consist of computable quantities related to each harmonic. Therefore, they not only evaluate the overall error, but also provide an information on errors associated with a certain harmonic. Moreover, since the respective quantities are integrals over Ω , their integrands serve as indicators of spatial errors. Thus, the majorants contain a rich amount of information to be utilized in various adaptive procedures.

Remark 4.10. Let f have a multiharmonic representation, i.e.,

$$f(\mathbf{x}, t) = f_0^c(\mathbf{x}) + \sum_{k=1}^{N_f} (f_k^c(\mathbf{x}) \cos(k\omega t) + f_k^s(\mathbf{x}) \sin(k\omega t)),$$

where $N_f \in \mathbb{N}$ is defined by f . If $N \geq N_f$, then η is the exact solution of problem (2.12) and $\boldsymbol{\tau}$ is the exact flux if and only if the error majorants vanish, i.e.,

$$\begin{aligned} \mathcal{R}_{1k}^c &= 0 \quad \text{and} \quad \mathcal{R}_{2k}^c = 0 \quad \text{for all } k = 0, 1, \dots, N_f, \\ \mathcal{R}_{1k}^s &= 0 \quad \text{and} \quad \mathcal{R}_{2k}^s = 0 \quad \text{for all } k = 1, 2, \dots, N_f. \end{aligned}$$

Indeed, let the error majorants vanish. Then, we deduce that $-\operatorname{div} \boldsymbol{\tau}_0^c = f_0^c$ and $\boldsymbol{\tau}_0^c = \nu \nabla \eta_0^c$, and furthermore we have, for all $k = 1, \dots, N_f$,

$$k\omega \sigma \eta_k^s - \operatorname{div} \boldsymbol{\tau}_k^c = f_k^c, \quad -k\omega \sigma \eta_k^c - \operatorname{div} \boldsymbol{\tau}_k^s = f_k^s, \quad \boldsymbol{\tau}_k^c = \nu \nabla \eta_k^c, \quad \boldsymbol{\tau}_k^s = \nu \nabla \eta_k^s$$

Therefore, collecting the harmonics, we find that

$$\begin{aligned} \boldsymbol{\tau}(\mathbf{x}, t) &= \boldsymbol{\tau}_0^c(\mathbf{x}) + \sum_{k=1}^{N_f} (\boldsymbol{\tau}_k^c(\mathbf{x}) \cos(k\omega t) + \boldsymbol{\tau}_k^s(\mathbf{x}) \sin(k\omega t)), \\ \eta(\mathbf{x}, t) &= \eta_0^c(\mathbf{x}) + \sum_{k=1}^{N_f} (\eta_k^c(\mathbf{x}) \cos(k\omega t) + \eta_k^s(\mathbf{x}) \sin(k\omega t)) \end{aligned}$$

and

$$\sigma \partial_t \eta - \operatorname{div} \boldsymbol{\tau} = f, \quad \boldsymbol{\tau} = \nu \nabla \eta.$$

Since η satisfies the boundary conditions and the equation, we conclude that $\eta = u$.

Another approach to derive a majorant is to insert the Fourier series ansatz directly into the bilinear form $a(u - \eta, v)$ and into the functional $\mathcal{F}_{\eta}(v)$ as defined in (4.3). Then, we obtain the following integral identities associated with every mode:

$$\begin{aligned} &\int_{\Omega} (\nu(\mathbf{x}) \nabla(\mathbf{u}_k(\mathbf{x}) - \boldsymbol{\eta}_k(\mathbf{x})) \cdot \nabla \mathbf{v}_k(\mathbf{x}) + k\omega \sigma(\mathbf{x})(\mathbf{u}_k(\mathbf{x}) - \boldsymbol{\eta}_k(\mathbf{x})) \cdot \mathbf{v}_k^{\perp}(\mathbf{x})) \, d\mathbf{x} \\ &= \int_{\Omega} (\mathbf{f}_k(\mathbf{x}) \cdot \mathbf{v}_k(\mathbf{x}) - \nu(\mathbf{x}) \nabla \boldsymbol{\eta}_k(\mathbf{x}) \cdot \nabla \mathbf{v}_k(\mathbf{x}) - k\omega \sigma(\mathbf{x}) \boldsymbol{\eta}_k(\mathbf{x}) \cdot \mathbf{v}_k^{\perp}(\mathbf{x})) \, d\mathbf{x}, \end{aligned} \quad (4.9)$$

which are valid for all $\mathbf{v}_k \in (H_0^1(\Omega))^2$. In the case $k = 0$, the integral identity

$$\int_{\Omega} v(\mathbf{x}) \nabla(u_0^c(\mathbf{x}) - \eta_0^c(\mathbf{x})) \cdot \nabla v_0^c(\mathbf{x}) \, d\mathbf{x} = \int_{\Omega} (f_0^c(\mathbf{x}) v_0^c(\mathbf{x}) - v(\mathbf{x}) \nabla \eta_0^c(\mathbf{x}) \cdot \nabla v_0^c(\mathbf{x})) \, d\mathbf{x} \quad (4.10)$$

is valid for all $v_0^c \in H_0^1(\Omega)$. We define the left-hand sides of (4.9) and (4.10) by

$$a_k(\mathbf{u}_k - \boldsymbol{\eta}_k, \mathbf{v}_k) \quad \text{and} \quad a_0(u_0^c - \eta_0^c, v_0^c),$$

and the right-hand sides by

$$\mathcal{F}_{\boldsymbol{\eta}_k}(\mathbf{v}_k) \quad \text{and} \quad \mathcal{F}_{\eta_0^c}(v_0^c),$$

respectively. Let us start with the case $k = 1, \dots, N$. Hence, an upper bound for the errors $\mathbf{e}_k := \mathbf{u}_k - \boldsymbol{\eta}_k$ in $(H_0^1(\Omega))^2$ has to be computed. The bilinear form $a_k(\cdot, \cdot)$ meets the inf-sup condition

$$\sup_{0 \neq \mathbf{v}_k \in (H_0^1(\Omega))^2} \frac{a_k(\mathbf{u}_k - \boldsymbol{\eta}_k, \mathbf{v}_k)}{\|\mathbf{v}_k\|_{1,\Omega}} \geq \underline{c}^k \|\mathbf{u}_k - \boldsymbol{\eta}_k\|_{1,\Omega} \quad (4.11)$$

with the inf-sup constant $\underline{c}^k = \min\{\underline{\nu}, k\omega \underline{\sigma}\} / \sqrt{2}$. By the same method as before, we reform the error functionals and obtain estimates for

$$\sup_{0 \neq \mathbf{v}_k \in (H_0^1(\Omega))^2} \frac{\mathcal{F}_{\boldsymbol{\eta}_k}(\mathbf{v}_k)}{\|\mathbf{v}_k\|_{1,\Omega}}.$$

We introduce a collection of vector-valued functions

$$\boldsymbol{\tau}_k = (\boldsymbol{\tau}_k^c, \boldsymbol{\tau}_k^s)^T, \quad \boldsymbol{\tau}_k^c, \boldsymbol{\tau}_k^s \in H(\operatorname{div}, \Omega) := \{\boldsymbol{\tau} \in [L^2(\Omega)]^d : \operatorname{div} \boldsymbol{\tau} \in L^2(\Omega)\},$$

and use the integral relations

$$\int_{\Omega} \operatorname{div} \boldsymbol{\tau} \, v \, d\mathbf{x} = - \int_{\Omega} \boldsymbol{\tau} \cdot \nabla v \, d\mathbf{x} \quad \text{for all } v \in H_0^1(\Omega).$$

It is easy to see that

$$\begin{aligned} \mathcal{F}_{\boldsymbol{\eta}_k}(\mathbf{v}_k) &= \int_{\Omega} (\mathbf{f}_k \cdot \mathbf{v}_k - k\omega \sigma(\mathbf{x}) \boldsymbol{\eta}_k \cdot \mathbf{v}_k^\perp + \operatorname{div} \boldsymbol{\tau}_k \cdot \mathbf{v}_k + \boldsymbol{\tau}_k \cdot \nabla \mathbf{v}_k - v(\mathbf{x}) \nabla \boldsymbol{\eta}_k \cdot \nabla \mathbf{v}_k) \, d\mathbf{x} \\ &= \int_{\Omega} ((\mathbf{f}_k + k\omega \sigma(\mathbf{x}) \boldsymbol{\eta}_k^\perp + \operatorname{div} \boldsymbol{\tau}_k) \cdot \mathbf{v}_k + (\boldsymbol{\tau}_k - v(\mathbf{x}) \nabla \boldsymbol{\eta}_k) \cdot \nabla \mathbf{v}_k) \, d\mathbf{x} \\ &\leq \|\mathcal{R}_{1k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_{\Omega} \|\mathbf{v}_k\|_{\Omega} + \|\mathcal{R}_{2k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_{\Omega} \|\nabla \mathbf{v}_k\|_{\Omega} \\ &\leq (\|\mathcal{R}_{1k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_{\Omega}^2 + \|\mathcal{R}_{2k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_{\Omega}^2)^{1/2} \|\mathbf{v}_k\|_{1,\Omega}, \end{aligned} \quad (4.12)$$

where

$$\begin{aligned} \mathcal{R}_{1k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k) &= k\omega \sigma \boldsymbol{\eta}_k^\perp + \operatorname{div} \boldsymbol{\tau}_k + \mathbf{f}_k \\ &= (-k\omega \sigma \boldsymbol{\eta}_k^s + \operatorname{div} \boldsymbol{\tau}_k^c + \mathbf{f}_k^c, k\omega \sigma \boldsymbol{\eta}_k^c + \operatorname{div} \boldsymbol{\tau}_k^s + \mathbf{f}_k^s)^T \\ &= (\mathcal{R}_{1k}^c(\boldsymbol{\eta}_k^s, \boldsymbol{\tau}_k^c), \mathcal{R}_{1k}^s(\boldsymbol{\eta}_k^c, \boldsymbol{\tau}_k^s))^T, \\ \mathcal{R}_{2k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k) &= \boldsymbol{\tau}_k - v \nabla \boldsymbol{\eta}_k \\ &= (\boldsymbol{\tau}_k^c - v \nabla \boldsymbol{\eta}_k^c, \boldsymbol{\tau}_k^s - v \nabla \boldsymbol{\eta}_k^s)^T \\ &= (\mathcal{R}_{2k}^c(\boldsymbol{\eta}_k^c, \boldsymbol{\tau}_k^c), \mathcal{R}_{2k}^s(\boldsymbol{\eta}_k^s, \boldsymbol{\tau}_k^s))^T. \end{aligned}$$

Hence, we have derived the same results as in (4.7) for every mode $k = 1, \dots, N$. Using the estimate (4.12) together with the inf-sup condition (4.11), we finally arrive at the following upper bounds for every single mode $k = 1, \dots, N$:

Theorem 4.11. Let $\boldsymbol{\eta}_k \in (H_0^1(\Omega))^2$ and the bilinear form $a_k(\cdot, \cdot)$ satisfy (4.11). Then,

$$\|\mathbf{u}_k - \boldsymbol{\eta}_k\|_{1,\Omega} \leq \frac{1}{\underline{c}^k} (\|\mathcal{R}_{1k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega^2 + \|\mathcal{R}_{2k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega^2)^{1/2} =: \mathcal{M}_{\|\cdot\|}^{\oplus k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k),$$

where $\underline{c}^k = \frac{1}{\sqrt{2}} \min\{\underline{\nu}, k\omega \underline{\sigma}\}$ and $\boldsymbol{\tau}_k = (\boldsymbol{\tau}_k^c, \boldsymbol{\tau}_k^s)^T$ with $\boldsymbol{\tau}_k^c, \boldsymbol{\tau}_k^s \in H(\operatorname{div}, \Omega)$.

Using the inf-sup condition

$$\begin{aligned} \sup_{0 \neq \mathbf{v}_k \in (H_0^1(\Omega))^2} \frac{a_k(\mathbf{u}_k, \mathbf{v}_k)}{|\mathbf{v}_k|_{1,\Omega}} &= \sup_{0 \neq \mathbf{v}_k \in (H_0^1(\Omega))^2} \frac{(\nu \nabla \mathbf{u}_k, \nabla \mathbf{v}_k)_\Omega + k\omega(\sigma \mathbf{u}_k, \mathbf{v}_k^\perp)_\Omega}{|\mathbf{v}_k|_{1,\Omega}} \\ &\geq \frac{(\nu \nabla \mathbf{u}_k, \nabla(\mathbf{u}_k - \mathbf{u}_k^\perp))_\Omega + k\omega(\sigma \mathbf{u}_k, (\mathbf{u}_k - \mathbf{u}_k^\perp)^\perp)_\Omega}{|\mathbf{u}_k - \mathbf{u}_k^\perp|_{1,\Omega}} \\ &= \frac{(\nu \nabla \mathbf{u}_k, \nabla \mathbf{u}_k)_\Omega + k\omega(\sigma \mathbf{u}_k, \mathbf{u}_k)_\Omega}{\sqrt{2} |\mathbf{u}_k|_{1,\Omega}} \geq \frac{\underline{\nu} \|\nabla \mathbf{u}_k\|_\Omega^2 + k\omega \underline{\sigma} \|\mathbf{u}_k\|_\Omega^2}{\sqrt{2} |\mathbf{u}_k|_{1,\Omega}} \\ &\geq \frac{\min\{\underline{\nu}, k\omega \underline{\sigma}\} \|\mathbf{u}_k\|_{1,\Omega}^2}{\sqrt{2} |\mathbf{u}_k|_{1,\Omega}} \geq \frac{\min\{\underline{\nu}, k\omega \underline{\sigma}\}}{\sqrt{2}} |\mathbf{u}_k|_{1,\Omega} \end{aligned} \quad (4.13)$$

together with the estimate

$$\begin{aligned} \mathcal{F}_{\boldsymbol{\eta}_k}(\mathbf{v}_k) &\leq \|\mathcal{R}_{1k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega \|\mathbf{v}_k\|_\Omega + \|\mathcal{R}_{2k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega \|\nabla \mathbf{v}_k\|_\Omega \\ &\leq (C_F \|\mathcal{R}_{1k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega + \|\mathcal{R}_{2k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega) |\mathbf{v}_k|_{1,\Omega} \end{aligned}$$

yields the following error majorant for $|\cdot|_{1,\Omega}$ with the same inf-sup constant $\underline{c}^k$:

Theorem 4.12. Let $\boldsymbol{\eta}_k \in (H_0^1(\Omega))^2$ and the bilinear form $a_k(\cdot, \cdot)$ satisfy (4.13). Then,

$$|\mathbf{u}_k - \boldsymbol{\eta}_k|_{1,\Omega} \leq \frac{1}{\underline{c}^k} (C_F \|\mathcal{R}_{1k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega + \|\mathcal{R}_{2k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k)\|_\Omega) =: \mathcal{M}_{|\cdot|}^{\oplus k}(\boldsymbol{\eta}_k, \boldsymbol{\tau}_k), \quad (4.14)$$

where $\underline{c}^k = \frac{1}{\sqrt{2}} \min\{\underline{\nu}, k\omega \underline{\sigma}\}$ and $\boldsymbol{\tau}_k = (\boldsymbol{\tau}_k^c, \boldsymbol{\tau}_k^s)^T$ with $\boldsymbol{\tau}_k^c, \boldsymbol{\tau}_k^s \in H(\operatorname{div}, \Omega)$.

Now, we consider the case $k = 0$. Here, an upper bound for the error $e_0^c := u_0^c - \eta_0^c$ in $H_0^1(\Omega)$ has to be computed. The inf-sup condition

$$\sup_{0 \neq v_0^c \in H_0^1(\Omega)} \frac{a_0(u_0^c - \eta_0^c, v_0^c)}{\|v_0^c\|_{1,\Omega}} \geq \underline{c}_{\|\cdot\|}^0 \|u_0^c - \eta_0^c\|_{1,\Omega}$$

with the inf-sup constant $\underline{c}_{\|\cdot\|}^0 = \underline{\nu}/(C_F^2 + 1)$ can be proved quite analogously to (4.11). Moreover, one can easily show that

$$\sup_{0 \neq v_0^c \in H_0^1(\Omega)} \frac{a_0(u_0^c - \eta_0^c, v_0^c)}{|v_0^c|_{1,\Omega}} \geq \frac{a_0(u_0^c - \eta_0^c, u_0^c - \eta_0^c)}{|u_0^c - \eta_0^c|_{1,\Omega}} \geq \underline{c}_{|\cdot|}^0 |u_0^c - \eta_0^c|_{1,\Omega}$$

with $\underline{c}_{|\cdot|}^0 = \underline{\nu}$, since v satisfies the assumptions (2.4). By arguments similar to those used above for the modes k , we deduce the following estimates:

$$\|u_0^c - \eta_0^c\|_{1,\Omega} \leq \frac{1}{\underline{c}_{\|\cdot\|}^0} (\|\mathcal{R}_{10}^c(\boldsymbol{\tau}_0^c)\|_\Omega^2 + \|\mathcal{R}_{20}^c(\eta_0^c, \boldsymbol{\tau}_0^c)\|_\Omega^2)^{1/2} =: \mathcal{M}_{\|\cdot\|}^{\oplus 0}(\eta_0^c, \boldsymbol{\tau}_0^c)$$

and

$$|u_0^c - \eta_0^c|_{1,\Omega} \leq \frac{1}{\underline{c}_{|\cdot|}^0} (C_F \|\mathcal{R}_{10}^c(\boldsymbol{\tau}_0^c)\|_\Omega + \|\mathcal{R}_{20}^c(\eta_0^c, \boldsymbol{\tau}_0^c)\|_\Omega) =: \mathcal{M}_{|\cdot|}^{\oplus 0}(\eta_0^c, \boldsymbol{\tau}_0^c), \quad (4.15)$$

where

$$\boldsymbol{\tau}_0^c \in H(\operatorname{div}, \Omega), \quad \mathcal{R}_{10}^c(\boldsymbol{\tau}_0^c) = f_0^c + \operatorname{div} \boldsymbol{\tau}_0^c, \quad \mathcal{R}_{20}^c(\eta_0^c, \boldsymbol{\tau}_0^c) = \boldsymbol{\tau}_0^c - \nu \nabla \eta_0^c.$$

5 Numerical Results

In this section, we present and discuss results of numerical experiments on computing functional *a posteriori* error estimates in the context of parabolic time-periodic boundary value problems discretized by the MhFEM. First, we present a numerical example with a given time-harmonic source term. In the second example, we consider a given time-periodic, but not time-harmonic source term. The computational domain $\Omega = (0, 1) \times (0, 1)$ is uniformly decomposed into triangles, and standard continuous, piecewise linear finite elements are used for the discretization in space. In this case, the Friedrichs constant is $C_F = 1/(\sqrt{2}\pi)$. In these two numerical experiments, we choose $\sigma = \nu = 1$.

The construction of η and τ is an important issue in order to obtain sharp guaranteed bounds from the majorants $\mathcal{M}_{\|\cdot\|}^{\oplus}$ or $\mathcal{M}_{|\cdot|}^{\oplus}$. As it has been already discussed in Section 4, we can choose multiharmonic finite element approximations (4.5) for η and τ . However, since the Fourier coefficients of η are constructed by continuous, piecewise linear approximations, their gradients are only piecewise constant. Then, $\nabla \eta_k^c, \nabla \eta_k^s \in L^2(\Omega)$, but $\nabla \eta_k^c, \nabla \eta_k^s \notin H(\operatorname{div}, \Omega)$, $k = 1, \dots, N$. Hence, a flux reconstruction is needed in order to obtain a suitable flux $\tau \in H(\operatorname{div}, Q_T)$. A good reconstruction of the flux is an important and nontrivial topic. We can regularize τ by a post-processing operator which maps the L^2 -functions into $H(\operatorname{div}, Q_T)$, see [26]. There are various techniques for realizing these post-processing steps such as local post-processing by an elementwise averaging procedure or by using Raviart–Thomas elements, see [23, 26] and references therein. In our numerical experiments, we use Raviart–Thomas elements of the lowest order, see, e.g., [5, 24, 28]. First, we define the normal fluxes on interior edges E_{mn} by

$$\begin{aligned}(\tau_k^c \cdot n_{E_{mn}})|_{E_{mn}} &= (\lambda_{mn}(\nabla \eta_k^c)|_{T_m} + (1 - \lambda_{mn})(\nabla \eta_k^c)|_{T_n}) \cdot n_{E_{mn}}, \\(\tau_k^s \cdot n_{E_{mn}})|_{E_{mn}} &= (\lambda_{mn}(\nabla \eta_k^s)|_{T_m} + (1 - \lambda_{mn})(\nabla \eta_k^s)|_{T_n}) \cdot n_{E_{mn}},\end{aligned}$$

for all $k = 1, \dots, N$, with $\lambda_{mn} = 1/2$ due to uniform discretization. Here, $(\nabla \eta_k^c)|_{T_m}$, $(\nabla \eta_k^s)|_{T_m}$, $(\nabla \eta_k^c)|_{T_n}$ and $(\nabla \eta_k^s)|_{T_n}$ are constant vectors on two arbitrary, neighboring elements T_m and T_n . On boundary edges, the only one existing flux is used. Hence, three normal fluxes are defined on the three sides of each element. Inside, we reconstruct the fluxes $\tau_k = (\tau_k^c, \tau_k^s)^T$ by the standard lowest-order Raviart–Thomas (RT^0 -) extension of normal fluxes with

$$\begin{aligned}\operatorname{RT}^0(\mathcal{T}_h) &:= \{\tau \in (L^2(T))^2 : \forall T \in \mathcal{T}_h \exists a, b, c \in \mathbb{R} \forall x \in T : \tau(x) = (a, b)^T + cx \\ &\text{and } [\tau]_E \cdot n_E = 0 \text{ for all interior edges } E\},\end{aligned}$$

where $[\tau]_E$ denotes the jump of τ across the edge E shared by two neighboring elements on a triangulation $\mathcal{T}_h$. Altogether, it follows an averaged flux from $H(\operatorname{div}, \Omega)$, i.e.,

$$\tau_k^c = G_{\operatorname{RT}}(\nabla \eta_k^c), \quad \tau_k^s = G_{\operatorname{RT}}(\nabla \eta_k^s), \quad G_{\operatorname{RT}} : L^2(\Omega) \rightarrow H(\operatorname{div}, \Omega).$$

In order to solve the saddle point systems (3.3) for $k = 1, \dots, N$, we use the AMLI preconditioner proposed by Kraus and Wolfmayr in [18] with a proper 3-refinement of the mesh as presented in [18] for an inexact realization of the block-diagonal preconditioner

$$\mathcal{P} = \begin{pmatrix} k\omega M_{h,\sigma} + K_{h,\nu} & 0 \\ 0 & k\omega M_{h,\sigma} + K_{h,\nu} \end{pmatrix} \quad (5.1)$$

in the MINRES method. The preconditioner (5.1) was presented and discussed in [34]. Here, we want to emphasize that the AMLI preconditioned MINRES solver is robust and of optimal complexity, see [18, 34]. This can be also observed in the numerical results of this paper. We mention that, in all tables where the number of MINRES iterations $n_{\operatorname{MINRES}}^{\operatorname{iter}}$ or of AMLI iterations $n_{\operatorname{AMLI}}^{\operatorname{iter}}$ is presented, the iteration was stopped after reducing the initial residual by a factor of 10^{-6} . In each MINRES iteration step, we have used the AMLI preconditioner according to [18] with 8 inner iterations. The presented CPU times in seconds t^{sec} include the computational times for computing the majorants, which are very small in comparison to the computational times of the solver. All computations were performed on a PC with Intel(R) Xeon(R) CPU W3680 @ 3.33GHz.

Grid	$n_{\text{MINRES}}^{\text{iter}}$	t^{sec}	$\ \mathcal{R}_1\ _{\Omega}$	$\ \mathcal{R}_2\ _{\Omega}$	$\mathcal{M}_{ \cdot }^{\oplus}$	I_{eff}
9×9	14	0.00	1.657e-01	4.604e-03	5.926e-02	1.976
27×27	14	0.03	6.381e-02	8.313e-05	2.043e-02	1.583
81×81	12	0.24	2.186e-02	7.545e-06	6.968e-03	1.530
243×243	12	2.43	7.334e-03	5.155e-07	2.335e-03	1.504
729×729	12	22.25	2.449e-03	3.298e-08	7.797e-04	1.498

Table 1. Majorant and its parts (Example 5.1).

Example 5.1. In the first example, we consider a given time-harmonic source term

$$f(\mathbf{x}, t) = 2(x_1(1 - x_1) + x_2(1 - x_2)) \cos(t) + x_1(1 - x_1)x_2(x_2 - 1) \sin(t),$$

where $T = 2\pi/\omega$ with $\omega = 1$. Hence, the Fourier coefficients of f are simply given by

$$f^c(\mathbf{x}) = 2(x_1(1 - x_1) + x_2(1 - x_2)), \quad f^s(\mathbf{x}) = x_1(1 - x_1)x_2(x_2 - 1),$$

and we have to consider only one single mode $k = 1$. For simplicity, we omit the index k in this example. The exact solution is given by

$$u(\mathbf{x}, t) = x_1(x_1 - 1)x_2(x_2 - 1) \cos(t).$$

Table 1 presents the number of MINRES iterations $n_{\text{MINRES}}^{\text{iter}}$, the CPU times in seconds t^{sec} , the norms of $\mathcal{R}_1$ and $\mathcal{R}_2$, i.e.,

$$\begin{aligned} \|\mathcal{R}_1\|_{\Omega}^2 &= \|\mathcal{R}_1^c(\eta^s, \boldsymbol{\tau}^c)\|_{\Omega}^2 + \|\mathcal{R}_1^s(\eta^c, \boldsymbol{\tau}^s)\|_{\Omega}^2 = \|-\eta^s + \operatorname{div} \boldsymbol{\tau}^c + f^c\|_{\Omega}^2 + \|\eta^c + \operatorname{div} \boldsymbol{\tau}^s + f^s\|_{\Omega}^2, \\ \|\mathcal{R}_2\|_{\Omega}^2 &= \|\mathcal{R}_2^c(\eta^s, \boldsymbol{\tau}^c)\|_{\Omega}^2 + \|\mathcal{R}_2^s(\eta^c, \boldsymbol{\tau}^s)\|_{\Omega}^2 = \|\boldsymbol{\tau}^c - \nabla \eta^c\|_{\Omega}^2 + \|\boldsymbol{\tau}^s - \nabla \eta^s\|_{\Omega}^2, \end{aligned}$$

as well as the majorants

$$\mathcal{M}_{|\cdot|}^{\oplus} = \frac{1}{\tilde{\mu}_1} (C_F \|\mathcal{R}_1\|_{\Omega} + \|\mathcal{R}_2\|_{\Omega}),$$

where $\tilde{\mu}_1 = \frac{1}{\sqrt{2}}$, and the corresponding efficiency indices

$$I_{\text{eff}} = \frac{\mathcal{M}_{|\cdot|}^{\oplus}}{\|\mathbf{u} - \boldsymbol{\eta}\|_{1,\Omega}},$$

obtained on grids of different mesh sizes. Here, $\mathbf{u} = \mathbf{u}(\mathbf{x}) = (u^c(\mathbf{x}), u^s(\mathbf{x}))^T$ denotes the vector of the exact solution's Fourier coefficients $u^c(\mathbf{x}) = x_1(x_1 - 1)x_2(x_2 - 1)$ and $u^s(\mathbf{x}) = 0$.

In Table 1, we observe the robustness and optimality of the AMLI preconditioned MINRES method as presented in [18, 34]. More precisely, the computational times increase with a factor of nine that exactly reveals the optimal computational complexity of the method according to the 3-refinement of the mesh. One can see that the norms of $\mathcal{R}_1$ reduce by a factor of three and the norms of $\mathcal{R}_2$ even better than by a factor of nine. Hence, the applied flux reconstruction is efficient. Altogether, the majorant reduces as a factor of three by trisection of the mesh size and is of the same order of convergence as of the exact error measured in the $H^1(\Omega)$ -seminorm. This is also observed in the efficiency index that is already quite small on the 27×27 -mesh and decreases up to a value of 1.498 on the (finest) 729×729 -mesh.

Example 5.2. In the second example, we consider a given time-analytic, but not time-harmonic source term

$$f(\mathbf{x}, t) = e^t \sin^2(t) \sin(x_1\pi) \sin(x_2\pi) ((1 + 2\pi^2) \sin(t) + 3 \cos(t)),$$

where $T = 2\pi/\omega$ with $\omega = 1$. The exact solution is given by

$$u(\mathbf{x}, t) = e^t \sin^3(t) \sin(x_1\pi) \sin(x_2\pi).$$

The Fourier coefficients of the Fourier series expansion of the source term f in time can be computed analytically. We truncate the Fourier series and approximate the Fourier coefficients by finite element functions as it

was presented before. Then, we solve the systems (3.3) and (3.4) for all $k \in \{0, \dots, N\}$ with $N = 8$, reconstruct the fluxes by an RT^0 -extension and then compute the corresponding majorants. Table 2 presents the number of AMLI iterations $n_{\text{AMLI}}^{\text{iter}}$, the CPU times in seconds t^{sec} , the norms of $\mathcal{R}_{10}^c$ and $\mathcal{R}_{20}^c$, i.e.,

$$\|\mathcal{R}_{10}^c\|_{\Omega}^2 = \|\operatorname{div} \boldsymbol{\tau}_0^c + f_0^c\|_{\Omega}^2, \quad \|\mathcal{R}_{20}^c\|_{\Omega}^2 = \|\boldsymbol{\tau}_0^c - \nabla \eta_0^c\|_{\Omega}^2,$$

as well as the majorants $\mathcal{M}_{|\cdot|}^{\oplus_0}$ as presented in (4.15) with $c_{|\cdot|}^0 = \underline{\nu} = 1$, and the corresponding efficiency indices

$$I_{\text{eff}}^0 = \frac{\mathcal{M}_{|\cdot|}^{\oplus_0}}{|\mathbf{u}_0^c - \boldsymbol{\eta}_0^c|_{1,\Omega}},$$

obtained on grids of different mesh sizes.

For $k = 0$, one has to solve the system (3.4). We observe in Table 2 that the AMLI solver presented by Kraus and Wolfmayr in [18] is of optimal computational complexity and the efficiency decreases up to a value of 1.002. Moreover, Tables 3 – 10 present the number of MINRES iterations $n_{\text{MINRES}}^{\text{iter}}$, the CPU times in seconds t^{sec} , the norms of $\mathcal{R}_{1k}$ and $\mathcal{R}_{2k}$, i.e.,

$$\begin{aligned} \|\mathcal{R}_{1k}\|_{\Omega}^2 &= \|\mathcal{R}_{1k}^c(\eta_k^s, \boldsymbol{\tau}_k^c)\|_{\Omega}^2 + \|\mathcal{R}_{1k}^s(\eta_k^c, \boldsymbol{\tau}_k^s)\|_{\Omega}^2 = \|-k\omega \eta_k^s + \operatorname{div} \boldsymbol{\tau}_k^c + f_k^c\|_{\Omega}^2 + \|k\omega \eta_k^c + \operatorname{div} \boldsymbol{\tau}_k^s + f_k^s\|_{\Omega}^2, \\ \|\mathcal{R}_{2k}\|_{\Omega}^2 &= \|\mathcal{R}_{2k}^c(\eta_k^s, \boldsymbol{\tau}_k^c)\|_{\Omega}^2 + \|\mathcal{R}_{2k}^s(\eta_k^c, \boldsymbol{\tau}_k^s)\|_{\Omega}^2 = \|\boldsymbol{\tau}_k^c - \nabla \eta_k^c\|_{\Omega}^2 + \|\boldsymbol{\tau}_k^s - \nabla \eta_k^s\|_{\Omega}^2, \end{aligned}$$

as well as the majorants $\mathcal{M}_{|\cdot|}^{\oplus_k}$ as presented in (4.14) with $\underline{c}^k = \frac{1}{\sqrt{2}} \min\{\underline{\nu}, k\omega \underline{\sigma}\} = \frac{1}{\sqrt{2}}$ for $k \in \{1, \dots, 8\}$, and, finally, the corresponding efficiency indices

$$I_{\text{eff}}^k = \frac{\mathcal{M}_{|\cdot|}^{\oplus_k}}{|\mathbf{u}_k - \boldsymbol{\eta}_k|_{1,\Omega}},$$

obtained on grids of different mesh sizes.

The results of Tables 3–10 regarding the number of MINRES iterations $n_{\text{MINRES}}^{\text{iter}}$ and the computational times are all similar and can be compared to our Example 5.1. Moreover, the reduction factors of $\|\mathcal{R}_{1k}\|_{\Omega}$, $\|\mathcal{R}_{2k}\|_{\Omega}$ and $\mathcal{M}_{|\cdot|}^{\oplus_k}$ as well as the values of the efficiency indices I_{eff}^k are approximately the same. This demonstrates the robustness of the method with respect to the modes k and the accurateness of the majorants $\mathcal{M}_{|\cdot|}^{\oplus_k}$. Moreover, the values of $\|\mathcal{R}_{1k}\|_{\Omega}$, $\|\mathcal{R}_{2k}\|_{\Omega}$ and $\mathcal{M}_{|\cdot|}^{\oplus_k}$ decrease for increasing k . This is also illustrated in Table 11. In this table, we finally compare the results from Tables 3–10 that were computed on the 729×729 -mesh. Hence, the results computed on the 729×729 -mesh are again presented for all $k \in \{0, \dots, 8\}$, and, then, for the overall functional error estimates. Here, the error majorant is given by (4.8) with $\tilde{\mu}_1 = \frac{1}{\sqrt{2}}$, and the remainder term (4.6) has to be computed in order to get $\|\mathcal{R}_1\|$. Remember that the remainder term $\mathcal{E}_N$ can be precomputed exactly as $\|f - f_N\|$, since f is the given data and f_N its truncated Fourier series. For $N = 8$, it is given by 1.197e-02. Altogether, we obtain a global efficiency index of 1.404 on the 729×729 -mesh.

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Grid	$n_{\text{AMLI}}^{\text{iter}}$	t^{sec}	$\ \mathcal{R}_{10}^c\ _{\Omega}$	$\ \mathcal{R}_{20}^c\ _{\Omega}$	$\mathcal{M}_{ \cdot }^{\oplus 0}$	I_{eff}^0
9×9	21	0.00	6.317e+01	1.773e+00	1.599e+01	1.315
27×27	23	0.00	2.349e+01	3.796e-02	5.325e+00	1.064
81×81	23	0.03	7.927e+00	2.865e-03	1.787e+00	1.020
243×243	22	0.27	2.646e+00	1.886e-04	5.957e-01	1.006
729×729	22	2.45	8.821e-01	1.183e-05	1.986e-01	1.002

Table 2. Majorant $\mathcal{M}_{|\cdot|}^{\oplus 0}$ and its parts (Example 5.2).

Grid	$n_{\text{MINRES}}^{\text{iter}}$	t^{sec}	$\ \mathcal{R}_{11}\ _{\Omega}$	$\ \mathcal{R}_{21}\ _{\Omega}$	$\mathcal{M}_{ \cdot }^{\oplus 1}$	I_{eff}^1
9×9	14	0.00	1.238e+02	3.444e+00	4.426e+01	1.908
27×27	12	0.02	4.566e+01	7.376e-02	1.464e+01	1.550
81×81	10	0.21	1.540e+01	5.560e-03	4.910e+00	1.485
243×243	10	2.08	5.140e+00	3.659e-04	1.637e+00	1.466
729×729	8	15.84	1.714e+00	2.295e-05	5.455e-01	1.460

Table 3. Majorant $\mathcal{M}_{|\cdot|}^{\oplus 1}$ and its parts (Example 5.2).

Grid	$n_{\text{MINRES}}^{\text{iter}}$	t^{sec}	$\ \mathcal{R}_{12}\ _{\Omega}$	$\ \mathcal{R}_{22}\ _{\Omega}$	$\mathcal{M}_{ \cdot }^{\oplus 2}$	I_{eff}^2
9×9	9	0.00	7.953e+01	2.209e+00	2.844e+01	1.880
27×27	9	0.02	2.930e+01	4.737e-02	9.394e+00	1.523
81×81	9	0.19	9.883e+00	3.555e-03	3.151e+00	1.460
243×243	8	1.73	3.299e+00	2.339e-04	1.050e+00	1.441
729×729	8	15.64	1.100e+00	1.467e-05	3.501e-01	1.435

Table 4. Majorant $\mathcal{M}_{|\cdot|}^{\oplus 2}$ and its parts (Example 5.2).

Grid	$n_{\text{MINRES}}^{\text{iter}}$	t^{sec}	$\ \mathcal{R}_{13}\ _{\Omega}$	$\ \mathcal{R}_{23}\ _{\Omega}$	$\mathcal{M}_{ \cdot }^{\oplus 3}$	I_{eff}^3
9×9	8	0.00	4.613e+01	1.277e+00	1.649e+01	1.905
27×27	8	0.02	1.696e+01	2.745e-02	5.437e+00	1.541
81×81	7	0.15	5.719e+00	2.046e-03	1.823e+00	1.477
243×243	7	1.56	1.909e+00	1.345e-04	6.079e-01	1.457
729×729	6	12.34	6.364e-01	8.436e-06	2.026e-01	1.451

Table 5. Majorant $\mathcal{M}_{|\cdot|}^{\oplus 3}$ and its parts (Example 5.2).

Grid	$n_{\text{MINRES}}^{\text{iter}}$	t^{sec}	$\ \mathcal{R}_{14}\ _{\Omega}$	$\ \mathcal{R}_{24}\ _{\Omega}$	$\mathcal{M}_{ \cdot }^{\oplus 4}$	I_{eff}^4
9×9	10	0.00	1.624e+01	4.474e-01	5.801e+00	1.958
27×27	9	0.02	5.950e+00	9.645e-03	1.908e+00	1.582
81×81	9	0.19	2.007e+00	7.120e-04	6.398e-01	1.516
243×243	9	1.90	6.698e-01	4.680e-05	2.133e-01	1.496
729×729	8	15.88	2.233e-01	2.934e-06	7.108e-02	1.490

Table 6. Majorant $\mathcal{M}_{|\cdot|}^{\oplus 4}$ and its parts (Example 5.2).

Grid	$n_{\text{MINRES}}^{\text{iter}}$	t^{sec}	$\ \mathcal{R}_{15}\ _{\Omega}$	$\ \mathcal{R}_{25}\ _{\Omega}$	$\mathcal{M}_{ \cdot }^{\oplus 5}$	I_{eff}^5
9×9	9	0.00	5.878e+00	1.611e-01	2.099e+00	1.970
27×27	9	0.02	2.146e+00	3.485e-03	6.880e-01	1.586
81×81	7	0.15	7.236e-01	2.542e-04	2.307e-01	1.520
243×243	7	1.54	2.415e-01	1.669e-05	7.690e-02	1.500
729×729	7	14.17	8.052e-02	1.046e-06	2.563e-02	1.493

Table 7. Majorant $\mathcal{M}_{|\cdot|}^{\oplus 5}$ and its parts (Example 5.2).

Grid	$n_{\text{MINRES}}^{\text{iter}}$	t^{sec}	$\ \mathcal{R}_{16}\ _{\Omega}$	$\ \mathcal{R}_{26}\ _{\Omega}$	$\mathcal{M}_{ \cdot }^{\oplus 6}$	I_{eff}^6
9×9	11	0.00	2.621e+00	7.132e-02	9.351e-01	1.991
27×27	10	0.02	9.522e-01	1.550e-03	3.053e-01	1.597
81×81	9	0.19	3.211e-01	1.114e-04	1.024e-01	1.529
243×243	8	1.73	1.072e-01	7.312e-06	3.412e-02	1.509
729×729	8	15.81	3.573e-02	4.583e-07	1.137e-02	1.503

Table 8. Majorant $\mathcal{M}_{|\cdot|}^{\oplus 6}$ and its parts (Example 5.2).

Grid	$n_{\text{MINRES}}^{\text{iter}}$	t^{sec}	$\ \mathcal{R}_{17}\ _{\Omega}$	$\ \mathcal{R}_{27}\ _{\Omega}$	$\mathcal{M}_{ \cdot }^{\oplus 7}$	I_{eff}^7
9×9	12	0.00	1.359e+00	3.670e-02	4.846e-01	2.023
27×27	11	0.02	4.913e-01	8.015e-04	1.575e-01	1.615
81×81	9	0.19	1.656e-01	5.669e-05	5.280e-02	1.546
243×243	9	1.89	5.528e-02	3.716e-06	1.760e-02	1.526
729×729	8	15.88	1.843e-02	2.329e-07	5.867e-03	1.520

Table 9. Majorant $\mathcal{M}_{|\cdot|}^{\oplus 7}$ and its parts (Example 5.2).

Grid	$n_{\text{MINRES}}^{\text{iter}}$	t^{sec}	$\ \mathcal{R}_{18}\ _{\Omega}$	$\ \mathcal{R}_{28}\ _{\Omega}$	$\mathcal{M}_{ \cdot }^{\oplus 8}$	I_{eff}^8
9×9	12	0.00	7.839e-01	2.098e-02	2.792e-01	2.064
27×27	11	0.02	2.816e-01	4.606e-04	9.028e-02	1.638
81×81	11	0.23	9.492e-02	3.200e-05	3.026e-02	1.569
243×243	10	2.05	3.168e-02	2.095e-06	1.009e-02	1.548
729×729	10	19.27	1.056e-02	1.312e-07	3.362e-03	1.542

Table 10. Majorant $\mathcal{M}_{|\cdot|}^{\oplus 8}$ and its parts (Example 5.2).

	$n_{\text{MINRES}}^{\text{iter}}$	t^{sec}	$\ \mathcal{R}_1\ $	$\ \mathcal{R}_2\ $	$\mathcal{M}_{ \cdot }^{\oplus}$	I_{eff}
$k = 0$	—	—	8.821e-01	1.183e-05	1.986e-01	1.002
$k = 1$	8	15.84	1.714e+00	2.295e-05	5.455e-01	1.460
$k = 2$	8	15.64	1.100e+00	1.467e-05	3.501e-01	1.435
$k = 3$	6	12.34	6.364e-01	8.436e-06	2.026e-01	1.451
$k = 4$	8	15.88	2.233e-01	2.934e-06	7.108e-02	1.490
$k = 5$	7	14.17	8.052e-02	1.046e-06	2.563e-02	1.493
$k = 6$	8	15.81	3.573e-02	4.583e-07	1.137e-02	1.503
$k = 7$	8	15.88	1.843e-02	2.329e-07	5.867e-03	1.520
$k = 8$	10	19.27	1.056e-02	1.312e-07	3.362e-03	1.542
overall	—	—	4.403e+00	5.886e-05	1.402e+00	1.404

Table 11. The overall majorant $\mathcal{M}_{|\cdot|}^{\oplus}$ and its parts computed on a 729×729 -mesh (Example 5.2).

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