Research Article

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# Estimates of the Distance to the Set of Solenoidal Vector Fields and Applications to A Posteriori Error Control

**Abstract:** The paper is concerned with computable estimates of the distance between a vector-valued function in the Sobolev space $W^{1,\gamma}(\Omega,\mathbb{R}^d)$ (where $\gamma \in (1,+\infty)$ and $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^d$) and the subspace $S^{1,\gamma}(\Omega,\mathbb{R}^d)$ containing all divergence-free (solenoidal) vector functions. Derivation of these estimates is closely related to the stability theorem that establishes existence of a bounded operator inverse to the operator div. The constant in the respective stability inequality arises in the estimates of the distance to the set $S^{1,\gamma}(\Omega,\mathbb{R}^d)$. In general, it is difficult to find a guaranteed and realistic upper bound of this global constant. We suggest a way to circumvent this difficulty by using weak (integral mean) solenoidality conditions and localized versions of the stability theorem. They are derived for the case where $\Omega$ is represented as a union of simple subdomains (overlapping or non-overlapping), for which estimates of the corresponding stability constants are known. These new versions of the stability theorem imply estimates of the distance to $S^{1,\gamma}(\Omega,\mathbb{R}^d)$ that involve only local constants associated with subdomains. Finally, the estimates are used for deriving fully computable a posteriori estimates for problems in the theory of incompressible viscous fluids.

**Keywords:** inf-sup Condition, Incompressible Viscous Fluids, Domain Decomposition, A Posteriori Error Estimates

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## 1 Introduction

Let $\Omega$ be an open bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) with Lipschitz boundary $\Gamma$. The inequality

$$\inf_{\substack{p\in\widetilde{L}^2(\Omega)\\ p\neq 0}} \sup_{\substack{w\in V_0(\Omega)\\ w\neq 0}} \frac{\int_\Omega p \operatorname{div} w\,dx}{\|p\|\|\nabla w\|} \geq c_\Omega > 0 \tag{1.1}$$

is one of the keystone relations in mathematical analysis of incompressible media problems. It is often called the LBB (Ladyzhenskaya–Babuska–Brezzi) or inf-sup condition. Here $V_0(\Omega)$ is a subspace of $H^1(\Omega,\mathbb{R}^d)$ containing vector-valued functions vanishing on $\Gamma$ and

$$\widetilde{L}^2(\Omega) := \Big\{ q \in L^2(\Omega) \mid \{q\}_\Omega := |\Omega|^{-1} \int_\Omega q\,dx = 0 \Big\},$$

where $|\Omega|$ denotes the Lebesgue measure of $\Omega$.

Another form of this result is known as Babuska–Aziz or Ladyzhenskaya–Solonnikov theorem. For the case $d = 2$ it was established in [3] and for $d = 3$ in [19], where this result was used in order to prove existence of a generalized solution to the Stokes problem [18].

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**Theorem 1.** *For any $f \in \widetilde{L}^2(\Omega)$, there exists a function $w_f \in W_0^{1,2}(\Omega, \mathbb{R}^d)$ such that*

$$\operatorname{div} w_f = f \quad \text{and} \quad \|\nabla w_f\| \le \mathbb{C}_\Omega \|f\|, \tag{1.2}$$

*where $\mathbb{C}_\Omega$ is a positive constant depending on $\Omega$.*

Theorem 1 states that inversion of the divergence operator is stable (with respect to the norm generated by $\nabla$). It is easy to see that (1.2) implies (1.1) with $c_\Omega = \mathbb{C}_\Omega^{-1}$. Henceforth, we will call $\mathbb{C}_\Omega$ and $c_\Omega$ the "stability" and "inf-sup" constants, respectively.

Another equivalent way, which leads to (1.1) and other similar conditions, comes from the saddle point theory where boundary value problems are considered as saddle point problems for a certain Lagrangians. This theory forms the basis of mixed methods for boundary value problems (see [2, 6] and a profoundly elaborated theory in the book [7]). Conditions analogous to (1.1) for various pairs of finite-dimensional spaces are often used for proving stability and convergence of numerical methods developed for viscous incompressible fluids (see, e.g., [15, 20, 35]).

Also (1.1) can be viewed as a form of the Nečas inequality [23] (for domains with Lipschitz boundaries a simple proof of this inequality can be found in [4]).

Theorem 1 has a principal meaning in the theory of viscous incompressible fluids and other problems related to incompressible media. Existence of a positive constant $c_\Omega$ and estimates of its values for various domains is of the same importance as estimates of the constant $K_\Omega$ in the Korn's inequality for elasticity problems. Moreover, in [16] it was shown that for simply connected domains in $d = 2$ the constants are joined by the relation $2\mathbb{C}_\Omega = K_\Omega = 2(1 + L_\Omega)$, where $L_\Omega$ is the constant in the Friedrichs inequality [11]

$$\|u\|^2 \le L_\Omega \|v\|^2, \tag{1.3}$$

which holds for an analytic function $u + iv$ provided that $\{u\}_\Omega = 0$.

Theorem 1 can be extended to $L^\gamma$ spaces for $1 < \gamma < +\infty$ (see [5, 14, 26, 27]).

**Theorem 2.** *Let $f \in L^\gamma(\Omega)$. If $\{f\}_\Omega = 0$, then there exists $v_f \in W_0^{1,\gamma}(\Omega, \mathbb{R}^d)$ such that*

$$\operatorname{div} v_f = f \quad \text{and} \quad \|\nabla v_f\|_{\Omega,\gamma} \le \mathbb{C}_{\Omega,\gamma} \|\operatorname{div} v_f\|_{\Omega,\gamma}, \tag{1.4}$$

*where $\mathbb{C}_{\Omega,\gamma}$ ($\mathbb{C}_{\Omega,2} = \mathbb{C}_\Omega$) is a positive constant, which depends only on $\Omega$.*

It is worth noting that for $\gamma = 1$ and $\gamma = +\infty$ similar results may be not true (see [8, 9]).

Finding sharp estimates of $\mathbb{C}_{\Omega,\gamma}$ is necessary if we wish to obtain computable estimates of the distance to the set of divergence-free fields (see Lemma 1). It is not difficult to see that the inf-sup constant $c_\Omega$ in (1.1) is nonnegative and cannot exceed 1 (so that $\mathbb{C}_\Omega \ge 1$). Also, it is known that $c_\Omega > 0$ for any bounded Lipschitz domain and, therefore, $\mathbb{C}_\Omega$ is bounded. Thus, for Lipschitz domains one has $1 \le \mathbb{C}_\Omega < +\infty$. For domains with caspidal tips, $c_\Omega$ may be equal to zero (a systematic analysis of these cases can be found in [1]).

First quantitative estimates of $c_\Omega$ and $\mathbb{C}_\Omega$ were obtained in [10, 24, 25, 34]. It is known that $c_\Omega = 1/\sqrt{d}$ for a ball in $\mathbb{R}^d$ (i.e., $\mathbb{C}_\Omega = \sqrt{d}$) and for an ellipse ($\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$, where $a < b$) $c_\Omega^2 = \frac{a^2}{a^2+b^2}$ (see [8, 17]).

Estimates are also known for Lipschitz domains in $\mathbb{R}^2$, which are star-shaped with respect to a ball with center $x_0$. Let $r$ be the ray from $x_0$ crossing $\Gamma$ at $x$. For almost all $x \in \Gamma$, there exists the unique tangent line, which forms a positive angle $\theta \le \frac{\pi}{2}$ with the ray $r$. The quantity $\Theta_\Omega := \inf_{x\in\Gamma} \theta(x)$ generates the estimate [16]

$$c_\Omega \ge \sin \frac{\Theta_\Omega}{2}.$$

However, these lower bounds of $c_\Omega$ (and respective upper bounds of $\mathbb{C}_\Omega$) may be rather coarse.

A significant improvement of the estimates was obtained in [8] for domains in $\mathbb{R}^2$, which are contained in a ball of radius $R$ and are star-shaped with respect to a concentric ball of radius $\rho$. It was shown that

$$c_\Omega \ge \frac{\kappa}{\sqrt{2}} \Big(1 + \sqrt{1 - \kappa^2}\Big)^{-1/2}, \tag{1.5}$$

where $\kappa = \frac{\rho}{R}$. This formula allows us to obtain guaranteed upper bounds of $\mathbb{C}_\Omega$ for simplexes, quadrilaterals, and other polygonal domains. In particular, it implies a simple upper bound $\mathbb{C}_\Omega \le \frac{2}{\kappa}$.

To the best of our knowledge, for $d = 3$, estimates of $c_\Omega$ are known only for domains with sufficiently regular boundaries (e.g., for an ellipsoid [17]). In [25], it was shown that for star-shaped domains in $\mathbb{R}^3$ with $C^1$ boundary described by the relation $r = r_0(\phi, \psi)$ (where $(r, \phi, \psi)$ denote coordinates of the spherical system) the value of $\mathbb{C}^2_\Omega$ is bounded from above by the quantity

$$1 + \max_\Gamma \frac{r_0^3}{a^3}(3 + 3Q + Q^2),$$

where $0 < a < \min r_0$ and

$$Q = \max_{\phi,\psi}\Big(\frac{1}{r_0^2}\Big(\frac{\partial r_0}{\partial\psi}\Big)^2 + \frac{1}{\sin^2\psi}\Big(\frac{\partial r_0}{\partial\phi}\Big)^2\Big)^{1/2}.$$

Attempts to find $\mathbb{C}_{\Omega,\gamma}$ numerically are faced with serious difficulties because the respective minimizers may expose highly singular behavior. This question was deeply studied in [17], where approximate values of $c_\Omega$ were computed for various domains (e.g., ring, cardoid, limacon, square, cube, cylinder). However, so far we do not have an efficient method able to compute guaranteed and realistic bounds of these constants for arbitrary Lipschitz domains in $\mathbb{R}^3$ or, at least, for arbitrary nondegenerate polyhedral domains.

From the viewpoint of numerical analysis, the constant $\mathbb{C}_\Omega$ is important to know by different reasons. In particular, it controls the distance to the set $S^{1,\gamma}(\Omega, \mathbb{R}^d)$ of divergence-free fields (e.g., see Lemma 1). Therefore, the question arises how to circumvent difficulties related to the fact that in general the constant $\mathbb{C}_\Omega$ (or $\mathbb{C}_{\Omega,\gamma}$ for $\gamma \neq 2$) is unknown and to obtain easily computable estimates of the distance to $S^{1,\gamma}(\Omega, \mathbb{R}^d)$ based on constants associated with a limited amount of simple basic domains.

Below we discuss a way to answer this question, which is based on the following idea:

> Estimates of the distance between $v \in W^{1,\gamma}(\Omega, \mathbb{R}^d)$ and the set $S^{1,\gamma}(\Omega, \mathbb{R}^d)$ are easier to obtain if $v$ satisfies "weak solenoidality conditions" globally (i.e., $\{\operatorname{div} v\}_\Omega = 0$) or locally ($\{\operatorname{div} v\}_{\Omega_i} = 0$ for a collection of subdomains $\Omega_i$). Estimates of the distance between $v$ and the set of weakly solenoidal fields can be deduced without stability constants $\mathbb{C}_{\Omega,\gamma}$. Jointly, these two estimates yield estimates of the distance to $S^{1,\gamma}(\Omega, \mathbb{R}^d)$ with computable constants.

For $\gamma = 2$ this idea was earlier suggested and used in [30–33].

The outline of the paper is as follows. In Section 2, we deduce estimates of the distance to the set of divergence-free fields for functions vanishing on a part $\Gamma_0$ of the boundary and show that regardless of the particular form of $\Gamma_0$ the corresponding estimate holds with the same constant as for $\Gamma_0 = \Gamma$ provided that the function has zero mean divergence (this result generalizes [30, Lemma 6.2.1]). After that, a more sophisticated estimate is derived, which provides an upper bound of the distance to the set of divergence-free fields with the same constant but without zero mean conditions. Section 3 presents estimates based on domain decomposition. They can be useful for polygonal domains decomposed into simplicial and polyhedral cells $\Omega_i$. If the constants $\mathbb{C}_{\Omega_i,\gamma}$ for these cells are known, then Lemmas 5 and 6 (derived for non-overlapping and overlapping decompositions, respectively) suggest a simple estimate of the distance to the set of divergence-free fields. Finally, in Section 4 we discuss applications of these results to a posteriori estimates for problems in the theory of viscous incompressible fluids.

## 2 Estimates of the Distance to the Set $S_0^{1,\gamma}$

Theorems 1 and 2 imply estimates of the distance between a vector-valued function $v \in W_0^{1,\gamma}(\Omega, \mathbb{R}^d)$ and the subspace $S_0^{1,\gamma}(\Omega, \mathbb{R}^d) \subset W_0^{1,\gamma}(\Omega, \mathbb{R}^d)$ containing solenoidal (divergence-free) functions. The distance is measured in terms of the quantity

$$d(v, S_0^{1,\gamma}(\Omega, \mathbb{R}^d)) := \inf_{v_0 \in S_0^{1,\gamma}(\Omega,\mathbb{R}^d)} \|\nabla(v - v_0)\|_{\Omega,\gamma}.$$

**Lemma 1.** *For any $v \in W_0^{1,\gamma}(\Omega, \mathbb{R}^d)$,*

$$d(v, S_0^{1,\gamma}(\Omega, \mathbb{R}^d)) \leq \mathbb{C}_{\Omega,\gamma} \|\operatorname{div} v\|_{\Omega,\gamma}. \tag{2.1}$$

This result directly follows from Theorem 2 if we set $f = \operatorname{div} v$. Since

$$\int_\Omega \operatorname{div} v dx = \int_\Gamma n \cdot v ds = 0, \tag{2.2}$$

there exists a function $v_f \in W_0^{1,\gamma}(\Omega, \mathbb{R}^d)$ such that (1.4) holds. We set $v_0 := v - v_f \in S_0^{1,\gamma}(\Omega)$ and obtain

$$\|\nabla(v - v_0)\|_{\Omega,\gamma} = \|\nabla v_f\|_{\Omega,\gamma} \leq \mathbb{C}_{\Omega,\gamma} \|\operatorname{div} v\|_{\Omega,\gamma}.$$

**Remark 1.** Lemma 1 implies an estimate that can be useful for error analysis of problems with nonhomogeneous boundary conditions. Consider $v \in W^{1,\gamma}(\Omega, \mathbb{R}^d)$ such that $v = g$ on $\Gamma$, where $g$ is a given function in $W^{1,\gamma}(\Omega, \mathbb{R}^d)$ satisfying the condition $\operatorname{div} g = 0$. Let $S_0^{1,\gamma}(\Omega, \mathbb{R}^d) + g$ denote the set of solenoidal fields satisfying the same boundary condition, i.e.,

$$S_0^{1,\gamma}(\Omega, \mathbb{R}^d) + g := \{v = w_0 + g,\ w_0 \in S_0^{1,\gamma}(\Omega, \mathbb{R}^d)\}.$$

Since $v - g \in W_0^{1,\gamma}(\Omega, \mathbb{R}^d)$, we obtain

$$\begin{aligned} d(v, S_0^{1,\gamma}(\Omega, \mathbb{R}^d) + g) &:= \inf_{\widetilde{w} \in S_0^{1,\gamma}(\Omega,\mathbb{R}^d)+g} \|\nabla(v - \widetilde{w})\|_{\Omega,\gamma} \\ &= \inf_{v_0 \in S_0^{1,\gamma}(\Omega,\mathbb{R}^d)} \|\nabla(v - g - w_0)\|_{\Omega,\gamma} \leq \mathbb{C}_{\Omega,\gamma} \|\operatorname{div} v\|_{\Omega,\gamma}. \end{aligned} \tag{2.3}$$

We see that the distance to the set of divergence-free fields is easy to estimate from above provided that the constant $\mathbb{C}_{\Omega,\gamma}$ (or a suitable upper bound of it) is known. However, this simple argumentation cannot be directly applied if $v$ vanishes only on a part of $\Gamma$ what happens if the boundary conditions are different on different parts of the boundary. Let $\Gamma_0$ be a part of $\Gamma$ such that $\operatorname{meas}_{d-1} \Gamma_0 > 0$. We consider functions in the set

$$W_{0,\Gamma_0}^{1,\gamma}(\Omega, \mathbb{R}^d) := \{v \in W^{1,\gamma}(\Omega, \mathbb{R}^d) \mid v = 0 \text{ on } \Gamma_0\}$$

and wish to estimate the distance between $v \in W_{0,\Gamma_0}^{1,\gamma}(\Omega, \mathbb{R}^d)$ and $S_{0,\Gamma_0}^{1,\gamma}(\Omega, \mathbb{R}^d)$, where

$$S_{0,\Gamma_0}^{1,\gamma}(\Omega, \mathbb{R}^d) = \{v \in W_{0,\Gamma_0}^{1,\gamma}(\Omega, \mathbb{R}^d) \mid \operatorname{div} v = 0\}.$$

Moreover, our goal is to deduce an estimate with the same constant $\mathbb{C}_{\Omega,\gamma}$ as in (1.3).

It is easy to see that the condition (2.2) may not hold and, therefore, we cannot directly use Theorem 2. However, if $v$ satisfies (2.2), then estimate (2.1) holds with the same constant $\mathbb{C}_{\Omega,\gamma}$.

**Lemma 2.** *Let*

$$v \in \widetilde{W}^{1,\gamma}(\Omega, \mathbb{R}^d) := \{w \in W^{1,\gamma}(\Omega, \mathbb{R}^d) \mid \{\operatorname{div} w\}_\Omega = 0\}.$$

*Then, there exists a function $v_0 \in S^{1,2}(\Omega, \mathbb{R}^d)$ satisfying the condition $v_0 = v$ on $\Gamma$ such that*

$$\|\nabla(v - v_0)\|_{\Omega,\gamma} \leq \mathbb{C}_{\Omega,\gamma} \|\operatorname{div} v\|_{\Omega,\gamma}.$$

Now our goal is to obtain similar estimates, which are valid for any function $v \in W_{0,\Gamma_0}^{1,\gamma}(\Omega, \mathbb{R}^d)$ vanishing on $\Gamma_0 \subset \Gamma$. First, we consider the most interesting case $\gamma = 2$ and find the distance

$$d(v, \widetilde{W}_{0,\Gamma_0}^{1,2}(\Omega, \mathbb{R}^d)) := \inf_{\widetilde{v} \in \widetilde{W}_{0,\Gamma_0}^{1,2}(\Omega,\mathbb{R}^d)} \|\nabla(\widetilde{v} - v)\|.$$

**Lemma 3.** *Let $v \in W^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)$. Then*

$$d(v, \widetilde{W}^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)) = \frac{1}{2\|\nabla u_1\|}\Big|\int\limits_\Omega \operatorname{div} v dx\Big|, \tag{2.4}$$

*where $u_1$ minimizes the functional*

$$J(w) := \|\nabla w\|^2 + \int\limits_\Omega \operatorname{div} w dx$$

*on the set $W^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)$.*

*Proof.* Let $w_\mu$ be a function in $W^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)$ such that

$$\{\operatorname{div} w_\mu\}_\Omega = \{\operatorname{div} v\}_\Omega = \frac{\mu}{|\Omega|}. \tag{2.5}$$

Then,

$$\{\operatorname{div}(w_\mu - v)\}_\Omega = 0 \quad \text{and} \quad \widetilde{v} = v - w_\mu \in \widetilde{W}^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d).$$

The relation $\widetilde{v} = v - w_\mu$ states an isomorphism between $\widetilde{W}^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)$ and the subset of $W^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)$ containing the functions subject to (2.5). Therefore,

$$d^2(v, \widetilde{W}^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)) = \inf_{\substack{w_\mu \in W^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d) \\ \{\operatorname{div} w_\mu\}_\Omega = \frac{\mu}{\Omega}}} \|\nabla w_\mu\|^2. \tag{2.6}$$

Due to standard theorems of convex analysis, the variational problem in the right-hand side of (2.6) possesses a unique solution.

It has a minimax form

$$\inf_{w \in W^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)} \sup_{\lambda \in \mathbb{R}} L(\lambda, w) \quad \text{where } L(\lambda, w) = \|\nabla w\|^2 + \lambda\Big(\int\limits_\Omega \operatorname{div} w dx - \mu\Big).$$

Since inf sup ≥ sup inf, we conclude that

$$d^2(v, \widetilde{W}^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)) \geq \sup_{\lambda \in \mathbb{R}} \inf_{w \in W^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)} L(\lambda, w).$$

This dual setting generates the functional

$$G(\lambda) := \inf_{w \in W^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)} \Big\{\|\nabla w\|^2 + \lambda \int\limits_\Omega \operatorname{div} w dx\Big\} - \lambda\mu. \tag{2.7}$$

The variational problem in the right-hand side of (2.7) is well posed and the respective minimizer $u_\lambda$ satisfies the integral identity

$$\int\limits_\Omega \nabla u_\lambda : \nabla w dx + \frac{\lambda}{2} \int\limits_{\Gamma \setminus \Gamma_0} n \cdot w ds = 0 \quad \text{for all } w \in W^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d).$$

It is easy to see that $u_\lambda = \lambda u_1$ and

$$\|\nabla u_1\|^2 + \frac{1}{2} \int\limits_\Omega \operatorname{div} u_1 dx = 0. \tag{2.8}$$

Now, we obtain an explicit form of the dual functional

$$G(\lambda) = \lambda^2 \|\nabla u_1\|^2 + \lambda\Big(\lambda \int\limits_\Omega \operatorname{div} u_1 dx - \mu\Big) = -\lambda^2 \|\nabla u_1\|^2 - \lambda\mu.$$

$\sup_\lambda G(\lambda)$ is attained at $\lambda = \lambda_* := -\mu/(2\|\nabla u_1\|^2)$. Since $u_1$ solves the problem with nonhomogeneous Neumann boundary condition, $\|\nabla u_1\| \neq 0$ and, therefore, $\lambda_*$ is a finite real number. Hence, we conclude that

$$d^2(v, \widetilde{W}^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)) \geq \sup_\lambda G(\lambda) = G(\lambda_*) = \frac{1}{4}\frac{\mu^2}{\|\nabla u_1\|^2}. \tag{2.9}$$

In view of (2.8), $\lambda_* \int_\Omega \operatorname{div} u_1 \, dx = \mu$. Hence, we set $w_\mu = \lambda_* u_1$ and obtain

$$d^2(v, \widetilde{W}^{1,2}_{0,\Gamma_0}) \leq \lambda_*^2 \|\nabla u_1\|^2 = \frac{1}{4}\frac{\mu^2}{\|\nabla u_1\|^2}. \tag{2.10}$$

Now (2.4) follows from (2.9) and (2.10). □

**Theorem 3.** *Let $v \in W^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)$. Then,*

$$d(v, S^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)) \leq \mathbb{C}_\Omega \|\operatorname{div} v\| + C_1 \Big| \int_\Omega \operatorname{div} v \, dx \Big|, \tag{2.11}$$

*where*

$$C_1 = \frac{1}{2\|\nabla u_1\|}\Big(\mathbb{C}_\Omega \frac{\|\operatorname{div} u_1\|}{\|\nabla u_1\|} + 1\Big)$$

*and $u_1$ is defined in Lemma 3.*

*Proof.* We set $\tilde{v} = v - \lambda_* u_1$ and find that

$$\begin{aligned}
\inf_{v_0 \in S^{1,2}_{0,\Gamma_0}(\Omega,\mathbb{R}^d)} \|\nabla(v - v_0)\| &\leq \|\nabla(v - \tilde{v})\| + \inf_{v_0 \in S^{1,2}_{0,\Gamma_0}(\Omega,\mathbb{R}^d)} \|\nabla(\tilde{v} - v_0)\| \\
&\leq \mathbb{C}_\Omega \|\operatorname{div} v - \lambda_* \operatorname{div} u_1\| + \frac{1}{2\|\nabla u_1\|}\Big|\int_\Omega \operatorname{div} v \, dx\Big| \\
&\leq \mathbb{C}_\Omega (\|\operatorname{div} v\| + |\lambda_*| \|\operatorname{div} u_1\|) + \frac{1}{2\|\nabla u_1\|}\Big|\int_\Omega \operatorname{div} v \, dx\Big| \\
&= \mathbb{C}_\Omega \|\operatorname{div} v\| + \Big(1 + \mathbb{C}_\Omega \frac{\|\operatorname{div} u_1\|}{\|\nabla u_1\|}\Big)\frac{1}{2\|\nabla u_1\|}\Big|\int_\Omega \operatorname{div} v \, dx\Big|.
\end{aligned}$$ □

**Remark 2.** It is not difficult to see that

$$C_1 \leq \widehat{C}_1 := \frac{1}{2\|\nabla u_1\|}\big(1 + \sqrt{d}\,\mathbb{C}_\Omega\big).$$

From the practical point of view, it is preferable to replace $u_1$ (exact solution of a boundary value problem) by a solution of some finite-dimensional problem. This can be done as follows. Let $V^h_{0,\Gamma_0} \subset W^{1,2}_{0,\Gamma_0}$ be a finite-dimensional space and $\widetilde{V}^h_{0,\Gamma_0}$ be the subset of functions with zero mean values. Instead of (2.6), we use the estimate

$$d^2(v, \widetilde{W}^{1,2}_{0,\Gamma_0}) \leq \inf_{\substack{w^h_\mu \in V^h_{0,\Gamma_0} \\ \{\operatorname{div} w^h_\mu\}_\Omega = \frac{\mu}{\Omega}}} \|\nabla w^h_\mu\|^2. \tag{2.12}$$

By repeating above arguments, we arrive at the finite-dimensional problem: find $u_{1,h} \in V^h_{0,\Gamma_0}$ such that

$$J(u_{1,h}) = \inf_{w_h \in V^h_{0,\Gamma_0}} J(w_h). \tag{2.13}$$

Instead of (2.8), we have

$$\int_\Omega \Big(\nabla u_{1,h} : \nabla w_h + \frac{1}{2}\operatorname{div} w_h\Big) dx = 0 \quad \text{for all } w_h \in V^h_{0,\Gamma_0}$$

and select $\lambda_*^h = -\mu/(2\|\nabla u_{1,h}\|^2)$. Then,

$$\lambda_*^h \int_\Omega \operatorname{div} u_{1,h} dx = \mu,$$

and $\widetilde{v}^h = v - \lambda_*^h u_{1,h}$ satisfies the condition $\{\operatorname{div} \widetilde{v}^h\}_\Omega = 0$. Therefore,

$$\inf_{w \in \widetilde{W}^{1,2}_{0,\Gamma_0}} \|\nabla(v - w_h)\| \leq \lambda_*^h \|\nabla u_{1,h}\| = \frac{1}{2\|\nabla u_{1,h}\|} \Big| \int_\Omega \operatorname{div} v dx \Big|.$$

Then, instead of Lemma 2 we have the following result.

**Lemma 4.** *Let $v \in W^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)$. Then*

$$d(v, \widetilde{W}^{1,2}_{0,\Gamma_0}) \leq \frac{1}{2\|\nabla u_{1,h}\|} \Big| \int_\Omega \operatorname{div} v dx \Big|, \tag{2.14}$$

*where $u_{1,h}$ is a minimizer of* (2.12).

Lemma 4 shows that estimate (2.11) holds with the constant

$$C_{1,h} = \frac{1}{2\|\nabla u_{1,h}\|} \Big( \mathbb{C}_\Omega \frac{\|\operatorname{div} u_{1,h}\|}{\|\nabla u_{1,h}\|} + 1 \Big) \leq \widehat{C}_{1,h} := \frac{1}{2\|\nabla u_{1,h}\|} \big( 1 + \sqrt{d}\mathbb{C}_\Omega \big).$$

It is easy to see that

$$\begin{aligned} \inf_{w \in W^{1,2}_{0,\Gamma_0}(\Omega, \mathbb{R}^d)} \Big\{ \|\nabla w\|^2 + \int_\Omega \operatorname{div} w dx \Big\} &= \|\nabla u_1\|^2 + \int_\Omega \operatorname{div} u_1 dx = -\|\nabla u_1\|^2 \\ &\leq \inf_{w_h \in V^h_{0,\Gamma_0}} \Big\{ \|\nabla w_h\|^2 + \int_\Omega \operatorname{div} w_h dx \Big\} \\ &= \|\nabla u_{1,h}\|^2 + \int_\Omega \operatorname{div} u_{1,h} dx = -\|\nabla u_{1,h}\|^2. \end{aligned}$$

Thus, $\|\nabla u_1\| \geq \|\nabla u_{1,h}\|$ and, therefore, the constant in (2.4) is smaller than in (2.14).

For $\gamma \in (1, +\infty)$, we deduce similar estimates by the same method. Let $u_1$ be the minimizer of the problem

$$\inf_{w \in W^{1,\gamma}_{0,\Gamma_0}(\Omega)} \Big\{ \|\nabla w\|^\gamma_{\Omega,\gamma} + \frac{1}{\gamma} \int_\Omega \operatorname{div} w dx \Big\},$$

which meets the integral identity

$$\int_\Omega \Big( |\nabla u_1|^{\gamma-2} \nabla u_1 : \nabla w + \frac{1}{\gamma} \operatorname{div} w \Big) dx = 0 \quad \text{for all } w \in W^{1,\gamma}_{0,\Gamma_0}(\Omega).$$

Then,

$$\|\nabla u_1\|^\gamma_{\Omega,\gamma} + \frac{1}{\gamma} \int_\Omega \operatorname{div} u_1 dx = 0.$$

Set $v_* = v - \lambda_* u_1$, where

$$\lambda_* = \frac{\int_\Omega \operatorname{div} v dx}{\int_\Omega \operatorname{div} u_1 dx} = -\frac{\int_\Omega \operatorname{div} v dx}{\gamma \|\nabla u_1\|^\gamma_{\Omega,\gamma}}.$$

Since

$$\int_\Omega \operatorname{div} v_* dx = \int_\Omega \operatorname{div} v dx - \lambda_* \int_\Omega \operatorname{div} u_1 dx = 0,$$

we conclude that $v_*$ belongs to $\widetilde{W}^{1,\gamma}_{0,\Gamma_0}(\Omega,\mathbb{R}^d)$. Then,

$$\begin{aligned}\inf_{v_0\in S^{1,\gamma}_{0,\Gamma_0}(\Omega,\mathbb{R}^d)}\|\nabla(v-v_0)\|_{\Omega,\gamma} &\le \inf_{v_0\in S^{1,\gamma}_{0,\Gamma_0}(\Omega,\mathbb{R}^d)}\|\nabla(v_*-v_0)\|_{\Omega,\gamma}+\|\lambda_*\nabla u_1\|_{\Omega,\gamma}\\ &\le \mathbb{C}_{\Omega,\gamma}\|\operatorname{div} v-\lambda_*\operatorname{div} u_1\|_{\Omega,\gamma}+\frac{1}{\gamma\|\nabla u_1\|^{\gamma-1}_{\Omega,\gamma}}\Big|\int_\Omega \operatorname{div} v\,dx\Big|\\ &\le \mathbb{C}_{\Omega,\gamma}\|\operatorname{div} v\|_{\Omega,\gamma}+C_{1,\gamma}\Big|\int_\Omega \operatorname{div} v\,dx\Big|,\end{aligned}$$

where

$$C_{1,\gamma}=\frac{1}{\|\nabla u_1\|^{\gamma-1}_{\Omega,\gamma}}\Big(\mathbb{C}_{\Omega,\gamma}\frac{\|\operatorname{div} u_1\|_{\Omega,\gamma}}{\gamma\|\nabla u_1\|_{\Omega,\gamma}}+1\Big).$$

**Remark 3.** By the same argumentation as in Lemma 4 we can show that an upper bound of $C_{1,\gamma}$ can be computed by means of a finite-dimensional problem analogous to (2.13).

Problems with divergence-free conditions are often associated with evolutionary equations in the space-time cylinder $Q_T := \Omega\times(0,T)$. At the end of this section we briefly consider this case and show that the above presented estimates yield estimates of the distance to the set of divergence-free fields defined in $Q_T$. Consider incremental approximations where the interval $(0,T)$ is split to $m$ time subintervals $(t_k,t_{k+1})$ ($t_0=0$ and $t_m=T$). Consider the simplest piecewise affine approximation

$$v(x,t)=\lambda(t)\widehat{v}_k(x)+(1-\lambda(t))\widehat{v}_{k+1},\quad \lambda(t)=\frac{t_{k+1}-t}{d_k},\ d_{k+1}=t_{k+1}-t_k, \tag{2.15}$$

where $\widehat{v}_k$ are some functions of spatial variables. Since the spatial divergence of $v$ satisfies the relation

$$\widehat{\operatorname{div}} v(x,t)=\operatorname{div}(\widehat{v}_k+v_{k+1})-\lambda(t)\operatorname{div}\widehat{v}_{k+1},$$

the function $v(x,t)$ belongs to $S^{1,2}(\Omega,\mathbb{R}^d)$ for almost all $t$ if $\widehat{v}_k(x)\in S^{1,2}(\Omega,\mathbb{R}^d)$ for $k=0,1,2,\dots,m$.

Let $\{\operatorname{div}\widehat{v}_k\}_\Omega=0$ for $k=0,1,2,\dots,m$. In view of the above presented results, there exist divergence-free functions $\widehat{v}_{0,k}$ such that $\widehat{v}_k=\widehat{v}_{0,k}$ on $\Gamma$ and

$$\|\nabla(\widehat{v}_k-\widehat{v}_{0,k})\|\le\mathbb{C}_\Omega\|\operatorname{div}\widehat{v}_k\|,\quad k=0,1,2,\dots,m. \tag{2.16}$$

By $\widehat{v}_{0,k}$ we construct a divergence-free function $v_0(x,t)$ in the form (2.15). It is easy to see that $v_0$ satisfies the same boundary conditions as $v$ and

$$\begin{aligned}\|\widehat{\nabla} v(x,t)-v_0(x,t)\|^2_{Q_T}&=\sum_{k=0}^m\int_{t_k}^{t_{k+1}}\big(\lambda\|\widehat{\nabla}(\widehat{v}_k-\widehat{v}_{0,k})\|_\Omega+(1-\lambda)\|\nabla(\widehat{v}_{k+1}-\widehat{v}_{0,k+1})\|_\Omega\big)^2dt\\ &\le\mathbb{C}^2_\Omega\sum_{k=0}^m\frac{2d_{k+1}}{3}(\|\operatorname{div}\widehat{v}_k\|^2+\|\operatorname{div}\widehat{v}_{k+1}\|^2),\end{aligned} \tag{2.17}$$

where

$$\widehat{\nabla}:=\Big\{\frac{\partial}{\partial x_j}\Big\}_{j=1}^d$$

is the spatial gradient. Hence the distance $d(v,S^{1,2}(Q_T,\mathbb{R}^d))$ is estimated from above by the right-hand side of (2.17).

In addition to (2.16), error majorants of the distance to exact solutions of evolutionary problems associated with incompressible media require upper bounds of $\|\frac{\partial(v-v_0)}{\partial t}\|_{Q_T}$. Note that

$$\frac{\partial(v-v_0)}{\partial t}=\frac{1}{d_k}((\widehat{v}_{k+1}-v_{0,k+1})-(\widehat{v}_k-v_{0,k})).$$

We apply (2.16) and the Friedrichs inequality and obtain

$$\Big\|\frac{\partial(v-v_0)}{\partial t}\Big\|^2_{Q_T}\le C_F^2\mathbb{C}^2_\Omega\sum_{k=0}^m\frac{1}{d_k^2}(\|\operatorname{div}\widehat{v}_k\|+\|\operatorname{div}\widehat{v}_{k+1}\|)^2.$$

# 3 Estimates Based on the Decomposition of Ω

## 3.1 Non-Overlapping Subdomains

Let $\Omega$ be divided into a collection of non-overlapping Lipschitz subdomains $\Omega_i$, $i = 1, 2, \dots, N$.

**Theorem 4.** *If $f \in L^\gamma(\Omega)$ satisfies the condition*

$$\{f\}_{\Omega_i} = 0, \quad i = 1, 2, \dots, N, \tag{3.1}$$

*then there exists $v_f \in \overset{\circ}{W}{}^{1,\gamma}(\Omega, \mathbb{R}^d)$ such that*

$$\operatorname{div} v_f = f \quad \textit{and} \quad \|\nabla v_f\|^\gamma_{\Omega,\gamma} \le \sum_{i=1}^N \mathbb{C}^\gamma_{\Omega_i,\gamma} \|f\|^\gamma_{\Omega_i,\gamma}, \tag{3.2}$$

*where $\mathbb{C}_{\Omega_i,\gamma}$ are positive constants associated with subdomains $\Omega_i$.*

*Proof.* In view of (3.1) and Theorem 2, for any $i$ there exists $v_{f,i} \in \overset{\circ}{W}{}^{1,\gamma}(\Omega_i, \mathbb{R}^d)$ such that

$$\operatorname{div} v_{f,i} = f \text{ in } \Omega_i \quad \text{and} \quad \|\nabla v_{f,i}\|_{\Omega,\gamma_i} \le \mathbb{C}_{\Omega_i,\gamma} \|f\|_{\Omega,\gamma_i}.$$

Set $v_f(x) = v_{f,i}(x)$ if $x \in \Omega_i$. Then, $v_f \in \overset{\circ}{W}{}^{1,\gamma}(\Omega, \mathbb{R}^d)$, $\operatorname{div} v_f = f$, and

$$\|\nabla v_f\|^\gamma_{\Omega,\gamma} = \sum_{i=1}^n \|v_{f,i}\|^\gamma_{\Omega,\gamma_i} \le \sum_{i=1}^n \mathbb{C}^\gamma_{\Omega_i,\gamma} \|f\|^\gamma_{\Omega,\gamma_i}. \qquad \square$$

Theorem 4 implies an estimate of the distance between $v \in W^{1,\gamma}(\Omega, \mathbb{R}^d)$ and the set of functions in $S^{1,\gamma}(\Omega, \mathbb{R}^d)$ satisfying the same boundary condition as $v$ (for the case $\gamma = 2$ a simple estimate of this type has been established in [31]). Assume that $v$ satisfies the conditions

$$\{\operatorname{div} v\}_{\Omega_i} = 0, \quad i = 1, 2, \dots, N. \tag{3.3}$$

It is worth noting that these additional integral type relations imposed on $v$ do not imply essential technical difficulties (if $N$ is not very large). Indeed, if an approximation $v$ does not satisfy (3.3), then we need to fix it by changing values of $v \cdot n$ on $\Gamma_{ij} = \Omega_i \cap \Omega_j$ and $\Gamma_1 \cap \Omega_i$. Respective procedures (changing $N$ parameters in the representation of $v$) can be easily constructed for approximations of a particular type.

**Lemma 5.** *Let $v \in W^{1,\gamma}(\Omega, \mathbb{R}^d)$ satisfy (3.3) and $\operatorname{div} v \in L^{\delta_i}(\Omega, \mathbb{R}^d)$, where $\delta_i \ge \gamma$, $i = 1, 2, \dots, N$. Then, there exists $v_0 \in S^{1,\gamma}(\Omega, \mathbb{R}^d)$ such that $v = v_0$ on $\Gamma$ and*

$$\|\nabla(v - v_0)\|_{\Omega,\gamma} \le \Big( \sum_{i=1}^N \mathbb{C}^\gamma_{\Omega_i,\gamma} |\Omega_i|^{1-\gamma/\delta_i} \|\operatorname{div} v\|^\gamma_{\Omega_i,\delta_i} \Big)^{1/\gamma}. \tag{3.4}$$

*Proof.* We set $f = \operatorname{div} v$ and use Theorem 4. There exists $v_f \in \overset{\circ}{W}{}^{1,\gamma}(\Omega_i, \mathbb{R}^d)$ satisfying (3.2). The function $v_0 = v - v_f$ is divergence free, it satisfies the same boundary condition as $v$, and

$$\|\nabla(v - v_0)\|^\gamma_{\Omega,\gamma} = \|\nabla v_f\|^\gamma_{\Omega,\gamma} \le \sum_{i=1}^N \mathbb{C}^\gamma_{\Omega,\gamma_i} \|\operatorname{div} v\|^\gamma_{\Omega_i,\gamma}. \tag{3.5}$$

Now (3.4) follows due to the Hölder inequality. $\square$

**Remark 4.** If $\operatorname{div} v$ is bounded almost everywhere (what is typical for piecewise polynomial approximations) then $\int_{\Omega_i} |\operatorname{div} v|^\gamma dx \le |\Omega_i| (\operatorname{ess\,sup}_{\Omega_i} |\operatorname{div} v|)^\gamma$ and (3.5) yields the estimate

$$\|\nabla(v - v_0)\|^\gamma_{\Omega,\gamma} \le \sum_{i=1}^N \mathbb{C}^\gamma_{\Omega_i,\gamma} |\Omega_i| \big(\operatorname*{ess\,sup}_{\Omega_i} |\operatorname{div} v|\big)^\gamma. \tag{3.6}$$

In particular, if all $\Omega_i$ are simplexes and $\operatorname{div} v \in P^1(\Omega_i)$, then

$$\operatorname*{ess\,sup}_{\Omega_i} |\operatorname{div} v| = \max_{N_i^t} |\operatorname{div} v(N_i^t)|,$$

where $N_i^t$, $t = 1, 2, \dots, d+1$ are nodal points of $\Omega_i$. The same relation can be used if $\Omega_i$ are convex polygons in $\mathbb{R}^d$.

## 3.2 Subdomains With Overlappings

Let $\Omega$ be decomposed into a collection of overlapping Lipschitz subdomains $D_k$, $k = 1, 2, \dots, K$. By $\mathbb{C}_{D_k,\gamma}$ we denote the respective constants. Subdomains $D_k$ may overlap, so that they generate a decomposition of $\Omega$ into a set of non-overlapping subdomains $\Omega_i$, $k = 1, 2, \dots, N$ (see Figure 1). In other words,

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{D}_k = \bigcup_{i=1}^{N} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j, \tag{3.7}$$

and $D_k \cap D_l$ is either empty or consists of one or several subdomains $\Omega_i$. For any $\Omega_i$ there exists at least one $D_k$ such that $\Omega_i \subset D_k$. We have the following localized version of Theorem 2.

**Theorem 5.** *Let $f$ satisfy the same conditions as in Theorem 4. Then, there exists a function $v_f \in W_0^{1,\gamma}(\Omega, \mathbb{R}^d)$ such that $\operatorname{div} v_f = f$ in $\Omega$ and*

$$\|\nabla v_f\|_\Omega \le \sum_{i=1}^{N} \mathbb{C}_i \|f\|_{\Omega_i},$$

*where*

$$\mathbb{C}_i = \min_{k=1,\dots,K} \rho_k, \quad \rho_k = \begin{cases} \mathbb{C}_{D_k,\gamma} & \text{if } \Omega_i \subset D_k, \\ +\infty & \text{if } \Omega_i \not\subset D_k. \end{cases} \tag{3.8}$$

*Proof.* Define

$$f_i(x) = \begin{cases} f & \text{if } x \in \Omega_i, \\ 0 & \text{if } x \notin \Omega_i. \end{cases}$$

There exists at least one $D_k$ such that $\Omega_i \subset D_k$. If there are several $D_k$ containing $\Omega_i$, then we select $k$ such that $\mathbb{C}_{D_k,\gamma}$ is minimal (see (3.8)). Since $\{f_i\}_{D_k} = 0$, and $D_k$ is a Lipschitz domain, we can find $v_{f_i} \in W_0^{1,\gamma}(D_k, \mathbb{R}^d)$ such that

$$\operatorname{div} v_{f_i} = f_i \quad \text{in } D_k \tag{3.9}$$

and

$$\|\nabla v_{f_i}\|_{\gamma,D_k} \le \mathbb{C}_i \|f_i\|_{\gamma,D_k} = \mathbb{C}_i \|f\|_{\Omega_i,\gamma}.$$

We extend $v_{f_i}$ by zero to $\Omega \setminus D_k$ and find that (3.9) holds in $\Omega$. Moreover,

$$\|\nabla v_{f_i}\|_{\Omega,\gamma} \le \mathbb{C}_i \|f\|_{\Omega,\gamma_i}. \tag{3.10}$$

Set $v_f = \sum_{i=1}^{N} v_{f_i} \in W_0^{1,\gamma}(\Omega, \mathbb{R}^d)$. Then $\operatorname{div} v_f = f$, and by (3.10) we obtain

$$\|\nabla v_f\|_{\Omega,\gamma} \le \sum_{i=1}^{N} \|\nabla v_{f_i}\|_{\Omega,\gamma} \le \sum_{i=1}^{N} \mathbb{C}_i \|f_i\|_{\Omega_i,\gamma}. \qquad \square$$

Theorem 5 implies another estimate of the distance to the set of divergence-free fields.

**Lemma 6.** *Assume that $v \in W^{1,\gamma}(\Omega, \mathbb{R}^d)$ satisfies* (3.3) *and* $\operatorname{div} v \in L^\delta(\Omega)$, *where $\delta \ge \gamma$. Then, there exists $v_0 \in W^{1,\gamma}(\Omega, \mathbb{R}^d)$ such that* $\operatorname{div} v_0 = 0$, *$v_0 = v$ on $\Gamma$, and*

$$\|\nabla(v - v_0)\|_{\Omega,\gamma} \le \sum_{i=1}^{N} \mathbb{C}_i |\Omega_i|^{\frac{1}{\gamma} - \frac{1}{\delta}} \|\operatorname{div} v\|_{\Omega_i,\delta}, \tag{3.11}$$

*where the constants $\mathbb{C}_i$ are defined by* (3.8).

### 3.3 Decomposition-Based Estimates in Abstract Form

It is clear that the same method can be applied to other operators with closed ranges. Below we shortly discuss formalities that yield decomposition-based estimates. Let $V(\Omega)$ be a reflexive Banach space and $V_0(\Omega)$ denote the subspace of $V$ containing functions vanishing on $\Gamma$. We consider a bounded linear operator $B : V_0(\Omega) \to H(\Omega)$, where $H(\Omega)$ is another reflexive Banach space. We assume that $\operatorname{Im} B \subset H$ is closed in $H$.

By the *closed range lemma* (see, e.g., [37]), we have the following result.

**Lemma 7.** *For any $g \in \operatorname{Im} B$ there exists $v_g \in V_0$ such that*

$$Bv_g = g \quad \textit{and} \quad \|v_g\|_{V(\Omega)} \le \mathbb{C}_\Omega \|g\|_{H(\Omega)},$$

*where $\mathbb{C}_\Omega > 0$ does not depend on $v$.*

Let $\Omega$ be divided into a collection of non-overlapping Lipschitz subdomains $\Omega_i$, the spaces $V_0(\Omega_i)$ are generated by the same norm as $V_0$ and contain functions vanishing on $\partial\Omega_i$.

Assume that any function $v(x)$ defined by the relation $v(x) = v_i(x)$ in $\Omega_i$ and $v(x) = 0$ in $\Omega \setminus \Omega_i$ belongs to $V_0(\Omega)$ provided that $v_i \in V_0(\Omega_i)$. Also, we assume that all the operators $B_i : V_0(\Omega_i) \to H(\Omega_i)$ (which are generated by restrictions of the operator $B$) are such that the sets $\operatorname{Im} B_i$ are closed in $H(\Omega_i)$ for $i = 1, 2, \dots, N$.

Let $g = g_i$ in $\Omega_i$ and $g_i \in \operatorname{Im} B_i$, $i = 1, 2, \dots, N$. In view of Lemma 7, there exists $v_{g_i}$ such that

$$Bv_{g_i} = g_i \quad \text{and} \quad \|v_{g_i}\|_{V(\Omega_i)} \le \mathbb{C}_{\Omega_i} \|g_i\|_{H(\Omega_i)}, \tag{3.12}$$

where $\mathbb{C}_{\Omega_i} > 0$ depends on $\Omega_i$. We extend all $v_{g_i}$ to $\Omega \setminus \Omega_i$ by zero and set $v_g = \sum_{i=1}^N v_{g_i} \in V_0(\Omega)$. Since $\|v_g\| \le \sum_{i=1}^N \|v_{g_i}\|_{V(\Omega_i)}$, we use (3.12) and conclude that

$$Bv_g = g \quad \text{and} \quad \|v_g\|_{V(\Omega)} \le \sum_{i=1}^N \mathbb{C}_{\Omega_i} \|g_i\|_{H(\Omega_i)}. \tag{3.13}$$

If a collection of subdomains satisfies (3.7), then similar arguments yield an analogue of Theorem 5. In this case, there exists $v_g \in V_0(\Omega)$ such that

$$Bv_g = g \quad \text{and} \quad \|v_g\|_{V(\Omega)} \le \sum_{i=1}^N \mathbb{C}_i \|g_i\|_{H(\Omega_i)}, \tag{3.14}$$

where $\mathbb{C}_i$ are defined by (3.8).

**Remark 5.** If $V(\Omega)$ is a Hilbert space and $\|v\|^2_{V(\Omega)} = (v, v)_{V(\Omega)}$, then

$$\|v\|^2_{V(\Omega)} = \sum_{i=1}^N \|v\|^2_{V(\Omega_i)},$$

and in addition to (3.13) we have the better estimate

$$\|v_g\|^2_{V(\Omega)} \le \sum_{i=1}^N \mathbb{C}^2_{\Omega_i} \|g_i\|^2_{H(\Omega_i)}. \tag{3.15}$$

Estimates (3.13)–(3.15) yield estimates of the distance to the set

$$W_0(\Omega) := \{v \in V_0(\Omega) \mid Bv = 0\}.$$

In particular, (3.15) yields the estimate

$$d(v, W_0(\Omega)) \le \sum_{i=1}^N \mathbb{C}^2_{\Omega_i} \|Bv\|^2_{H(\Omega_i)},$$

where $d(v, W_0(\Omega)) = \inf_{w_0 \in W_0(\Omega)} \|v - w_0\|_{V(\Omega)}$.

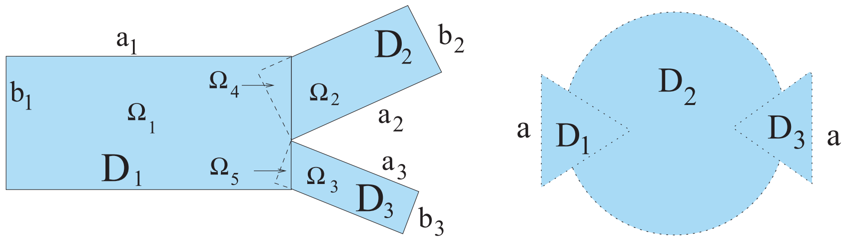


**Figure 1.** Domains composed of overlapping subdomains $D_1$, $D_2$, and $D_3$.

## 3.4 Examples

Consider two simple examples related to the case where $\overline{\Omega} = \overline{D}_1 \cup \overline{D}_2 \cup \overline{D}_3$ (the respective domains are depicted in Figure 1).

In the first example, $D_i$ are rectangles with sides $a_i$ and $d_i$, $\overline{D}_2 = \overline{\Omega}_2 \cup \overline{\Omega}_4$, and $\overline{D}_3 = \overline{\Omega}_3 \cup \overline{\Omega}_5$.

Let $v \in V_0(\Omega)$ be such that

$$\{\operatorname{div} v\}_{\Omega_i} = 0, \quad i = 1, 2, 3, 4, 5. \tag{3.16}$$

From (1.5) it follows that for a rectangular domain $\square_{a,b} := (0, a) \times (0, b)$ $(a, b > 0,\ a > b)$, the stability constant meets the estimate

$$\mathbb{C}_{\square_{ab}} \leq \frac{1}{b}\sqrt{2d(a + d)}, \tag{3.17}$$

where $d = \sqrt{a^2 + b^2}$ is the length of the diagonal. In particular, for the unit square, estimate (3.17) gives $\mathbb{C}_{\square_{11}} < 2.6131$ which is in good correspondence with [17], where by accurate computations it was found that $2.347 \leq \mathbb{C}_{\square_{11}} < 2.611$.

In view of Lemma 6, there exists a divergence-free field $v_0$ such that $v_0 = v$ on $\Gamma$ and

$$\|\nabla(v - v_0)\| \leq \mathbb{C}_{D_1}\|\operatorname{div} v\|_{\Omega_1} + \mathbb{C}_{D_2}\big(\|\operatorname{div} v\|_{\Omega_2} + \|\operatorname{div} v\|_{\Omega_4}\big) + \mathbb{C}_{D_3}\big(\|\operatorname{div} v\|_{\Omega_3} + \|\operatorname{div} v\|_{\Omega_5}\big),$$

where

$$\mathbb{C}_{D_k} = \frac{1}{b_k}\sqrt{2d_k^2 + 2a_k d_k}, \quad k = 1, 2, 3.$$

Hence, the distance between $v$ and the set of divergence-free fields is estimated from above by the expression in the right-hand side.

Another example is related to the domain depicted in Figure 1 (right). Here $D_1$ and $D_3$ are isosceles triangles and $D_2$ is a circle. Let

$$\Omega_2 = D_1 \cap D_2, \quad \overline{\Omega}_1 + \overline{\Omega}_2 = \overline{D}_1, \quad \overline{\Omega}_2 + \overline{\Omega}_3 + \overline{\Omega}_4 = \overline{D}_2 \quad (\operatorname{meas} \Omega_1 > 0),$$
$$\Omega_4 = D_3 \cap D_2, \quad \overline{\Omega}_4 + \overline{\Omega}_5 = \overline{D}_3 \quad (\operatorname{meas} \Omega_5 > 0),$$

and $v$ satisfy (3.16). In view of Lemma 6 (for $\gamma = \delta = 2$), there exists $v_0$ such that $\operatorname{div} v_0 = 0$, $v = v_0$ on $\Gamma$ and

$$\|\nabla(v - v_0)\| \leq \mathbb{C}_{D_1}\|\operatorname{div} v\|_{\Omega_1} + \mathbb{C}_{D_3}\|\operatorname{div} v\|_{\Omega_5} + \mathbb{C}_{D_2}\big(\|\operatorname{div} v\|_{\Omega_2} + \|\operatorname{div} v\|_{\Omega_3} + \|\operatorname{div} v\|_{\Omega_4}\big).$$

Since $\mathbb{C}_{D_2} = \sqrt{2}$, it remains to find estimates of $\mathbb{C}_{D_1}$ and $\mathbb{C}_{D_2}$. Note that for a simplex $\Delta_{abc}$ with sides $a \geq b \geq c > 0$, we have

$$\rho = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}} \quad \text{and} \quad R^2 = \rho^2 + \frac{(a + b - c)^2}{4},$$

where $s$ is the semiperimeter. By (1.5) we find that

$$\mathbb{C}_{\Delta_{abc}} \leq \frac{R}{\rho}\Big(2 + \frac{a + b - c}{R}\Big)^{1/2}. \tag{3.18}$$

If $a = b = c = 1$, then $\mathbb{C}_{\Delta_{111}} \leq 3.8637$ (compare this result with [17], where $3.401 \leq \mathbb{C}_{\Delta_{111}} < 3.861$ was found). Hence, we can set $\mathbb{C}_{D_1} = \mathbb{C}_{D_3} = \mathbb{C}_{\Delta_{111}}$. By (3.18) it is not difficult to find stability constants if $D_1$ and $D_3$ are arbitrary nondegenerate triangles. We note that the upper bound in (3.18) is minimal for equilateral triangles. For other triangles the estimate generates larger bounds, which tend to infinity if $b + c$ tends to $a$.

# 4 Applications to A Posteriori Estimates

Guaranteed bounds of the distance to the exact solution of a boundary value problem usually contain constants in functional inequalities (e.g., Poincaré, Friedrichs, Korn, trace inequalities). Such bounds are often called a posteriori estimates of functional type (or deviation estimates). The reader can find a systematic exposition of the respective theory and many references in [30]. Computational aspects related to efficient use of these estimates for various problems are discussed in [21] (see also [36]). Here, we briefly recall results related to the Stokes problem, which is the basic model in the theory of incompressible viscous fluids and show how the constant $\mathbb{C}_\Omega$ enters these estimates. The problem is to find $u$ (velocity vector function), $\sigma$ (stress tensor function), and $p$ (pressure field) satisfying the system

$$\begin{cases} -\operatorname{Div}\sigma = f - \nabla p & \text{in } \Omega, \\ \sigma = \nu\nabla u & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = g & \text{on } \Gamma. \end{cases} \tag{4.1}$$

Here $\nu$ is a positive constant (viscosity), $f \in L^2(\Omega, \mathbb{R}^d)$, and $g \in H^1(\Omega, \mathbb{R}^d)$ is a given vector function, which must be selected such that the compatibility condition

$$\int_\Gamma g \cdot n\,ds = 0$$

holds. Guaranteed and computable bounds of the distance between any (energy admissible) approximation $v$ and the exact solution $u$ were firstly derived in [28] (see also [12, 13, 22, 29, 31]). It was shown that if $v \in S^{1,2}(\Omega, \mathbb{R}^d) + g$, then the following *error identity* holds:

$$\int_\Omega (\nu|\nabla(u - v)|^2 + \nu^{-1}|\sigma - \tau|^2)\,dx = 2(J(v) - I^*(\tau)), \tag{4.2}$$

where $J(v) := \int_\Omega (\frac{\nu}{2}|\nabla v|^2 - f \cdot v)\,dx$ is the energy functional of the Stokes problem, $I^*(\tau) := -\frac{1}{2}\|\tau\|^2$ is the dual energy functional, $\sigma = \nu\nabla u$, and

$$\tau \in Q_f := \{\tau \in L^2(\Omega, \mathbb{M}^{d\times d}_{\mathrm{sym}}) \mid \operatorname{Div}\tau + f = 0\}.$$

Moreover, in [28] it was shown that (4.2) can be extended to classes of functions which are much wider than $v \in S^{1,2}(\Omega, \mathbb{R}^d) + g$ and $Q_f$ (what is important from the practical point of view). The respective results are presented by the estimates

$$\nu\|\nabla(u - v)\| \le 2\nu R_1(v) + R_2(v, \tau) + C_{F\Omega} R_3(\tau, q) =: M_v(q, \tau), \tag{4.3}$$

$$\frac{1}{2\mathbb{C}_\Omega}\|p - q\| \le \nu R_1(v) + R_2(v, \tau) + 2C_{F\Omega} R_3(\tau, q) =: M_q(v, \tau), \tag{4.4}$$

$$\|\tau - \sigma\| \le \nu R_1(v) + R_2(v, \tau) + C_{F\Omega} R_3(\tau, q) =: M_\tau(v, q), \tag{4.5}$$

where (cf. (2.3))

$$\begin{aligned} R_1(v) &:= d(v, S_0^{1,2} + g) \le \mathbb{C}_\Omega\|\operatorname{div} v\|, \\ R_2(v, \tau) &:= \|\tau - \nu\nabla v\|, \\ R_3(v, q) &:= \|\operatorname{Div}\tau + f - \nabla q\|, \end{aligned}$$

$v$ is any function in the set

$$V + g : \{v = w_0 + g,\ w_0 \in V_0 := H^1_0(\Omega, \mathbb{R}^d)\}$$

satisfying the last equation in (4.1), $\tau$ is any function in $H(\Omega, \operatorname{Div})$, $q$ is any function in $\widetilde{L}^2(\Omega)$, and $C_{F\Omega}$ is the constant in the Friedrichs inequality for the functions in $V_0$.

By setting $\tau = \eta + q\mathbb{I}$ (where $\eta \in H(\Omega, \mathrm{Div})$), we obtain a slightly different form of (4.3)–(4.5) where $R_2(v, \tau)$ and $R_3(\tau, q)$ are replaced by

$$R_2(v, \eta, q) := \|\eta - \nu\nabla v + q\mathbb{I}\| \quad \text{and} \quad R_3(\eta) := \|\mathrm{Div}\,\tau + f\|,$$

respectively. It is easy to show that combined norms containing errors in the left-hand sides of (4.3)–(4.5) are bounded from below by weighted sums analogous to the majorants $M_v(q, \tau)$ or $M_\tau(v, q)$. Therefore, the majorants indeed present adequate separate measures of errors in terms of velocity, pressure, and stress, and also a measure of the combined error containing all of them. Since

$$\inf_{\substack{q\in\widetilde{L}^2(\Omega)\\ \tau\in H(\Omega,\mathrm{Div})}} M_v(q, \tau) = M_v(p, \sigma) = \nu\|\nabla(u - v)\|,$$
$$\inf_{\substack{q\in\widetilde{L}^2(\Omega)\\ v\in V_0+g}} M_\tau(q, \tau) = M_\tau(p, u) = \|\sigma - \tau\|,$$
$$\inf_{\substack{\eta\in H(\Omega,\mathrm{Div})\\ v\in V_0+g}} M_q(q, \eta) = M_\tau(p, u) = \sqrt{d}\|p - q\|,$$

the majorants always provide realistic error bounds if approximate solutions are close to the exact ones.

Estimates (4.3) and (4.5) involve the constant $\mathbb{C}_\Omega$, which appears if the distance to the set of divergence-free fields is measured by Lemma 1. If $v$ satisfies condition (3.3) for a non-overlapping collection of subdomains $\Omega_i$, then we use Lemma 5 and obtain a somewhat different error majorant for the Stokes problem:

$$\nu\|\nabla(u - v)\| \le 2\nu\Big(\sum_{i=1}^{N} \mathbb{C}^2_{\Omega_i}\|\mathrm{div}\,v\|^2_{\Omega_i}\Big)^{1/2} + \|\eta + q\mathbb{I} - \nu\nabla v\| + C_{F\Omega}\|\mathrm{Div}\,\eta + f\|. \tag{4.6}$$

Here, the functions $\eta \in H(\Omega, \mathrm{Div})$ and $q \in \widetilde{L}^2(\Omega)$ can be viewed as approximations of the stress and pressure functions, respectively. If $\mathrm{div}\,v$ has higher regularity $\mathrm{div}\,v \in L^\delta(\Omega)$, $\delta > 2$, then the sum in round brackets (which reflects the distance to divergence-free fields) could be replaced with the help of estimates (3.4) or (3.6). If $\Omega_i$ are formed by intersecting subdomains, then this term should be replaced by the right-hand side of (3.11) with $\gamma = 2$.

**Remark 6.** Estimates (4.3)–(4.5) could be helpful in selecting suitable weights if approximate solutions to the Stokes problem are computed by the least squares finite element method. For example, if our analysis is focused on the velocity field, then (4.3) shows that the weights of $R_1^2$, $R_2^2$, and $R_3^2$ should be close to $4\nu^2\mathbb{C}^2_\Omega$, 1, and $C_F^2$, respectively. Analogously, estimate (4.6) suggests a “decomposed” version of the least square complex

$$\sum_{i=1}^{N} 4\nu^2\mathbb{C}^2_{\Omega_i}\|\mathrm{div}\,v\|^2_{\Omega_i} + R^2(v, \eta, q) + C^2_{F\Omega}R^2(\eta),$$

where the weights are presented by local stability constants $\mathbb{C}_{\Omega_i}$.

If we have an overlapping collection of subdomains $D_k$ and the corresponding set of $\Omega_i$ satisfying (3.7), then Lemma 6 yields another estimate:

$$\nu\|\nabla(u - v)\| \le 2\nu\sum_{i=1}^{N} \mathbb{C}_i\|\mathrm{div}\,v\|_{\Omega_i} + \|\eta + q\mathbb{I} - \nu\nabla v\| + C_{F\Omega}\|\mathrm{Div}\,\eta + f\|, \tag{4.7}$$

where the constants $\mathbb{C}_i$ are defined in (3.8). Similar estimates (based on decomposition of $\Omega$) for other problems related to incompressible fluids can be found in [31, 32].

**Remark 7.** There is an obvious way to obtain computable estimates of the distance to divergence-free fields without the condition (3.3). For this purpose, we need to construct a suitable correction function $w \in W_0^{1,\gamma}(\Omega, \mathbb{R}^d)$ such that

$$\int_{\Omega_i} \mathrm{div}\,w\,dx = \delta_i := \int_{\Omega_i} \mathrm{div}\,v\,dx \quad \text{for } i = 1, 2, \dots, N.$$

Then using (4.5), we conclude that there exists a solenoidal vector-valued function $v_0$ such that $v = v_0$ on $\Gamma$ and

$$\|\nabla(v - v_0)\|_{\Omega,\gamma} \leq \Big(\sum_{i=1}^{N} \mathbb{C}^{\gamma}_{\Omega_i,\gamma}\|\mathrm{div}(v - w)\|^{\gamma}_{\Omega_i,\gamma}\Big)^{1/\gamma} + \|\nabla w\|_{\Omega,\gamma}.$$

This estimate provides an upper bound of the distance to the set of divergence-free fields for any $w \in W^{1,\gamma}_0(\Omega, \mathbb{R}^d)$. Certainly the quality of this estimate depends on the choice of $w$, which should be selected such that $\|\nabla w\|_{\Omega_i,\gamma}$ is small and $\mathrm{div}\, v$ does not differ much from $\mathrm{div}\, w$. In certain cases, finding such $w$ may generate a special and not an easy task. We believe that conceptually it is more logical to view (3.3) as a natural condition for any "good" (physically suitable) approximation and use (4.6) or (4.7).

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