# Projection methods for ill-posed problems revisited 

Stefan Kindermann*


#### Abstract

The discretization of least-squares problems for linear ill-posed operator equations in Hilbert spaces is considered. The main subject of this article concerns conditions for convergence of the associated discretized minimum-norm least-squares solution to the exact solution using exact attainable data. The two cases of global convergence (convergence for all exact solution) or local convergence (convergence for a specific exact solution) are investigated. We review the existing results and prove new equivalent condition when the discretized solution always converges to the exact solution. An important tool is to recognize the discrete solution operator as oblique projection. Hence, global convergence can be characterized by certain subspaces having uniformly bounded angles. We furthermore derive practically useful conditions when this holds and put them into the context of known results. For local convergence we generalize results on the characterization of weak or strong convergence and state some new sufficient conditions. We furthermore provide an example of a bounded sequence of discretized solutions which does not converge at all, not even weakly.


## 1 Introduction

We study the role of discretization in the use of solving ill-posed linear operator equations in Hilbert spaces. Consider an ill-posed problem in Hilbert spaces

$$
\begin{equation*}
A x=y, \tag{1}
\end{equation*}
$$

where $A: X \rightarrow Y$ is continuous and equation (11) for solving $x$ from given data $y$ is ill-posed. In the following, $N(A)$ and $R(A)$ denote the nullspace and the range of an operator $A$, respectively. By $A^{\dagger}$ we denote the pseudoinverse of $A$; cf., e.g., [7. We symbolize norm-convergence by $\rightarrow$ and weak convergence by $\rightarrow$. We denote the weak limit by the symbol ${ }_{\mathrm{w}} \mathrm{lim}$, and, for a closed subspace $Z, \Pi_{Z}$ denotes the associated orthogonal projector onto $Z$. In the following, we assume (unless specified otherwise) the attainable case for problem (1), i.e., that $y$ is in $R(A)$. In this case, we can set $y$ being the image of an element in $N(A)^{\perp}$.

$$
A x^{\dagger}=y, \quad x^{\dagger} \in N(A)^{\perp}
$$

It is the unique element $x^{\dagger}$, which we want to reconstruct from given data $y$.

[^0]We are interested in projection methods acting as a regularization, i.e., in approximating the pseudoinverse of $A$ by solving discrete least-squares problems related to (11). For this task we introduce discretizations in the spaces $X$ and $Y$. Precisely, we assume given an increasing sequence of finite-dimensional spaces $X_{n} \subset X$ and $Y_{m} \subset Y, n, m \in \mathbb{N}$, with the property

$$
\begin{equation*}
X_{n} \subset X_{n+1}, \quad \overline{\bigcup_{n} X_{n}}=X, \quad Y_{m} \subset Y_{m+1}, \quad \overline{\bigcup_{m} Y_{m}}=Y \tag{2}
\end{equation*}
$$

For the discretization spaces we always denote the associated orthogonal projector onto $X_{n}$ by $P_{n}:=\Pi_{X_{n}}$ and onto $Y_{m}$ by $Q_{m}:=\Pi_{Y_{m}}$

$$
P_{n}: X \rightarrow X_{n}, \quad Q_{m}: Y \rightarrow Y_{m}
$$

The discretization of (1) by a general projection method involves the operator

$$
\begin{equation*}
A_{n, m}:=Q_{m} A P_{n} \tag{3}
\end{equation*}
$$

and we define the associated solutions by (assuming attainability)

$$
\begin{equation*}
x_{n, m}:=A_{n, m}^{\dagger} y=A_{n, m}^{\dagger} A x^{\dagger} . \tag{4}
\end{equation*}
$$

It is well-known that $x_{n, m}$ is the unique solution of minimum norm under all least-squares solutions of the projected problem, i.e.,

$$
\begin{aligned}
& x_{n, m}=\operatorname{argmin}_{x \in X_{n}}\left\|Q_{m} A x-y\right\|^{2}=\operatorname{argmin}_{x \in X_{n}}\left\|Q_{m} A x-Q y\right\|^{2}, \quad \text { and } \\
& x_{n, m} \in N\left(A_{n, m}\right)^{\perp} .
\end{aligned}
$$

It follows that $A_{n, m}^{\dagger}=P_{n} A_{n, m}^{\dagger} Q_{m}$. The general projection method (4) embraces two special well-known methods: if we put formally $m=\infty$, and hence $Q_{m}=I$, we obtain the projected least-squares method involving

$$
\begin{equation*}
x_{n}:=A_{n}^{\dagger} A x^{\dagger}, \quad A_{n}=A P_{n} . \tag{5}
\end{equation*}
$$

Conversely if we set $n=\infty$ and formally put $n=\infty$, we obtain the dual least-squares method,

$$
x_{\infty, m}:=A_{\infty, m}^{\dagger} A x^{\dagger}, \quad A_{\infty, m}=Q_{m} A
$$

We distinguish these important special cases by labeling them with only one index for the first method and by the index $\infty, m$ for the second one. However, the dual least-squares method is not so much of interest for this paper (although it is of practical importance) as it always leads to a convergent method.

It is clear that $A_{n, m}^{\dagger}$ is a bounded operator and hence $x_{n, m}$ can be computed in a stable way. Moreover, the usual rules for adjoints and inverses hold: $\left(A_{n, m}^{\dagger}\right)^{*}=\left(A_{n, m}^{*}\right)^{\dagger}$.

The immediate question that arises from this setup is, if $x_{n, m}$ in (4) converges to $x^{\dagger}$ as $n, m \rightarrow \infty$, in what sense does this convergence happen, and for which $x^{\dagger}$ does this hold.

More precisely, we study two different subjects:

- Local convergence. Fix $x^{\dagger}$. Find conditions such that

$$
\begin{array}{ll}
x_{n, m} \rightharpoonup x^{\dagger} & \text { as } n, m \rightarrow \infty, \\
x_{n, m} \rightarrow x^{\dagger} & \text { as } n, m \rightarrow \infty .
\end{array}
$$

or

- Global convergence. Find conditions such that

$$
x_{n, m} \rightarrow x^{\dagger} \quad \text { as } n, m \rightarrow \infty \quad \forall x^{\dagger} \in N(A)^{\perp}
$$

The second issue concerns convergence not only for one fixed $x^{\dagger}$ but for all $x^{\dagger} \in N(A)^{\perp}$.

Both questions are relevant for the general projection method (4) and the projected least-squares method (5). Note that the distinction between weak and strong convergence is not relevant for global convergence because the corresponding conditions are identical [9].

Of course, these question have been discussed and partly answered in literature, but often only for the projected least-squares method or even with further restriction like injective operators; see Section 2.1 for a review. It is observed that many authors in different articles use different conditions to prove convergence of a specific scheme, for instance, (6) or (7) below. The relation between different conditions in different papers is not always obvious. It is one of the purposes of this paper to clarify this situation and to unify the convergence conditions at best, to generalize known results to the general projection case using the operator (3) and hereby avoiding unnecessary assumptions like injectivity.

Let us mention that the convergence of $x_{n, m}$ to $x^{\dagger}$ is the most important requirement for the projection methods discussed here to act as regularization. The second one, the stability of the regularization, is automatically satisfied since we are dealing with finite-dimensional problems. Indeed, if convergence of $x_{n, m}$ to $x^{\dagger}$ is verified, it is not difficult to find error estimates for noisy data as well and with appropriate parameter choice rules (where the index of the approximation spaces $n, m$ act as "regularization parameter"), convergence of $x_{n, m}$ to $x^{\dagger}$ can be proven even for the case of noisy data. We do not dwell further on this matter since it can be treated by standard methods; for results on the noisy case or also nonlinear problems, see, e.g., [1, 3, 8, 11, 13, 14, 15, 16, 23; for combination with regularization, see e.g., [25, 30, 31, 34]. For results with focus on the analysis of specific advanced method of choosing the discretization spaces (like adaptivity or multilevel-type), we refer to [17, 18, 22].

This paper is organized as follows: in Section 2, we review existing convergence results and prove some important lemmas. In Section 3 we provide new conditions for global or local convergence and relate them to results in literature. In Section 4 we state a nontrivial example of a non-convergent sequence $x_{n, m}$ which is bounded. We summarize with a conclusion in Section 5 .

## 2 Known and preliminary results

In this section we give an extensive literature review of know results related to the questions raised in the previous section. Moreover, we present some lemmas needed later for the convergence analysis.

### 2.1 A review of known results

The question of local or global convergence has, of course, been addressed in several articles. However, as stated above, quite often only injective operators, i.e., $N(A)=\emptyset$, or the case of projected least-squares problems, i.e., $Q_{m}=I$,
have been addressed. Moreover, although those results are useful, they are not always completely sharp.

Before we come to the positive results, we remind of a well-known negative result of non-convergence. The following statement is the famous counterexample of Seidman [32] for the projected least-squares problem.

Example 1 (Seidman). There exists a compact injective linear operator $A$ and a $x^{\dagger}$ such that $x_{n}$ as given by (5) is a unbounded sequence. Thus, in particular, we have non-convergence $x_{n} \nrightarrow x^{\dagger}$. Moreover there also exists $A, x^{\dagger}$ as before such that $x_{n}$ is bounded but $x_{n} \nrightarrow x^{\dagger}$.

The operator used for this example is a diagonal operator in the $l^{2}$-sequence space with a rank-1 perturbation:

$$
A: l^{2} \rightarrow l^{2} \quad A=\operatorname{diag}(\gamma)+\beta \otimes e_{1}
$$

where $\gamma$ and $\beta$ are appropriate sequences and $e_{1}$ is the sequence with all 0 except at the first position, where it is 1 . By an appropriate (constructive) choice of $x^{\dagger}$ and $\beta, \gamma$, the unboundedness of $x_{n}$ can be shown; see 32] or [7]. The last statement in this theorem of a bounded (strongly-) non-convergence sequence is stated in [32] but not explicitly proven.

Concerning the question of finding conditions for global convergence, the problem is well-studied. The following result is proven by Nashed [26], (for $A_{n}$ ), see also [27], in [20, Theorem 3.7] for $A_{n}$ being injective, and for the general case with $A_{n, m}$ by Du [5] (see also [6]). It gives a necessary and sufficient condition for global convergence.

## Theorem 1.

$$
x_{n, m} \rightarrow x^{\dagger} \quad \text { as } m, n \rightarrow \infty \quad \forall x^{\dagger} \in N(A)^{\perp}
$$

if and only if there exists a constant $C$ such that

$$
\begin{equation*}
\sup _{n, m}\left\|A_{n, m}^{\dagger} A\right\| \leq C \tag{6}
\end{equation*}
$$

Below, we will also reprove the corresponding result (Theorem 15) and, in particular, study characterizations of the uniform boundedness condition (6); see Theorems 17 and 18 .

Note that in [5, Theorem 2.6], Theorem 1] has been generalized to the case of nonatainable data, i.e., when $y=A x^{\dagger}+R(A)^{\dagger}$. In this case the necessary and sufficient conditions for $x_{n}$ being strongly (weakly) convergent to $A^{\dagger} y$ is (6) and $A_{n}^{*} Q_{n} y \rightarrow 0 \Rightarrow A_{n}^{\dagger} Q_{n} y \rightarrow(-) 0$.

If follows immediately from Theorem 1 for the dual projection case, i.e., $A_{n, m}=A_{\infty, m}$, by $A_{\infty, m}^{\dagger}=A_{\infty, m}^{\dagger} Q_{m}$, that condition (6) is always satisfied, i.e., this method always globally converges. This is well-known and has been shown, e.g., in 7 .

A widely used sufficient condition for uniform boundedness and hence global convergence of $x_{n}$ has been presented by Natterer [27] using a result by Nitsche [29].
Theorem 2 (Natterer). Let $A$ be injective. Suppose that there exists a constant $C$ such that for all $x^{\dagger} \in X$ there exists a $u_{n} \in X_{n}$ :

$$
\begin{equation*}
\left\|x^{\dagger}-u_{n}\right\|+\left\|A_{n}^{\dagger}\right\|\left\|A\left(x^{\dagger}-u_{N}\right)\right\| \leq C\left\|x^{\dagger}\right\| \tag{7}
\end{equation*}
$$

Then $x_{n} \rightarrow x^{\dagger}$ as $n \rightarrow \infty$ for all $x^{\dagger} \in X$.

In this theorem,

$$
\left\|A_{n}^{\dagger}\right\|=\sup _{\left\|A_{n} x_{n}\right\|=1, x_{n} \in X_{n}}\left\|x_{n}\right\|=\sigma_{\min }^{-1}\left(A P_{n}\right)
$$

It is not difficult to verify that (7) implies (6), We will generalize this result by giving a condition resembling (77) which is equivalent to (6) and hence yields global convergence in the general case (including non-injective operators and for the general projection case); see below Proposition 19

Furthermore a quite general condition has been proposed by Vainikko and Hämarik [35] (see also [11, 14, 15] and [12] and the references therein).

Theorem 3 (Vainikko and Hämarik). Suppose that $N\left(Q_{m} A P_{n} A^{*}\right)=\{0\}$. If there is a constants $C$ such that

$$
\begin{equation*}
\left\|A^{*} Q_{m} A P_{n} z\right\| \leq C\left\|P_{n} A^{*} Q_{m} A P_{n} z\right\| \quad \forall z \in X_{n} \tag{8}
\end{equation*}
$$

then $x_{n, m} \rightarrow x^{\dagger}$ for all $x^{\dagger} \in N(A)^{\perp}$.
Further results, e.g., on appropriate parameter choice rules, are proven in [35] as well. We will show below (Theorem (18) that (8) is actually equivalent to (6).

A simple condition involving the product of the ill-posedness and approximation rate has been used by several authors (e.g., [19, 24])

Theorem 4. If $\left\|A\left(I-P_{n}\right)\right\|\left\|A_{n, m}^{\dagger}\right\|<\infty$, then $x_{n, m} \rightarrow x^{\dagger}$ as $m, n \rightarrow \infty$ for all $x^{\dagger} \in N(A)^{\perp}$. If for a specific $x^{\dagger}, \lim _{m, n \rightarrow \infty}\left\|A\left(I-P_{n}\right) x^{\dagger}\right\|\left\|A_{n, m}^{\dagger}\right\|=0$, then $x_{n, m} \rightarrow x^{\dagger}$ as $m, n \rightarrow \infty$.

The previous results are ones that hold uniformly for all $x^{\dagger}$ (except for the very last one), and convergence for all $x^{\dagger} \in N(A)^{\perp}$ is obtained. However, it is of high interest to study conditions for convergence for one specific $x^{\dagger}$, when we do not care about global convergence. There are some statements concerning local convergence in literature.

In a quite general situation, necessary and sufficient conditions for local convergence have been established by Groetsch and Neubauer 7, 9.

Theorem 5 (Groetsch and Neubauer, also Du). We have the following local convergence conditions for strong convergence:

$$
x_{n} \rightarrow x^{\dagger} \Longleftrightarrow \limsup _{n}\left\|x_{n}\right\| \leq\left\|x^{\dagger}\right\|
$$

Moreover, suppose that

$$
\begin{equation*}
\overline{\bigcup_{n}\left(N(A) \cap X_{n}\right)}=N(A) . \tag{9}
\end{equation*}
$$

Then we have the following local convergence conditions for weak convergence:

$$
\begin{equation*}
x_{n} \rightharpoonup x^{\dagger} \Longleftrightarrow \sup _{n}\left\|x_{n}\right\|<\infty \tag{10}
\end{equation*}
$$

Note that the part on weak convergence in Theorem [5 identity (10) is erroneously stated in [7, 9] without the space condition (9), as has been noted by Du [5]. The characterization of strong convergence is valid without (9) and was already stated in [9] (using the incomplete result for weak convergence). It has been rigorously proved by Du and Du [6, Remark 4.3]. We will extend Theorem 5 to general projection methods with $x_{n, m}$; see Theorem 25 ,

Besides the convergence result of Neubauer and Groetsch, a sufficient condition for strong local convergence without requiring information about $x^{\dagger}$ has been stated by Luecke and Hickey [21].

Theorem 6 (Luecke, Hickey). Suppose that

$$
\begin{equation*}
\sup _{n}\left\|\left(A_{n}^{\dagger}\right)^{*} x_{n}\right\|<\infty \tag{11}
\end{equation*}
$$

then $x_{n} \rightarrow x^{\dagger}$.
This result is also proven in [7, where it is also explained that (11) is quite strong (and thus not a necessary condition for convergence) as it leads to a convergence rate of $x_{n}-x^{\dagger}$. In Proposition 27 we provide a similar result but by employing weaker conditions.

A subtle and important point is the space condition (9). In case of injective operators, of course, (91) holds true but in the general case not always, not even if $N(A)$ is finite-dimensional. (Think, for instance, of a discretization space $X_{n}$ that is disjoint to $N(A)$.) We note that $\left(N(A) \cap X_{n}\right)$ is an increasing family of closed subspaces, thus the following identity holds, (cf., e.g., [10, Chpt. 1, § 12])

$$
\bigcup_{n}\left(N(A) \cap X_{n}\right) \quad \perp \bigcap_{n}\left(N(A) \cap X_{n}\right)^{\perp}
$$

so that (9) is equivalent to

$$
\begin{equation*}
\bigcap_{n}\left(N(A) \cap X_{n}\right)^{\perp}=N(A)^{\perp} \tag{12}
\end{equation*}
$$

A recent preprint [6, Theorem 1.1] discusses equivalent conditions to (9) (respectively, (12)).

Theorem 7. The condition (9) is equivalent to each of the following conditions, Here $\mathcal{G}(A)$ denotes the graph of an operator $A$.
-

$$
\forall x \in N(A): \lim _{n \rightarrow \infty} \inf _{z_{n} \in N\left(A_{n}\right)}\left\|x-z_{n}\right\| \rightarrow 0
$$

- 

$$
\forall(x, y) \in \mathcal{G}\left(A^{\dagger}\right): \lim _{n \rightarrow \infty} \inf _{\left(z_{n}, w_{n}\right) \in \mathcal{G}\left(A_{n}^{\dagger}\right)}\left\|(x, y)-\left(z_{n}, y_{n}\right)\right\| \rightarrow 0
$$

- for all sequences $y_{n}$ :

$$
\sup _{n}\left\|A_{n}^{\dagger} y_{n}\right\|<\infty \text { and } y_{n} \rightharpoonup y: \Rightarrow y \in \mathcal{D}\left(A^{\dagger}\right) \text { and } A^{\dagger} y=A_{n}^{\dagger} y
$$

Moreover (6) is equivalent to the following two conditions holding simultaneously, (9) and $\operatorname{gap}\left(R\left(A^{*} A P_{n}\right), R\left(A^{\dagger} A P_{n}\right)<1\right.$, where the gap between two spaces $M, N$ is defined as (see [6, Lemma 3.2] $\operatorname{gap}(M, N)=\left\|P_{M}-P_{N}\right\|$.

We will extend the second part of this result and we show that (6) can be equivalently be characterized as a certain angle (or gap) between subspaces (but not those in this theorem) to be smaller then one; cf. Lemma 16. Moreover, we also study local convergence also when (9) is not satisfied.

The subtle fact that boundedness of $x_{n}$ is not enough for weak convergence and that an additional condition, e.g., like (9), is needed, is not very wellknown. Du [5, Example 2.10] gave a counterexample of a sequence of $\left\|x_{n}\right\|$ being uniformly bounded but which does not converge weakly to $x^{\dagger}$.

Example $2(\mathrm{Du})$. There exists a linear operator $A$ and $a x^{\dagger} \in N(A)^{\perp}$ such that

$$
\sup _{n}\left\|x_{n}\right\|<\infty
$$

but $x_{n} \nRightarrow x^{\dagger}$. In this example, however, $x_{n}$ converges strongly to some element $\neq x^{\dagger}$.

The operator in this counterexample is actually not ill-posed but a simple projection operator onto the complement of a one-dimensional subspace, $A=$ $I-(., e) e$ with some appropriately chosen $e$. Failure of convergence happens because $x_{n}$ converges (even strongly) but to the "wrong" solution.

Below in Theorem [28, we give a counterexample that is even more extreme: a situation like in the previous result, Example 2 but where the sequence $x_{n}$ does not converge at all (not even weakly). This example has been devised by Neubauer [28].
Example 3 (Neubauer). There exists a linear operator $A$ and a $x^{\dagger} \in N(A)^{\perp}$ such that

$$
\sup _{n}\left\|x_{n}\right\|<\infty
$$

but $x_{n}$ does not converge weakly.
Moreover, the sequence $x_{n}$ has a subsequence, which converges weakly but with limit $u \neq x^{\dagger}$, and no weakly convergent subsequence has limit $x^{\dagger}$.

### 2.2 Preliminary lemmas

Since $x_{n, m}$ is always in $N\left(A_{n, m}\right)^{\perp}$, it is important to study these spaces. We note that by discretization, $A_{n, m}$ is an operator with closed range. We have the well-known duality relations,

$$
\begin{equation*}
R\left(A_{n, m}^{*}\right)=N\left(A_{n, m}\right)^{\perp} \quad \text { and } \quad N\left(A_{n, m}^{*}\right)=R\left(A_{n, m}\right)^{\perp} \tag{13}
\end{equation*}
$$

and all these spaces are closed.
The following characterizations follow easily.

## Lemma 8.

$$
\begin{align*}
N\left(A_{n, m}\right) & =\left\{x \in X \mid x=w_{n}+q_{n}: w_{n} \in X_{n} \cap N\left(Q_{m} A\right), q_{n} \in X_{n}^{\perp}\right\}  \tag{14}\\
N\left(A_{n, m}\right)^{\perp} & =\left\{x \in X_{n} \cap\left(N\left(Q_{m} A\right) \cap X_{n}\right)^{\perp}\right\} \\
& =\left\{x \in X_{n} \mid \exists v_{n}: x=P_{n} A^{*} Q_{m} v_{n}\right\},
\end{align*}
$$

Proof. The first identity (14) follows easily by $x=P_{n} x+\left(I-P_{n}\right) x$. It is straightforward to proof that any $x \in X_{n} \cap\left(X_{n} \cap N\left(Q_{m} A\right)\right)^{\perp}$ is in $N\left(A_{n, m}\right)^{\perp}$. Conversely, by taking either $w_{n}$ or $q_{n}$ to be 0 , it follows that any $x \in N\left(A_{n, m}\right)^{\perp}$ must be in both $X_{n}$ and $\left(X_{n} \cap N\left(Q_{m} A\right)\right)^{\perp}$. The last identity is (13).

Note that these spaces are not necessarily nested. However, the following inclusions can be verified:

$$
\begin{align*}
& N\left(Q_{m} A\right) \cap X_{n} \subset N\left(Q_{m} A\right) \cap X_{n+1}  \tag{15}\\
& N\left(Q_{m} A\right) \cap X_{n} \supset N\left(Q_{m+1} A\right) \cap X_{n}  \tag{16}\\
& \supset \ldots N\left(Q_{m} A\right),  \tag{17}\\
& N(A) \cap X_{n}=\bigcap_{m}\left(N\left(Q_{m} A\right) \cap X_{n}\right)
\end{align*}
$$

A simple consequence is that, for a fixed $x$, the norm of the projection $\left\|\Pi_{N\left(Q_{m} A\right) \cap X_{n}} x\right\|$, is increasing in $m$ (for fixed $n$ ) and decreasing in $n$ (for fixed $m$ ).

We now state some approximation results, namely that elements in $N(A)^{\perp}$, i.e., the space where $x^{\dagger}$ lives, can be approximated arbitrary well by elements in the corresponding discrete space $N\left(A_{n, m}\right)^{\perp}$.
Lemma 9. For all $x \in N(A)^{\perp}$,

$$
\begin{equation*}
\lim _{n, m \rightarrow \infty} \inf _{z \in R\left(A_{n, m}^{*}\right)}\|x-z\|=\lim _{n, m \rightarrow \infty} \inf _{z \in N\left(A_{n, m}\right)^{\perp}}\|x-z\|=0 \tag{18}
\end{equation*}
$$

Proof. Let $x \in N(A)^{\perp}=\overline{R\left(A^{*}\right)}=\overline{R\left(A^{*} A\right)}$. Then for any $\epsilon>0$ fixed, we can find a $w_{\epsilon}$ such that

$$
\left\|x-A^{*} w_{\epsilon}\right\| \leq \epsilon,
$$

Moreover by (2), with $\epsilon$ and $w_{\epsilon}$ as before, we can find a $n_{0}$ such that for all $n \geq n_{0}$ and $m \geq n_{0}$

$$
\left\|P_{n} A^{*} w_{\epsilon}-A^{*} w_{\epsilon}\right\| \leq \epsilon, \quad\left\|Q_{m} w_{\epsilon}-w_{\epsilon}\right\| \leq \frac{\epsilon}{\left\|A^{*}\right\|}
$$

thus,

$$
\left\|x-P_{n} A^{*} Q_{m} w_{\epsilon}\right\| \leq\left\|x-P_{n} A^{*} w_{\epsilon}\right\|+\left\|P_{n} A^{*}\left(I-Q_{m}\right) w_{\epsilon}\right\| \leq 2 \epsilon
$$

which yields (18). Note that $P_{n} A^{*} Q_{m} w_{\epsilon}$ is in $R\left(A_{n}^{*}\right)=N\left(A_{n}\right)^{\perp}$.
We remark that $N\left(A_{n, m}\right)^{\perp}$ is not necessarily a subspace of $N(A)^{\perp}$ so that it is not correct to say that $N\left(A_{n, m}\right)^{\perp}$ is dense in $N(A)^{\perp}$.

Furthermore, the operators $A^{*} A_{n, m}$ and $A_{n, m}^{*} A$ will play an important role in the subsequent analysis.

## Lemma 10.

$$
N\left(A_{n, m}\right)=N\left(A^{*} A_{n, m}\right) \quad R\left(A_{n, m}^{*} A\right)=R\left(A_{n, m}^{*}\right)
$$

Proof. Clearly $N\left(A_{n, m}\right) \subset N\left(A^{*} A_{n, m}\right)=N\left(A^{*} A_{n, m}\right)$. Conversely for $x \in$ $N\left(A^{*} A_{n, m}\right)$ we have $A^{*} Q_{m} A P_{n} x=0$, and hence also $P_{n} A^{*} Q_{m} Q_{m} A P_{n} x=0$, thus $x \in N\left(A_{n, m}\right)$, which shows $N\left(A_{n, m}\right)=N\left(A^{*} A_{n, m}\right)$, and by (131) the lemma follows.

Using $A^{*} A_{n, m}$ and $A^{*} A_{n, m}$, we have a characterization of the solution operator $A_{n, m}^{\dagger} A$ as a certain nonorthogonal projection operator.

Proposition 11. For any $x \in X$ and any $n, m \in \mathbb{N}$, we have the unique decomposition

$$
\begin{equation*}
x=v_{n, m}+u_{n, m} \quad v_{n, m} \in R\left(A_{n, m}^{*} A\right), u_{n, m} \in N\left(A_{n, m}^{*} A\right) \tag{19}
\end{equation*}
$$

Moreover the mapping $x \rightarrow v_{n, m}$ is given by $A_{n, m}^{\dagger} A$, i.e.,

$$
v_{n, m}=A_{n, m}^{\dagger} A x
$$

For any $x \in X$ any $n, m \in \mathbb{N}$, we have the unique decomposition

$$
\begin{align*}
& x=\bar{v}_{n, m}+\bar{u}_{n, m} \quad \bar{v}_{n, m} \in R\left(A^{*} A_{n, m}\right), \bar{u}_{n, m} \in N\left(A^{*} A_{n, m}\right)  \tag{20}\\
&=\bar{v}_{n, m}+\bar{w}_{n, m}+\bar{q}_{n, m} \quad \bar{w}_{n, m} \in R\left(A^{*} A_{n, m}\right) \\
& \bar{q}_{n, m} \in X_{n}^{\perp} \tag{21}
\end{align*}
$$

Here, $\bar{v}_{n, m}, \bar{u}_{n, m}, \bar{w}_{n, m}, \bar{q}_{n, m}$ are uniquely determined. Moreover, the mapping $x \rightarrow \bar{v}_{n, m}$ is given by $A^{*}\left(A_{n, m}^{*}\right)^{\dagger}$, i.e.,

$$
\bar{v}_{n}=A^{*}\left(A_{n, m}^{*}\right)^{\dagger} x,
$$

and the mapping $x \rightarrow \bar{w}_{n, m}$ is given by

$$
\begin{equation*}
\bar{w}_{n, m}=\Pi_{N\left(Q_{m} A\right) \cap X_{n}} x . \tag{22}
\end{equation*}
$$

Proof. Define $v_{n, m}=A_{n, m}^{\dagger} A x$, then $v_{n, m} \in N\left(A_{n, m}\right)^{\perp}=N\left(A^{*} A_{n, m}\right)^{\perp}=$ $R\left(A_{n, m}^{*} A\right)$. In particular, we have $v_{n, m} \in X_{n}$. Moreover $v_{n}$ satisfies the normal equations

$$
0=P_{n} A^{*} Q_{m}\left(A P_{n} v_{n}-A x\right)=P_{n} A^{*} Q_{m} A\left(v_{n}-x\right)
$$

Thus $v_{n}-x \in N\left(A_{n, m}^{*} A\right)$ yielding the desired decomposition (19). Conversely, for any other decomposition as above, it follows that $v_{n} \in N\left(A_{n, m}\right)^{\perp}$, and it satisfies the normal equations. By uniqueness of the minimal-norm least-squares solution, if follows that $v_{n}=A_{n, m}^{\dagger} A x$. Thus the decomposition is unique. For the second part, define $z_{n, m}=\left(A_{n, m}^{*}\right)^{\dagger} x$, then $z_{n, m} \in N\left(A_{n, m}^{*}\right)^{\perp}=R\left(A_{n, m}\right)$. In particular $z_{n, m} \in Y_{m}$. The normal equation implies that $Q_{m} A P_{n} A^{*} z_{n, m}-$ $Q_{m} A P_{n} x=0$, thus $\bar{v}_{n, m}-x=A^{*} z_{n, m}-x \in N\left(A_{n, m}\right)=N\left(A^{*} A_{n, m}\right)$, which gives the decomposition. Any other decomposition of the form $x=A^{*} A_{n, m} p+$ $N\left(A_{n, m}\right)$ implies that $A_{n, m} p$ satisfies the same normal equation as $z_{n, m}$ and it clearly is in $R\left(A_{n, m}\right)=N\left(A_{n, m}^{*}\right)^{\dagger}$, thus by the uniqueness of the minimal-norm least-squares solution, we have $A_{n, m} p=\left(A_{n}^{*}\right)^{\dagger} x$. Hence, $A^{*} A_{n, m} p=A^{*}\left(A_{n}^{*}\right)^{\dagger} x$, which implies the unique decomposition (20). The decomposition of $\bar{u}_{n, m}$, into $\bar{w}_{n, m}+\bar{q}_{n, m}$ exists by the characterization of $N\left(A_{n, m}\right)$ in (14) and is clearly unique since $\bar{w}_{n, m}$ is orthogonal to $\bar{q}_{n, m}$. Since $\bar{v}_{n, m}=A^{*} z_{n, m}=A^{*} Q_{m} z_{n, m}$, it follows that $\bar{v}_{n, m}$ is orthogonal to $N\left(Q_{m} A\right)$ and clearly $\bar{q}_{n, m}$ is orthogonal to $X_{n}$, hence applying $\Pi_{N(A) \cap X_{n}}$ to the decomposition gives the representation for $\bar{w}_{n, m}$.

Remark 1. This proposition will be used widely; in particular, we recognize that $x_{n, m}=A_{n, m}^{\dagger} A x^{\dagger}$ is the first element in (19) in the decomposition of $x^{\dagger}$ and thus $x_{n, m}$ is the result of a nonorthogonal (oblique) projection applied to $x^{\dagger}$.

From (19), (20) we also obtain the nontrivial fact that

$$
N\left(A_{n, m}^{*} A\right) \cap R\left(A_{n, m}^{*} A\right)=\emptyset, \quad \text { and } \quad N\left(A^{*} A_{n, m}\right) \cap R\left(A^{*} A_{n, m}\right)=\emptyset .
$$

As a corollary we have a formula for $\Pi_{N\left(Q_{m} A\right) \cap X_{n}}$.
Corollary 12. For any $k \leq n$,

$$
\begin{gathered}
P_{k} \Pi_{N\left(Q_{m} A\right) \cap X_{n}}=P_{k}-A_{k}^{*}\left(A_{n, m}^{*}\right)^{\dagger} \\
\Pi_{N\left(Q_{m} A\right) \cap X_{n}} P_{k}=P_{k}-A_{n, m}^{\dagger} A_{k} .
\end{gathered}
$$

Proof. Applying $P_{k}$ to (21) and using the fact that orthogonal projectors are selfadjoint and $\left(A_{k}^{*}\left(A_{n, m}^{*}\right)^{\dagger}\right)^{*}=A_{n, m}^{\dagger} A_{k}$ yields the result.

As another illustration of the usefulness of Proposition 11, we can prove a similar characterization of the space condition (9) as in Theorem 7 .
Proposition 13. We have that (9) is satisfied if and only if

$$
\begin{equation*}
\left(\lim _{n, m \rightarrow \infty} \inf _{z_{n, m} \in R\left(A_{n, m}^{*}\right)}\left\|x-z_{n, m}\right\|=0\right) \Longrightarrow x \in \overline{R\left(A^{*}\right)} \tag{23}
\end{equation*}
$$

Proof. Let (23) hold and suppose that (9) does not hold. Then there exists a $x \neq 0$ and $x \in N(A)$ and $x \in \bigcap_{n}\left(N(A) \cap X_{n}\right)^{\perp}$. For such a $x$ using (21), (22), and (17), it follows that for all $n \lim _{m \rightarrow \infty} \bar{w}_{n, m}=0$. Thus, by Corollary 12 with $k=n$,

$$
\lim _{m \rightarrow \infty}\left\|\left(P_{n}-A_{n}^{*}\left(A_{n, m}^{*}\right)^{\dagger}\right) x\right\|=\lim _{m \rightarrow \infty}\left\|\left(P_{n}-A_{n, m}^{\dagger} A_{n, m}\right) x\right\|=0
$$

Taking $n$ such that $\left\|x-P_{n} x\right\| \leq \epsilon$, we find a $m$ such that

$$
\left\|x-A_{n, m}^{\dagger} A_{n, m} x\right\| \leq\left\|x-P_{n} x\right\|+\left\|\left(P_{n}-A_{n, m}^{\dagger} A_{n, m}\right) x\right\| \leq 2 \epsilon
$$

Thus by (23), since $A_{n, m}^{\dagger} A_{n, m} \in R\left(A_{n, m}^{*}\right)$, it follows that $x \in \overline{R\left(A^{*}\right)}=N(A)^{\perp}$. Since $x \in N(A)$, we have a contradiction, thus (9) must hold. Conversely, if (9) holds, suppose that (23) does not hold. Then we have a $x \in N(A)$ and $z_{n, m} \in R\left(A_{n, m}^{*}\right)$ with $\left\|x-\Pi_{R\left(A_{n, m}^{*}\right)} x\right\| \rightarrow_{n, m} 0$. As $\Pi_{R\left(A_{n, m}^{*}\right)}=A_{n, m}^{\dagger} A_{n, m}=$ $A_{n, m}^{*}\left(A_{n, m}^{*}\right)^{\dagger}$, we have that $x-A_{n, m}^{*}\left(A_{n, m}^{*}\right)^{\dagger} \rightarrow_{n, m} 0$. Applying $P_{n}$ to (21), it follows that $\bar{w}_{n, m}=P_{n} x-A_{n, m}^{*}\left(A_{n, m}^{*}\right)^{\dagger} x \rightarrow_{n, m} 0$; in particular $\lim _{n} \lim _{m} \bar{w}_{n, m}=$ $\lim _{n} \Pi_{N(A) \cap X_{n}} x=0$. By (15), $\Pi_{N(A) \cap X_{n}} x$ is increasing, hence $\Pi_{N(A) \cap X_{n}} x=0$ for all $n$. In other words, $x \in \bigcap_{n}\left(N(A) \cap X_{n}\right)^{\perp}$. Using (9) implies that $x \in N(A)^{\perp}$, which is a contradiction to $x \in N(A)$.

In view of Lemma [9 we always have that

$$
\begin{equation*}
N(A)^{\perp} \subset\left\{x: \operatorname{dist}\left(x, N\left(A_{n, m}\right)^{\perp}\right) \rightarrow_{n, m} 0\right\} \tag{24}
\end{equation*}
$$

but according to (23), equality holds only if the space condition (9) holds. The corresponding result for the projected least-squares case using $A_{n}$ (and more) has already been proven in 6].

## 3 Convergence results

We now study necessary and sufficient conditions for local and global convergence of $x_{n, m}$ to $x^{\dagger}$, thus extending the known results of the Section 2.1.

### 3.1 Conditions for global convergence

At first we consider convergence for all $x^{\dagger} \in N(A)^{\perp}$. We reprove the statement of Theorem 1 based on the following lemma.

Lemma 14. For all $x^{\dagger} \in N(A)^{\perp}$ and $x_{n, m}=A_{n}^{\dagger} A x^{\dagger}$, we have

$$
\limsup _{n, m \rightarrow \infty}\left\|x_{n, m}-x^{\dagger}\right\| \leq \limsup _{n, m \rightarrow \infty}\left\|A_{n, m}^{\dagger} A\left(I-P_{n}\right) x^{\dagger}\right\|
$$

Proof. Noting that $A_{n, m}^{\dagger} A_{n, m}=I-\Pi_{N\left(A_{n, m}\right)}=\Pi_{N\left(A_{n, m}\right)^{\perp}}=\Pi_{R\left(A_{n, m}^{*}\right)}$,

$$
\begin{aligned}
x_{n, m} & -x^{\dagger}=A_{n, m}^{\dagger} A x^{\dagger}-x^{\dagger}=\left(A_{n, m}^{\dagger} Q_{m} A P_{n}-I\right) x^{\dagger}+A_{n, m}^{\dagger} Q_{m} A\left(I-P_{n}\right) x^{\dagger} \\
& =-\left(I-\Pi_{R\left(A_{n, m}^{*}\right)}\right) x^{\dagger}+A_{n, m}^{\dagger} A\left(I-P_{n}\right) x^{\dagger} .
\end{aligned}
$$

Thus we have

$$
\left\|x_{n, m}-x^{\dagger}\right\| \leq \inf _{z \in R\left(A_{n, m}^{*}\right)}\left\|x^{\dagger}-z\right\|+\left\|A_{n, m}^{\dagger} A\left(I-P_{n}\right) x^{\dagger}\right\|
$$

Now, Lemma 9 and (18) yields the result
Remark 2. In the above result we can easily replace $A_{n, m}^{\dagger} A\left(I-P_{n}\right) x^{\dagger}$ by the expression $A_{n, m}^{\dagger} Q_{m} A \Pi_{N\left(Q_{m} A\right)^{\perp}}\left(I-P_{n}\right) x^{\dagger}$ or by $A_{n, m}^{\dagger} Q_{m} A \Pi_{N\left(Q_{m} A\right)^{\perp} \cap X_{n}^{\perp}} x^{\dagger}$.

We obtain the first (well-known) result on global convergence; cf. Theorem 1
Theorem 15. The approximations $x_{n, m}$ converge to $x^{\dagger}$ for all $x^{\dagger} \in N(A)^{\perp}$ if and only if there exists a constant $C$ such that

$$
\begin{equation*}
\sup _{n, m}\left\|A_{n, m}^{\dagger} A\right\| \leq C \tag{25}
\end{equation*}
$$

Equivalent to (25) is that there exists a constant $C^{\prime}$ such that

$$
\begin{equation*}
\sup _{n, m}\left\|A_{n, m}^{\dagger} A\left(I-P_{n}\right)\right\| \leq C \tag{26}
\end{equation*}
$$

Proof. Let $x_{n, m} \rightarrow x^{\dagger}$ for all $x^{\dagger} \in N(A)^{\perp}$ as $m, n \rightarrow \infty$. Then we have by $x_{n, m}=A_{n, m}^{\dagger} A x^{\dagger}$ that $A_{n, m}^{\dagger} A \rightarrow I$ pointwise on $N(A)^{\perp}$. By the uniform boundedness principle this implies that $\left.A_{n, m}^{\dagger} A\right|_{N(A) \perp}$ must be uniformly bounded. But it is easy to see that this is equivalent to (25).

Conversely let (25) hold, then

$$
\left\|A_{n, m}^{\dagger} A\left(I-P_{n}\right) x^{\dagger}\right\| \leq\left\|A_{n, m}^{\dagger} A\right\|\left\|\left(I-P_{n}\right) x^{\dagger}\right\|,
$$

and by (25) and (2), the first result follows. Since $A_{n}^{\dagger} A\left(I-P_{n}\right)=A_{n, m}^{\dagger} A-$ $A_{n, m}^{\dagger} A_{n, m}$, and $A_{n, m}^{\dagger} A_{n, m}$ is always bounded, (26) follows.

Next, we study condition (25) in depth and rewrite it in other forms. Clearly it holds that

$$
\begin{equation*}
\sup _{m, n}\left\|A_{n, m}^{\dagger} A\right\| \leq C \Leftrightarrow \sup _{m, n}\left\|A^{*}\left(A_{n, m}^{*}\right)^{\dagger}\right\| \leq C \tag{27}
\end{equation*}
$$

We show that (25) is equivalent to the fact that angles between certain subspaces are uniformly bounded. More precisely, we consider the norm of the product of orthogonal projectors onto two subspaces, which is related to the (minimal canonical) angle; cf. [33, Lemma 5.1].

Lemma 16. For a sequence of closed subspaces $X_{n}, Y_{n}$, let $\Pi_{X_{n}}, \Pi_{Y_{n}}$ be the corresponding orthogonal projectors.

Then we have the following equivalent conditions

$$
\begin{align*}
& \exists \rho<1: \sup _{n}\left\|\Pi_{X_{n}} \Pi_{Y_{n}}\right\| \leq \rho,  \tag{28}\\
\Leftrightarrow & \exists \tau>0 \forall x \in X_{n}, y \in Y_{n}:\|(x+y)\|^{2} \geq \tau\|x\|^{2},  \tag{29}\\
\Leftrightarrow & \exists \tau^{\prime}>0 \forall x \in X_{n}, y \in Y_{n}:\|(x+y)\|^{2} \geq \tau\|y\|^{2} . \tag{30}
\end{align*}
$$

Proof. If (28) holds, then by Young's inequality for any $\epsilon>0 x \in X_{n}, y \in Y_{n}$,

$$
\|(x+y)\|^{2} \geq\|x\|^{2}+\|y\|^{2}-2 \rho\|x\|\|y\| \geq\|x\|^{2}(1-\rho \epsilon)+\|y\|^{2}\left(1-\frac{\rho}{\epsilon}\right)
$$

With $\epsilon=\rho$ or $\epsilon=\rho^{-1}$ either of (30) or (29) follows. Conversely let (29) hold, then then for any $x \in X_{n}, y \in y_{n}$ with $\|x\|=\|y\|=1$ and any $\epsilon>0$

$$
\epsilon^{2} \tau^{2} \leq\left\|\epsilon x+\frac{1}{\epsilon} y\right\|^{2}=\epsilon^{2}+\frac{1}{\epsilon^{2}}-2(x, y) .
$$

Taking $\epsilon^{2}=\left(1-\tau^{2}\right)^{-\frac{1}{2}}$ gives the bound

$$
2(x, y) \leq 2\left(1-\rho^{2}\right)^{\frac{1}{2}}<2
$$

thus with (cf. [33, Lemma 5.1])

$$
\left\|\Pi_{X} \Pi_{Y}\right\|=\sup _{\|x\| \leq 1,\|y\| \leq 1, x \in X, y \in Y}(x, y)
$$

the result follows.
Combining Proposition 11 and Lemma 16 yields the following
Theorem 17. The uniform boundedness condition (25) is equivalent to one of the following (and hence all) conditions:

$$
\begin{array}{rr}
\exists \eta<1: \forall n: & \left\|\Pi_{N\left(A_{n, m}^{*} A\right)} \Pi_{R\left(A_{n, m}^{*} A\right)}\right\|<\eta, \\
\exists \eta<1: \forall n: & \left\|\Pi_{N\left(A^{*} A_{n, m}\right)} \Pi_{R\left(A^{*} A_{n, m}\right)}\right\|<\eta, \\
\exists \eta<1: \forall n: & \left\|\left(I-P_{n}\right) \Pi_{R\left(A^{*} A_{n, m}\right)}\right\|<\eta . \tag{33}
\end{array}
$$

Proof. The boundedness condition can be rephrased as the condition that a constant $C$ exists with (using the notation in Proposition [11)

$$
\left\|v_{n, m}\right\| \leq C\|x\|=C\left\|v_{n, m}+u_{n, m}\right\| .
$$

for all $v_{n, m} \in R\left(A_{n, m}^{*} A\right)$ and $u_{n, m} \in N\left(A_{n, m}^{*} A\right)$. However, this is (29) with the spaces $N\left(A_{n, m}^{*} A\right)$ and $R\left(A_{n, m}^{*} A\right)$, thus Lemma 16 gives (31). By (27) we have the equivalent characterization of uniform boundedness using (29) that a constant exists, such that

$$
\left\|\bar{v}_{n, m}\right\| \leq C\|x\|=C\left\|\bar{v}_{n, m}+\bar{u}_{n, m}\right\|,
$$

which yields (32). By (30), this is equivalent to the existence of a constant such that

$$
\begin{equation*}
\left\|\bar{w}_{n, m}+\bar{q}_{n, m}\right\| \leq C\|x\|, \tag{34}
\end{equation*}
$$

with $\bar{w}_{n, m}, \bar{q}_{n, m}$ as in Proposition [11 However, $\bar{w}_{n, m}$ is always uniformly bounded for bounded $x$ by (22), and it is orthogonal to $q_{n, m}$. Thus, this condition is satisfied if and only if
$\left\|\bar{q}_{n, m}\right\|^{2} \leq C^{\prime}\|x\|^{2}=C^{\prime}\left(\left\|\bar{v}_{n, m}+\bar{q}_{n, m}+\bar{w}_{n, m}\right\|^{2}\right)=C^{\prime}\left(\left\|\bar{v}_{n, m}+\bar{q}_{n, m}\right\|^{2}+\left\|\bar{w}_{n, m}\right\|^{2}\right)$.
Since we can take $x=\bar{v}_{n, m}+\bar{q}_{n, m}+\bar{w}_{n, m}$ with arbitrary chosen elements $\bar{v}_{n, m}, \bar{q}_{n, m}, \bar{w}_{n, m}$ out of the corresponding spaces, we have that (34) holds if and only if for all $\bar{v}_{n, m} \in R\left(A^{*} A_{n, m}\right)$ and all $\bar{q}_{n, m} \in X_{n}^{\perp}$

$$
\left\|q_{n, m}\right\| \leq C^{\prime}\left(\left\|\bar{v}_{n, m}+\bar{q}_{n, m}\right\|\right)
$$

which is equivalent to (33).
Remark 3. We remark that for closed subspaces $X, Y$, in Hilbert spaces the identity $\left\|\Pi_{X} \Pi_{Y}\right\|<1 \Leftrightarrow\left\|\Pi_{X} \perp \Pi_{Y \perp}\right\|<1$ usually does not hold [2, 4].

These conditions can be rewritten in more convenient form.
Theorem 18. The uniform boundedness condition (25) is is equivalent to one (and hence all) of the following conditions:

$$
\begin{array}{lr}
\exists C>0: \forall n, m, x & \left\|\left(I-P_{n}\right) A^{*} Q_{m} A P_{n} x\right\| \leq C^{\prime}\left\|P_{n} A^{*} Q_{m} A P_{n} x\right\|, \\
\exists \eta<1: \forall n, m, x & \left\|\left(I-P_{n}\right) A^{*} Q_{m} A P_{n} x\right\| \leq \eta\left\|A^{*} Q_{m} A P_{n} x\right\|, \\
\exists \eta<1: \forall n, m, w & \inf _{v}\left\|P_{n} A^{*} Q_{m} A w-A^{*} Q_{m} A P_{n} v\right\| \leq \eta\left\|P_{n} A^{*} Q_{m} A w\right\|, \\
\exists \eta<1: \forall n, m, v & \inf _{w}\left\|P_{n} A^{*} Q_{m} A w-A^{*} Q_{m} A P_{n} v\right\| \leq \eta\left\|A^{*} Q_{m} A P_{n} v\right\| . \tag{38}
\end{array}
$$

Proof. Condition (33) can be rewritten as (36). By splitting the terms using the complementary orthogonal projectors $P$ and $I-P$, it is easy to see that this is equivalent to (35). The identities (37) and (38) are (31) and (32), respectively, when writing the projectors onto the nullspaces as complementary projectors onto the ranges of the adjoints and using the minimization property of such orthogonal projectors.

It is not difficult to verify that (35) is equivalent to Vainikko and Hämarik's condition (8). Note that a characterization over angles of subspaces has also been used by Du and $\mathrm{Du}\left[6\right.$, Theorem 1.2] for the case $Q_{m}=I$ and with different spaces, which do not yield an equivalent condition to (6) but need additionally the space condition (9).

### 3.1.1 Necessary and sufficient conditions for convergence

In this section we investigate practically useful conditions such that (25) is satisfied and necessary conditions for (25).

We find a condition of Natterer's type that is equivalent to the uniform boundedness condition extending Natterer's result to the cases of non-injective operators and $Q_{m} \neq I$.

Proposition 19. The uniform boundedness condition (25) and hence $x_{n, m} \rightarrow$ $x^{\dagger}$ for all $x^{\dagger} \in N(A)^{\perp}$ holds if and only if there exists a constant $C$ such that for all $x^{\dagger} \in X$ there exists a $u_{n} \in X_{n}$ such that

$$
\begin{equation*}
\left\|x^{\dagger}-u_{n}\right\|+\left\|A_{n, m}^{\dagger} A\left(x^{\dagger}-u_{n}\right)\right\| \leq C\left\|x^{\dagger}\right\| \tag{39}
\end{equation*}
$$

In particular for the case $Q_{m}=I$, if Natterer's condition (17) holds for all $x^{\dagger} \in N(A)^{\perp}$, then (25) holds, thus $x_{n} \rightarrow x^{\dagger}$ for all $x^{\dagger} \in N(A)^{\perp}$.

Proof. If (25) and hence global convergence holds, then $u_{n}=x_{n, m}$ satisfies (39). Conversely if (39) holds, then since $Q_{m} A u_{n}=A_{n, m} u_{n}$

$$
\begin{aligned}
\left\|A_{n, m}^{\dagger} A x\right\| & \leq\left\|A_{n, m}^{\dagger} Q_{m} A\left(x-u_{n}\right)\right\|+\left\|A_{n, m}^{\dagger} A_{n, m} u_{n}\right\| \\
& \leq\left\|A_{n, m}^{\dagger} A\left(x-u_{n}\right)\right\|+c_{1}\left\|u_{n}\right\| \\
& \leq\left\|A_{n, m}^{\dagger} A\left(x-u_{n}\right)\right\|+c_{1}\left\|u_{n}-x^{\dagger}\right\|+c_{1}\left\|x^{\dagger}\right\| \leq C^{\prime}\left\|x^{\dagger}\right\| .
\end{aligned}
$$

It is easy to see that (7) implies (39).
From the conditions (35)-(38), probably (35) is the most useful. We introduce the norm of the pseudoinverse of the discretized forward operator

$$
\left\|A_{n, m}^{\dagger}\right\|=\frac{1}{\sigma_{\min }\left(Q_{m} A P_{n}\right)}=\sup _{x \in P_{n}, Q_{m} A P_{n} x \neq 0} \frac{(x, x)}{\left(x, P_{n} A^{*} Q_{m} A P_{n} x\right)}
$$

where $\sigma_{\min }$ denotes the smallest (by definition nonzero) singular value. We have the following result:

Lemma 20. If there exists a constant $C$ such that

$$
\begin{equation*}
\forall n, m \quad\left\|\left(I-P_{n}\right) A^{*} Q_{m}\right\|\left\|A_{n, m}^{\dagger}\right\| \leq C \tag{40}
\end{equation*}
$$

then (35) and hence (25) is satisfied.
Proof. In view of (35), we observe that

$$
\left\|P_{n} A^{*} Q_{m} z\right\| \geq \sigma_{\min }\left(Q_{m} A P_{n}\right)\|z\| .
$$

Taking $z=A P_{n} x$ and $\left\|\left(I-P_{n}\right) A^{*} Q_{m} A P_{n} x\right\| \leq\left\|\left(I-P_{n}\right) A^{*} Q_{m}\right\|\left\|Q_{m} A P_{n} x\right\|$ proves the assertion.

Note that this result implies in particular Theorem 4. In the same way we could prove the result by replacing (40) by

$$
\forall n, m \quad\left\|\left(I-P_{n}\right) A^{*} Q_{m} A\right\|\left\|A_{n, m}^{\dagger}\right\|^{2} \leq C .
$$

Natterer [27] has outlined how to prove conditions like (39) in practical situations, namely from inverse inequalities of approximation spaces combined with error estimates for the approximation. Using (40) we can do a similar thing.

Proposition 21. Let $\left(H_{s},\|x\|_{s}\right)_{s \in \mathbb{R}}$ be a Hilbert scale generated by a densely defined unbounded selfadjoint strictly positive operator L, i.e.,

$$
\|x\|_{s}=\left\|L^{s} x\right\| .
$$

Suppose that $A: H_{0} \rightarrow H_{0}$ is such that for some numbers $l, r>0$

$$
\begin{equation*}
c_{1}\|x\|_{-l} \leq\|A x\|_{L^{2}} \leq c_{2}\|x\|_{-r} \quad \forall x \in H_{0} \tag{41}
\end{equation*}
$$

and that $X_{n}$ is a discrete subspace satisfying the approximation condition

$$
\left\|\left(I-P_{n}\right) z\right\| \leq \gamma_{n}\|z\|_{l} \quad \forall z \in H_{l}
$$

and the inverse inequality

$$
\left\|z_{n}\right\|_{L^{2}} \leq \frac{1}{\beta_{n}}\left\|z_{n}\right\|_{-r} \quad \forall z_{n} \in X_{n}
$$

holds. Then if

$$
\limsup _{n} \frac{\gamma_{n}}{\beta_{n}} \leq C
$$

the uniform boundedness condition (25) holds.
Proof. We have that

$$
\left\|A P_{n} x\right\| \geq c_{1}\left\|P_{n} x\right\|_{-l} \geq c_{1} \beta_{n}\left\|P_{n} x\right\| .
$$

Thus,

$$
\left\|A_{n}^{\dagger}\right\| \leq \frac{1}{c_{1} \beta_{n}}
$$

Moreover with $z=A^{*} x$ we find that

$$
\left\|\left(I-P_{n}\right) A^{*} x\right\| \leq \gamma_{n}\left\|A^{*} x\right\|_{r}
$$

From the right hand side of (41), we see that $A L^{r}$ is a bounded linear operator and so is its adjoint $L^{r} A^{*}$, i.e., $\left\|A^{*} x\right\|_{r} \leq C\|x\|$. Thus (40) is satisfied by

$$
\left\|\left(I-P_{n}\right) A^{*} x\right\|\left\|A_{n}^{\dagger}\right\| \leq C \frac{\gamma_{n}}{\beta_{n}} \leq C
$$

In a typical case of finite-element spaces or spline spaces and if we consider a Hilbert scale of Sobolev spaces, then the inverse inequality is usually satisfied with $\beta_{n}=\frac{1}{n^{t}}$ and the approximation condition with $\gamma_{n}=\frac{1}{n^{r}}$. Thus if $r=l$ is applicable, then we obtain convergence. A similar argument has been utilized by Natterer using condition (7).

The next result concerns the dual variant of (33).
Proposition 22. If

$$
\begin{equation*}
\exists: \eta<1: \forall n: \quad\left\|P_{n} \Pi_{N\left(A_{n, m}^{*} A\right)}\right\|<\eta \tag{42}
\end{equation*}
$$

then (25) is satisfied. Moreover, (42) holds if

$$
\begin{equation*}
\exists: \eta<1: \forall n, \forall w: \quad \inf _{v}\left\|A^{*} Q_{m} A P_{n} v-P_{n} w\right\| \leq \eta\left\|P_{n} w\right\| \tag{43}
\end{equation*}
$$

Proof. Inequality (42) can be written as

$$
\eta>\left\|\Pi_{N\left(A_{n, m}^{*} A\right)} P_{n}\right\|=\left\|\left(I-\Pi_{R\left(A^{*} A_{n, m}\right)}\right) P_{n}\right\|
$$

By the characterization of orthogonal projectors as minimizers we have that this is equivalent to

$$
\inf _{v \in R\left(A^{*} A_{n, m}\right)}\left\|v-P_{n} x\right\| \leq \eta\left\|P_{n} x\right\|
$$

which is exactly (43). Setting $x=A^{*} Q_{m} A w$ we obtain that this implies (37).

Note that (42) is not equivalent to (25) because (42) can only hold if the intersection of the corresponding spaces is empty. However, if $X_{n} \cap N\left(A_{n, m}^{*} A\right) \neq \emptyset$, then (42) cannot hold but (25) still can.

Let us now come to a necessary condition for uniform boundedness. We show that the uniform boundedness (25) implies the space condition (9). In the case $Q_{m}=I$, this has already been observed by Du [5].
Proposition 23. Let (25) hold, then

$$
\bigcap_{n}\left(N(A) \cap X_{n}\right)^{\perp}=N(A)^{\perp},
$$

i.e., the space condition (19) holds.

Proof. Since $\bigcup_{n}\left(N(A) \cap X_{n}\right) \subset N(A)$ it follows that $N(A)^{\perp} \subset \bigcap_{n}\left(N(A) \cap X_{n}\right)^{\perp}$. Thus, we only need to proof the opposite inclusion. Let $x \in \bigcap_{n}\left(N(A) \cap X_{n}\right)^{\perp}$. In view of (17) we have that for all $n$,

$$
\lim _{m \rightarrow \infty}\left\|x-\Pi_{\left(N\left(Q_{m} A\right) \cap X_{n}\right)^{\perp}} x\right\| \rightarrow 0
$$

thus using (21) for $x$, we have that

$$
\forall n: \quad \lim _{m \rightarrow \infty} \bar{w}_{n, m}=0
$$

By (16), we have that the double sequence $\left\|\bar{w}_{n, m}\right\|$ is decreasing in $m$ for all $n$. Since

$$
\|x\|^{2}=\left\|\bar{w}_{n, m}\right\|^{2}+\left\|\bar{v}_{n, m}+\bar{q}_{n, m}\right\|^{2}
$$

we have that for all $n,\left\|\bar{v}_{n, m}+\bar{q}_{n, m}\right\|$ is increasing in $m$ and that $\lim _{m \rightarrow \infty} \bar{v}_{n, m}+\bar{q}_{n, m}=x^{\dagger}$. Thus for all $n$, $\sup _{m}\left\|\bar{v}_{n, m}+\bar{q}_{n, m}\right\|=x^{\dagger}$, and hence $\left\|\bar{v}_{n, m}+\bar{q}_{n, m}\right\|$ is bounded uniformly in $n, m$. Since (25) implies (33) using (30), we have a constant $C$ such that

$$
\left\|\bar{q}_{n, m}\right\| \leq C\left\|\bar{v}_{n, m}+\bar{q}_{n, m}\right\| \leq C\|x\| .
$$

Thus, $\left\|\bar{q}_{n, m}\right\|$ is uniformly bounded, it has a weakly convergent subsequence as $n, m \rightarrow \infty$, and as $q_{n, m} \in X_{n}^{\perp}$, it follows that this limit can only be 0 . By a subsequence argument we conclude that ${ }_{\mathrm{w}} \lim _{n, m \rightarrow \infty} \bar{q}_{n, m}=0$. It follows that the iterated limit ${ }_{\mathrm{w}} \lim _{n \rightarrow \infty}\left({ }_{\mathrm{w}} \lim _{m \rightarrow \infty} \bar{q}_{n, m}\right)=0$. Thus,

$$
x=\underset{n \rightarrow \infty}{\operatorname{w}} \lim _{n \rightarrow \infty}\left(\underset{m \rightarrow \infty}{\operatorname{w}} \lim _{m, m} \bar{v}_{n, m}\right) .
$$

Since each $v_{n, m}$ is in $N(A)^{\perp}$, and this space is weakly closed, all the limits are in $N(A)^{\perp}$ as well, thus $x \in N(A)^{\perp}$.

### 3.2 Conditions for local convergence

We are now interested in local convergence results, i.e., to study the question if for a given a specific element $x^{\dagger} \in N(A)^{\perp}$ the corresponding sequence $x_{n, m}$ converges (weakly or strongly). The difference to the previous section is that the conditions imposed here are not "uniform" in $x^{\dagger}$ but depend on the specific $x^{\dagger}$.

A practically useful sufficient condition for strong convergence is a simple consequence of Lemma 14 (compare Theorem (4)).

Proposition 24. If

$$
\lim _{m, n \rightarrow \infty}\left\|A_{n, m}^{\dagger}\right\|\left\|A\left(I-P_{n}\right) x^{\dagger}\right\| \rightarrow 0
$$

then $x_{n, m} \rightarrow x^{\dagger}$ as $m, n \rightarrow \infty$.
Hence if $x^{\dagger}$ can be approximated well in $X_{n}$, we can hope for strong convergence. This result is quite crude compared to Theorem where (except for weak convergence) equivalent conditions to convergence are established. However, the mentioned theorem of Groetsch and Neubauer (with the supplementary result of Du ) can also be extended with minor modifications to the case of $Q_{m} \neq I$.
Theorem 25. We have the following local convergence conditions for strong convergence.

$$
\begin{equation*}
x_{n, m} \rightarrow x^{\dagger} \Longleftrightarrow \limsup _{n, m}\left\|x_{n, m}\right\| \leq\left\|x^{\dagger}\right\| . \tag{44}
\end{equation*}
$$

Suppose that the space condition (9) holds. Then we have the following local convergence conditions for weak convergence.

$$
x_{n, m} \rightharpoonup x^{\dagger} \Longleftrightarrow \sup _{n, m}\left\|x_{n, m}\right\|<\infty
$$

Proof. Consider first the part on weak convergence. By boundedness, $x_{n, m}$ has a weakly convergent subsequence with limit $u$. As in 9 it follows immediately that $u-x^{\dagger} \in N(A)$. Moreover each $x_{n, m} \in\left(N\left(Q_{m} A\right) \cap X_{n}\right)^{\perp} \subset\left(N(A) \cap X_{n}\right)^{\perp}$, thus $x_{n, m} \in\left(N(A) \cap X_{k}\right)^{\perp}$ for all $k \geq n$. This implies that $u \in \bigcap_{n}\left(N(A) \cap X_{n}\right)^{\perp}$, and by (9), $u \in N(A)^{\perp}$. Thus $u-x^{\dagger} \in N(A) \cap N(A)^{\perp}$, hence $u=x^{\dagger}$. By a subsequence argument, $x_{n, m} \rightharpoonup x^{\dagger}$. For (44), we do not need (9). The proof follows [5]: from (44), we again find a weakly convergence subsequence with limit $u$ and $u-x^{\dagger} \in N(A)$ and $x^{\dagger} \in N(A)^{\perp}$. Thus,

$$
\left\|u-x^{\dagger}\right\|^{2}+\left\|x^{\dagger}\right\|^{2}=\|u\|^{2} \leq \liminf _{n, m}\left\|x_{n, m}\right\|^{2} \leq \limsup _{n, m}\left\|x_{n, m}\right\|^{2} \leq\left\|x^{\dagger}\right\|^{2},
$$

thus $u=x^{\dagger}$ and as before $x_{n, m} \rightharpoonup x^{\dagger}$. From (44) we also find that $\left\|x_{n, m}\right\| \rightarrow$ $\left\|x^{\dagger}\right\|$, which together with weak convergence implies strong convergence. The other directions of the implications are trivial.

In a next step, we replace (9) by other "local" conditions.
Lemma 26. We have that

$$
x_{n, m} \rightharpoonup x^{\dagger} \quad \Longleftrightarrow\left\{\begin{array}{l}
\sup _{n, m}\left\|x_{n, m}\right\|<\infty \quad \text { and }  \tag{45}\\
\Pi_{N(A)} x_{n, m} \rightharpoonup 0
\end{array}\right.
$$

Proof. Suppose that $x_{n, m}$ converges weakly to $x^{\dagger}$. Then for arbitrary $z$

$$
\begin{aligned}
\lim _{n, m \rightarrow \infty}\left(\Pi_{N(A)} x_{n, m}, z\right) & =\lim _{n, m \rightarrow \infty}\left(x_{n, m}, \Pi_{N(A)} z\right) \\
& =\lim _{n, m \rightarrow \infty}\left(x_{n, m}-x^{\dagger}, \Pi_{N(A)} z\right)=0,
\end{aligned}
$$

where we used that $\left(x^{\dagger}, \Pi_{N(A)} z\right)=0$ as $x^{\dagger} \in N(A)^{\perp}$. Conversely, let $x_{n, m}$ be bounded and $\Pi_{N(A)} x_{n, m} \longrightarrow 0$. As in the proof before, $x_{n, m}$ has a weakly convergent subsequence with limit $u$ such that $u-x^{\dagger} \in N(A)$. Thus with $z=$ $u-x^{\dagger}$ we have that

$$
\begin{aligned}
0 & =\lim _{k}\left(\Pi_{N(A)} x_{n_{k}, m_{k}}, u-x^{\dagger}\right)=\lim _{k}\left(x_{n_{k}, m_{k}}, \Pi_{N(A)}\left(u-x^{\dagger}\right)\right) \\
& =\lim _{k}\left(x_{n_{k}, m_{k}}-x^{\dagger}, \Pi_{N(A)}\left(u-x^{\dagger}\right)\right)=\left\|u-x^{\dagger}\right\|^{2},
\end{aligned}
$$

thus $u=x^{\dagger}$. By a subsequence argument $x_{n, m} \rightharpoonup x^{\dagger}$.
As a consequence, we can find sufficient conditions for weak and strong local convergence generalizing the results of Luecke and Hickey.

Proposition 27. Suppose that with some fixed constant $C$, there exists for each $n, m$ an index pair $\left(\tilde{m}_{n, m}, \tilde{n}_{n, m}\right) \in \mathbb{N} \times \mathbb{N}$ with $\lim _{n, m \rightarrow \infty} \tilde{n}_{n, m}=\infty$, and $\tilde{m}_{n, m} \geq m$, and $\tilde{n}_{n, m} \leq n$, such that

$$
\begin{equation*}
\sup _{n, m}\left\|x_{n, m}\right\|<C \quad \text { and } \quad \sup _{n, m}\left\|A^{*}\left(A_{\tilde{m}_{n, m}, \tilde{n}_{n, m}}^{*}\right)^{\dagger} x_{n, m}\right\|<C \tag{46}
\end{equation*}
$$

Then $x_{n, m} \rightharpoonup x^{\dagger}$ as $m, n \rightarrow \infty$.
If we can choose $\left(\tilde{m}_{n, m}, \tilde{n}_{n, m}\right)=(n, m)$, i.e.,

$$
\begin{equation*}
\sup _{n, m}\left\|x_{n, m}\right\|<C \quad \text { and } \quad \sup _{n, m}\left\|A^{*}\left(A_{n, m}^{*}\right)^{\dagger} x_{n, m}\right\|<C \tag{47}
\end{equation*}
$$

then $x_{n, m} \rightarrow x^{\dagger}$ as $m, n \rightarrow \infty$.
Proof. We apply (21) and get for all $n, m$

$$
x_{n, m}=A^{*}\left(A_{\tilde{m}_{n, m}, \tilde{n}_{n, m}}^{*}\right)^{\dagger} x_{n, m}+\bar{w}_{\tilde{m}_{n, m}, \tilde{n}_{n, m}}+\bar{q}_{\tilde{m}_{n, m}, \tilde{n}_{n, m}} .
$$

By (46) it follows that $\bar{w}_{\tilde{m}_{n, m}, \tilde{n}_{n, m}}+\bar{q}_{\tilde{m}_{n, m}, \tilde{n}_{n, m}}$ is uniformly bounded, and since these two elements are orthogonal to each other it follows that both components are uniformly bounded as well, hence they have weakly convergent subsequences as $n, m \rightarrow \infty$ with limit $w, q$.

For fixed $k, P_{k} q={ }_{\mathrm{w}} \lim _{n, m \rightarrow \infty} P_{k} \bar{q}_{\tilde{m}_{n, m}, \tilde{n}_{n, m}}=0$ since $\tilde{n}_{n, m} \rightarrow \infty$, and thus it follows that $q=0$. Since $x_{n, m} \in\left(N\left(Q_{\tilde{m}_{n, m}} A\right) \cap X_{n}\right)^{\perp}$, for $\tilde{m}_{n, m} \geq m$, and $\left(N\left(Q_{\tilde{m}_{n, m}} A\right) \cap X_{n}\right)^{\perp} \subset\left(N\left(Q_{\tilde{m}_{n, m}} A\right) \cap X_{\tilde{n}_{n, m}}\right)^{\perp}$, we have that $\bar{w}_{\tilde{m}_{n, m}, \tilde{n}_{n, m}}=0$. Thus we have for a subsequence

$$
\underset{k \rightarrow \infty}{\operatorname{w}} \lim \left(x_{n_{k}, m_{k}}-A^{*}\left(A_{\tilde{m}_{m_{k}, n_{k}}, \tilde{n}_{m_{k}, n_{k}}^{*}}\right)^{\dagger} x_{n_{k}, m_{k}}\right)=0
$$

By a subsequence argument we have that this holds for the whole sequence.

$$
\begin{equation*}
\lim _{n, m \rightarrow \infty}\left(x_{n, m}-A^{*}\left(A_{\tilde{m}_{n, m}, \tilde{n}_{n, m}}^{*}\right)^{\dagger} x_{n, m}\right)=0 \tag{48}
\end{equation*}
$$

It follows that

$$
\Pi_{N(A)} x_{n}=\Pi_{N(A)}\left(x_{n, m}-A^{*}\left(A_{\tilde{m}_{n, m}, \tilde{n}_{n, m}}^{*}\right)^{\dagger} x_{n, m}\right) \rightharpoonup 0, \quad \text { as } m, n \rightarrow \infty
$$

thus, by (45) we obtain the result that $x_{n} \rightharpoonup x^{\dagger}$ as $m, n \rightarrow \infty$. Since (47) is a special case of (46), we have that under (47), $x_{n, m}$ converges weakly to $x^{\dagger}$ and furthermore by (48) also that $A^{*}\left(A_{n, m}^{*}\right)^{\dagger} x_{n, m} \rightharpoonup x^{\dagger}$ as $m, n \rightarrow \infty$. Thus, by weak convergence,

$$
\lim _{n, m \rightarrow \infty}\left\|x_{n, m}\right\|^{2}=\lim _{n, m \rightarrow \infty}\left(A^{*}\left(A_{n, m}^{*}\right)^{\dagger} x_{n, m}, x^{\dagger}\right)=\left(x^{\dagger}, x^{\dagger}\right)=\left\|x^{\dagger}\right\|^{2} .
$$

By the Radon-Riesz property, we obtain that $\left\|x_{n, m}-x^{\dagger}\right\| \rightarrow 0$.
Remark 4. Proposition 27 includes Theorem 6 as a special case. Indeed, setting $Q_{m}=I$, from (11), the boundedness of $x_{n}$ and $\left\|A^{*}\left(A_{n}^{*}\right)^{\dagger} x_{n}\right\|$ follows immediately, and thus by (47) we obtain Theorem [6 as a corollary.

## 4 A counterexample

In this section we provide a nontrivial example of a sequence of projected leastsquares solutions, $x_{n}$, which is bounded but non even weakly convergent. Note that Du's example considers a similar situation but the sequence $x_{n}$ is strongly convergent (but not to $x^{\dagger}$ ). The example again stresses the importance of the space conditions (9) and the fact that the part on weak convergence in Theorem 5 is false without the space condition (9).

Theorem 28. There exists an operator $A$ and $x^{\dagger}$ and a sequence of finitedimensional spaces $\left(X_{n}\right)_{n}$ satisfying (2) such that $x_{2 n} \rightharpoonup u$ but $x_{2 n} \nrightarrow u$ and $x_{2 n+1} \rightharpoonup v$, but $x_{2 n+1} \nrightarrow v$ and $u, v \neq x^{\dagger}$. In particular the sequence $x_{n}$ neither converges weakly nor strongly and it has no weakly convergent subsequence has limit $x^{\dagger}$.

Proof. Let $X$ be a separable Hilbert space with orthonormal basis $e_{i j},(i, j) \in$ $\mathbb{N} \times \mathbb{N}$, i.e., all elements $x \in X$ may be represented via

$$
x=\sum_{i, j=1}^{\infty} \xi_{i j} e_{i j} \quad \text { with } \quad\|x\|^{2}=\sum_{i, j=1}^{\infty} \xi_{i j}^{2}<\infty .
$$

We define a linear bounded operator $A: X \rightarrow X$ via

$$
A x:=\sum_{i=1}^{\infty} \sum_{j=2}^{\infty}\left(\xi_{i j}+q^{j} \xi_{i 1}\right) e_{i j},
$$

where $q \in(0,1)$ is fixed. Obviously,

$$
\|A x\|^{2}=\sum_{i=1}^{\infty} \sum_{j=2}^{\infty}\left(\xi_{i j}+q^{j} \xi_{i 1}\right)^{2} \leq 2 \max \left\{1, \frac{q^{4}}{1-q^{2}}\right\}\|x\|^{2} .
$$

It is easy to see that $A$ has an infinite-dimensional nullspace. It holds that

$$
\begin{equation*}
x \in N(A) \Longleftrightarrow \forall i \geq 1, j>1: \xi_{i j}=-q^{j} \xi_{i 1} \text { and } \sum_{i=1}^{\infty} \xi_{i 1}^{2}<\infty \tag{49}
\end{equation*}
$$

A generalized solution $x^{\dagger}=A^{\dagger} y$ of the equation $A x=y$ is always an element of $N(A)^{\perp}$; we may characterize these elements as follows:

$$
z=\sum_{i, j=1}^{\infty} \eta_{i j} e_{i j} \in N(A)^{\perp} \Longleftrightarrow \forall i \geq 1: \begin{align*}
& \eta_{i 1}=\sum_{j=2}^{\infty} q^{j} \eta_{i j} \text { and }  \tag{50}\\
& \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} \eta_{i j}^{2}<\infty
\end{align*}
$$

Now we choose finite-dimensional subspaces of $X$ :

$$
X_{n}:=\operatorname{span}\left\{e_{i j}: 1 \leq i, j \leq n\right\} .
$$

Obviously, (2) holds. Let $x^{\dagger} \in N(A)^{\perp}$ with $x^{\dagger}=\sum_{i, j=1}^{\infty} \zeta_{i j} e_{i j} \in N(A)^{\perp}$ and $y:=A x^{\dagger}$, and set $x_{n}:=A_{n}^{\dagger} y$, where $A_{n}:=A P_{n}$.

Since, due to (49), $N(A) \cap X_{n}=\{0\}$, we get by (14) that $N\left(A_{n}\right)=X_{n}^{\perp}$. Therefore,

$$
\begin{equation*}
x_{n}=\sum_{i, j=1}^{n} \xi_{i j}^{n} e_{i j} \tag{51}
\end{equation*}
$$

is the unique minimizer in $X_{n}$ of the problem

$$
\left\|A x_{n}-y\right\|^{2}=\sum_{i=1}^{\infty} \sum_{j=2}^{\infty}\left(\left(\xi_{i j}^{n}-\zeta_{i j}\right)+q^{j}\left(\xi_{i 1}^{n}-\zeta_{i 1}\right)\right)^{2} \rightarrow \min
$$

where $\xi_{i j}^{n}:=0$ if $i>n$ or $j>n$.
From the first order necessary conditions for a minimum we obtain the solution:

$$
\xi_{i j}^{n}=\zeta_{i j}-q^{j}\left(\xi_{i 1}^{n}-\zeta_{i 1}\right), \quad \text { if } 1 \leq i \leq n, 2 \leq j \leq n
$$

and

$$
\begin{aligned}
\xi_{i 1}^{n} & =\zeta_{i 1}+\left(\sum_{j=n+1}^{\infty} q^{2 j}\right)^{-1} \sum_{j=n+1}^{\infty} q^{j} \zeta_{i j} \\
& =\zeta_{i 1}+\left(1-q^{2}\right) q^{-(n+1)} \sum_{j=0}^{\infty} q^{j} \zeta_{i, n+1+j}, \quad 1 \leq i \leq n .
\end{aligned}
$$

Now we choose a concrete element $x^{\dagger} \in N(A)^{\perp}$ :

$$
\zeta_{i j}:=\frac{q^{j}}{1-q^{2}}\left(c_{i} \rho_{j}+r_{i j}\right), \quad i \geq 1, j>1
$$

with

$$
\rho_{j}:=\left\{\begin{array}{ll}
1, & j \text { even },  \tag{52}\\
0, & j \text { odd },
\end{array} \quad r_{i j}:=\left\{\begin{array}{ll}
1, & j=i+1, \\
0, & \text { else },
\end{array} \quad \sum_{i=1}^{\infty} c_{i}^{2}<\infty\right.\right.
$$

and the extension (50) for $j=1$. The condition on the coefficients $c_{i}$ guarantees that

$$
\sum_{i=1}^{\infty} \sum_{j=2}^{\infty} \zeta_{i j}{ }^{2}<\infty
$$

It then holds that

$$
\begin{equation*}
\xi_{i 1}^{n}=\zeta_{i 1}+c_{i} e_{n}+r_{i, n+1} \quad \text { with } \quad e_{n}:=\sum_{j=0}^{\infty} q^{2 j} \rho_{n+1+j} . \tag{53}
\end{equation*}
$$

Noting that

$$
e_{n}=\frac{1}{1-q^{4}} \cdot \begin{cases}q^{2}, & n \text { even }  \tag{54}\\ 1, & n \text { odd }\end{cases}
$$

(53) implies that

$$
\begin{align*}
\lim _{l \rightarrow \infty} \xi_{i 1}^{2 l} & =\zeta_{i 1}+\frac{c_{i} q^{2}}{1-q^{4}}  \tag{55}\\
\lim _{l \rightarrow \infty} \xi_{i 1}^{2 l+1} & =\zeta_{i 1}+\frac{c_{i}}{1-q^{4}} \tag{56}
\end{align*}
$$

Let us now define the two elements

$$
u:=\sum_{i, j=1}^{\infty} u_{i j} e_{i j} \quad \text { and } \quad v:=\sum_{i, j=1}^{\infty} u_{i j} e_{i j}
$$

with

$$
\begin{array}{ll}
u_{i 1}:=\zeta_{i 1}+\frac{c_{i} q^{2}}{1-q^{4}} & u_{i j}:=\zeta_{i j}-q^{j}\left(u_{i 1}-\zeta_{i 1}\right), \\
v_{i 1}:=\zeta_{i 1}+\frac{c_{i}}{1-q^{4}} & v_{i j}:=\zeta_{i j}-q^{j}\left(v_{i 1}-\zeta_{i 1}\right),
\end{array} \quad j>1 .
$$

Obviously, due to (49) and (52), $u-x^{\dagger}$ and $v-x^{\dagger} \in N(A)$, and thus $u$ and $v$ are least-squares solutions of $A x=y=A x^{\dagger}$ with $u, v \neq x^{\dagger}$. Together with (52), (53), and (54) we immediately obtain that (remember that $x_{n}=A_{n}^{\dagger} y$ is given by (51), (531)

$$
\begin{equation*}
\left\|x_{2 l}-P_{2 l} u\right\|^{2}=\frac{q^{4}-q^{2 n+2}}{1-q^{2}}=\left\|x_{2 l+1}-P_{2 l+1} v\right\|^{2} \tag{57}
\end{equation*}
$$

Now (55) and (56) imply that

$$
x_{2 l} \rightharpoonup u, \quad x_{2 l+1} \rightharpoonup v,
$$

but (57) implies $x_{2 l} \nrightarrow u$ and $x_{2 l+1} \nrightarrow v$. Thus, it is possible that $x_{n}$ has different weakly convergent subsequences, but it neither converges weakly nor strongly towards $x^{\dagger}$.

## 5 Conclusion

We have studied global and local convergence of general projection schemes for ill-posed problems. For global convergence, we have established the uniform boundedness condition (25) as being necessary and sufficient and have found concrete conditions in Theorems 17 and 18 when this holds. Several practically useful sufficient condition were given in Section 3.1.1. Concerning local convergence, we have generalized the well-known results of Groetsch and Neubauer and

Du giving an equivalent characterization of local convergence by norm bounds in Theorem 25. Further sufficient conditions of the type of Luecke and Hickey were given in Proposition 27.

In the analysis, we point out two important findings: the recognition of the $x_{n, m}$ as oblique projection of $x^{\dagger}$, which leads to a study of angles of a sequence of subspaces. The second point is the question if the intuitive identity " $N(A)^{\perp}=\lim _{m, n} N\left(A_{n, m}\right)^{\perp "}$ is valid, understood in the sense as (24). As the inclusion " $\subset$ " always holds, this gives a way of applying the uniform boundedness principle. However it is important to notice that this identity does only hold unless the additional space condition (9) holds. While for injective operators this is trivially true, for noninjective operators (9) has to be taken into account when studying local (weak) convergence.

The issue that this condition is needed for weak convergence is illustrated by a nontrivial counterexample in Theorem 28 of a bounded sequence $x_{n}$ which does not converge at all, thus generalizing the examples of Seidman and Du.

## Acknowledgments

The author would like to thank Andreas Neubauer for useful discussions and for providing the counterexample in Theorem 28 .

## References

[1] G. Bruckner and S. Pereverzev, Self-regularization of projection methods with a posteriori discretization level choice for severely ill-posed problems, Inverse Problems, 19 (2003), pp. 147-156.
[2] G. Corach and A. Maestripieri, Redundant decompositions, angles between subspaces and oblique projections, Publ. Mat., 54 (2010), pp. 461484.
[3] W. Dahmen and M. Jürgens, Error controlled regularization by projection, Electron. Trans. Numer. Anal., 25 (2006), pp. 67-100.
[4] F. Deutsch, The angle between subspaces of a Hilbert space, in Approximation theory, wavelets and applications (Maratea, 1994), vol. 454 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Kluwer Acad. Publ., Dordrecht, 1995, pp. 107-130.
[5] N. Du, Finite-dimensional approximation settings for infinite-dimensional Moore-Penrose inverses, SIAM J. Numer. Anal., 46 (2008), pp. 1454-1482.
[6] S. Du and N. Du, On the two mutally independent factors that determine the convergence of least-squares projection method. Preprint on arXiv, arXiv:1406.0578v2, 24 pages, 2014.
[7] H. W. Engl, M. Hanke, and A. Neubauer, Regularization of inverse problems, vol. 375 of Mathematics and its Applications, Kluwer Academic Publishers Group, Dordrecht, 1996.
[8] C. W. Groetsch and M. Hanke, Regularization by projection for unbounded operators arising in inverse problems, in Inverse problems and applications to geophysics, industry, medicine and technology (Ho Chi Minh City, 1995), vol. 2 of Publ. HoChiMinh City Math. Soc., HoChiMinh City Math. Soc., Ho Chi Minh City, 1995, pp. 61-70.
[9] C. W. Groetsch and A. Neubauer, Convergence of a general projection method for an operator equation of the first kind, Houston J. Math., 14 (1988), pp. 201-208.
[10] P. R. Halmos, Introduction to Hilbert Space and the theory of Spectral Multiplicity, Chelsea Publishing Company, New York, N. Y., 1951.
[11] U. Hämarik, On the self-regularization by solving ill-posed problems by projection methods, Tartu Riikl. Ül. Toimetised, (1990), pp. 65-72.
[12] U. Hämarik, E. Avi, and A. Ganina, On the solution of ill-posed problems by projection methods with a posteriori choice of the discretization level, Math. Model. Anal., 7 (2002), pp. 241-252.
[13] B. Hofmann, P. Mathé, and S. V. Pereverzev, Regularization by projection: approximation theoretic aspects and distance functions, J. Inverse Ill-Posed Probl., 15 (2007), pp. 527-545.
[14] U. HÄmarik, Projection methods for regularization of linear ill-posed problems, Trudy Vychisl. Tsentra Tartu. Gos. Univ., (1983), pp. 69-90.
[15] __, Self-regularization in solving ill-posed problems by projection methods, Tartu Riikl. Ül. Toimetised, (1988), pp. 91-96.
[16] B. Kaltenbacher, Regularization by projection with a posteriori discretization level choice for linear and nonlinear ill-posed problems, Inverse Problems, 16 (2000), pp. 1523-1539.
[17] __ On the regularizing properties of a full multigrid method for ill-posed problems, Inverse Problems, 17 (2001), pp. 767-788.
[18] _,,$~ V$-cycle convergence of some multigrid methods for ill-posed problems, Math. Comp., 72 (2003), pp. 1711-1730.
[19] B. Kaltenbacher and J. Offtermatt, A convergence analysis of regularization by discretization in preimage space, Math. Comp., 81 (2012), pp. 2049-2069.
[20] A. Kirsch, An introduction to the mathematical theory of inverse problems, vol. 120 of Applied Mathematical Sciences, Springer, New York, second ed., 2011.
[21] G. R. Luecke and K. R. Hickey, Convergence of approximate solutions of an operator equation, Houston J. Math., 11 (1985), pp. 345-354.
[22] P. Maass, S. V. Pereverzev, R. Ramlau, and S. G. Solodky, An adaptive discretization for Tikhonov-Phillips regularization with a posteriori parameter selection, Numer. Math., 87 (2001), pp. 485-502.
[23] P. Mathé and S. V. Pereverzev, Optimal discretization of inverse problems in Hilbert scales. Regularization and self-regularization of projection methods, SIAM J. Numer. Anal., 38 (2001), pp. 1999-2021.
[24] P. Mathé and N. Schöne, Regularization by projection in variable Hilbert scales, Appl. Anal., 87 (2008), pp. 201-219.
[25] S. Morigi, L. Reichel, and F. Sgallari, Orthogonal projection regularization operators, Numer. Algorithms, 44 (2007), pp. 99-114.
[26] M. Z. Nashed, Perturbations and approximations for generalized inverses and linear operator equations, in Generalized inverses and applications (Proc. Sem., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1973), Academic Press, New York, 1976, pp. 325-396. Publ. Math. Res. Center Univ. Wisconsin, No. 32.
[27] F. Natterer, Regularisierung schlecht gestellter Probleme durch Projektionsverfahren, Numer. Math., 28 (1977), pp. 329-341.
[28] A. Neubauer, Personal communication.
[29] J. Nitsche, Zur Konvergenz von Näherungsverfahren bezüglich verschiedener Normen, Numer. Math., 15 (1970), pp. 224-228.
[30] R. Plato and G. Vainikko, On the regularization of projection methods for solving ill-posed problems, Numer. Math., 57 (1990), pp. 63-79.
[31] T. Regińska, Two-parameter discrepancy principle for combined projection and Tikhonov regularization of ill-posed problems, J. Inverse Ill-Posed Probl., 21 (2013), pp. 561-577.
[32] T. I. Seidman, Nonconvergence results for the application of least-squares estimation to ill-posed problems, J. Optim. Theory Appl., 30 (1980), pp. 535-547.
[33] D. B. Szyld, The many proofs of an identity on the norm of oblique projections, Numer. Algorithms, 42 (2006), pp. 309-323.
[34] A. N. Tikhonov and V. Y. Arsenin, Solutions of ill-posed problems, V. H. Winston \& Sons, Washington, D.C.: John Wiley \& Sons, New YorkToronto, Ont.-London, 1977.
[35] G. M. Vaĭnikko and U. A. Hämarik, Projection methods and selfregularization in ill-posed problems, Izv. Vyssh. Uchebn. Zaved. Mat., (1985), pp. 3-17, 84.


[^0]:    *Industrial Mathematics Institute, Johannes Kepler University Linz, Alternbergergstraße 69, 4040 Linz (kindermann@indmath.uni-linz.ac.at).

