

A difference method of solving the Steklov nonlocal boundary value problem of the second kind for the time-fractional diffusion equation

Anatoly A. Alikhanov

e-mail: aaalikhanov@gmail.com

We consider difference schemes for the time-fractional diffusion equation with variable coefficients and nonlocal boundary conditions containing real parameters α , β and γ . By the method of energy inequalities, for the solution of the difference problem, we obtain a priori estimates, which imply the stability and convergence of these difference schemes. The obtained results are supported by the numerical calculations carried out for some test problems.

1. Introduction. Consider the nonlocal boundary value problem

$$\partial_{0t}^\nu u = \frac{\partial}{\partial x} \left(k(x, t) \frac{\partial u}{\partial x} \right) + f(x, t), \quad 0 < x < 1, \quad 0 < t \leq T, \quad (1)$$

$$\begin{cases} u(0, t) = \alpha u(1, t), \\ k(1, t) u_x(1, t) = \beta k(0, t) u_x(0, t) + \gamma u(1, t) + \mu(t), \end{cases} \quad 0 \leq t \leq T, \quad (2)$$

$$u(x, 0) = u_0(x), \quad 0 \leq x \leq 1, \quad (3)$$

where $k(x, t)$ and $f(x, t)$ are given sufficiently smooth functions, $0 < c_1 \leq k(x, t) \leq c_2$, $k(x, t) = k(1 - x, t)$ for all $(x, t) \in [0, 1] \times [0, T]$; α , β , γ are real numbers; $\mu(t) \in C[0, T]$; $\partial_{0t}^\nu u(x, t) = \int_0^t u_\tau(x, \tau) (t - \tau)^{-\nu} d\tau / \Gamma(1 - \nu)$ is a Caputo fractional derivative of order ν , $0 < \nu < 1$.

We introduce the space grid $\bar{\omega}_h = \{x_i = ih\}_{i=0}^N$, and the time grid $\bar{\omega}_\tau = \{t_n = n\tau\}_{n=0}^{N_T}$ with increments $h = 1/N$ and $\tau = T/N_T$. Set $a_i^n = k(x_i - 0.5h, t_n + \sigma\tau)$, $\varphi_i^n = f(x_i, t_n + \sigma\tau)$, $y_i^n = y(x_i, t_n)$, $y_{\bar{x},i}^n = (y_i^n - y_{i-1}^n)/h$, $y_{x,i}^n = (y_{i+1}^n - y_i^n)/h$, $(ay_{\bar{x}})_{x,i} = (a_{i+1}y_{i+1} - (a_{i+1} + a_i)y_i + a_i y_{i-1})/h^2$, $y_{t,i} = (y_i^{n+1} - y_i^n)/\tau$, $y_i^{(\sigma)} = \sigma y_i^{n+1} + (1 - \sigma)y_i^n$, $\sigma = 1 - \nu/2$.

Let us approximate the Caputo fractional derivative of order $\nu \in (0, 1)$ by the $L2-1_\sigma$ formula [1]:

$$\Delta_{0t_n+\sigma}^\nu y_i = \frac{\tau^{1-\nu}}{\Gamma(2-\nu)} \sum_{s=0}^n c_{n-s}^{(\nu,\sigma)} y_{t,i}^s,$$

where

$$a_0^{(\nu,\sigma)} = \sigma^{1-\nu}, \quad a_l^{(\nu,\sigma)} = (l + \sigma)^{1-\nu} - (l - 1 + \sigma)^{1-\nu},$$

$$b_l^{(\nu,\sigma)} = \frac{1}{2-\nu} [(l + \sigma)^{2-\nu} - (l - 1 + \sigma)^{2-\nu}] - \frac{1}{2} [(l + \sigma)^{1-\nu} + (l - 1 + \sigma)^{1-\nu}], \quad l \geq 1;$$

$c_0^{(\nu,\sigma)} = a_0^{(\nu,\sigma)}$, for $n = 0$; and for $n \geq 1$,

$$c_s^{(\nu,\sigma)} = \begin{cases} a_0^{(\nu,\sigma)} + b_1^{(\nu,\sigma)}, & s = 0, \\ a_s^{(\nu,\sigma)} + b_{s+1}^{(\nu,\sigma)} - b_s^{(\nu,\sigma)}, & 1 \leq s \leq n-1, \\ a_n^{(\nu,\sigma)} - b_n^{(\nu,\sigma)}, & s = n. \end{cases} \quad (4)$$

Lemma 1. [1] For any $\nu \in (0, 1)$ and $u(t) \in C^3[0, t_{n+1}]$

$$|\partial_{0t_{n+\sigma}}^\nu u - \Delta_{0t_{n+\sigma}}^\nu u| = O(\tau^{3-\nu}). \quad (5)$$

Consider the scheme

$$\Delta_{0t_{n+\sigma}}^\nu y_i - (ay_{\bar{x}}^{(\sigma)})_{x,i} = \varphi_i^n, \quad i = 1, 2, \dots, N-1, \quad (6)$$

$$\begin{cases} y_0^{n+1} - \alpha y_N^{n+1} = 0, \\ \beta \Delta_{0t_{n+\sigma}}^\nu y_0^n + \Delta_{0t_{n+\sigma}}^\nu y_N^n + \frac{2}{h} \left(a_N^n y_{\bar{x},N}^{(\sigma)} - \beta a_1^n y_{x,0}^{(\sigma)} - \gamma y_N^{(\sigma)} \right) = \frac{2}{h} \mu(t_{n+\sigma}) + \varphi_N^n + \beta \varphi_0^n, \end{cases} \quad (7)$$

$$y_i^0 = u_0(x_i). \quad (8)$$

The difference scheme (6)–(8) has approximation order $O(\tau^2 + h^2)$ [1, 2].

The nonlocal boundary value problem with the boundary conditions $u(b, t) = \rho u(a, t)$, $u_x(b, t) = \sigma u_x(a, t) + \tau u(a, t)$ for the simplest equations of mathematical physics, referred to as conditions of the second class, was studied in the monograph [3]. Results in the case in which $\rho\sigma - 1 = 0$ and $\rho\tau \leq 0$ were obtained there. Difference schemes for problem (1)–(3) with $\alpha = \beta$, $\gamma = 0$ and $\nu = 1$ (the classical diffusion equation) were studied in [4]. In this case, the operator occurring in the elliptic part is self-adjoint. Self-adjointness permits one to use general theorems on the stability of two-layer difference schemes in energy spaces and consider difference schemes for equations with variable coefficients. Stability criteria for difference schemes for the heat equation with nonlocal boundary conditions were studied in [5, 6, 7, 8, 9]. The difference schemes considered in these papers have the specific feature that the corresponding difference operators are not self-adjoint. The method of energy inequalities was developed in [10, 11, 12] for the derivation of a priori estimates for solutions of difference schemes for the classical diffusion equation with variable coefficients in the case of nonlocal boundary conditions. Using the energy inequality method, a priori estimates for the solution of the Dirichlet and Robin boundary value problems for the fractional, variable and distributed order diffusion equation with Caputo fractional derivative have been obtained [1, 13, 14, 15]. A priori estimates for the difference problems analyzed in [16] by using the maximum principle imply the stability and convergence of these difference schemes.

The method proposed in this paper requires symmetry of the coefficient: $k(x, t) = k(1-x, t)$. In the case $\nu = 1$, $\alpha = 0$, $\beta = 1$, $\gamma = 0$ and the symmetric coefficients, the stability and convergence of the difference schemes in the mesh C-norm have been

proved [17]. A priori estimates for the solution of the Steklov nonlocal boundary value problem of the second kind for the simplest differential equations of mathematical physics have been obtained [18].

In the present paper, a difference scheme of the second approximation order for all $\nu \in (0, 1)$ is constructed. A priori estimates for the solutions of differential as well as difference problems are obtained. A theorem stating that the corresponding difference scheme converges with the rate equal to the order of the approximation error is proved. The obtained results are supported by numerical calculations carried out for some test problems.

2. A priori estimate for the differential problem

Lemma 2. [14] For any function $v(t)$ absolutely continuous on $[0, T]$ the following equality takes place:

$$v(t)\partial_{0t}^\nu v(t) = \frac{1}{2}\partial_{0t}^\nu v^2(t) + \frac{\nu}{2\Gamma(1-\nu)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\nu}} \left(\int_0^\xi \frac{v'(\eta)d\eta}{(t-\eta)^\nu} \right)^2, \quad (9)$$

where $0 < \nu < 1$.

Theorem 1. If the conditions $\alpha = \beta \neq 1$, $\gamma \leq 0$ are satisfied, then the solution of problem (1)–(3) satisfies the estimate

$$\|u\|_0^2 + D_{0t}^{-\nu}\|u_x\|_0^2 \leq M (D_{0t}^{-\nu}\|f(x, t)\|_0^2 + D_{0t}^{-\nu}\mu^2(t) + \|u_0(x)\|_0^2), \quad (10)$$

where $\|u\|_0^2 = \int_0^1 u^2(x, t)dx$, $D_{0t}^{-\nu}u(x, t) = \int_0^t (t-s)^{\nu-1}u(x, s)ds/\Gamma(\nu)$ is the fractional Riemann–Liouville integral of order ν , $M > 0$ is a known constant independent of T .

Proof. Let us multiply (1) by $u(x, t)$ and integrate the resulting relation over x from 0 to 1:

$$\int_0^1 u(x, t)\partial_{0t}^\gamma u(x, t)dx - \int_0^1 (k(x, t)u_x(x, t))_x u(x, t)dx = \int_0^1 u(x, t)f(x, t)dx. \quad (11)$$

This, together with the nonlocal boundary conditions (2) and equality (9), implies the relation

$$\begin{aligned} & \frac{1}{2}\partial_{0t}^\gamma \int_0^1 u^2(x, t)dx + \frac{\gamma}{2\Gamma(1-\gamma)} \int_0^1 dx \int_0^t \frac{d\xi}{(t-\xi)^{1-\gamma}} \left(\int_0^\xi \frac{\partial u}{\partial \eta}(x, \eta)d\eta \right)^2 + \\ & + \int_0^1 k(x, t)u_x^2(x, t)dx = \int_0^1 u(x, t)f(x, t)dx + \gamma u^2(1, t) + u(1, t)\mu(t). \end{aligned} \quad (12)$$

Since $\alpha \neq 1$ than

$$u^2(1, t) = \left(\frac{1}{1 - \alpha} \int_0^1 u_x(x, t) dx \right)^2 \leq \frac{1}{(1 - \alpha)^2} \|u_x\|_0^2.$$

Let us estimate the values of $\int_0^1 u(x, t) f(x, t) dx$ and $u(1, t) \mu(t)$. Since $u(x, t) = u(1, t) - \int_x^1 u_s(s, t) ds$, one has

$$\begin{aligned} \int_0^1 u(x, t) f(x, t) dx &= \int_0^1 f(x, t) \left(u(1, t) - \int_x^1 u_s(s, t) ds \right) dx = \\ &= u(1, t) \int_0^1 f(x, t) dx - \int_0^1 u_x(x, t) dx \int_0^x f(s, t) ds \leq \\ &\leq \frac{\varepsilon_1}{2} u^2(1, t) + \frac{1}{2\varepsilon_1} \int_0^1 f^2(x, t) dx + \int_0^1 |u_x(x, t)| dx \int_0^1 |f(s, t)| ds \leq \\ &\leq \frac{\varepsilon_1}{2} u^2(1, t) + \frac{1}{2\varepsilon_1} \int_0^1 f^2(x, t) dx + \frac{\varepsilon_1}{2} \int_0^1 u_x^2(x, t) dx + \frac{1}{2\varepsilon_1} \int_0^1 f^2(x, t) dx \leq \\ &\leq \varepsilon_1 \left(\frac{1}{2(1 - \alpha)^2} + \frac{1}{2} \right) \|u_x\|_0^2 + \frac{1}{\varepsilon_1} \|f\|_0^2, \end{aligned}$$

$$u(1, t) \mu(t) \leq \frac{\varepsilon_1}{2} u^2(1, t) + \frac{1}{2\varepsilon_1} \mu^2(t) \leq \frac{\varepsilon_1}{2(1 - \alpha)^2} \|u_x\|_0^2 + \frac{1}{2\varepsilon_1} \mu^2(t), \quad \varepsilon_1 > 0.$$

Taking into account these inequalities, from (12) one finds that

$$\begin{aligned} &\frac{1}{2} \partial_{0t}^\nu \|u\|_0^2 + c_1 \|u_x\|_0^2 \leq \\ &\leq \varepsilon_1 \left(\frac{1}{(1 - \alpha)^2} + \frac{1}{2} \right) \|u_x\|_0^2 + \gamma u^2(1, t) + \frac{1}{\varepsilon_1} \|f\|_0^2 + \frac{1}{2\varepsilon_1} \mu^2(t). \end{aligned} \quad (13)$$

By applying the fractional differentiation operator $D_{0t}^{-\nu}$ to both sides of inequality (13) at $\varepsilon_1 = c_1(1 - \alpha)^2(2 + (1 - \alpha)^2)^{-1}$, we obtain the inequality (10) with constant $M = (2c_1)^{-1}(2 + (1 - \alpha)^2)(1 - \alpha)^{-2} / \min\{1, c_1\}$. The proof of Theorem 1 is complete.

Let $u(x, t)$ is the solution of problem (1)–(3), then the function $v(x, t) = \delta u(x, t) + u(1 - x, t)$, at $\delta \neq \pm 1, -\alpha, \beta$, be the solution of the following problem:

$$\partial_{0t}^\nu v = \frac{\partial}{\partial x} \left(k(x, t) \frac{\partial v}{\partial x} \right) + f_1(x, t), \quad 0 < x < 1, \quad 0 < t \leq T, \quad (14)$$

$$\begin{cases} v(0, t) = \alpha_1 v(1, t), \\ k(1, t)v_x(1, t) = \beta_1 k(0, t)v_x(0, t) + \gamma_1 v(1, t) + \mu_1(t), \quad 0 \leq t \leq T, \end{cases} \quad (15)$$

$$v(x, 0) = v_0(x), \quad 0 \leq x \leq 1, \quad (16)$$

where

$$\alpha_1 = \frac{\delta\alpha + 1}{\delta + \alpha}, \quad \beta_1 = \frac{\delta\beta - 1}{\delta - \beta}, \quad \gamma_1 = \frac{\gamma(\delta^2 - 1)}{(\delta + \alpha)(\delta - \beta)}, \quad \mu_1(t) = \frac{\delta^2 - 1}{\delta - \beta} \mu(t),$$

$$f_1(x, t) = \delta f(x, t) + f(1 - x, t), \quad v_0(x) = \delta u_0(x) + u_0(1 - x).$$

Let us find such a value of δ that for the problem (14)–(16) the conditions of the Theorem 1 are fulfilled. The condition $\alpha_1 = \beta_1$ leads to a quadratic equation:

$$\delta^2 - 2\frac{\alpha\beta - 1}{\alpha - \beta}\delta + 1 = 0,$$

which, at $(\alpha^2 - 1)(\beta^2 - 1) > 0$, has two real roots

$$\delta_1 = \frac{\alpha\beta - 1 - \sqrt{(\alpha^2 - 1)(\beta^2 - 1)}}{\alpha - \beta}, \quad \delta_2 = \frac{\alpha\beta - 1 + \sqrt{(\alpha^2 - 1)(\beta^2 - 1)}}{\alpha - \beta}.$$

At $\alpha^2 - 1 < 0$ and $\beta^2 - 1 < 0$ let us take $\delta = \delta_1$, but at $\alpha^2 - 1 > 0$ and $\beta^2 - 1 > 0$ we take $\delta = \delta_2$. This will guarantee the fulfillment of the condition $\delta \neq -\alpha, \beta$.

Let us consider these two cases:

1) $\alpha^2 - 1 < 0$, $\beta^2 - 1 < 0$ и $\delta = \delta_1$. The second condition of the Theorem 1 leads to

$$\frac{\gamma(\delta^2 - 1)}{(\delta + \alpha)(\delta - \beta)} \leq 0,$$

which at $\delta = \delta_1$ takes the form

$$\gamma \frac{\left(\left(\sqrt{1 - \alpha^2} + \sqrt{1 - \beta^2} \right)^2 + (\alpha - \beta)^2 \right)}{\left(\sqrt{1 - \alpha^2} + \sqrt{1 - \beta^2} \right)^2} \leq 0$$

and equal to $\gamma \leq 0$ at $|\alpha| < 1$, $|\beta| < 1$.

2) $\alpha^2 - 1 > 0$, $\beta^2 - 1 > 0$ и $\delta = \delta_2$. In this case the inequality $\gamma_1 \leq 0$ reads

$$\gamma \frac{\left(\left(\sqrt{\alpha^2 - 1} + \sqrt{\beta^2 - 1} \right)^2 - (\alpha - \beta)^2 \right)}{\left(\sqrt{\alpha^2 - 1} + \sqrt{\beta^2 - 1} \right)^2} \leq 0$$

and equivalent to $\alpha\beta\gamma \leq 0$ at $|\alpha| > 1$, $|\beta| > 1$.

Theorem 2. If

1) $|\alpha| < 1$, $|\beta| < 1$ and $\gamma \leq 0$; or **2)** $|\alpha| > 1$, $|\beta| > 1$ and $\alpha\beta\gamma \leq 0$, than for the solution of the problem (1)–(3) a priori (10) is valid.

Proof. At the mentioned conditions, the conditions of the Theorem 1 for the problem (14)–(16) are fulfilled. Therefore, for its solution the a priori estimate is valid

$$\|v\|_0^2 + D_{0t}^{-\nu}\|v_x\|_0^2 \leq M_1 (D_{0t}^{-\nu}\|f_1(x, t)\|_0^2 + D_{0t}^{-\nu}\mu_1^2(t) + \|v_0(x)\|_0^2), \quad (17)$$

where $M_1 > 0$ is a known number independent on T .

Since $v(x, t) = \delta u(x, t) + u(1 - x, t)$, $f_1(x, t) = \delta f(x, t) + f(1 - x, t)$, $v_0(x) = \delta u_0(x) + u_0(1 - x)$, $\mu_1(t) = (\delta^2 - 1)(\delta - \beta)^{-1}\mu(t)$, then

$$u(x, t) = \frac{\delta}{\delta^2 - 1}v(x, t) - \frac{1}{\delta^2 - 1}v(1 - x, t), \quad \|u\|_0^2 \leq \frac{2(\delta^2 + 1)}{(\delta^2 - 1)^2}\|v\|_0^2,$$

$$u_x(x, t) = \frac{\delta}{\delta^2 - 1}v_x(x, t) + \frac{1}{\delta^2 - 1}v_x(1 - x, t), \quad \|u_x\|_0^2 \leq \frac{2(\delta^2 + 1)}{(\delta^2 - 1)^2}\|v_x\|_0^2,$$

$$\|f_1\|_0^2 \leq 2(\delta^2 + 1)\|f\|_0^2, \quad \|v_0(x)\|_0^2 \leq 2(\delta^2 + 1)\|u_0(x)\|_0^2.$$

From (17), taking into account these inequalities with $\delta = \delta_1$ for the first case and $\delta = \delta_2$ for second one, we obtain the a priori estimate (10).

The proof of the Theorem 2 is complete.

3. A priori estimate for the difference problem.

Lemma 3. [1] For any function $y(t)$ defined on the grid $\bar{\omega}_\tau$ one has the equality

$$y^{(\sigma)}\Delta_{0t_{n+\sigma}}^\nu y \geq \frac{1}{2}\Delta_{0t}^\nu(y^2) \quad (18)$$

Theorem 3. If $\alpha = \beta \neq 1$ and $\gamma \leq 0$, then the difference scheme (6)–(8) is absolutely stable and its solution satisfies the following a priori estimate:

$$\|y^{n+1}\|_0^2 \leq \|y^0\|_0^2 + M \max_{0 \leq n \leq N_T-1} (\|[\varphi^{n+1}]\|_0^2 + \mu^2(t_{n+\sigma})), \quad (19)$$

where $\|y\|_0^2 = \sum_{i=0}^N y_i^2 h$, $M > 0$ is a known number independent of h , τ and T .

Proof. Taking the inner product of the equation (6) with $y^{(\sigma)}$, we have

$$(y^{(\sigma)}, \Delta_{0t_{n+\sigma}}^\nu y) - (y^{(\sigma)}, (ay_{\bar{x}}^{(\sigma)})_x) = (y^{(\sigma)}, \varphi^{n+1}), \quad (20)$$

where $(y, v) = \sum_{i=1}^{N-1} y_i v_i h$.

Using inequality (18) and Green's first difference formula, we get

$$\begin{aligned} & \frac{1}{2}\Delta_{0t_{n+\sigma}}^\nu \|y\|_0^2 + c_1 \|y_{\bar{x}}^{(\sigma)}\|_0^2 - \gamma (y_N^{(\sigma)})^2 \leq \\ & \leq y_N^{(\sigma)} \mu(t_{n+\sigma}) + \frac{h}{2} (\varphi_N + \beta \varphi_0) y_N^{(\sigma)} + (y^{(\sigma)}, \varphi^{n+1}). \end{aligned} \quad (21)$$

Since $\alpha \neq 1$ than

$$(y_N^{(\sigma)})^2 = \left(\frac{1}{1-\alpha} \sum_{i=1}^N y_{\bar{x},i}^{(\sigma)} h \right)^2 \leq \frac{1}{(1-\alpha)^2} \|y_{\bar{x}}^{(\sigma)}\|_0^2.$$

Let us estimate the values of $(\varphi, y^{(\sigma)})$ and $y_N^{(\sigma)} \tilde{\mu}(t_{n+\sigma})$, where $\tilde{\mu}(t_{n+\sigma}) = \mu(t_{n+\sigma}) + (\varphi_N + \beta\varphi_0)h/2$. Since $y_i^{(\sigma)} = y_N^{(\sigma)} - \sum_{s=i+1}^N y_{\bar{x},s}^{(\sigma)} h$, $i = 0, 1, \dots, N-1$, one has

$$\begin{aligned} (\varphi, y^{(\sigma)}) &= \sum_{i=1}^{N-1} \varphi_i h \left(y_N^{(\sigma)} - \sum_{s=i+1}^N y_{\bar{x},s}^{(\sigma)} h \right) = y_N^{(\sigma)} \sum_{i=1}^{N-1} \varphi_i h - \sum_{i=1}^{N-1} \varphi_i h \sum_{s=i+1}^N y_{\bar{x},s}^{(\sigma)} h \leq \\ &\leq |y_N^{(\sigma)}| \sum_{i=1}^{N-1} |\varphi_i| h + \sum_{i=1}^{N-1} |\varphi_i| h \sum_{i=1}^N |y_{\bar{x},i}^{(\sigma)}| h \leq \left(\frac{1}{|1-\alpha|} + 1 \right) \sum_{i=1}^N |y_{\bar{x},i}^{(\sigma)}| h \sum_{i=1}^{N-1} |\varphi_i| h \leq \\ &\leq \left(\frac{1}{|1-\alpha|} + 1 \right) \|y_{\bar{x}}^{(\sigma)}\|_0 \|\varphi\|_0 \leq \frac{\varepsilon_1}{2} \|y_{\bar{x}}^{(\sigma)}\|_0^2 + \left(\frac{1}{|1-\alpha|} + 1 \right)^2 \frac{1}{2\varepsilon_1} \|\varphi\|_0^2, \\ y_N^{(\sigma)} \tilde{\mu}(t_{n+\sigma}) &\leq \frac{\varepsilon_1(1-\alpha)^2}{2} (y_N^{(\sigma)})^2 + \frac{1}{2\varepsilon_1(1-\alpha)^2} \tilde{\mu}^2(t_{n+\sigma}) \leq \\ &\leq \frac{\varepsilon_1}{2} \|y_{\bar{x}}^{(\sigma)}\|_0^2 + \frac{1}{2\varepsilon_1(1-\alpha)^2} \tilde{\mu}^2(t_{n+\sigma}), \quad \varepsilon_1 > 0. \end{aligned}$$

Taking into account these inequalities, from (21), at $\varepsilon_1 = c_1$, one finds that

$$\Delta_{0t_{n+\sigma}}^\nu (|[y]|_0^2) \leq M_2 (|\varphi|_0^2 + \mu^2(t_{n+\sigma})), \quad (22)$$

where $M_2 > 0$ is a known number independent of h , τ and T .

Let us rewrite inequality (22) in the form

$$g_n^{n+1} |[y^{n+1}]|_0^2 \leq \sum_{s=1}^n (g_s^{n+1} - g_{s-1}^{n+1}) |[y^s]|_0^2 + g_0^{n+1} |[y^0]|_0^2 + M_2(\varepsilon_1) (|\varphi|_0^2 + \mu^2(t_{n+\sigma})), \quad (23)$$

where

$$g_s^{n+1} = \frac{c_{n-s}^{(\alpha,\beta)}}{\tau^\alpha \Gamma(2-\alpha)}, \quad 0 \leq s \leq n \leq N_T - 1.$$

Noticing that [1]

$$g_0^{n+1} = \frac{c_n^{(\alpha,\beta)}}{\tau^\alpha \Gamma(2-\alpha)} > \frac{1}{2t_{n+\sigma}^\alpha \Gamma(1-\alpha)} > \frac{1}{2T^\alpha \Gamma(1-\alpha)},$$

we get

$$g_n^{n+1} |[y^{n+1}]|_0^2 \leq \sum_{s=1}^n (g_s^{n+1} - g_{s-1}^{n+1}) |[y^s]|_0^2 + g_0^{n+1} E, \quad (24)$$

where

$$E = \|y^0\|_0^2 + 2T^\alpha \Gamma(1 - \alpha) M_2 \max_{0 \leq n \leq N_T - 1} (\|\varphi^{n+1}\|_0^2 + \mu^2(t_{n+\sigma})).$$

It is obvious that at $n = 0$ the a priori estimate (19) follows from (24). Let us prove that (19) holds for $n = 1, 2, \dots$ by using the mathematical induction method. For this purpose, let us assume that the a priori estimate (19) takes place for all $n = 0, 1, \dots, k - 1$:

$$\|y^{n+1}\|_0^2 \leq E, \quad n = 0, 1, \dots, k - 1.$$

From (24) at $n = k$ one has

$$\begin{aligned} g_k^{k+1} \|y^{k+1}\|_0^2 &\leq \sum_{s=1}^k (g_s^{k+1} - g_{s-1}^{k+1}) \|y^s\|_0^2 + g_0^{k+1} E \leq \\ &\leq \sum_{s=1}^k (g_s^{k+1} - g_{s-1}^{k+1}) E + g_0^{k+1} E = g_k^{k+1} E. \end{aligned} \quad (25)$$

The proof of Theorem 3 is complete.

Let y_i^n is the solution of problem (6)–(8), then the function $v_i^n = \delta y_i^n + y_{N-i}^n$, at $\delta \neq \pm 1, -\alpha, \beta$, be the solution of the following problem:

$$\Delta_{0t_n}^\nu v_i - (av_{\bar{x}}^{(\sigma)})_{x,i} = \tilde{\varphi}_i^n, \quad i = 1, 2, \dots, N - 1, \quad (26)$$

$$\begin{cases} v_0^{n+1} - \alpha_1 v_N^{n+1} = 0, \\ \beta_1 \Delta_{0t_n}^\nu v_0 + \Delta_{0t_n}^\nu v_N + \frac{2}{h} \left(a_N v_{\bar{x},N}^{(\sigma)} - \beta_1 a_1 v_{x,0}^{(\sigma)} - \gamma_1 v_N^{(\sigma)} \right) = \frac{2}{h} \mu_1(t_{n+1/2}) + \tilde{\varphi}_N + \beta_1 \tilde{\varphi}_0, \end{cases} \quad (27)$$

$$v_i^0 = v_0(x_i), \quad (28)$$

where

$$\alpha_1 = \frac{\delta\alpha + 1}{\delta + \alpha}, \quad \beta_1 = \frac{\delta\beta - 1}{\delta - \beta}, \quad \gamma_1 = \frac{\gamma(\delta^2 - 1)}{(\delta + \alpha)(\delta - \beta)}, \quad \mu_1(t) = \frac{\delta^2 - 1}{\delta - \beta} \mu(t),$$

$$\tilde{\varphi}_i^n = \delta \varphi_i^n + \varphi_{N-i}^n, \quad v_0(x_i) = \delta u_0(x_i) + u_0(1 - x_i).$$

Similarly of the differential problem, at $|\alpha| < 1, |\beta| < 1$ and $\gamma \leq 0$ let us take $\delta = \delta_1$, but at $|\alpha| > 1, |\beta| > 1$ and $\alpha\beta\gamma \leq 0$ we take $\delta = \delta_2$. This will guarantee the fulfillment of the conditions $\alpha_1 = \beta_1 \neq 1, \gamma_1 \leq 0$ and $\delta \neq -\alpha, \beta$ for the problem (26)–(28).

Theorem 4. If

1) $|\alpha| < 1, |\beta| < 1$ and $\gamma \leq 0$; or **2)** $|\alpha| > 1, |\beta| > 1$ and $\alpha\beta\gamma \leq 0$, then the difference scheme (6)–(8) is absolutely stable and its solution satisfies the following a priori estimate:

$$\|y^{n+1}\|_0^2 \leq M_3 \left(\|y^0\|_0^2 + \max_{0 \leq n \leq N_T - 1} (\|\varphi^{n+1}\|_0^2 + \mu^2(t_{n+\sigma})) \right), \quad (29)$$

where $M_3 > 0$ is a known number independent of h , τ and T .

Proof. At the mentioned conditions, the conditions of the Theorem 3 for the problem (26)–(28) are fulfilled. Therefore, for its solution the a priori estimate is valid

$$\|v^{n+1}\|_0^2 \leq \|v^0\|_0^2 + M \max_{0 \leq n \leq N_T-1} (\|\varphi^{n+1}\|_0^2 + \mu^2(t_{n+\sigma})). \quad (30)$$

Since $v_i^n = \delta y_i^n + y_{N-i}^n$, $\tilde{\varphi}_i^n = \delta \varphi_i^n + \varphi_{N-i}^n$, $v_0(x_i) = \delta u_0(x_i) + u_0(1 - x_i)$, $\mu_1(t) = (\delta^2 - 1)(\delta - \beta)^{-1}\mu(t)$, then

$$y_i^n = \frac{\delta}{\delta^2 - 1} v_i^n - \frac{1}{\delta^2 - 1} v_{N-i}^n, \quad \|y^n\|_0^2 \leq \frac{2(\delta^2 + 1)}{(\delta^2 - 1)^2} \|v^n\|_0^2,$$

$$\|[\tilde{\varphi}^n]\|_0^2 \leq 2(\delta^2 + 1) \|\varphi^n\|_0^2, \quad \|v_0(x_i)\|_0^2 \leq 2(\delta^2 + 1) \|u_0(x_i)\|_0^2.$$

From (30), taking into account these inequalities with $\delta = \delta_1$ for the first case and $\delta = \delta_2$ for second one, we obtain the a priori estimate (29).

The proof of the Theorem 4 is complete.

To probe the convergence of the difference scheme (6)–(8) let us introduce the mesh function $z(x, t) = y(x, t) - u(x, t)$. It is obvious that $z(x, t)$ is a solution of the following problem

$$\Delta_{0t_{n+\sigma}}^\nu z_i - (az_{\bar{x}}^{(\sigma)})_{x,i} = \psi_i^n, \quad i = 1, 2, \dots, N - 1, \quad (31)$$

$$\begin{cases} z_0^{n+1} - \alpha z_N^{n+1} = 0, \\ \beta \Delta_{0t_{n+\sigma}}^\nu z_0 + \Delta_{0t_{n+\sigma}}^\nu z_N + \frac{2}{h} \left(a_N z_{\bar{x},N}^{(\sigma)} - \beta a_1 z_{x,0}^{(\sigma)} - \gamma z_N^{(\sigma)} \right) = \frac{2}{h} \nu^n + \psi_N^n + \beta \psi_0^n, \end{cases} \quad (32)$$

$$z_i^0 = 0, \quad (33)$$

where $\psi(x, t) = O(h^2 + \tau^2)$, $\nu(t) = O(h^2 + \tau^2)$, for all $(x, t) \in \bar{\omega}_{h\tau}$.

Theorem 5. Suppose that a sufficiently smooth solution of the problem (1)–(3) exists and the conditions of the Theorem 5 are fulfilled. Then the solution of the difference problem in the L_2 -norm converges to the solution of the differential problem with the rate equal to the order of the approximation error of the scheme (6)–(8).

Proof. The uniqueness of the solution of the problem (1)–(3) follows from the a priori estimate (10). According to the Theorem 5, for the solution of the difference problem (31)–(33), the estimate is valid

$$\|z^{n+1}\|_0^2 \leq M_3 \max_{0 \leq n \leq N_T-1} (\|\psi^{n+1}\|_0^2 + (\nu^n)^2), \quad (34)$$

from which the statement of the theorem follows.

4. Numerical Results.

Numerical calculations are performed for a test problem when the function

$$u(x, t) = (\alpha + 1 + \sin(\pi x) + (\alpha - 1) \cos(\pi x))(t^2 + t + 1)$$

is the exact solution of the problem (1)–(3) with the coefficient $k(x) = 2 - \sin(\pi x)$.

The errors ($z = y - u$) and convergence order (CO) in the norms $[\cdot]_0$ and $\|\cdot\|_{C(\bar{\omega}_{h\tau})}$ are given in tables 1–5.

Each of tables 1–5 shows that when we take $h = \tau$, as the number of spatial subintervals and time steps is decreased, a reduction in the maximum error takes place, as expected and the convergence order of the approximate scheme is $O(h^2)$, where the convergence order is given by the formula: $\text{CO} = \log_{\frac{h_1}{h_2}} \frac{\|z_1\|}{\|z_2\|}$.

Table 1

$\nu = 0.5, \alpha = 3, \beta = 2, \gamma = -5, T = 1, h = \tau$

h	$\max_{0 \leq n \leq N_T} \ [z^n]\ _0$	CO in $[\cdot]_0$	$\ z\ _{C(\bar{\omega}_{h\tau})}$	CO in $\ \cdot\ _{C(\bar{\omega}_{h\tau})}$
1/160	$3.33916e - 005$		$5.25440e - 005$	
1/320	$8.34728e - 006$	2.000	$1.31382e - 005$	2.000
1/640	$2.08672e - 006$	2.000	$3.28445e - 006$	2.000

Table 2

$\nu = 0.7, \alpha = 2, \beta = -5, \gamma = 10, T = 1, h = \tau$

h	$\max_{0 \leq n \leq N_T} \ [z^n]\ _0$	CO in $[\cdot]_0$	$\ z\ _{C(\bar{\omega}_{h\tau})}$	CO $\ \cdot\ _{C(\bar{\omega}_{h\tau})}$
1/160	$1.97469e - 004$		$2.40953e - 004$	
1/320	$4.93670e - 005$	2.000	$6.02370e - 005$	2.000
1/640	$1.23418e - 005$	2.000	$1.50593e - 005$	2.000

Table 3

$\nu = 0.3, \alpha = 0.7, \beta = 0.1, \gamma = -3, T = 1, h = \tau$

h	$\max_{0 \leq n \leq N_T} \ [z^n]\ _0$	CO in $[\cdot]_0$	$\ z\ _{C(\bar{\omega}_{h\tau})}$	CO in $\ \cdot\ _{C(\bar{\omega}_{h\tau})}$
1/160	$7.17620e - 005$		$1.23543e - 004$	
1/320	$1.79401e - 005$	2.000	$3.08862e - 005$	2.000
1/640	$4.48502e - 006$	2.000	$7.72159e - 006$	2.000

Table 4

$\nu = 0.9, \alpha = 0.1, \beta = -0.9, \gamma = -7, T = 1, h = \tau$

h	$\max_{0 \leq n \leq N_T} \ [z^n]\ _0$	CO in $[\cdot]_0$	$\ z\ _{C(\bar{\omega}_{h\tau})}$	CO in $\ \cdot\ _{C(\bar{\omega}_{h\tau})}$
1/160	$1.03913e - 004$		$1.43555e - 004$	
1/320	$2.59783e - 005$	2.000	$3.58883e - 005$	2.000
1/640	$6.49458e - 006$	2.000	$8.97203e - 006$	2.000

Tabel 5 $\nu = 0.1, \alpha = 100, \beta = -200, \gamma = 300, T = 1, h = \tau$

h	$\max_{0 \leq n \leq N_T} \ z^n\ _0$	CO in $\ \cdot\ _0$	$\ z\ _{C(\bar{\omega}_{h\tau})}$	CO in $\ \cdot\ _{C(\bar{\omega}_{h\tau})}$
1/160	$3.01867e - 002$		$5.35210e - 002$	
1/320	$7.54659e - 003$	2.000	$1.33801e - 002$	2.000
1/640	$1.88664e - 003$	2.000	$3.34503e - 003$	2.000

It is notable that in the case of an arbitrary and non-symmetric coefficient of the equation (1), results of the numerical investigation of the stability and convergence of the difference schemes (6)–(8) are compatible with statements of the Theorems 4 and 5.

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