

On Well-Posedness for a Piezo-Electromagnetic Coupling Model with Boundary Dynamics.

Rainer Picard

June 16, 2021

Abstract

We consider a coupled system of Maxwell's equations and the equations of elasticity, which is commonly used to model piezo-electric material behavior. The boundary influence is encoded as a separate dynamics on the boundary data spaces coupled to the partial differential equations. Evolutionary well-posedness, i.e. Hadamard well-posedness and causal dependence on the data, is shown for the resulting model system.

1 Introduction.

There is an abundance of applications for piezoelectric materials. Their primary use is in ultrasonic transducers. Typical applications can be found in medical imaging and non-destructive testing of safety critical structures. The well-posedness of corresponding models, which is the focus of this paper, is of interest in the evaluation of respective models and in particular as a basis for inverse problems. A useful summary of the literature that has examined well-posedness issues for a range of boundary conditions can be found in [1].

In this paper we will consider a model class discussed in [2] and generalize it to a broader class of problems, where for example the operator coefficients could be non-local, e.g. of convolution type, and will not be restricted to just multiplication operators. To be concrete a coefficient operator α may for example be given in the form

$$(\alpha f)(x) := \alpha_0(x) f(x) + \int_{\Omega} \alpha_1(x, y) f(y) dy,$$

where Ω is the underlying spatial domain carrying the material properties described by α . Another common way non-locality of coefficients can come into play is via orthogonal projectors entering the coefficient operators, e.g. Helmholtz projectors.

More importantly, there will be *no* constraints on the boundary quality of Ω so that more complex configurations such as materials with fractal boundaries, which have been considered and even prototyped more recently, see e.g. [10], come into reach. We shall propose a generalized boundary condition, which in fact takes on the form of an extra equation describing the dynamics on the topological boundary set $\dot{\Omega}$ of the underlying non-empty open set Ω . For computational purposes one would have to assume approximations by domains with better boundary quality such as a Lipschitz boundary in which case classical boundary trace operators can be utilized. To pave the way a discussion of classical boundary trace arguments and our abstract characterization of boundary data spaces is also included.

Since in the general situation we consider here boundary trace theorems are not available, the analysis is based on an alternative characterization of boundary data, which makes no reference to the boundary quality.

Our discussion will be based on the space-time Hilbert space framework developed in [6], see also [7], for what we shall call *evo-systems*. After briefly recalling the conceptual building blocks of the theoretical framework in Section 2 we then establish the classical system of piezo-electro-magnetism with standard homogeneous boundary conditions as such a system (Section 3). In Section 4 we initially discuss standard inhomogeneous boundary conditions to introduce

the boundary data characterization utilized in our general setting, in particular Subsection 4.2. Then the more complex situation of a Leontovich type boundary condition is explored within this boundary data space framework in Subsections 4.3. Rather than implementing this type of boundary constraint into the differential operator domain, as is standard for the classical Dirichlet and Neumann type boundary condition, this mixed type boundary condition is described via additional dynamic equations in the boundary data spaces.

2 A Brief Summary of Evo-Systems

The solution theory of the class of so-called *evolutionary* equations (evo-systems for short) introduced in [6] is based on the fact that the (time) derivative ∂_0 is, in a suitable setting, a normal operator with a strictly positive real part. Indeed, in the space $H_{\nu,0}(\mathbb{R}, H)$, $\nu \in]0, \infty[$, of H -valued $L^{2,\text{loc}}$ -functions (H a Hilbert space with inner product $\langle \cdot | \cdot \rangle_H$) equipped with the inner product $\langle \cdot | \cdot \rangle_{\nu,0,H}$

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \langle \varphi(t) | \psi(t) \rangle_H \exp(-2\nu t) dt,$$

we have that ∂_0 is a normal operator, i.e. commuting with its adjoint on $D(\partial_0^2)$, and

$$\Re \partial_0 = \nu > 0.$$

Throughout, we denote by ∂_0 this derivative as a derivative with respect to time. Under suitable assumptions the latter property of ∂_0 can be carried over to problems of the general form

$$\overline{\partial_0 M (\partial_0^{-1})} + A U = F, \quad (1)$$

where now $A : D(A) \subseteq H \rightarrow H$ is a closed densely defined linear operator and $(M(z))_{z \in B_{\mathbb{C}}(r,r)}$ ($B_{\mathbb{C}}(r,r)$ denotes the open ball in \mathbb{C} of radius $r \in]0, \infty[$ centered at $r \in]0, \infty[$) is a uniformly bounded analytic operator family. The well-posedness of (1) can be based on strict (real) positive definiteness of $(\partial_0 M (\partial_0^{-1}) + A)$ and its adjoint for all sufficiently large weight parameters $\nu \in]0, \infty[$. The resulting problem class is referred to as *evolutionary equations* to contrast it with classical evolution equations in Hilbert space, which are a special case. For emphasis we shall use the term “evo-system” for problems of this class, since classical evolution equations are sometimes also referred to as “evolutionary”.

In this paper we shall be dealing with a rather special and so also more easily accessible case. We only need to consider the case, where A is skew-selfadjoint and $z \mapsto M(z)$ is actually a rational (operator-valued) function defined in a neighborhood of the origin.

To recall the solution theory (as described in the last chapter of [7]) for our somewhat simpler situation the needed requirement is that $M(0)$ is selfadjoint and that

$$\nu M(0) + \Re M'(0) \geq c_0 > 0 \text{ for all sufficiently large } \nu \in]0, \infty[. \quad (2)$$

Equation (2) is for example satisfied if $M(0)$ is strictly positive definite on its range and $\Re M'(0)$ strictly positive definite on the null space of $M(0)$, which will turn out to be valid in our present application.

Remark 2.1. Whenever we are not interested in the actual constant $c_0 \in]0, \infty[$ we shall write for the strict positive definiteness constraint

$$\Re T \geq c_0$$

simply

$$T \gg 0.$$

If we want to state that there is such a constant c_0 for a whole family of operators $(T_\nu)_{\nu \in I}$, we say that

$$T_\nu \gg 0$$

holds uniformly with respect to ν .

So, the general requirement for the problem class under consideration would be stated as $M(0)$ selfadjoint and

$$\nu M(0) + \Re M'(0) \gg 0 \quad (3)$$

uniformly for all sufficiently large $\nu \in]0, \infty[$.

Definition 2.2. A problem class is called Hadamard well-posed, if we have existence, uniqueness of a solution as well as continuous dependence of the solution on the data. For dynamic problems we also want causal dependence on the data. We shall call the problem class described by an evo-system as well-posed, if there exists a continuous linear solution operator \mathcal{S} (Hadamard-wellposedness), which moreover satisfies the causality condition¹

$$\chi_{]-\infty, a[}(m_0) \mathcal{S} \chi_{[a, \infty[}(m_0) = 0$$

for all $a \in \mathbb{R}$ (causality).

For sake of reference we record the corresponding well-posedness result for evo-systems.

Theorem 2.3. *Let $A : D(A) \subseteq H \rightarrow H$ be skew-selfadjoint and $z \mapsto M(z)$ be a uniformly bounded, linear-operator-valued rational function in a neighborhood of zero such that (3) is satisfied uniformly for all $\nu \in [\nu_0, \infty[$, for some $\nu_0 \in]0, \infty[$. Then the evo-system (1) is well-posed.*

Thus, we have not only that for every $F \in H_{\nu,0}(\mathbb{R}, H)$ there is a unique solution $U \in D(\overline{\partial_0 M(\partial_0^{-1}) + A})$, but also that the solution operator $\overline{\partial_0 M(\partial_0^{-1}) + A}^{-1} : H_{\nu,0}(\mathbb{R}, H) \rightarrow H_{\nu,0}(\mathbb{R}, H)$ is a continuous linear mapping, which, moreover, is also causal in the sense that

$$\chi_{]-\infty, a[}(m_0) \overline{\partial_0 M(\partial_0^{-1}) + A}^{-1} \chi_{[a, \infty[}(m_0) = 0$$

for all $a \in \mathbb{R}$.

On occasion, we also want to use some additional regularity observations, which we therefore also introduce here. For this we need some dual spaces. We choose to identify

$$H = H'$$

and

$$H_{\nu,0}(\mathbb{R}, H) = (H_{\nu,0}(\mathbb{R}, H))',$$

and we define $H_{\nu,1}(\mathbb{R}, H)$ as the domain of ∂_0 equipped with the norm induced by the inner product $\langle \cdot | \cdot \rangle_{\nu,1,H} := \langle \partial_0 \cdot | \partial_0 \cdot \rangle_{\nu,0,H}$ as well as

$$H_{\nu,-1}(\mathbb{R}, H) := (H_{\nu,1}(\mathbb{R}, H))'.$$

We also shall make use of the Hilbert space

$$H_{\nu,-1}(\mathbb{R}, D(A^*))' := H_{\nu,1}(\mathbb{R}, D(A^*))',$$

where we canonically consider $D(C)$ with a closed operator C as a Hilbert space with respect to the graph inner product. Denoting again by A the continuous extension of

$$\begin{aligned} D(A) \subseteq H &\rightarrow D(A^*)' \\ x &\mapsto Ax \end{aligned}$$

we have with this, letting $M_0 := M(0)$, $M_1(\partial_0^{-1}) := \partial_0(M(\partial_0^{-1}) - M(0))$, that

$$\partial_0 M_0 U = F - M_1(\partial_0^{-1})U - AU \in H_{\nu,0}(\mathbb{R}, D(A^*))'$$

¹Here $\chi_I(m_0)$ denotes the temporal cut-off by the characteristic function of I

$$(\chi_I(m_0) f)(t) = \begin{cases} f(t) & \text{for } t \in I, \\ 0 & \text{otherwise.} \end{cases}$$

and so we read off that

$$M_0 U \in H_{\nu,1}(\mathbb{R}, D(A^*)'). \quad (4)$$

We similarly have

$$AU = F - M_1 (\partial_0^{-1}) U - M_0 \partial_0 U \in H_{\nu,-1}(\mathbb{R}, H)$$

and so

$$U \in H_{\nu,-1}(\mathbb{R}, D(A)). \quad (5)$$

Note that for the solution U according to Theorem 2.3 we not only have the regularity statements (4), (5), but also that the equation

$$\partial_0 M_0 U + M_1 (\partial_0^{-1}) U + AU = F$$

holds in $H_{\nu,-1}(\mathbb{R}, D(A^*)')$. We shall use the latter fact as motivation to drop henceforth the closure bar for equations of the form (1).

One of the foremost complications in practical applications is that the evo-system structure is frequently obscured. This is mostly the case due to purely formal, i.e. informal, calculations performed under unclear assumptions in the modeling process. To address rigorous ways to produce equations equivalent to evo-systems we recall the following linear algebra terminology.

Definition 2.4. If continuous, linear Hilbert space bijections \mathcal{W}, \mathcal{V} exist such that

$$\mathcal{B} = \mathcal{W}\mathcal{A}\mathcal{V},$$

then \mathcal{A} and \mathcal{B} are called equivalent. If $\mathcal{V} = \mathcal{W}^*$ then \mathcal{A} and \mathcal{B} are called congruent. If $\mathcal{V} = \mathcal{W}^{-1}$ then \mathcal{A} and \mathcal{B} are called similar. If \mathcal{V}, \mathcal{W} are unitary then \mathcal{A} and \mathcal{B} are called unitarily equivalent, unitarily congruent (or unitarily similar), respectively.

Remark 2.5. Equivalence in the stated sense preserves Hadamard well-posedness.² For an equivalent equation it may, however, be hard to detect further structural properties, since for example (skew-)selfadjointness gets easily lost in the process.

In contrast, *spatial congruence*, i.e. \mathcal{W} only acts on the spatial Hilbert space H , is, if lifted to the time-dependent case, a structure preserving operation for evo-systems. Indeed, for $\mathcal{W} : H \rightarrow X$

$$\begin{aligned} \mathcal{W}F &= \mathcal{W}(\partial_0 M_0 + M_1 (\partial_0^{-1}) + A) \mathcal{W}^* \left((\mathcal{W}^{-1})^* U \right) \\ &= (\partial_0 \mathcal{W}M_0 \mathcal{W}^* + \mathcal{W}M_1 (\partial_0^{-1}) \mathcal{W}^* + \mathcal{W}A\mathcal{W}^*) \left((\mathcal{W}^{-1})^* U \right) \end{aligned}$$

where now $\mathcal{W}A\mathcal{W}^*$ is still skew-selfadjoint and $\mathcal{W}M_0\mathcal{W}^*$ is still selfadjoint. Assuming that (2) holds, we find

$$\begin{aligned} \langle U | \nu \mathcal{W}M_0 \mathcal{W}^* U + \Re(\mathcal{W}M_1(0) \mathcal{W}^*) U \rangle_X &= \langle \mathcal{W}^* U | (\nu M_0 + \Re(M_1(0))) \mathcal{W}^* U \rangle_H \\ &\geq c_0 \langle \mathcal{W}^* U | \mathcal{W}^* U \rangle_H \\ &\geq c_0 \left\| (\mathcal{W}^{-1})^* \right\|^{-2} \langle U | U \rangle_X \end{aligned}$$

where we have used

$$\begin{aligned} \langle U | U \rangle_X &= \left\langle (\mathcal{W}^{-1})^* \mathcal{W}^* U | (\mathcal{W}^{-1})^* \mathcal{W}^* U \right\rangle_X \\ &\leq \left\| (\mathcal{W}^{-1})^* \right\|^2 \langle \mathcal{W}^* U | \mathcal{W}^* U \rangle_H. \end{aligned}$$

In particular, (3) remains satisfied, i.e. $(\partial_0 \mathcal{W}M_0 \mathcal{W}^* + \mathcal{W}M_1 (\partial_0^{-1}) \mathcal{W}^* + \mathcal{W}A\mathcal{W}^*)$ is an evo-system operator in $H_{\nu,0}(\mathbb{R}, X)$, where we originally had an evo-system in $H_{\nu,0}(\mathbb{R}, H)$.

²This fact is actually the reason for the choice of the term ‘‘equivalence’’.

3 The Evo-System of Piezo-Electro-Magnetism

3.1 The Basic System

Let $\Omega \subseteq \mathbb{R}^3$ be an arbitrary non-empty open set. The system of Piezo-Electro-Magnetism in a medium occupying Ω is a coupled system consisting of the equation of elasticity and Maxwell's equations. The equation of elasticity is given by

$$\partial_0^2 \rho_* u - \operatorname{Div} T = F_0, \quad (6)$$

where $u : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^3$ describes the displacement of the elastic body Ω , $T : \mathbb{R} \times \Omega \rightarrow \operatorname{sym} [\mathbb{R}^{3 \times 3}]$ denotes the stress tensor, which is assumed to attain values in the space of symmetric matrices. The function $\rho_* : \Omega \rightarrow \mathbb{R}$ stands for the density of Ω and $F_0 : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^3$ is an external force term. Maxwell's equations are given by

$$\begin{aligned} \partial_0 B + \operatorname{curl} E &= F_1, \\ \partial_0 D - \operatorname{curl} H &= -J_0 - \sigma E. \end{aligned} \quad (7)$$

Here, $B, D, E, H : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^3$ denote the magnetic flux density, the electric displacement, the electric field and the magnetic field, respectively. The functions $J_0, F_1 : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^3$ are given source terms and $\sigma : \Omega \rightarrow \mathbb{R}^{3 \times 3}$ denotes the resistance tensor. Of course, all these equations need to be completed by suitably modified material laws, where also the coupling will occur. As it will turn out, the system can be written in the following abstract form

$$\begin{aligned} \left(\partial_0 M_0 + M_1 (\partial_0^{-1}) + \begin{pmatrix} 0 & -\operatorname{Div} & 0 & 0 \\ -\operatorname{Grad} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\operatorname{curl} \\ 0 & 0 & \operatorname{curl} & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ T \\ E \\ H \end{pmatrix} &= \\ &= \begin{pmatrix} F_0 \\ G_0 \\ -J_0 \\ F_1 \end{pmatrix}, \end{aligned} \quad (8)$$

for a suitable bounded operator M_0 and a uniformly bounded rational operator family $(M_1(z))_{z \in U}$, U a neighborhood of zero, on the Hilbert space $H := L^2(\Omega)^3 \oplus \operatorname{sym} [L^2(\Omega)^{3 \times 3}] \oplus L^2(\Omega)^3 \oplus L^2(\Omega)^3$. Here, $v := \partial_0 u$.

Of course, we also need to impose boundary constraints. To make this precise, we need to properly define the spatial differential operators.

Definition 3.1. We denote by $\mathring{C}_\infty(\Omega)$ the space of arbitrarily differentiable functions with compact support in Ω . Then we define the operator curl as the closure of

$$\begin{aligned} \mathring{C}_\infty(\Omega)^3 \subseteq L^2(\Omega)^3 &\rightarrow L^2(\Omega)^3 \\ (\phi_1, \phi_2, \phi_3) &\mapsto \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \end{aligned}$$

and $\operatorname{curl} := (\mathring{\operatorname{curl}})^* \supseteq \mathring{\operatorname{curl}}$. We also define $\mathring{\operatorname{Grad}}$ and $\mathring{\operatorname{Div}}$ as the closures of

$$\begin{aligned} \mathring{C}_\infty(\Omega)^3 \subseteq L^2(\Omega)^3 &\rightarrow \operatorname{sym} [L^2(\Omega)^{3 \times 3}] \\ (\phi_1, \phi_2, \phi_3) &\mapsto \frac{1}{2} (\partial_j \phi_i + \partial_i \phi_j)_{i,j \in \{1,2,3\}} \end{aligned}$$

and of

$$\begin{aligned} \operatorname{sym} [\mathring{C}_\infty(\Omega)^{3 \times 3}] \subseteq \operatorname{sym} [L^2(\Omega)^{3 \times 3}] &\rightarrow L^2(\Omega)^3 \\ (\phi_{ij})_{i,j \in \{1,2,3\}} &\mapsto \left(\sum_{j=1}^3 \partial_j \phi_{ij} \right)_{i \in \{1,2,3\}}, \end{aligned}$$

respectively, and set $\mathring{\text{Grad}} := -(\mathring{\text{Div}})^*$ as well as $\mathring{\text{Div}} := -(\mathring{\text{Grad}})^*$. Here $L^2(\Omega)^{3 \times 3}$ has a Hilbert space structure unitarily equivalent to $L^2(\Omega)^9$. Elements in the domain of the operators marked by a overset circle satisfy an abstract homogeneous boundary condition, which, in case of a sufficiently smooth boundary $\partial\Omega$ (e.g. a Lipschitz boundary), can be written as

$$u = 0 \text{ on } \partial\Omega$$

for $u \in D(\mathring{\text{Grad}})$,

$$Tn = 0 \text{ on } \partial\Omega$$

for $T \in D(\mathring{\text{Div}})$, where n denotes the exterior unit normal vector field on $\partial\Omega$ and

$$E \times n = 0 \text{ on } \partial\Omega,$$

for $E \in D(\mathring{\text{curl}})$.

Not to incur unnecessary constraints on the boundary quality we shall, however, use the generalized homogeneous boundary conditions of containment in $D(\mathring{\text{Grad}})$, $D(\mathring{\text{Div}})$, $D(\mathring{\text{curl}})$, respectively.

For sake of definiteness we shall focus for now on the classical Dirichlet case: $n \times E = 0$, $v = 0$ on the boundary $\partial\Omega$, i.e. in generalized terms on the system

$$\left(\partial_0 M_0 + M_1 (\partial_0^{-1}) + \begin{pmatrix} 0 & -\mathring{\text{Div}} & 0 & 0 \\ -\mathring{\text{Grad}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathring{\text{curl}} \\ 0 & 0 & \mathring{\text{curl}} & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ T \\ E \\ H \end{pmatrix} = \begin{pmatrix} F_0 \\ G_0 \\ -J_0 \\ F_1 \end{pmatrix}. \quad (9)$$

We still need to specify the material law of interest.

3.2 The Material Relations of Piezo-Electro-Magnetism

In this section we discuss material relations suggested in [4] and derive the structure of the corresponding operators M_0 and M_1 . Furthermore, we give sufficient conditions on the parameters involved to warrant the solvability condition (3).

The material relations described in [4] are initially given (ignoring for simplicity thermal couplings) in the form (where we write $\mathcal{E} = \text{Grad } u$ as usual for the strain tensor)

$$\begin{aligned} T &= C \mathcal{E} - eE, \\ D &= e^* \mathcal{E} + \varepsilon E, \\ B &= \mu H. \end{aligned}$$

Here $C \in L(\text{sym}[L^2(\Omega)^{3 \times 3}])$ is the (invertible) elasticity “tensor”³, $\varepsilon, \mu \in L(L^2(\Omega)^3)$ are the permittivity and permeability, respectively, all assumed to be selfadjoint and non-negative.

³Since every linear mapping $F : X \rightarrow Y$ can be interpreted as a bilinear functional $((x, y) \mapsto y(Fx)) \in (X \otimes Y)'$ the term tensor for C is not completely misplaced. It supports, however, a common misunderstanding that C is considered to be a tensor *field*, where it is indeed just a *mapping* between symmetric tensor field. The mapping C can only be considered as a tensor field if we would restrict our attention to multiplicative mappings, i.e.

$$(C\mathcal{E})(x) := \tilde{C}(x) \mathcal{E}(x) \text{ a.e.}$$

for an L^∞ -function \tilde{C} from Ω to $L(\text{sym}[\mathbb{R}^{3 \times 3}])$, which expressly we do not want to limit ourselves to, then C itself could also be *interpreted* as a tensor field $(C_{ij}^{kl}(x))_{i,j,k,l}$ so that

$$(C\mathcal{E})(x) = (C_{ij}^{kl}(x) \mathcal{E}_{kl}(x))_{i,j}.$$

Since C is supposed to map symmetric tensor fields to symmetric tensor fields we must have – in this case – the well-known symmetry relations for the real-valued functions (g denotes the metric tensor)

$$C^{ijkl}(x) := \sum_{s,t=1}^3 g^{is}(x) g^{jt}(x) C_{st}^{kl}(x) \text{ a.e.}, \quad i, j, k, l = 1, 2, 3,$$

The notation $L(X_0, X_1)$ is used to denote the Banach space of continuous linear mappings from the Hilbert space X_0 to the Hilbert space X_1 . In the case $X_0 = X_1$ we write, as done here, more concisely $L(X_0)$. The operator

$$e \in L(L^2(\Omega)^3, \text{sym}[L^2(\Omega)^{3 \times 3}])$$

acts as a coupling ‘‘parameter’’. To adapt the material relations to our framework we solve for \mathcal{E} and obtain

$$\begin{aligned} \mathcal{E} &= C^{-1}T + C^{-1}eE, \\ D &= e^*C^{-1}T + (\varepsilon + e^*C^{-1}e)E, \\ B &= \mu H. \end{aligned}$$

Thus, we arrive at a material law equation of the form

$$\begin{pmatrix} \varrho_* v \\ \mathcal{E} \\ D \\ B \end{pmatrix} = M_0 \begin{pmatrix} v \\ T \\ E \\ H \end{pmatrix} + \partial_0^{-1} M_1 \begin{pmatrix} v \\ T \\ E \\ H \end{pmatrix}.$$

with

$$\begin{aligned} M_0 &= \begin{pmatrix} \varrho_* & 0 & 0 & 0 \\ 0 & C^{-1} & C^{-1}e & 0 \\ 0 & e^*C^{-1} & \varepsilon + e^*C^{-1}e & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}, \\ M_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{10}$$

Here $\sigma \in L(L^2(\Omega)^3)$ represents an additional conductivity coefficient.

We need to ensure the solvability condition (3) with these material relations to obtain our first result.

Theorem 3.2. *Assume that $\varrho_*, \varepsilon, \mu, C$ are selfadjoint and non-negative. Furthermore, we assume $\varrho_*, \mu, C \gg 0$ and $\nu\varepsilon + \Re\sigma \gg 0$ uniformly for all sufficiently large $\nu \in]0, \infty[$. Then, M_0 and M_1 given by (10) satisfy the condition (3) and hence, the corresponding problem of piezo-electro-magnetism is a well-posed evo-system.*

Proof. Obviously, M_0 is selfadjoint. Moreover, since $\varrho_*, \varepsilon, \mu \gg 0$, the only thing, which is left to show, is that

$$\nu \begin{pmatrix} C^{-1} & C^{-1}e \\ e^*C^{-1} & \varepsilon + e^*C^{-1}e \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \Re\sigma \end{pmatrix} \gg 0$$

for all sufficiently large ν . By symmetric Gauss steps as congruence transformations we get that the above operator is congruent to

$$\nu \begin{pmatrix} C^{-1} & 0 \\ 0 & \varepsilon \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \Re\sigma \end{pmatrix}.$$

The latter operator is then strictly positive definite by assumption and so the assertion follows. \square

namely that

$$C^{ijkl}(x) = C^{ijlk}(x) = C^{jikl}(x) \text{ a.e.}, \quad i, j, k, l = 1, 2, 3.$$

The also assumed selfadjointness of C clearly results in another set of symmetry relations

$$C^{ijkl}(x) = C^{klij}(x) \text{ a.e.}, \quad i, j, k, l = 1, 2, 3,$$

which is like-wise a standard requirement in this context.

4 Inhomogeneous Boundary Conditions

4.1 Boundary Data Spaces

Using that domains of closed, linear Hilbert space operators are themselves in a canonical sense Hilbert spaces with respect to the associated graph inner product we see that with

$$\begin{aligned} D(\overset{\circ}{\text{Grad}}) &\subseteq D(\text{Grad}), \\ D(\overset{\circ}{\text{Div}}) &\subseteq D(\text{Div}), \\ D(\overset{\circ}{\text{curl}}) &\subseteq D(\text{curl}), \end{aligned}$$

we may consider the ortho-complements of the vanishing boundary data spaces $D(\overset{\circ}{\text{Grad}})$, $D(\overset{\circ}{\text{Div}})$, $D(\overset{\circ}{\text{curl}})$ in the larger Hilbert spaces $D(\text{Grad})$, $D(\text{Div})$, $D(\text{curl})$, respectively. Prescribing boundary data for $D(\text{Grad})$, $D(\text{Div})$, $D(\text{curl})$ can now be done conveniently by choosing elements of these ortho-complements, which are

$$N(1 - \text{Div Grad}), N(1 - \text{Grad Div}), N(1 + \text{curl curl}),$$

respectively. If $\iota_{\text{Grad}}, \iota_{\text{Div}}, \iota_{\text{curl}}$ denote the canonical isometric, embeddings (i.e. via the identity) of these null spaces into $D(\text{Grad})$, $D(\text{Div})$, $D(\text{curl})$, respectively, then ι_{Grad}^* , ι_{Div}^* , ι_{curl}^* perform the orthogonal projection⁴ onto the respective null spaces. With

$$\overset{\bullet}{\text{Grad}} := \iota_{\text{Div}}^* \text{Grad} \iota_{\text{Grad}}, \quad \overset{\bullet}{\text{Div}} := \iota_{\text{Grad}}^* \text{Div} \iota_{\text{Div}}, \quad \overset{\bullet}{\text{curl}} := \iota_{\text{curl}}^* \text{curl} \iota_{\text{curl}},$$

we get that these are unitary mappings and

$$\begin{aligned} \overset{\bullet}{\text{Grad}}^* &= \overset{\bullet}{\text{Div}}, \\ \overset{\bullet}{\text{curl}}^* &= -\overset{\bullet}{\text{curl}}. \end{aligned}$$

Note that in contrast we have for example in \mathbb{R}^3

$$\text{Grad}^* = -\text{Div}, \quad \text{curl}^* = \text{curl}.$$

This apparent contrast stems from the different choice of inner product with respect to which the adjoints are constructed. To understand this point let us recall from [9] the case of $\overset{\bullet}{\text{curl}} : N(1 + \text{curl curl}) \rightarrow N(1 + \text{curl curl})$ (the argument for $\overset{\bullet}{\text{Grad}}$ being analogous). As a closed subspace of the Hilbert space $D(\text{curl})$ the inner product of $N(1 + \text{curl curl})$ is the graph inner product of curl and so – indicating inner product by the respective spaces – we have for all $\phi, \psi \in N(1 + \text{curl curl})$, i.e. with $\text{curl curl} \phi = -\phi$ and $\psi = -\text{curl curl} \psi$, indeed that

$$\begin{aligned} \left\langle \overset{\bullet}{\text{curl}} \phi | \psi \right\rangle_{N(1 + \text{curl curl})} &= \langle \iota_{\text{curl}}^* \text{curl} \iota_{\text{curl}} \phi | \psi \rangle_{N(1 + \text{curl curl})} \\ &= \langle \text{curl} \phi | \psi \rangle_{D(\text{curl})} \\ &:= \langle \text{curl} \phi | \psi \rangle_{L^2(\Omega)^3} + \langle \text{curl curl} \phi | \text{curl} \psi \rangle_{L^2(\Omega)^3} \\ &= -\langle \text{curl} \phi | \text{curl curl} \psi \rangle_{L^2(\Omega)^3} - \langle \phi | \text{curl} \psi \rangle_{L^2(\Omega)^3} \\ &= -\langle \phi | \text{curl} \psi \rangle_{D(\text{curl})} \\ &= -\langle \phi | \iota_{\text{curl}}^* \text{curl} \iota_{\text{curl}} \psi \rangle_{N(1 + \text{curl curl})} \\ &= -\left\langle \phi | \overset{\bullet}{\text{curl}} \psi \right\rangle_{N(1 + \text{curl curl})}. \end{aligned}$$

⁴The more familiar corresponding orthogonal projectors from the projection theorem context are

$$P_{N(1 - \text{Div Grad})} = \iota_{\text{Grad}} \iota_{\text{Grad}}^*, \quad P_{N(1 - \text{Grad Div})} = \iota_{\text{Div}} \iota_{\text{Div}}^*, \quad P_{N(1 + \text{curl curl})} = \iota_{\text{curl}} \iota_{\text{curl}}^*.$$

4.2 Inhomogeneous Initial Boundary Value Problems

With the above boundary space set-up we can for example discuss now inhomogeneous boundary conditions in the sense that we are looking for a solution

$$\left(\partial_0 M_0 + M_1 + \begin{pmatrix} 0 & -\text{Div} & 0 & 0 \\ -\text{Grad} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\text{curl} \\ 0 & 0 & \text{curl} & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ T \\ E \\ H \end{pmatrix} = \begin{pmatrix} F_0 \\ G_0 \\ -J_0 \\ F_1 \end{pmatrix} \quad (11)$$

with

$$\begin{aligned} v - \iota_{\text{Grad}} v_{\dot{\Omega}} &\in H_{\nu,-1} \left(\mathbb{R}, D \left(\overset{\circ}{\text{Grad}} \right) \right) \cap H_{\nu,0} \left(\mathbb{R}, L^2 \left(\Omega, \mathbb{C}^3 \right) \right), \\ E - \iota_{\text{curl}} E_{\dot{\Omega}} &\in H_{\nu,-1} \left(\mathbb{R}, D \left(\overset{\circ}{\text{curl}} \right) \right) \cap H_{\nu,0} \left(\mathbb{R}, L^2 \left(\Omega, \mathbb{C}^3 \right) \right), \end{aligned} \quad (12)$$

where

$$\begin{aligned} v_{\dot{\Omega}} &\in H_{\nu,1} \left(\mathbb{R}, N \left(1 - \text{Div Grad} \right) \right), \\ E_{\dot{\Omega}} &\in H_{\nu,1} \left(\mathbb{R}, N \left(1 + \text{curl curl} \right) \right), \end{aligned} \quad (13)$$

are given (generalized) boundary data. The solution theory of this problem can be obtained from solving the evo-system

$$\begin{aligned} &\left(\partial_0 M_0 + M_1 + \begin{pmatrix} 0 & -\text{Div} & 0 & 0 \\ -\text{Grad} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\text{curl} \\ 0 & 0 & \text{curl} & 0 \end{pmatrix} \right) \begin{pmatrix} v - \iota_{\text{Grad}} v_{\dot{\Omega}} \\ T \\ E - \iota_{\text{curl}} E_{\dot{\Omega}} \\ H \end{pmatrix} = \\ &= \begin{pmatrix} F_0 \\ G_0 \\ -J_0 \\ F_1 \end{pmatrix} - (\partial_0 M_0 + M_1) \begin{pmatrix} \iota_{\text{Grad}} v_{\dot{\Omega}} \\ 0 \\ \iota_{\text{curl}} E_{\dot{\Omega}} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \overset{\bullet}{\iota_{\text{Div}} \text{Grad}} v_{\dot{\Omega}} \\ 0 \\ -\overset{\bullet}{\iota_{\text{curl}} \text{curl}} E_{\dot{\Omega}} \end{pmatrix}, \end{aligned}$$

where we note that

$$\begin{aligned} \overset{\bullet}{\iota_{\text{Div}} \text{Grad}} v_{\dot{\Omega}} &= \iota_{\text{Div}} \overset{\bullet}{\iota_{\text{Div}}^* \text{Grad}} \iota_{\text{Grad}} v_{\dot{\Omega}}, \\ &= \text{Grad} \iota_{\text{Grad}} v_{\dot{\Omega}}, \end{aligned}$$

$$\begin{aligned} \overset{\bullet}{\iota_{\text{curl}} \text{curl}} E_{\dot{\Omega}} &= \iota_{\text{curl}} \overset{\bullet}{\iota_{\text{curl}}^* \text{curl}} \iota_{\text{curl}} E_{\dot{\Omega}}, \\ &= \text{curl} \iota_{\text{curl}} E_{\dot{\Omega}}. \end{aligned}$$

Remark 4.1. A similar approach can be used to implement initial conditions by simply solving the evo-system⁵

$$\begin{aligned} &= \begin{pmatrix} F_0 \\ G_0 \\ -J_0 \\ F_1 \end{pmatrix} - M_1 (\partial_0^{-1}) \left(\chi_{]0, \infty[} \otimes \begin{pmatrix} v_0 \\ T_0 \\ E_0 \\ H_0 \end{pmatrix} \right) + \chi_{]0, \infty[} \otimes A \begin{pmatrix} v_0 \\ T_0 \\ E_0 \\ H_0 \end{pmatrix}, \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & -\text{Div} & 0 & 0 \\ -\text{Grad} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\text{curl} \\ 0 & 0 & \text{curl} & 0 \end{pmatrix}$$

⁵Here $(\chi_{]0, \infty[} \otimes U_0) := \chi_{]0, \infty[}(t) U_0$ for $t \in \mathbb{R}$.

and $M_0 \begin{pmatrix} v_0 \\ T_0 \\ E_0 \\ H_0 \end{pmatrix}$ with $\begin{pmatrix} v_0 \\ T_0 \\ E_0 \\ H_0 \end{pmatrix} \in D(A)$ describe the initial data. The desired solution $\begin{pmatrix} v \\ T \\ E \\ H \end{pmatrix}$ can now be easily reconstructed from

$$\begin{pmatrix} v \\ T \\ E \\ H \end{pmatrix} = U + \chi_{]0, \infty[} \otimes \begin{pmatrix} v_0 \\ T_0 \\ E_0 \\ H_0 \end{pmatrix}.$$

It is for this reason that we have simplified the discussion to vanishing initial data, compare [7, Chapter 6].

4.3 Leontovich Type Boundary Conditions as Dynamics on Boundary Data Spaces

4.3.1 Translating Particular Model Boundary Conditions

We recall from [2] the two boundary conditions:

$$\begin{aligned} n \times H_t - n \times Q^* v + E_t &= 0 \text{ on } \partial\Omega, \\ Tn - Q(n \times E_t) + (1 + \alpha \partial_0^{-1}) v &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where E_t, H_t denote the tangential components of E, H , respectively, and Q, α are certain matrix-valued functions.

With $n \times$ replaced by $\mathring{\text{curl}}$, Tn by $\mathring{\text{Div}} \iota_{\text{Div}}^* T$ and E_t, H_t replaced by $\iota_{\text{curl}}^* E, \iota_{\text{curl}}^* H$, we get

$$\begin{aligned} \mathring{\text{curl}} \iota_{\text{curl}}^* H - \mathring{\text{curl}} Q^* \iota_{\text{Grad}}^* v + \iota_{\text{curl}}^* E &= 0 \\ \mathring{\text{Div}} \iota_{\text{Div}}^* T - Q \mathring{\text{curl}} \iota_{\text{curl}}^* E + (1 + \alpha \partial_0^{-1}) \iota_{\text{Grad}}^* v &= 0 \end{aligned} \quad (14)$$

In this now

$$\begin{aligned} Q &: N(1 + \mathring{\text{curl}} \mathring{\text{curl}}) \rightarrow N(1 - \mathring{\text{Div}} \mathring{\text{Grad}}) \\ \alpha &: N(1 - \mathring{\text{Div}} \mathring{\text{Grad}}) \rightarrow N(1 - \mathring{\text{Div}} \mathring{\text{Grad}}) \end{aligned}$$

are boundary operators. This translation yields boundary conditions (14) which are in a form that allows again generalization to arbitrary non-empty open sets for Ω , which is one of our main goals here.

To motivate this translation process we note that for all $\Phi \in D(\mathring{\text{curl}})$

$$\begin{aligned} \left\langle \iota_{\text{curl}}^* \Phi | \mathring{\text{curl}} \iota_{\text{curl}}^* H \right\rangle_{N(1 + \mathring{\text{curl}} \mathring{\text{curl}})} &= \\ = \langle \iota_{\text{curl}} \iota_{\text{curl}}^* \Phi | \mathring{\text{curl}} \iota_{\text{curl}} \iota_{\text{curl}}^* H \rangle_{D(\mathring{\text{curl}})} & \\ = \langle \iota_{\text{curl}} \iota_{\text{curl}}^* \Phi | \mathring{\text{curl}} \iota_{\text{curl}} \iota_{\text{curl}}^* H \rangle_0 + & \\ + \langle \mathring{\text{curl}} \iota_{\text{curl}} \iota_{\text{curl}}^* \Phi | \mathring{\text{curl}} \mathring{\text{curl}} \iota_{\text{curl}} \iota_{\text{curl}}^* H \rangle_0 & \\ = \langle \iota_{\text{curl}} \iota_{\text{curl}}^* \Phi | \mathring{\text{curl}} \iota_{\text{curl}} \iota_{\text{curl}}^* H \rangle_0 - \langle \mathring{\text{curl}} \iota_{\text{curl}} \iota_{\text{curl}}^* \Phi | \iota_{\text{curl}} \iota_{\text{curl}}^* H \rangle_0 & \\ = \langle \Phi | \mathring{\text{curl}} H \rangle_0 - \langle \mathring{\text{curl}} \Phi | H \rangle_0 & \\ = \langle \Phi | n \times H \rangle_{L^2(\partial\Omega)} & \\ = \langle (\gamma_{-n \times n \times \iota_{\text{curl}}}) \iota_{\text{curl}}^* \Phi | (\gamma_{n \times \iota_{\text{curl}}}) \iota_{\text{curl}}^* H \rangle_{L^2(\partial\Omega)} & \\ = \langle (\gamma_{-n \times n \times \iota_{\text{curl}}}) \iota_{\text{curl}}^* \Phi | R_X (\gamma_{n \times \iota_{\text{curl}}}) \iota_{\text{curl}}^* H \rangle_X & \\ = \langle \iota_{\text{curl}}^* \Phi | (\gamma_{-n \times n \times \iota_{\text{curl}}})^* R_X (\gamma_{n \times \iota_{\text{curl}}}) \iota_{\text{curl}}^* H \rangle_{N(1 + \mathring{\text{curl}} \mathring{\text{curl}})} & \end{aligned} \quad (15)$$

and so

$$\begin{aligned} \mathring{\text{curl}} \iota_{\text{curl}}^* H &= (\gamma_{-n \times n \times \iota_{\text{curl}}})^* R_X (\gamma_{n \times \iota_{\text{curl}}}) \iota_{\text{curl}}^* H \\ R_X^* ((\gamma_{-n \times n \times \iota_{\text{curl}}})^{-1})^* \mathring{\text{curl}} \iota_{\text{curl}}^* H &= (\gamma_{n \times \iota_{\text{curl}}})^* \iota_{\text{curl}}^* H \\ &= \gamma_{n \times} H \end{aligned}$$

Here $R_X : Y \rightarrow X$ denotes the associated Riesz mapping and

$$\begin{aligned} \gamma_{-n \times n \times} : D(\text{curl}) &\rightarrow X \\ \gamma_{n \times} : D(\text{curl}) &\rightarrow Y \end{aligned}$$

are suitable continuous boundary trace surjections with X, Y being $L^2(\partial\Omega)$ -dual Hilbert spaces (we avoid the intricate details here, see e.g. [3], for more specifics) and

$$N(\gamma_{-n \times n \times}) = N(\gamma_{n \times}) = D(\mathring{\text{curl}}).$$

Then

$$\begin{aligned} \gamma_{-n \times n \times} \iota_{\text{curl}} : N(1 + \text{curl curl}) &\rightarrow X \\ \gamma_{n \times} \iota_{\text{curl}} : N(1 + \text{curl curl}) &\rightarrow Y \end{aligned}$$

are continuous bijections.

Similarly, for all $\Phi \in D(\text{Grad})$

$$\begin{aligned} \left\langle \iota_{\text{Grad}}^* \Phi | \mathring{\text{Div}} \iota_{\text{Div}}^* T \right\rangle_{N(1 - \text{Div Grad})} &= \\ &= \langle \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \text{Div} \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle_{D(\text{Grad})} \\ &= \langle \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \text{Div} \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle_0 \\ &+ \langle \text{Grad} \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \text{Grad Div} \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle_0 \\ &= \langle \Phi | \text{Div} \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle_0 + \langle \text{Grad} \Phi | \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle_0 \\ &= \langle \Phi | \text{Div} T \rangle_0 + \langle \text{Grad} \Phi | T \rangle_0 \\ &= \langle \Phi | T n \rangle_{L^2(\partial\Omega)} \\ &= \langle (\gamma_1 \iota_{\text{Grad}}) \iota_{\text{Grad}}^* \Phi | (\gamma \cdot n \iota_{\text{Div}}) \iota_{\text{Div}}^* T \rangle_{L^2(\partial\Omega)} \\ &= \langle (\gamma_1 \iota_{\text{Grad}}) \iota_{\text{Grad}}^* \Phi | R_{\tilde{X}} (\gamma \cdot n \iota_{\text{Div}}) \iota_{\text{Div}}^* T \rangle_{\tilde{X}} \\ &= \langle \iota_{\text{Grad}}^* \Phi | (\gamma_1 \iota_{\text{Grad}})^* R_{\tilde{X}} (\gamma \cdot n \iota_{\text{Div}}) \iota_{\text{Div}}^* T \rangle_{N(1 - \text{Div Grad})} \end{aligned} \tag{16}$$

and so

$$\begin{aligned} \mathring{\text{Div}} &= (\gamma_1 \iota_{\text{Grad}})^* R_{\tilde{X}} (\gamma \cdot n \iota_{\text{Div}}) \\ R_{\tilde{X}}^* ((\gamma_1 \iota_{\text{Grad}})^{-1})^* \mathring{\text{Div}} \iota_{\text{Div}}^* T &= (\gamma \cdot n \iota_{\text{Div}}) \iota_{\text{Div}}^* T \\ &= \gamma \cdot n T \end{aligned}$$

Here $R_{\tilde{X}} : \tilde{Y} \rightarrow \tilde{X}$ denotes the corresponding associated Riesz mapping and

$$\begin{aligned} \gamma_1 : D(\text{Grad}) &\rightarrow \tilde{X} \\ \gamma \cdot n : D(\text{Div}) &\rightarrow \tilde{Y} \end{aligned}$$

are suitable continuous boundary trace surjections with \tilde{X}, \tilde{Y} being $L^2(\partial\Omega)$ -dual Hilbert spaces and

$$N(\gamma_1) = D(\mathring{\text{Grad}}), \quad N(\gamma \cdot n) = D(\mathring{\text{Div}}).$$

Then

$$\begin{aligned} \gamma_1 \iota_{\text{Grad}} : N(1 - \text{Div Grad}) &\rightarrow \tilde{X} \\ \gamma \cdot n \iota_{\text{Div}} : N(1 - \text{Grad Div}) &\rightarrow \tilde{Y} \end{aligned}$$

are continuous bijections.

Both instances are showing a close, formal connection, which we have taken as a justification for the proposed generalization for boundary terms.

4.3.2 An Evo-System Set-Up

We shall, however, implement the boundary constraints (14) not as typical boundary conditions but by appending, in the spirit of abstract grad – div - systems, see [8], the differential equations in Ω by dynamical equations on the boundary spaces. Hence, we consider a system of the form

$$(\partial_0 M_0 + M_1 (\partial_0^{-1}) + A) \begin{pmatrix} v \\ \begin{pmatrix} T \\ \tau_T \end{pmatrix} \\ E \\ \begin{pmatrix} H \\ \tau_H \end{pmatrix} \end{pmatrix} = \begin{pmatrix} F_0 \\ \begin{pmatrix} G_0 \\ g_0 \end{pmatrix} \\ -J_0 \\ \begin{pmatrix} F_1 \\ f_1 \end{pmatrix} \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & -\begin{pmatrix} -\text{Grad} \\ \iota_{\text{Grad}}^* \end{pmatrix}^* & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -\text{Grad} \\ \iota_{\text{Grad}}^* \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & -\begin{pmatrix} \text{curl} \\ \iota_{\text{curl}}^* \end{pmatrix}^* \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \text{curl} \\ \iota_{\text{curl}}^* \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

is by construction – as desired – skew-selfadjoint and $M_0, M_1 (\partial_0^{-1})$ are to be specified later.

To analyze the operator A closer we need to obtain a better understanding of $\begin{pmatrix} -\text{Grad} \\ \iota_{\text{Grad}}^* \end{pmatrix}^*$ and $\begin{pmatrix} -\text{curl} \\ \iota_{\text{curl}}^* \end{pmatrix}^*$. We first observe that

$$\begin{pmatrix} -\overset{\circ}{\text{Grad}} \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} -\text{Grad} \\ \iota_{\text{Grad}}^* \end{pmatrix}, \quad \begin{pmatrix} \overset{\circ}{\text{curl}} \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} \text{curl} \\ \iota_{\text{curl}}^* \end{pmatrix},$$

which implies that

$$\begin{pmatrix} -\text{Grad} \\ \iota_{\text{Grad}}^* \end{pmatrix}^* \subseteq \begin{pmatrix} -\overset{\circ}{\text{Grad}} \\ 0 \end{pmatrix}^* = (\text{Div} \quad 0), \\ \begin{pmatrix} \text{curl} \\ \iota_{\text{curl}}^* \end{pmatrix}^* \subseteq \begin{pmatrix} \overset{\circ}{\text{curl}} \\ 0 \end{pmatrix}^* = (\text{curl} \quad 0).$$

Thus, for all $\Phi \in D(\text{Grad})$ and $\begin{pmatrix} T \\ \tau_T \end{pmatrix} \in D\left(\begin{pmatrix} -\text{Grad} \\ \iota_{\text{Grad}}^* \end{pmatrix}^*\right)$

$$\begin{aligned} \langle -\text{Grad} \Phi | T \rangle + \langle \iota_{\text{Grad}}^* \Phi | \tau_T \rangle &= \left\langle \begin{pmatrix} -\text{Grad} \\ \iota_{\text{Grad}}^* \end{pmatrix} \Phi \middle| \begin{pmatrix} T \\ \tau_T \end{pmatrix} \right\rangle \\ &= \left\langle \Phi \middle| \begin{pmatrix} -\text{Grad} \\ \iota_{\text{Grad}}^* \end{pmatrix}^* \begin{pmatrix} T \\ \tau_T \end{pmatrix} \right\rangle \\ &= \langle \Phi | \text{Div} T \rangle. \end{aligned}$$

Since

$$\langle \text{Grad} \Phi | T \rangle + \langle \Phi | \text{Div} T \rangle = 0$$

for $\Phi \in D(\mathring{\text{Grad}})$ or $T \in D(\mathring{\text{Div}})$ we have for $\Phi \in D(\text{Grad})$ and $T \in D(\text{Div})$

$$\begin{aligned}
& \langle \text{Grad } \Phi | T \rangle + \langle \Phi | \text{Div } T \rangle = \\
& = \langle \text{Grad } \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle + \langle \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \text{Div } \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle + \\
& + \langle \text{Grad } \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | (1 - \iota_{\text{Div}} \iota_{\text{Div}}^*) T \rangle + \\
& + \langle (1 - \iota_{\text{Grad}} \iota_{\text{Grad}}^*) \Phi | \text{Div } \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle + \\
& + \langle \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \text{Div } (1 - \iota_{\text{Div}} \iota_{\text{Div}}^*) T \rangle + \\
& + \langle \text{Grad } (1 - \iota_{\text{Grad}} \iota_{\text{Grad}}^*) \Phi | \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle + \\
& + \langle (1 - \iota_{\text{Grad}} \iota_{\text{Grad}}^*) \Phi | \text{Div } (1 - \iota_{\text{Div}} \iota_{\text{Div}}^*) T \rangle + \\
& + \langle \text{Grad } (1 - \iota_{\text{Grad}} \iota_{\text{Grad}}^*) \Phi | (1 - \iota_{\text{Div}} \iota_{\text{Div}}^*) T \rangle \\
& = \langle \text{Grad } \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle + \langle \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \text{Div } \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle
\end{aligned}$$

and recalling (16) we calculate with this for all $\Phi \in D(\text{Grad})$ and $\begin{pmatrix} T \\ \tau_T \end{pmatrix} \in D\left(\begin{pmatrix} -\text{Grad} \\ \iota_{\text{Grad}}^* \end{pmatrix}^*\right) \subseteq D(\text{Div})$

$$\begin{aligned}
& \langle \iota_{\text{Grad}}^* \Phi | \tau_T \rangle_{N(1-\text{Div Grad})} = \\
& = \langle \text{Grad } \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle + \langle \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \text{Div } \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle, \\
& = \frac{1}{2} \langle \text{Grad } \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle + \frac{1}{2} \langle \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \text{Div } \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle + \\
& + \frac{1}{2} \langle \text{Grad } \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \text{Grad Div } \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle + \\
& + \frac{1}{2} \langle \text{Div Grad } \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \text{Div } \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle, \\
& = \frac{1}{2} \langle \text{Grad } \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle_{D(\text{Div})} + \\
& + \frac{1}{2} \langle \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \text{Div } \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle_{D(\text{Grad})}, \\
& = \frac{1}{2} \langle \iota_{\text{Div}}^* \text{Grad } \iota_{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \iota_{\text{Div}}^* T \rangle_{N(1-\text{Grad Div})} + \\
& + \frac{1}{2} \langle \iota_{\text{Grad}}^* \Phi | \iota_{\text{Grad}}^* \text{Div } \iota_{\text{Div}} \iota_{\text{Div}}^* T \rangle_{N(1-\text{Div Grad})} \\
& = \frac{1}{2} \langle \mathring{\text{Grad}} \iota_{\text{Grad}}^* \Phi | \iota_{\text{Div}}^* T \rangle_{N(1-\text{Grad Div})} + \frac{1}{2} \langle \iota_{\text{Grad}}^* \Phi | \mathring{\text{Div}} \iota_{\text{Div}}^* T \rangle_{N(1-\text{Div Grad})}, \\
& = \langle \iota_{\text{Grad}}^* \Phi | \mathring{\text{Div}} \iota_{\text{Div}}^* T \rangle_{N(1-\text{Div Grad})}
\end{aligned}$$

and so

$$\tau_T = \mathring{\text{Div}} \iota_{\text{Div}}^* T.$$

Similarly, we find for all $\Phi \in D(\text{curl})$ and $\begin{pmatrix} H \\ \tau_H \end{pmatrix} \in D\left(\begin{pmatrix} \text{curl} \\ \iota_{\text{curl}}^* \end{pmatrix}^*\right) \subseteq D(\text{curl})$

$$\begin{aligned}
& \langle \text{curl } \Phi | E \rangle + \langle \iota_{\text{curl}}^* \Phi | \tau_H \rangle_{N(1+\text{curl curl})} = \left\langle \begin{pmatrix} \text{curl} \\ \iota_{\text{curl}}^* \end{pmatrix} \Phi \middle| \begin{pmatrix} H \\ \tau_H \end{pmatrix} \right\rangle \\
& = \left\langle \Phi \middle| \begin{pmatrix} \text{curl} \\ \iota_{\text{curl}}^* \end{pmatrix}^* \begin{pmatrix} H \\ \tau_H \end{pmatrix} \right\rangle \\
& = \langle \Phi | \text{curl } H \rangle
\end{aligned}$$

leading with (15) to

$$\begin{aligned}
& \left\langle \iota_{\text{curl}}^* \Phi \middle| \tau_H \right\rangle_{N(1+\text{curl curl})} = \\
& = - \left\langle \text{curl } \iota_{\text{curl}} \iota_{\text{curl}}^* \Phi \middle| \iota_{\text{curl}} \iota_{\text{curl}}^* E \right\rangle + \left\langle \iota_{\text{curl}} \iota_{\text{curl}}^* \Phi \middle| \text{curl } \iota_{\text{curl}} \iota_{\text{curl}}^* E \right\rangle \\
& = -\frac{1}{2} \left\langle \text{curl } \iota_{\text{curl}} \iota_{\text{curl}}^* \Phi \middle| \iota_{\text{curl}} \iota_{\text{curl}}^* E \right\rangle + \\
& + \frac{1}{2} \left\langle \iota_{\text{curl}} \iota_{\text{curl}}^* \Phi \middle| \text{curl } \iota_{\text{curl}} \iota_{\text{curl}}^* E \right\rangle + \\
& + \frac{1}{2} \left\langle \text{curl } \iota_{\text{curl}} \iota_{\text{curl}}^* \Phi \middle| \text{curl curl } \iota_{\text{curl}} \iota_{\text{curl}}^* E \right\rangle + \\
& - \frac{1}{2} \left\langle \text{curl curl } \iota_{\text{curl}} \iota_{\text{curl}}^* \Phi \middle| \text{curl } \iota_{\text{curl}} \iota_{\text{curl}}^* E \right\rangle \\
& = -\frac{1}{2} \left\langle \text{curl } \iota_{\text{curl}} \iota_{\text{curl}}^* \Phi \middle| \iota_{\text{curl}} \iota_{\text{curl}}^* E \right\rangle_{D(\text{curl})} + \\
& + \frac{1}{2} \left\langle \iota_{\text{curl}} \iota_{\text{curl}}^* \Phi \middle| \text{curl } \iota_{\text{curl}} \iota_{\text{curl}}^* E \right\rangle_{D(\text{curl})} \\
& = -\frac{1}{2} \left\langle \overset{\bullet}{\text{curl}} \iota_{\text{curl}}^* \Phi \middle| \iota_{\text{curl}}^* E \right\rangle_{N(1-\text{curl curl})} + \\
& + \frac{1}{2} \left\langle \iota_{\text{curl}}^* \Phi \middle| \overset{\bullet}{\text{curl}} \iota_{\text{curl}}^* E \right\rangle_{N(1-\text{curl curl})} \\
& = \left\langle \iota_{\text{curl}}^* \Phi \middle| \overset{\bullet}{\text{curl}} \iota_{\text{curl}}^* E \right\rangle_{N(1-\text{curl curl})}
\end{aligned}$$

and so

$$\tau_H = \overset{\bullet}{\text{curl}} \iota_{\text{curl}}^* H.$$

With this the boundary constraints take the form

$$\begin{aligned}
\tau_H - \overset{\bullet}{\text{curl}} Q^* \iota_{\text{Grad}}^* v + \iota_{\text{curl}}^* E &= 0, \\
\tau_T - Q \overset{\bullet}{\text{curl}} \iota_{\text{curl}}^* E + (1 + \alpha \partial_0^{-1}) \iota_{\text{Grad}}^* v &= 0,
\end{aligned}$$

or

$$\begin{pmatrix} \tau_H \\ \tau_T \end{pmatrix} + \begin{pmatrix} 1 & -\overset{\bullet}{\text{curl}} Q^* \\ -Q \overset{\bullet}{\text{curl}} & (1 + \alpha \partial_0^{-1}) \end{pmatrix} \begin{pmatrix} \iota_{\text{curl}}^* E \\ \iota_{\text{Grad}}^* v \end{pmatrix} = 0.$$

We calculate

$$\begin{aligned}
& \begin{pmatrix} 1 & -\overset{\bullet}{\text{curl}} Q^* \\ -Q \overset{\bullet}{\text{curl}} & (1 + \alpha \partial_0^{-1}) \end{pmatrix}^{-1} = \\
& = \begin{pmatrix} 1 + \overset{\bullet}{\text{curl}} Q^* (1 + Q Q^* + \alpha \partial_0^{-1})^{-1} Q \overset{\bullet}{\text{curl}} & \overset{\bullet}{\text{curl}} Q^* (1 + Q Q^* + \alpha \partial_0^{-1})^{-1} \\ (1 + Q Q^* + \alpha \partial_0^{-1})^{-1} Q \overset{\bullet}{\text{curl}} & (1 + Q Q^* + \alpha \partial_0^{-1})^{-1} \end{pmatrix}
\end{aligned}$$

and thus obtain equivalently

$$S(\partial_0^{-1}) \begin{pmatrix} \tau_H \\ \tau_T \end{pmatrix} + \begin{pmatrix} \iota_{\text{curl}}^* E \\ \iota_{\text{Grad}}^* v \end{pmatrix} = 0. \quad (17)$$

with $S(\partial_0^{-1})$ given by

$$\begin{pmatrix} 1 + \overset{\bullet}{\text{curl}} Q^* (1 + Q Q^* + \alpha \partial_0^{-1})^{-1} Q \overset{\bullet}{\text{curl}} & \overset{\bullet}{\text{curl}} Q^* (1 + Q Q^* + \alpha \partial_0^{-1})^{-1} \\ (1 + Q Q^* + \alpha \partial_0^{-1})^{-1} Q \overset{\bullet}{\text{curl}} & (1 + Q Q^* + \alpha \partial_0^{-1})^{-1} \end{pmatrix}.$$

We are now ready to formulate the material law operators

$$M_0 = \begin{pmatrix} \begin{pmatrix} \varrho_* \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ C^{-1} & 0 \\ 0 & 0 \\ e^* C^{-1} & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ C^{-1} e & 0 \\ 0 & 0 \\ \mu & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

and

$$M_1(\partial_0^{-1}) = \begin{pmatrix} 0 & (0 \ 0) & 0 & (0 \ 0) \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{1,22}(\partial_0^{-1}) & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & -M_{1,24}(\partial_0^{-1}) \\ 0 & (0 \ 0) & \sigma & (0 \ 0) \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{1,42}(\partial_0^{-1}) & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{1,44}(\partial_0^{-1}) \end{pmatrix}$$

with

$$\begin{aligned} M_{1,44}(\partial_0^{-1}) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 + \mathop{\text{curl}}\mathop{Q^*} (1 + QQ^* + \alpha\partial_0^{-1})^{-1} Q \mathop{\text{curl}} \end{pmatrix} \\ M_{1,42}(\partial_0^{-1}) &= \begin{pmatrix} 0 & 0 \\ 0 & (1 + QQ^* + \alpha\partial_0^{-1})^{-1} Q \mathop{\text{curl}} \end{pmatrix} \\ M_{1,24}(\partial_0^{-1}) &= \begin{pmatrix} 0 & \mathop{\text{curl}}\mathop{Q^*} (1 + QQ^* + \alpha\partial_0^{-1})^{-1} \\ 0 & 0 \end{pmatrix} \\ M_{1,22}(\partial_0^{-1}) &= \begin{pmatrix} 0 & 0 \\ 0 & (1 + QQ^* + \alpha\partial_0^{-1})^{-1} \end{pmatrix}. \end{aligned}$$

Theorem 4.2. *Assume that $\varrho_*, \varepsilon, \mu, C$ are selfadjoint and non-negative, $Q : N(1 + \text{curl curl}) \rightarrow N(1 - \text{Div Grad})$. Furthermore, we assume $\varrho_*, \mu, C \gg 0$ and $\nu\varepsilon + \Re\sigma \gg 0$ uniformly for all sufficiently large $\nu \in]0, \infty[$. Then, M_0 and $M_1(\partial_0^{-1})$ satisfy the condition (3) and hence, the corresponding problem of piezo-electricity with dynamics on the boundary data space is also a well-posed evo-system.*

Proof. Obviously, M_0 is selfadjoint. Moreover, since

$$\nu \begin{pmatrix} C^{-1} & C^{-1}e \\ e^* C^{-1} & \varepsilon + e^* C^{-1}e \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \Re\sigma \end{pmatrix} \gg 0$$

uniformly for all sufficiently large $\nu \in]0, \infty[$, the assertion follows.

Indeed, noting that

$$\begin{aligned} & \Re S(0) = \\ &= \Re \begin{pmatrix} 1 + \mathop{\text{curl}}\mathop{Q^*} (1 + QQ^*)^{-1} Q \mathop{\text{curl}} & \mathop{\text{curl}}\mathop{Q^*} (1 + QQ^*)^{-1} \\ (1 + QQ^*)^{-1} Q \mathop{\text{curl}} & (1 + QQ^*)^{-1} \end{pmatrix}, \\ &= \begin{pmatrix} 1 + \mathop{\text{curl}}\mathop{Q^*} (1 + QQ^*)^{-1} Q \mathop{\text{curl}} & 0 \\ 0 & (1 + QQ^*)^{-1} \end{pmatrix} \geq 1 \end{aligned}$$

the desired result follows from the general result of Theorem 2.3. \square

Remark 4.3.

1. For simplicity we have assumed that there is no thermal interaction. There is, however, no major obstacle to incorporate such interaction along the lines of [5]. Similarly, more complex boundary constraints of abstract grad – div-type could be implemented following the lead of the present framework.

2. Although we have merely generalized a known model system, it is clear from the set-up that more complicated situations are easily incorporated. For example
 - (a) apart from the generalized coefficients we can of course allow inhomogeneous data with no extra provision, since the “boundary conditions” are built into the system as part of the evo-system,
 - (b) the material laws can be even more general as long as requirement (3) remains satisfied.
3. As stated in Remark 2.5, equivalence is a common way of obscuring the basic structure of evo-systems. In the above we have in fact encountered such a situation. If we may assume that boundary trace mappings are available, another pertinent case is given in our present context by

$$\mathcal{W} (\partial_0 M_0 + M_1 (\partial_0^{-1}) + A) \mathcal{V} (\mathcal{V}^{-1} U) = \mathcal{W} F$$

with

$$\mathcal{W} = \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \gamma_1 \iota_{\text{Grad}} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & \gamma_{-n \times n \times \iota_{\text{curl}}} \end{pmatrix} \end{pmatrix},$$

$$\mathcal{V} = \mathcal{W}^* \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & R_{\bar{X}} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & R_X \end{pmatrix} \end{pmatrix}.$$

The unknown is

$$\mathcal{V}^{-1} U = \begin{pmatrix} \begin{pmatrix} v \\ T \\ R_{\bar{X}}^* ((\gamma_1 \iota_{\text{Grad}})^{-1})^* \tau_T \\ E \\ H \\ R_X^* ((\gamma_{-n \times n \times \iota_{\text{curl}}})^{-1})^* \tau_H \end{pmatrix} \end{pmatrix} \in H_{\nu,0}(\mathbb{R}, \mathcal{Y})$$

with

$$\mathcal{Y} = L^2(\Omega, \mathbb{C}^3) \oplus \left(L^2(\Omega, \text{sym}[\mathbb{C}^{3 \times 3}]) \oplus \tilde{Y} \right) \oplus L^2(\Omega, \mathbb{C}^3) \oplus (L^2(\Omega, [\mathbb{C}^3]) \oplus Y)$$

and $\mathcal{W} F \in H_{\nu,0}(\mathbb{R}, \mathcal{X})$ with

$$\mathcal{X} = L^2(\Omega, \mathbb{C}^3) \oplus \left(L^2(\Omega, \text{sym}[\mathbb{C}^{3 \times 3}]) \oplus \tilde{X} \right) \oplus L^2(\Omega, \mathbb{C}^3) \oplus (L^2(\Omega, [\mathbb{C}^3]) \oplus X).$$

This is now the corresponding situation utilizing classical boundary trace spaces. To obtain a structure preserving congruence we could instead replace \mathcal{V} by \mathcal{W}^* in which case

$$(\mathcal{W}^{-1})^* U = \begin{pmatrix} \begin{pmatrix} v \\ T \\ ((\gamma_1 \iota_{\text{Grad}})^{-1})^* \tau_T \\ E \\ H \\ ((\gamma_{-n \times n \times \iota_{\text{curl}}})^{-1})^* \tau_H \end{pmatrix} \end{pmatrix} \in H_{\nu,0}(\mathbb{R}, \mathcal{X})$$

is now the new unknown.

5 Summary

We have generalized a piezo-electromagnetism model with Dirichlet type boundary conditions to arbitrary non-empty open sets, as well as to include operator coefficients, indeed to general material laws. The resulting evo-system in a non-empty open set Ω and on boundary data spaces, which includes inhomogeneous volume and boundary data, has been investigated for evolutionary well-posedness, i.e. Hadamard well-posedness and causality. Based on this the model has been extended to include also a Leontovich type boundary coupling via an additional set of dynamic equations on spaces characterizing boundary data.

References

- [1] M. Akamatsu and G. Nakamura, Well-Posedness of Initial-Boundary Value Problems for Piezoelectric Equations, *Applicable Analysis*, **81**, 129–141, (2002).
- [2] K. Ammari and S. Nicaise. Stabilization of a piezoelectric system. *Asymptotic Anal.*, 73(3):125–146, (2011).
- [3] A. Buffa, M. Costabel, and D. Sheen. On traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains. *J. Math. Anal. Appl.*, 276(2):845–867, (2002).
- [4] R.D. Mindlin, Equations Of High Frequency Vibrations of Thermo-Piezoelectric Crystal Plates, *Intl. J. Solids& Structures*, 10, 625–637, (1974).
- [5] A. J. Mulholland, R. Picard, S. Trostorff, and M. Waurick. On well-posedness for some thermo-piezoelectric coupling models. *Mathematical Methods in the Applied Sciences*, , vol. 39(15), pp. 4375–4384, (2016).
- [6] R. Picard. A Structural Observation for Linear Material Laws in Classical Mathematical Physics. *Math. Methods Appl. Sci.*, 32(14):1768–1803, (2009).
- [7] R. Picard and D. F. McGhee. *Partial Differential Equations: A unified Hilbert Space Approach*, volume 55 of *De Gruyter Expositions in Mathematics*. De Gruyter. Berlin, New York. 518 p., (2011).
- [8] R. Picard, St. Seidler, S. Trostorff, and M. Waurick. On abstract grad-div systems. *Journal of Differential Equations*, 260(6):4888 – 4917, (2016).
- [9] R. Picard, S. Trostorff, and M. Waurick. On a comprehensive class of linear control problems. *IMA Journal of Mathematical Control and Information*, 33(2):257–291, (2016).
- [10] A. J. Walker and A. J. Mulholland, Piezoelectric Ultrasonic Transducers with Fractal Geometry, *Fractals*, 19(4), 469–479, (2011).