# An Optimal Embedded Discontinuous Galerkin Method for Second-Order Elliptic Problems * 

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#### Abstract

The embedded discontinuous Galerkin (EDG) method by Cockburn et al. [SIAM J. Numer. Anal., 2009, 47(4), 2686-2707] is obtained from the hybridizable discontinuous Galerkin method by changing the space of the Lagrangian multiplier from discontinuous functions to continuous ones, and adopts piecewise polynomials of equal degrees on simplex meshes for all variables. In this paper, we analyze a new EDG method for second order elliptic problems on polygonal/polyhedral meshes. By using piecewise polynomials of degrees $k+1, k+1, k$ $(k \geq 0)$ to approximate the potential, numerical trace and flux, respectively, the new method is shown to yield optimal convergence rates for both the potential and flux approximations. Numerical experiments are provided to confirm the theoretical results.


Keywords. embedded discontinuous Galerkin method, hybridizable discontinuous Galerkin method, optimal convergence rate

## 1 Introduction

Let $\Omega \subset \mathbb{R}^{d}(d=2,3)$ be a polyhedral domain with boundary $\partial \Omega$. We consider the following second-order elliptic problem: Find the potential $u$ and the flux $\boldsymbol{\sigma}$ such that

$$
\left\{\begin{align*}
& c \boldsymbol{\sigma}-\nabla u=0, \text { in } \Omega  \tag{1.1}\\
&-\operatorname{div} \boldsymbol{\sigma}=f, \\
& \text { in } \Omega \\
& u=g, \\
& \text { on } \partial \Omega
\end{align*}\right.
$$

where the diffusion-dispersion tensor $\boldsymbol{c} \in\left[L^{2}(\Omega)\right]^{d \times d}$ is a matrix valued function that is symmetric and uniformly positive definite on $\Omega, f \in L^{2}(\Omega)$, and $g \in L^{2}(\partial \Omega)$.

[^0]In [5], Cockburn et al. first proposed a unifying framework for hybridization of finite element methods for second-order elliptic problems. The unifying framework includes as particular cases hybridized versions of mixed methods [1, 2, 4], the continuous Galerkin (CG) method [8], and a wide class of hybridizable discontinuous Galerkin (HDG) methods. In the HDG framework, the constraint of function continuity on the inter-element boundaries is relaxed by introducing numerical traces (Lagrange multipliers) defined on the inter-element boundaries, thus allowing for piecewise-independent approximation to the potential or flux solution. By local elimination of the unknowns defined in the interior of elements, the HDG methods finally lead to symmetric and positive definite (SPD) systems where the unknowns are only the globally coupled degrees of freedom describing the numerical traces. We refer to $[6,9,12]$ for some relevant analyses for the HDG methods.

The EDG methods were first proposed in [11] for linear shell problems, and then were further studied in [7] for second-order elliptic problems. The methods are obtained from the HDG methods by simply reducing the space of the numerical traces, from piecewise independent to continuous on the whole inter-element boundaries. Since the only degrees of freedom that are globally coupled are precisely those of the numerical traces, such reduction leads to smaller computational cost of an EDG method than that of the corresponding HDG method. Recently, the EDG methods have been extended to solving several types of fluid flow problems $[10,13,14]$.

However, as shown in [7], the EDG methods using piecewise polynomials of degree $k(k \geq 1)$ to approximate all kinds of variables results in loss of convergence rate for the approximation of flux. On the other hand, so far all the EDG methods $[7,10,11,13,14]$ are based on simplex meshes, and there is no such work on general polygonal/polyhedral meshes. We note that the classical analysis of HDG methods on simplex meshes [5, 6] is hard to extend to polygonal meshes; one can see [9] for more details.

In this paper, we shall develop a class of new EDG methods for the model problem (1.1) on polygonal/polyhedral meshes. Compared with the original EDG methods in [7], our methods are of the following features.

- The new methods use piecewise polynomials of degrees $k+1, k+1, k(k \geq 0)$ to approximate the potential, numerical trace and flux, respectively.
- Optimal error estimates are derived for both the potential and flux approximations.
- Our analysis is based on polygonal/polyhedral meshes. The analysis technique here is due to [12], where a family of HDG methods for (1.1) on simplex meshes were analyzed under the minimal regularity condition.

The rest of this paper is organized as follows. In Section 2 we introduce notation. Section 3 describes the EDG scheme. Section 4 is devoted to the error estimation of the proposed EDG
methods. Finally, Section 5 provides some numerical results to verify the theoretical analysis.

## 2 Notation

For an arbitrary open set $D \subset \mathbb{R}^{d}$, we denoted by $H^{1}(D)$ the Sobolev space of scalar functions on $D$ whose derivatives up to order 1 are square integrable, with the norm $\|\cdot\|_{1, D}$. The notation $|\cdot|_{1, D}$ denotes the semi-norm derived from the partial derivatives of order equal to 1 . The space $H_{0}^{1}(D)$ denotes the closure in $H^{1}(D)$ of the set of infinitely differentiable functions with compact supports in $D$. We use $(\cdot, \cdot)_{D}$ and $\langle\cdot, \cdot\rangle_{\partial D}$ to denote the $L^{2}$-inner products on the square integrable function spaces $L^{2}(D)$ and $L^{2}(\partial D)$, respectively, with $\|\cdot\|_{D}$ and $\|\cdot\|_{\partial D}$ representing the corresponding induced $L^{2}$-norms. Let $P_{k}(D)$ denote the set of polynomials of degree $\leq k$ defined on $D$.

Let $\mathcal{T}_{h}=\bigcup\{T\}$ be a conforming and shape regular subdivision of $\Omega$ into convex polygons ( $d=2$ ) or polyhedron ( $d=3$ ), with $h_{T}$ being the diameter of $T$ and $h:=\max _{T \in \mathcal{T}_{h}}\left\{h_{T}\right\}$. Here 'shape regular' is in the sense that the following two assumptions M1-M2 hold [3].

- M1 (Star-shaped elements). There exists a positive constant $\theta_{*}$ such that the following holds: for each element $T \in \mathcal{T}_{h}$, there exists a point $M_{T} \in T$ such that $T$ is star-shaped with respect to every point in the circle (or sphere) of center $M_{T}$ and radius $\theta_{*} h_{T}$.
- M2 (Edges or faces). There exists a positive constant $l_{*}$ such that: every element $T \in \mathcal{T}_{h}$, the distance between any two vertexes is no less than $l_{*} h_{T}$.

The regularity parameter of $\mathcal{T}_{h}$ is defined by $\rho:=\max _{T \in \mathcal{T}_{h}}\left\{h_{T}^{d} /|T|\right\}$, where $|T|$ is the $d$-dimension Lebesgue measure of $T$. Let $\mathcal{F}_{h}$ denote the set of all edges/faces of $\mathcal{T}_{h}$, and set $\partial \mathcal{T}_{h}:=\{\partial T: T \in$ $\left.\mathcal{T}_{h}\right\}$.

Based on the subdivision $\mathcal{T}_{h}$, we introduce an auxiliary simplicial mesh $\mathcal{T}_{h}^{*}$ as follows:

- When $d=2$, for any $T \in \mathcal{T}_{h}$, we connect $M_{T}$ and all $T$ 's vertexes to divide $T$ into a set of triangles, denoted by $w(T)$.
- When $d=3$, for any $T, T^{\prime} \in \mathcal{T}_{h}$ and every face $F \subset \partial T \cap \partial T^{\prime}$, we choose any vertex $A$ on $F$ and connect A to the rest of $F$ 's vertexes to get a set of triangles, $v(F)$, and if $F \cap \partial \Omega \neq \varnothing$, we can get a set of triangles $v(F)$ by the same way. Finally we connect $M_{T}$ and every $v(F)$ to get a set of tetrahedrons, $w(T)$.
- We set $\mathcal{T}_{h}^{*}:=\bigcup_{T \in \mathcal{T}_{h}} w(T)$ for $d=2,3$. We note that $\mathcal{T}_{h}^{*}$ is shape regular due to M1 and M2.

For any $T \in \mathcal{T}_{h}$, set $\partial T^{*}:=\bigcup_{T^{\prime} \in w(T)}\left\{\partial T^{\prime} \cap \partial T\right\}$, and define

$$
\partial \mathcal{T}_{h^{*}}:=\left\{\partial T^{*}: T \in \mathcal{T}_{h}\right\}
$$

$$
\mathcal{F}_{h}^{*}:=\left\{F: F \text { is an edge } / \text { face of } \mathcal{T}_{h}^{*} \text { and } F \subset F^{\prime} \text { for some } F^{\prime} \in \mathcal{F}_{h}\right\}
$$

Notice that when $d=2$ or $\mathcal{T}_{h}$ is a tetrahedron mesh for $d=3$, it holds

$$
\partial T^{*}=\partial T, \quad \partial \mathcal{T}_{h^{*}}=\partial \mathcal{T}_{h}, \quad \mathcal{F}_{h}^{*}=\mathcal{F}_{h}
$$

And, when $d=3$ and $\mathcal{T}_{h}$ is a polyhedral mesh, $\partial T^{*}$ is the set of triangles, into which each face $F \subset \partial T$ is subdivided.

We also need the broken Soblev space

$$
H^{s}\left(\mathcal{T}_{h}\right):=\left\{v \in L^{2}(\Omega):\left.v\right|_{T} \in H^{s}(T), \forall T \in \mathcal{T}_{h}\right\}
$$

with the norm $\|\cdot\|_{s, \mathcal{T}_{h}}$ defined by

$$
\|v\|_{s, \mathcal{T}_{h}}^{2}:=\sum_{T \in \mathcal{T}_{h}}\|v\|_{s, T}^{2}, \quad \forall v \in H^{s}\left(\mathcal{T}_{h}\right) .
$$

The broken Soblev space $H^{s}\left(\mathcal{T}_{h}^{*}\right)$ is defined similarly.
Throughout this paper, $x \lesssim y(x \gtrsim y)$ means $\mathrm{x} \leq C y(x \geq C y)$, where C denotes a positive constant that only depends on $d, k, \Omega$, the regularity parameter $\rho$, and the coefficient matrix $\boldsymbol{c}$. The notation $x \sim y$ abbreviates $x \lesssim y \lesssim x$.

## 3 EDG method

For any $T \in \mathcal{T}_{h}$ and $F \in \mathcal{F}_{h}^{*}$, let $V(T) \subset L^{2}(T), \mathbf{W}(T) \subset\left[L^{2}(T)\right]^{d}$ and $M(F) \subset L^{2}(F)$ be local finite dimensional spaces. Then we define

$$
\begin{align*}
V_{h} & :=\left\{v_{h} \in L^{2}(\Omega):\left.v_{h}\right|_{T} \in V(T), \forall T \in \mathcal{T}_{h}\right\},  \tag{3.1}\\
\mathbf{W}_{h} & :=\left\{\boldsymbol{\tau}_{h} \in\left[L^{2}(\Omega)\right]^{d}:\left.\boldsymbol{\tau}_{h}\right|_{T} \in \mathbf{W}(T), \forall T \in \mathcal{T}_{h}\right\},  \tag{3.2}\\
M_{h} & :=\left\{\mu_{h} \in L^{2}\left(\mathcal{F}_{h}^{*}\right):\left.\mu_{h}\right|_{F} \in M(F), \forall F \in \mathcal{F}_{h}^{*}\right\},  \tag{3.3}\\
\widetilde{M}_{h} & :=\left\{\mu_{h} \in C^{0}\left(\mathcal{F}_{h}^{*}\right):\left.\mu_{h}\right|_{F} \in M(F), \forall F \in \mathcal{F}_{h}^{*}\right\},  \tag{3.4}\\
\widetilde{M}_{h}(g) & :=\left\{\mu_{h} \in \widetilde{M}_{h}:\left.\mu_{h}\right|_{\partial \Omega}=\Pi_{h}^{\partial} g\right\}, \tag{3.5}
\end{align*}
$$

where $\Pi_{h}^{\partial}$ is a continuous interpolation operator from $L^{2}(\partial \Omega)$ to $C^{0}(\partial \Omega) \cap P_{k+1}\left(\mathcal{F}_{h}^{*} \cap \partial \Omega\right)$, which will be defined in the next section.

Then the variational formulations of the EDG method are given as follows: Seek $\left(u_{h}, \widetilde{\lambda}_{h}, \boldsymbol{\sigma}_{h}\right) \in$ $V_{h} \times \widetilde{M}_{h}(g) \times \mathbf{W}_{h}$ such that

$$
\begin{gather*}
\left(\boldsymbol{c} \boldsymbol{\sigma}_{h}, \boldsymbol{\tau}_{h}\right)+\left(u_{h}, \operatorname{div}_{h} \boldsymbol{\tau}_{h}\right)-\sum_{T \in \mathcal{T}_{h}}\left\langle\widetilde{\lambda}_{h}, \boldsymbol{\tau}_{h} \cdot n\right\rangle_{\partial T^{*}}=0  \tag{3.6}\\
-\left(v_{h}, \operatorname{div}_{h} \boldsymbol{\sigma}_{h}\right)+\sum_{T \in \mathcal{T}_{h}}\left\langle\alpha_{T}\left(u_{h}-\widetilde{\lambda}_{h}\right), v_{h}\right\rangle_{\partial T^{*}}=\left(f, v_{h}\right)  \tag{3.7}\\
\sum_{T \in \mathcal{T}_{h}}\left\langle\boldsymbol{\sigma}_{h} \cdot n-\alpha_{T}\left(u_{h}-\widetilde{\lambda}_{h}\right), \widetilde{\mu}_{h}\right\rangle_{\partial T^{*}}=0 \tag{3.8}
\end{gather*} \quad \forall \widetilde{\mu}_{h} \in \widetilde{M}_{h}(0) .
$$

Here the broken operator $\operatorname{div}_{h}$ is defined by $\left.\operatorname{div}_{h} \boldsymbol{\tau}_{h}\right|_{T}:=\operatorname{div}\left(\left.\boldsymbol{\tau}_{h}\right|_{T}\right)$ for any $\boldsymbol{\tau}_{h} \in \mathbf{W}_{h}, T \in \mathcal{T}_{h}$.
In this paper we choose the local spaces $V(T), M(F), \mathbf{W}(T)$ and the penalty parameter $\alpha_{T}$ as following: for integer $k \geq 0$,

$$
\begin{align*}
V(T) & =P_{k+1}(T), \quad M(F)=P_{k+1}(F), \quad \mathbf{W}(T)=\left[P_{k}(T)\right]^{d}  \tag{3.9}\\
\left.\alpha_{T}\right|_{F} & =h_{T}^{-1}, \quad \forall \text { face } F \text { of } T . \tag{3.10}
\end{align*}
$$

We have the following existence and uniqueness result:
Lemma 3.1. The $E D G$ method (3.6)-(3.8) admits a unique solution $\left(u_{h}, \widetilde{\lambda}_{h}, \sigma_{h}\right) \in V_{h} \times \widetilde{M}_{h}(g) \times$ $\mathbf{W}_{h}$.

Proof. It suffices to prove the uniqueness, or equivalently, to show that the system has the trivial solution when $f=g=0$.

In fact, $f=g=0$ implies $\widetilde{\lambda}_{h} \in \widetilde{M}_{h}^{0}$. By taking $\left(\boldsymbol{\tau}_{h}, v_{h}, \widetilde{\mu}_{h}\right)=\left(\boldsymbol{\sigma}_{h}, u_{h}, \widetilde{\lambda}_{h}\right)$ in (3.6)-(3.8), and summing all equations together, one can obtain

$$
\left(\boldsymbol{c} \boldsymbol{\sigma}_{h}, \boldsymbol{\sigma}_{h}\right)+\sum_{T \in \mathcal{T}_{h}} \alpha_{T}\left\|u_{h}-\widetilde{\lambda}_{h}\right\|_{\partial T^{*}}^{2}=0
$$

Since $\boldsymbol{c}$ is uniformly positive and $\alpha_{T}$ is nonnegative, the above equation implies $\boldsymbol{\sigma}_{h}=0$ and $u_{h}=\widetilde{\lambda}_{h}$ on $\partial T$ for all $T \in \mathcal{T}_{h}$. Then, taking $\boldsymbol{\tau}_{h}=\nabla u_{h}$ in (3.6) yields

$$
0=\left(u_{h}, \operatorname{div}_{h} \nabla u_{h}\right)_{\mathcal{T}_{h}}-\sum_{T \in \mathcal{T}_{h}}\left\langle\tilde{\lambda}_{h}, \nabla u_{h} \cdot n\right\rangle_{\partial T^{*}}=-\left(\nabla u_{h}, \nabla u_{h}\right)
$$

This means $\nabla u_{h}=0$ on each $T \in \mathcal{T}_{h}$, i.e. $u_{h}$ is piecewise constant. Recalling that $u_{h}=\widetilde{\lambda}_{h}$ on $\mathcal{F}_{h}^{*}$ and $\widetilde{\lambda}_{h}=0$ on $\partial \Omega$, we finally obtain $u_{h}=0$ and $\widetilde{\lambda}_{h}=0$.

Introduce the following two local problems:
For any $T \in \mathcal{T}_{h}$ and $\lambda_{h} \in L^{2}\left(\partial T^{*}\right)$, seek $\left(u_{\lambda_{h}}, \boldsymbol{\sigma}_{\lambda_{h}}\right) \in V(T) \times \mathbf{W}(T)$ such that

$$
\begin{align*}
& \left(\boldsymbol{c} \boldsymbol{\sigma}_{\lambda_{h}}, \boldsymbol{\tau}_{h}\right)_{T}+\left(u_{\lambda_{h}}, \operatorname{div} \boldsymbol{\tau}_{h}\right)_{T}=\left\langle\lambda_{h}, \boldsymbol{\tau}_{h} \cdot n\right\rangle_{\partial T^{*}} \quad \forall \boldsymbol{\tau}_{h} \in \mathbf{W}(T),  \tag{3.11}\\
& -\left(v_{h}, \operatorname{div} \boldsymbol{\sigma}_{\lambda_{h}}\right)_{T}+\left\langle\alpha_{T} u_{\lambda_{h}}, v_{h}\right\rangle_{\partial T}=\left\langle\alpha_{T} \lambda_{h}, v_{h}\right\rangle_{\partial T^{*}} \quad \forall v_{h} \in V(T) . \tag{3.12}
\end{align*}
$$

For any $T \in \mathcal{T}_{h}$ and $f \in L^{2}(T)$, seek $\left(u_{f}, \boldsymbol{\sigma}_{f}\right) \in V(T) \times \mathbf{W}(T)$ such that

$$
\begin{align*}
\left(\boldsymbol{c} \boldsymbol{\sigma}_{f}, \boldsymbol{\tau}_{h}\right)_{T}+\left(u_{f}, \operatorname{div} \boldsymbol{\tau}_{h}\right)_{T}=0 & \forall \boldsymbol{\tau}_{h} \in \mathbf{W}(T),  \tag{3.13}\\
-\left(v_{h}, \operatorname{div} \boldsymbol{\sigma}_{f}\right)_{T}+\left\langle\alpha_{T} u_{f}, v_{h}\right\rangle_{\partial T}=\left(f, v_{h}\right)_{T} & \forall v_{h} \in V(T) . \tag{3.14}
\end{align*}
$$

Similar to the HDG method, after the local elimination of unknowns $u_{h}$ and $\boldsymbol{\sigma}_{h}$, the EDG method leads to the following reduced system: seek $\widetilde{\lambda}_{h} \in \widetilde{M}_{h}(g)$ such that

$$
\begin{equation*}
a_{h}\left(\widetilde{\lambda}_{h}, \widetilde{\mu}_{h}\right)=\left(f, v_{\widetilde{\mu}_{h}}\right) \quad \forall \widetilde{\mu}_{h} \in \widetilde{M}_{h}(0) \tag{3.15}
\end{equation*}
$$

where $a_{h}(\cdot, \cdot): \widetilde{M}_{h}(g) \times \widetilde{M}_{h}^{0} \longrightarrow \mathbb{R}$ is defined by

$$
\begin{equation*}
a_{h}\left(\widetilde{\lambda}_{h}, \widetilde{\mu}_{h}\right):=\sum_{T \in \mathcal{T}_{h}}\left(\boldsymbol{c} \boldsymbol{\sigma}_{\widetilde{\lambda}_{h}}, \boldsymbol{\sigma}_{\widetilde{\mu}_{h}}\right)_{T}+\sum_{T \in \mathcal{T}_{h}}\left\langle\alpha_{T}\left(u_{\widetilde{\lambda}_{h}}-\widetilde{\lambda}_{h}\right), u_{\widetilde{\mu}_{h}}-\widetilde{\mu}_{h}\right\rangle_{\partial T^{*}} . \tag{3.16}
\end{equation*}
$$

Remark 3.1. Follow from [5], we can define an $H D G$ method: $\operatorname{Seek}\left(u_{h}, \lambda_{h}, \boldsymbol{\sigma}_{h}\right) \in V_{h} \times \widehat{M}_{h}(g) \times \mathbf{W}_{h}$ such that

$$
\begin{gathered}
\left(\boldsymbol{c} \boldsymbol{\sigma}_{h}, \boldsymbol{\tau}_{h}\right)+\left(u_{h}, \operatorname{div_{h}} \boldsymbol{\tau}_{h}\right)-\sum_{T \in \mathcal{T}_{h}}\left\langle\lambda_{h}, \boldsymbol{\tau}_{h} \cdot n\right\rangle_{\partial T}=0 \\
-\left(v_{h}, d i v_{h} \boldsymbol{\sigma}_{h}\right)+\sum_{T \in \mathcal{T}_{h}}\left\langle\alpha_{T}\left(u_{h}-\lambda_{h}\right), v_{h}\right\rangle_{\partial T}=\left(f, v_{h}\right) \\
\sum_{T \in \mathcal{T}_{h}}\left\langle\boldsymbol{\sigma}_{h} \cdot n-\alpha_{T}\left(u_{h}-\lambda_{h}\right), \mu_{h}\right\rangle_{\partial T}=0
\end{gathered}
$$

Here
$\widehat{M}_{h}(g):=\left\{\mu \in L^{2}\left(\mathcal{F}_{h}\right):\left.\mu\right|_{F} \in P_{k+1}(F), \forall F \in \mathcal{F}_{h}\right.$ and $\langle\mu, \eta\rangle_{F}=\langle g, \eta\rangle_{F}$ if $\left.F \subset \partial \Omega, \forall \eta \in P_{k+1}(F)\right\}$.

Remark 3.2. We can see that the EDG method is a modification of the corresponding HDG method by simply replacing the discontinuous numerical trace space $\widehat{M}_{h}(g)$ with the continuous trace space $\widetilde{M}_{h}(g)$. In particular, when $d=2$ or $\mathcal{T}_{h}$ is a tetrahedron mesh, $\widetilde{M}_{h}(g)$ is much smaller than $\widehat{M}_{h}(g)$. In such cases, the EDG method leads to a smaller system than the corresponding HDG method.

## 4 Error analysis

This section is devoted to the estimation of the flux error $\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}$ and the potential error $u-u_{h}$ for the EDG scheme (3.6)-(3.8). In subsections 4.1 and 4.2 we carry out the analysis for the flux and potential approximations, respectively on $2 \mathrm{D} / 3 \mathrm{D}$ polygon meshes.

### 4.1 Estimation for flux approximation

This subsection is devoted to the error estimation of the flux approximation $\boldsymbol{\sigma}_{h}$ for the EDG scheme (3.6)-(3.8).

Let $P_{V}: L^{2}(\Omega) \longrightarrow V_{h}, P_{\mathbf{W}}:\left[L^{2}(\Omega)\right]^{d} \longrightarrow \mathbf{W}_{h}$, and $P_{M}: L^{2}\left(\mathcal{F}_{h}^{*}\right) \longrightarrow M_{h}$ be the standard $L^{2}$-orthogonal projection operators. Then the following estimates are standard.

Lemma 4.1. For any $T \in \mathcal{T}_{h}$ and $(v, \boldsymbol{\tau}) \in H^{k+2}(T) \times\left[H^{k+1}(T)\right]^{d}$, it holds

$$
\begin{align*}
\left\|v-P_{V} v\right\|_{T}+h_{T}^{\frac{1}{2}}\left\|v-P_{V} v\right\|_{\partial T} & \lesssim h_{T}^{k+2}|v|_{k+2, T}, \\
\left\|\boldsymbol{\tau}-P_{\mathbf{W}} \boldsymbol{\tau}\right\|_{T}+h_{T}^{\frac{1}{2}}\left\|\boldsymbol{\tau}-P_{\mathbf{W}} \boldsymbol{\tau}\right\|_{\partial T} & \lesssim h_{T}^{k+1}|\boldsymbol{\tau}|_{k+1, T}  \tag{4.1}\\
\left\|v-P_{M} v\right\|_{\partial T^{*}} & \lesssim h_{T}^{k+\frac{3}{2}}|v|_{k+2, T}
\end{align*}
$$

For any $d$-simplex element $T \in \mathcal{T}_{h}$ with vertices $\boldsymbol{a}_{j}=\left(x_{1 j}, x_{2 j}, \ldots x_{d j}\right)^{T}(1 \leq j \leq d+1)$, denote by

$$
\begin{equation*}
S_{T}:=\left\{\boldsymbol{x} \left\lvert\, \boldsymbol{x}=\sum_{i=1}^{d+1} \frac{j_{i}}{k+1} \boldsymbol{a}_{i}\right., \sum_{i=1}^{d+1} j_{i}=k+1, j_{i} \in\{0,1, \ldots, k+1\}, 1 \leq i \leq d+1\right\} \tag{4.2}
\end{equation*}
$$

the set of nodes of $T$ and by $S_{\mathcal{T}_{h}}:=\bigcup_{T \in \mathcal{T}_{h}} S_{T}$ the set of nodes of $\mathcal{T}_{h}$. Note that $S_{T}$ is the set of nodes for the $C^{0}$ Lagrange finite element of order $k+1$. We let $\boldsymbol{a}_{i, T}$ be a node of Lagrange finite element of $S_{T}$ and $\boldsymbol{a}_{i, F} \in S_{\mathcal{T}_{h}}$, which lays on some edge/face $F \in \mathcal{F}_{h}$.

Given a point $\boldsymbol{a} \in \mathbb{R}^{d}$, we define

$$
\begin{aligned}
S_{h}^{c}(\boldsymbol{a}) & :=\left\{T: T \in \mathcal{T}_{h}, \boldsymbol{a} \in T\right\} \\
S_{h}^{F}(\boldsymbol{a}) & :=\left\{F: F \in \mathcal{F}_{h}^{*} \cap \partial \Omega, \boldsymbol{a} \text { is on } F\right\}
\end{aligned}
$$

and let $\# S_{h}^{c}(\boldsymbol{a})$ and $\# S_{h}^{F}(\boldsymbol{a})$ be the number of elements in $S_{h}^{c}(\boldsymbol{a})$ and $S_{h}^{F}(\boldsymbol{a})$, respectively.
Now we define the continuous interpolation operator $\Pi_{h}^{\partial}: L^{2}(\partial \Omega) \longrightarrow C^{0}(\partial \Omega) \cap P_{k+1}\left(\mathcal{F}_{h}^{*} \cap \partial \Omega\right)$ as follows: For any $g \in L^{2}(\partial \Omega)$ and $F \in \mathcal{F}_{h}^{*} \cap \partial \Omega,\left.\Pi_{h}^{\partial} g\right|_{F} \in M(F)$ satisfies

$$
\begin{aligned}
& \left.\Pi_{h}^{\partial} g\right|_{F}\left(\boldsymbol{a}_{i, F}\right)=\left.P_{M} g\right|_{F}\left(\boldsymbol{a}_{i, F}\right), \quad \text { if } \boldsymbol{a}_{i, F} \text { is in the interior of } F, \\
& \left.\Pi_{h}^{\partial} g\right|_{F}\left(\boldsymbol{a}_{i, F}\right)=\left.\frac{1}{\# S_{h}^{F}\left(\boldsymbol{a}_{i, F}\right)} \sum_{F^{\prime} \in S_{h}^{F}\left(a_{i, F}\right)} P_{M} g\right|_{F^{\prime}}\left(\boldsymbol{a}_{i, F}\right), \quad \text { if } \boldsymbol{a}_{i, F} \text { is a vertex of } F .
\end{aligned}
$$

Following Chapter 3 of [15], we introduce the projection mean operator $\Pi_{h}^{P}: L^{2}(\Omega) \longrightarrow$ $V_{h} \bigcap H^{1}(\Omega)$, defined as follows: for any $T \in \mathcal{T}_{h}, u \in L^{2}(\Omega),\left.\Pi_{h}^{P} u\right|_{T} \in V(T)$ and

$$
\begin{array}{ll}
\left.\Pi_{h}^{P} u\right|_{T}\left(\boldsymbol{a}_{i, T}\right)=\left.\left(P_{V} u\right)\right|_{T}\left(\boldsymbol{a}_{i, T}\right) & \text { for any } \boldsymbol{a}_{i, T} \text { in the interior of } T, \\
\left.\Pi_{h}^{P} u\right|_{T}\left(\boldsymbol{a}_{i, T}\right)=\left.\frac{1}{\# S_{h}^{c}\left(\boldsymbol{a}_{i, T}\right)} \sum_{T^{\prime} \in S_{h}^{c}\left(\boldsymbol{a}_{i, T}\right)}\left(P_{V} u\right)\right|_{T^{\prime}}\left(\boldsymbol{a}_{i, T^{\prime}}\right) & \text { for any } \boldsymbol{a}_{i, T} \text { on } \partial T, \partial T \cap \partial \Omega=\varnothing \\
\left.\Pi_{h}^{P} u\right|_{T}\left(\boldsymbol{a}_{i, T}\right)=\Pi_{h}^{\partial} g\left(\boldsymbol{a}_{i, T}\right) & \text { for any } \boldsymbol{a}_{i, T} \text { on } \partial \Omega .
\end{array}
$$

When $\mathcal{T}_{h}$ is a polygonal/polyhedral subdivision, we can first define the projection mean operator $\Pi_{h^{*}}^{P}$ on the auxilliary mesh $\mathcal{T}_{h}^{*}$ whose elements are simplexes. Then we define $\Pi_{h}^{P}$ as follows:

$$
\Pi_{h}^{P} u(\boldsymbol{a})=\Pi_{h^{*}}^{P} u(\boldsymbol{a}) \quad \forall \boldsymbol{a} \in S_{\mathcal{T}_{h^{*}}} \cap \mathcal{F}_{h}^{*}, \forall u \in H^{1}(\Omega)
$$

From [15] we have the following approximation result.
Lemma 4.2. For any $u \in H^{k+2}\left(\mathcal{T}_{h}\right)$ and $T \in \mathcal{T}_{h}$, it holds

$$
\begin{equation*}
\left\|u-\Pi_{h^{*}}^{P} u\right\|_{\partial T^{*}} \lesssim h_{T}^{k+\frac{3}{2}}\left(\sum_{T^{\prime} \in \omega_{T}}|v|_{k+2, T^{\prime}}^{2}\right)^{\frac{1}{2}}, \tag{4.3}
\end{equation*}
$$

where $\omega_{T}:=\left\{T^{\prime} \in \mathcal{T}_{h}: T^{\prime} \cap T \neq \emptyset\right\}$.
With the above projection operators, we set

$$
\begin{align*}
& \delta^{\boldsymbol{\sigma}}:=\boldsymbol{\sigma}-P_{\mathbf{W}} \boldsymbol{\sigma}, \quad \delta^{u}:=u-P_{V} u, \quad \delta^{\tilde{\lambda}}:=u-\Pi_{h} u, \\
& e_{h}^{\boldsymbol{\sigma}}:=P_{\mathbf{W}} \boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}, \quad e_{h}^{u}:=P_{V} u-u_{h}, \quad e_{h}^{\tilde{\lambda}}:=\Pi_{h} u-\widetilde{\lambda}_{h} . \tag{4.4}
\end{align*}
$$

For any given $(\boldsymbol{\tau}, v) \in\left[L^{2}(\Omega)\right]^{d} \times H^{1}(\Omega)$, we define

$$
\begin{equation*}
L_{\boldsymbol{\tau}, v}(\psi):=\sum_{T \in \mathcal{T}_{h}}\left\langle\left(P_{W} \boldsymbol{\tau}-\boldsymbol{\tau}\right) \cdot \boldsymbol{n}-\alpha_{T}\left(P_{V} v-\Pi_{h}^{P} v\right), \psi\right\rangle_{\partial T^{*}}, \forall \psi \in H^{1}(\Omega) \bigcup \widetilde{M}_{h} \bigcup V_{h} \tag{4.5}
\end{equation*}
$$

Then we have the following error equations.
Lemma 4.3. For all $\left(\boldsymbol{\tau}_{h}, v_{h}, \widetilde{\mu}_{h}\right) \in \mathbf{W}_{h} \times V_{h} \times \widetilde{M}_{h}(0)$ it holds

$$
\begin{gather*}
\left(\boldsymbol{c} e_{h}^{\boldsymbol{\sigma}}, \boldsymbol{\tau}_{h}\right)+\left(e_{h}^{u}, d i v_{h} \boldsymbol{\tau}_{h}\right)-\sum_{T \in \mathcal{T}_{h}}\left\langle e_{h}^{\widetilde{\lambda}}, \boldsymbol{\tau}_{h} \cdot n\right\rangle_{\partial T^{*}}=-\left(\boldsymbol{c} \delta^{\boldsymbol{\sigma}}, \boldsymbol{\tau}_{h}\right)+\sum_{T \in \mathcal{T}_{h}}\left\langle P_{M} u-\Pi_{h}^{P} u, \boldsymbol{\tau}_{h} \cdot \boldsymbol{n}\right\rangle_{\partial T^{*}}  \tag{4.6}\\
-\left(d i v_{h} e_{h}^{\boldsymbol{\sigma}}, v_{h}\right)+\sum_{T \in \mathcal{T}_{h}}\left\langle\alpha_{T}\left(e_{h}^{u}-e_{h}^{\widetilde{\lambda}}\right), v_{h}\right\rangle_{\partial T^{*}}=L_{\boldsymbol{\sigma}, u}\left(v_{h}\right)  \tag{4.7}\\
\sum_{T \in \mathcal{T}_{h}}\left\langle e_{h}^{\boldsymbol{\sigma}} \cdot n-\alpha_{T}\left(e_{h}^{u}-e_{h}^{\widetilde{\lambda}}\right), \widetilde{\mu}_{h}\right\rangle_{\partial T^{*}}=-L_{\boldsymbol{\sigma}, u}\left(\widetilde{\mu}_{h}\right) \tag{4.8}
\end{gather*}
$$

Proof. In light of (1.1) and the definitions of $L^{2}$-orthogonal projection operators, we have, for all $\left(\boldsymbol{\tau}_{h}, v_{h}\right) \in \mathbf{W}_{h} \times V_{h}$,

$$
\begin{aligned}
\left(\boldsymbol{c} P_{\mathbf{W}} \boldsymbol{\sigma}, \boldsymbol{\tau}_{h}\right)+\left(P_{V} u, \operatorname{div}_{h} \boldsymbol{\tau}_{h}\right)-\sum_{T \in \mathcal{T}_{h}}\left\langle\Pi_{h}^{P} u, \boldsymbol{\tau}_{h} \cdot n\right\rangle_{\partial T^{*}}= & \left(\boldsymbol{c}\left(P_{\mathbf{W}} \boldsymbol{\sigma}-\boldsymbol{\sigma}\right), \boldsymbol{\tau}_{h}\right) \\
& +\sum_{T \in \mathcal{T}_{h}}\left\langle P_{M} u-\Pi_{h}^{P} u, \boldsymbol{\tau}_{h} \cdot \boldsymbol{n}\right\rangle_{\partial T^{*}} \\
\left(P_{\mathbf{W}} \boldsymbol{\sigma}, \nabla v_{h}\right)_{\mathcal{T}_{h}}+\sum_{T \in \mathcal{T}_{h}}\left\langle P_{\mathbf{W}} \boldsymbol{\sigma} \cdot \boldsymbol{n}, v_{h}\right\rangle_{\partial T^{*}}= & \left(f, v_{h}\right)_{\mathcal{T}_{h}}+\sum_{T \in \mathcal{T}_{h}}\left\langle\left(P_{\mathbf{W}} \boldsymbol{\sigma}-\boldsymbol{\sigma}\right) \cdot \boldsymbol{n}, v_{h}\right\rangle_{\partial T}
\end{aligned}
$$

By subtracting the above two equations from (3.6) and (3.7), respectively, we then obtain (4.6) and (4.7). Finally, equation (4.8) follows form (3.8) and the relation

$$
\begin{equation*}
\sum_{T \in \mathcal{T}_{h}}\left\langle\boldsymbol{\sigma} \cdot \boldsymbol{n}, \widetilde{\mu}_{h}\right\rangle_{\partial T^{*}}=0, \forall \widetilde{\mu}_{h} \in \widetilde{M}_{h}(0) \tag{4.9}
\end{equation*}
$$

Introduce a seminorm $\|\|\cdot\|\|: V_{h} \times \widetilde{M}_{h}(0) \times \mathbf{W}_{h} \longrightarrow \mathbb{R}$ with

$$
\begin{equation*}
\left\|\left\|\left(v_{h}, \widetilde{\mu}_{h}, \boldsymbol{\tau}_{h}\right)\right\|\right\|^{2}:=\left(\boldsymbol{c} \tau_{h}, \boldsymbol{\tau}_{h}\right)+\sum_{T \in \mathcal{T}_{h}}\left\|\alpha_{T}^{\frac{1}{2}}\left(v_{h}-\widetilde{\mu}_{h}\right)\right\|_{\partial T^{*}}^{2}, \forall\left(v_{h}, \widetilde{\mu}_{h}, \boldsymbol{\tau}_{h}\right) \in V_{h} \times \widetilde{M}_{h}(0) \times \mathbf{W}_{h} \tag{4.10}
\end{equation*}
$$

then we easily get the following lemma.

Lemma 4.4. It holds

$$
\begin{equation*}
\left\|\left\|\left(e_{h}^{u}, e_{h}^{\widetilde{\lambda}}, e_{h}^{\boldsymbol{\sigma}}\right)\right\|\right\|^{2} \lesssim\left\|\boldsymbol{\sigma}-P_{\mathbf{W}} \boldsymbol{\sigma}\right\|_{\mathcal{T}_{h}}^{2}+\sum_{T \in \mathcal{T}_{h}} h_{T}\left\|\boldsymbol{\sigma}-P_{\mathbf{W}} \boldsymbol{\sigma}\right\|_{\partial T^{*}}^{2}+\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\|\Pi_{h}^{P} u-P_{M} u\right\|_{\partial T^{*}}^{2} \tag{4.11}
\end{equation*}
$$

Proof. We first show

$$
\begin{equation*}
\left\|\left\|\left(e_{h}^{u}, e_{h}^{\widetilde{\lambda}}, e_{h}^{\boldsymbol{\sigma}}\right)\right\|\right\|^{2}=I_{1}+I_{2}+I_{3} \tag{4.12}
\end{equation*}
$$

where $I_{1}:=-\left(\boldsymbol{c} \delta^{\boldsymbol{\sigma}}, e_{h}^{\boldsymbol{\sigma}}\right), I_{2}:=\sum_{T \in \mathcal{T}_{h}}\left\langle P_{M} u-\Pi_{h}^{P} u, \boldsymbol{e}_{h}^{\boldsymbol{\sigma}} \cdot \boldsymbol{n}\right\rangle_{\partial T^{*}}$, and $I_{3}:=L_{\sigma, u}\left(e_{h}^{u}-e_{h}^{\widetilde{\lambda}}\right)$.
In fact, taking $\tau_{h}=e_{h}^{\boldsymbol{\sigma}}$ in (4.6), $v_{h}=e_{h}^{u}$ in (4.7), $\widetilde{\mu}_{h}=e_{h}^{\widetilde{\lambda}}$ in (4.8), and adding the resultant three equations together, we obtain

$$
\begin{aligned}
& \left(\boldsymbol{c} e_{h}^{\boldsymbol{\sigma}}, e_{h}^{\boldsymbol{\sigma}}\right)+\left(e_{h}^{u}, \operatorname{div}_{h} e_{h}^{\boldsymbol{\sigma}}\right)-\sum_{T \in \mathcal{T}_{h}}\left\langle e_{h}^{\tilde{\lambda}}, e_{h}^{\boldsymbol{\sigma}} \cdot n\right\rangle_{\partial T^{*}}-\left(\operatorname{div}_{h} e_{h}^{\boldsymbol{\sigma}}, e_{h}^{u}\right) \\
& \quad+\sum_{T \in \mathcal{T}_{h}}\left\langle\alpha_{T}\left(e_{h}^{u}-e_{h}^{\widetilde{\lambda}}\right), e_{h}^{u}\right\rangle_{\partial T^{*}}+\sum_{T \in \mathcal{T}_{h}}\left\langle e_{h}^{\boldsymbol{\sigma}} \cdot n-\alpha_{T}\left(e_{h}^{u}-e_{h}^{\widetilde{\lambda}}\right), e_{h}^{\widetilde{\lambda}}\right\rangle_{\partial T^{*}} \\
& =\left\|\left(e_{h}^{u}, e_{h}^{\widetilde{\lambda}}, e_{h}^{\boldsymbol{\sigma}}\right) \mid\right\|^{2}
\end{aligned}
$$

which, together with Lemma 4.3, yields (4.12).
In view of Cauchy-Schwarz inequality and the trace inequality, it is easy to get

$$
\begin{aligned}
& I_{1} \lesssim\left\|\delta^{\boldsymbol{\sigma}}\right\| \mathcal{T}_{h}\| \|\left(e_{h}^{u}, e_{h}^{\tilde{\lambda}}, e_{h}^{\boldsymbol{\sigma}}\right) \| \\
& I_{2} \\
& \lesssim\left(\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1}\left\|\left(P_{M} u-\Pi_{h}^{P} u\right)\right\|_{\partial T^{*}}^{2}\right)^{\frac{1}{2}}\| \|\left(e_{h}^{u}, e_{h}^{\tilde{\lambda}}, e_{h}^{\boldsymbol{\sigma}}\right) \| \\
& I_{3} \\
& \lesssim\left(\sum_{T \in \mathcal{T}_{h}} h_{T}\left\|\delta^{\boldsymbol{\sigma}}\right\|_{\partial T^{*}}^{2}+h_{T}^{-1}\left\|P_{V} u-\Pi_{h}^{P} u\right\|_{\partial T^{*}}^{2}\right)^{\frac{1}{2}}\| \|\left(e_{h}^{u}, e_{h}^{\widetilde{\lambda}}, e_{h}^{\boldsymbol{\sigma}}\right)\| \|
\end{aligned}
$$

Finally, the desired estimate (4.11) follows from (4.12) and the above three inequalities.
Based on the above lemmas, we easily derive the following error estimate for the flux approximation.

Theorem 4.1. Let $(u, \boldsymbol{\sigma}) \in H^{k+2}\left(\mathcal{T}_{h}\right) \times\left[H^{k+1}\left(\mathcal{T}_{h}\right)\right]^{d}$ be the weak solution to the model (1.1) with $k \geq 0$, and let $\left(u_{h}, \widetilde{\lambda}_{h}, \sigma_{h}\right) \in V_{h} \times \widetilde{M}_{h}(g) \times \mathbf{W}_{h}$ be the solution to the EDG scheme (3.6)-(3.8). Then we have

$$
\begin{equation*}
\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\| \lesssim h^{k+1}\left(\|\boldsymbol{\sigma}\|_{k+1, \mathcal{T}_{h}}+\|u\|_{k+2, \mathcal{T}_{h}}\right) \tag{4.13}
\end{equation*}
$$

Proof. The desired estimate (4.13) follows from the triangle inequality

$$
\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\| \leqslant\left\|\boldsymbol{\sigma}-P_{\mathbf{W}} \boldsymbol{\sigma}\right\|+\left\|P_{\mathbf{W}} \boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\|
$$

the defintion (4.10) of the seminorm $\|\|\cdot\|\|$, and Lemmas 4.1, 4.2 and 4.4.

### 4.2 Estimation for potential approximation

Based on the error estimation for the flux approximation $\sigma_{h}$ in the previous subsection, we shall use the Aubin-Nitsche's technique of duality argument to derive the estimation for the potential
approximation $u_{h}$. First, we introduce the following auxilliary problem:

$$
\begin{cases}c \boldsymbol{\Phi}-\nabla \Psi=0 & \text { in } \Omega  \tag{4.14}\\ \nabla \cdot \boldsymbol{\Phi}=e_{h}^{u} & \text { in } \Omega \\ \Psi=0, & \text { on } \partial \Omega\end{cases}
$$

where, as defined in (4.4), $e_{h}^{u}=P_{V} u-u_{h}$. In addition, we assume the following regularity property holds:

$$
\begin{equation*}
\|\boldsymbol{\Phi}\|_{1, \Omega}+\|\Psi\|_{2, \Omega} \lesssim\left\|e_{h}^{u}\right\|_{0, \Omega} \tag{4.15}
\end{equation*}
$$

We have the following equality.

## Lemma 4.5. It holds

$$
\begin{align*}
\left\|e_{h}^{u}\right\|^{2}= & \left(\boldsymbol{c} e_{h}^{\boldsymbol{\sigma}}, \delta^{\boldsymbol{\Phi}}\right)+\left(\boldsymbol{c} \delta^{\boldsymbol{\sigma}}, P_{\mathbf{W}} \boldsymbol{\Phi}\right)+\left\langle e_{h}^{u}-e_{h}^{\tilde{\lambda}}, \delta^{\boldsymbol{\Phi}} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h^{*}}}+\sum_{T \in \mathcal{T}_{h}}\left\langle\alpha_{T}\left(e_{h}^{u}-e_{h}^{\tilde{\lambda}}\right), P_{V} \Psi-\Pi_{h}^{P} \Psi\right\rangle_{\partial T^{*}} \\
& -\left\langle e_{h}^{\boldsymbol{\sigma}} \cdot \boldsymbol{n}, \delta^{\widetilde{\Psi}}\right\rangle_{\partial \mathcal{T}_{h^{*}}}+\left\langle P_{M} u-\Pi_{h}^{P} u, \delta^{\boldsymbol{\Phi}} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h^{*}}}-L_{\boldsymbol{\sigma}, u}\left(\Pi_{h}^{P} \Psi-P_{V} \Psi\right) \tag{4.16}
\end{align*}
$$

where $\delta^{\Phi}:=\boldsymbol{\Phi}-P_{\mathbf{W}} \boldsymbol{\Phi}, \delta^{\Psi}:=\Psi-P_{V} \Psi, \delta^{\widetilde{\Psi}}:=\Psi-\Pi_{h}^{P} \Psi$, and $e_{h}^{\boldsymbol{\sigma}}, \delta^{\sigma}, e_{h}^{\widetilde{\lambda}}$ are defined in (4.4).
Proof. By taking $\boldsymbol{\tau}_{h}=-P_{\mathbf{W}} \boldsymbol{\Phi}, v_{h}=P_{V} \Psi$, and $\widetilde{\mu}_{h}=\Pi_{h}^{P} \Psi$ in the error equations (4.6)-(4.8), we can get
$-\left(\boldsymbol{c} e_{h}^{\boldsymbol{\sigma}}, P_{\mathbf{W}} \boldsymbol{\Phi}\right)-\left(e_{h}^{u}, \nabla \cdot P_{\mathbf{W}} \boldsymbol{\Phi}\right)+\sum_{T \in \mathcal{T}_{h}}\left\langle e_{h}^{\widetilde{\lambda}}, P_{\mathbf{W}} \boldsymbol{\Phi} \cdot n\right\rangle_{\partial T^{*}}=\left(\boldsymbol{c} \delta^{\boldsymbol{\sigma}}, P_{\mathbf{W}} \boldsymbol{\Phi}\right)-\left\langle P_{M} u-\Pi_{h}^{P} u, P_{\mathbf{W}} \boldsymbol{\Phi} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h^{*}}}$,
$-\left(\nabla \cdot e_{h}^{\boldsymbol{\sigma}}, P_{V} \Psi\right)+\sum_{T \in \mathcal{T}_{h}}\left\langle\alpha_{T}\left(e_{h}^{u}-e_{h}^{\tilde{\lambda}}\right), P_{V} \Psi\right\rangle_{\partial T^{*}}=L_{\boldsymbol{\sigma}, u}\left(P_{V} \Psi\right)$,

$$
\begin{equation*}
\sum_{T \in \mathcal{T}_{h}}\left\langle e_{h}^{\boldsymbol{\sigma}} \cdot n-\alpha_{T}\left(e_{h}^{u}-e_{h}^{\tilde{\lambda}}\right), \Pi_{h}^{P} \Psi\right\rangle_{\partial T^{*}}=-L_{\boldsymbol{\sigma}, u}\left(\Pi_{h}^{P} \Psi\right) \tag{4.19}
\end{equation*}
$$

Integration by parts gives

$$
\begin{aligned}
-\left(e_{h}^{u}, \nabla \cdot P_{\mathbf{W}} \boldsymbol{\Phi}\right) & =-\left\langle e_{h}^{u}, P_{\mathbf{W}} \boldsymbol{\Phi} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h^{*}}}+\left(\nabla e_{h}^{u}, P_{\mathbf{W}} \boldsymbol{\Phi}\right)_{\mathcal{T}_{h}} \\
& =-\left\langle e_{h}^{u}, P_{\mathbf{W}} \boldsymbol{\Phi} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h^{*}}}+\left(\nabla e_{h}^{u}, \boldsymbol{\Phi}\right)_{\mathcal{T}_{h}} \\
& =\left\langle e_{h}^{u}, \delta^{\boldsymbol{\Phi}} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h^{*}}}-\left(e_{h}^{u}, \nabla \cdot \boldsymbol{\Phi}\right)_{\mathcal{T}_{h}} \\
& =\left\langle e_{h}^{u}, \delta^{\boldsymbol{\Phi}} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h^{*}}}-\left\|e_{h}^{u}\right\|_{\mathcal{T}_{h}}^{2} .
\end{aligned}
$$

Similarly, we can get

$$
-\left(\nabla \cdot e_{h}^{\boldsymbol{\sigma}}, P_{V} \Psi\right)_{\mathcal{T}_{h}}=\left(e_{h}^{\boldsymbol{\sigma}}, \nabla \Psi\right)_{\mathcal{T}_{h}}-\left\langle e_{h}^{\boldsymbol{\sigma}} \cdot \boldsymbol{n}, \Psi\right\rangle_{\partial \mathcal{T}_{h^{*}}}
$$

Inserting the two equations above into (4.17)-(4.18), we have

$$
\begin{align*}
-\left(\boldsymbol{c} e_{h}^{\boldsymbol{\sigma}}, \boldsymbol{\Phi}\right)_{\mathcal{T}_{h}} & +\left\langle e_{h}^{u}, \delta^{\boldsymbol{\Phi}} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h^{*}}}-\left\|e_{h}^{u}\right\|_{\mathcal{T}_{h}}^{2}+\left\langle e_{h}^{\widetilde{\lambda}}, P_{\mathbf{W}} \boldsymbol{\Phi} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h^{*}}} \\
& =\left(\boldsymbol{c} \delta^{\boldsymbol{\sigma}}, P_{\mathbf{W}} \boldsymbol{\Phi}\right)_{\mathcal{T}_{h}}-\left(\boldsymbol{c} e_{h}^{\boldsymbol{\sigma}}, \delta^{\boldsymbol{\Phi}}\right)_{\mathcal{T}_{h}}+\left\langle P_{M} u-\Pi_{h}^{P} u, P_{\mathbf{W}} \boldsymbol{\Phi} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h^{*}}},  \tag{4.20}\\
\left(e_{h}^{\boldsymbol{\sigma}}, \nabla \Psi\right)_{\mathcal{T}_{h}} & -\left\langle e_{h}^{\boldsymbol{\sigma}} \cdot \boldsymbol{n}, \Psi\right\rangle_{\partial \mathcal{T}_{h^{*}}}+\sum_{T \in \mathcal{T}_{h}}\left\langle\alpha_{T}\left(e_{h}^{u}-e_{h}^{\tilde{\lambda}}\right), P_{V} \Psi\right\rangle_{\partial T^{*}}=L_{\boldsymbol{\sigma}, u}\left(P_{V} \Psi\right) . \tag{4.21}
\end{align*}
$$

Adding equations (4.19), (4.20), and (4.21) together, and using the facts that $\boldsymbol{\Phi}-\nabla \Psi=0$ and $\left\langle e_{h}^{\widetilde{\lambda}}, \boldsymbol{\Phi} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h^{*}}}=0$, we obtain

$$
\begin{aligned}
& \left(e_{h}^{\boldsymbol{\sigma}},-\boldsymbol{c} \boldsymbol{\Phi}+\nabla \Psi\right)_{\mathcal{T}_{h}}+\left\langle e_{h}^{u}-e_{h}^{\tilde{\lambda}}, \delta^{\boldsymbol{\Phi}} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h^{*}}}-\left\|e_{h}^{u}\right\|_{\mathcal{T}_{h}}^{2} \\
& \quad+\sum_{T \in \partial \mathcal{T}_{h}}\left\langle\alpha_{T}\left(e_{h}^{u}-e_{h}^{\widetilde{\lambda}}\right), P_{V} \Psi-\Pi_{h}^{P} \Psi\right\rangle_{\partial T^{*}}-\left\langle e_{h}^{\boldsymbol{\sigma}} \cdot \boldsymbol{n}, \delta^{\widetilde{\Psi}}\right\rangle_{\partial \mathcal{T}_{h^{*}}} \\
& =-\left(\boldsymbol{c} \delta^{\boldsymbol{\sigma}}, P_{\mathbf{W}} \boldsymbol{\Phi}\right)_{\mathcal{T}_{h}}-\left(\boldsymbol{c} e_{h}^{\boldsymbol{\sigma}}, \delta^{\Phi}\right)_{\mathcal{T}_{h}}+L_{\boldsymbol{\sigma}, u}\left(P_{V} \Psi-\Pi_{h}^{P} \Psi\right) \\
& \quad-\left\langle P_{M} u-\Pi_{h}^{P} u, \delta^{\boldsymbol{\Phi}} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h^{*}}},
\end{aligned}
$$

which yields the desired conclusion.
In light of Lemma 4.5, we further have the following estimate.
Lemma 4.6. Under the regularity assumption (4.15), it holds

$$
\begin{equation*}
\left\|e_{h}^{u}\right\|_{\mathcal{T}_{h}} \lesssim h\left\|\mid\left(e_{h}^{u}, e_{h}^{\widetilde{\lambda}}, e_{h}^{\boldsymbol{\sigma}}\right)\right\|\|+h\| \delta^{\boldsymbol{\sigma}}\left\|_{\mathcal{T}_{h}}+h^{\frac{3}{2}}\right\| \delta^{\boldsymbol{\sigma}}\left\|_{\partial \mathcal{T}_{h^{*}}}+h^{\frac{1}{2}}\right\| P_{V} u-\Pi_{h}^{P} u \|_{\partial \mathcal{T}_{h^{*}}} \tag{4.22}
\end{equation*}
$$

Proof. Set $\left\|e_{h}^{u}\right\|^{2}=: \Pi_{1}+\Pi_{2}+\Pi_{3}+\Pi_{4}+\Pi_{5}+\Pi_{6}+\Pi_{7}$ with

$$
\begin{aligned}
& \Pi_{1}:=\left(\boldsymbol{c} e_{h}^{\boldsymbol{\sigma}}, \delta^{\boldsymbol{\Phi}}\right)_{\mathcal{T}_{h}}, \quad \Pi_{2}:=\left(\boldsymbol{c} \delta^{\boldsymbol{\sigma}}, P_{\mathbf{W}} \boldsymbol{\Phi}\right), \quad \Pi_{3}:=\left\langle e_{h}^{u}-e_{h}^{\widetilde{\lambda}}, \delta^{\boldsymbol{\Phi}} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h^{*}}} \\
& \Pi_{4}:=\sum_{T \in \mathcal{T}_{h}}\left\langle\alpha_{T}\left(e_{h}^{u}-e_{h}^{\widetilde{\lambda}}\right), P_{V} \Psi-\Pi_{h}^{P} \Psi\right\rangle_{\partial T^{*}}, \quad \Pi_{5}:=-\left\langle e_{h}^{\boldsymbol{\sigma}} \cdot \boldsymbol{n}, \delta^{\widetilde{\Psi}}\right\rangle_{\partial \mathcal{T}_{h^{*}}} \\
& \Pi_{6}:=\left\langle P_{M} u-\Pi_{h}^{P} u, \delta^{\boldsymbol{\Phi}} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h^{*}}}, \quad \Pi_{7}:=-L_{\boldsymbol{\sigma}, u}\left(\Pi_{h}^{P} \Psi-P_{V} \Psi\right)
\end{aligned}
$$

In view of Lemmas 4.1-4.2, the assumption (4.15), and Cauchy-Schwartz inequality, we obtain

$$
\begin{aligned}
& \left|\Pi_{1}\right| \lesssim\left\|e_{h}^{\boldsymbol{\sigma}}\right\|_{\mathcal{T}_{h}}\left\|\delta^{\boldsymbol{\Phi}}\right\|_{\mathcal{T}_{h}} \lesssim h\left\|e_{h}^{\boldsymbol{\sigma}}\right\|_{\mathcal{T}_{h}}\|\boldsymbol{\Phi}\|_{1, \Omega} \lesssim h\left|\left\|\left(e_{h}^{u}, e_{h}^{\widetilde{\lambda}}, e_{h}^{\boldsymbol{\sigma}}\right) \mid\right\|\left\|e_{h}^{u}\right\|_{\mathcal{T}_{h}},\right. \\
& \left.\left|\Pi_{3}\right| \leq\left\|e_{h}^{u}-e_{h}^{\widetilde{\lambda}}\right\|_{\partial \mathcal{T}_{h^{*}}}\left\|\delta^{\boldsymbol{\Phi}}\right\|_{\partial \mathcal{T}_{h}} \lesssim h^{\frac{1}{2}}\left\|e_{h}^{u}-e_{h}^{\widetilde{\lambda}}\right\|_{\partial \mathcal{T}_{h^{*}}}\|\boldsymbol{\Phi}\|_{1, \Omega} \lesssim h \right\rvert\,\left\|\left(e_{h}^{u}, e_{h}^{\widetilde{\lambda}}, e_{h}^{\boldsymbol{\sigma}}\right)\right\|\| \| e_{h}^{u} \|_{\mathcal{T}_{h}}, \\
& \left|\Pi_{4}\right| \leq h^{-1}\left\|e_{h}^{u}-e_{h}^{\widetilde{\lambda}}\right\|_{\partial \mathcal{T}_{h^{*}}}\left\|P_{V} \Psi-\Pi_{h}^{P} \Psi\right\|_{\partial \mathcal{T}_{h^{*}}} \lesssim h \mid\left\|\left(e_{h}^{u}, e_{h}^{\widetilde{\lambda}}, e_{h}^{\sigma}\right)\right\|\| \| e_{h}^{u} \|_{\mathcal{T}_{h}}, \\
& \left|\Pi_{5}\right| \leq\left\|e_{h}^{\boldsymbol{\sigma}}\right\|_{\partial \mathcal{T}_{h^{*}}}\left\|\delta^{\widetilde{\Psi}}\right\|_{\partial \mathcal{T}_{h^{*}}} \lesssim h\left\|e_{h}^{\boldsymbol{\sigma}}\right\|_{\mathcal{T}_{h}}\|\Psi\|_{2, \Omega} \lesssim h\left|\left\|\left(e_{h}^{u}, e_{h}^{\widetilde{\lambda}}, e_{h}^{\boldsymbol{\sigma}}\right) \mid\right\|\left\|e_{h}^{u}\right\|_{\mathcal{T}_{h}},\right. \\
& \left|\Pi_{6}\right| \lesssim\left\|P_{M} u-\Pi_{h}^{P} u\right\|_{\partial \mathcal{T}_{h^{*}}}\left\|\delta^{\Phi}\right\|_{\partial \mathcal{T}_{h^{*}}} \lesssim h^{\frac{1}{2}}\left\|P_{M} u-\Pi_{h}^{P} u\right\|_{\partial \mathcal{T}_{h^{*}}}\left\|e_{h}^{u}\right\|_{\mathcal{T}_{h}} \\
& \left|\Pi_{7}\right| \lesssim\left(\left\|\delta^{\sigma}\right\|_{\partial \mathcal{T}_{h^{*}}}+h^{-1}\left\|P_{V} u-\Pi_{h}^{P} u\right\|_{\partial \mathcal{T}_{h^{*}}}\right)\left(\left\|P_{V} \Psi-\Pi_{h}^{P} \Psi\right\|_{\partial \mathcal{T}_{h^{*}}}\right) \\
& \lesssim h^{\frac{3}{2}}\left\|\delta^{\sigma}\right\|_{\partial \mathcal{T}_{h^{*}}}\|\Psi\|_{2, \Omega}+h^{\frac{1}{2}}\left\|P_{V} u-\Pi_{h}^{P} u\right\|_{\partial \mathcal{T}_{h^{*}}}\|\Psi\|_{2, \Omega} \\
& \lesssim\left(h^{\frac{3}{2}}\left\|\delta^{\boldsymbol{\sigma}}\right\|_{\partial \mathcal{T}_{h^{*}}}+h^{\frac{1}{2}}\left\|P_{V} u-\Pi_{h}^{P} u\right\|_{\partial \mathcal{T}_{h^{*}}}\right)\left\|e_{h}^{u}\right\|_{\mathcal{T}_{h}}
\end{aligned}
$$

The thing left is to estimate $\Pi_{2}$, and we have

$$
\begin{aligned}
\Pi_{2} & =\left(P_{\mathbf{W}} \boldsymbol{\sigma}-\boldsymbol{\sigma}, \boldsymbol{c} P_{\mathbf{W}} \boldsymbol{\Phi}\right) \\
& =\left(P_{\mathbf{W}} \boldsymbol{\sigma}-\boldsymbol{\sigma}, \boldsymbol{c}\left(P_{\mathbf{W}} \boldsymbol{\Phi}-\boldsymbol{\Phi}\right)\right)+\left(P_{\mathbf{W}} \boldsymbol{\sigma}-\boldsymbol{\sigma}, \boldsymbol{c} \boldsymbol{\Phi}\right) \\
& =\left(P_{\mathbf{W}} \boldsymbol{\sigma}-\boldsymbol{\sigma}, \boldsymbol{c}\left(P_{\mathbf{W}} \boldsymbol{\Phi}-\boldsymbol{\Phi}\right)\right)+\left(P_{\mathbf{W}} \boldsymbol{\sigma}-\boldsymbol{\sigma}, \nabla \phi-\nabla P_{V} \phi\right) \\
& \lesssim h\left\|\delta^{\boldsymbol{\sigma}}\right\|_{\mathcal{T}_{h}}\|\boldsymbol{\Phi}\|_{1, \Omega}+h\left\|\delta^{\boldsymbol{\sigma}}\right\|_{\mathcal{T}_{h}}\|\Psi\|_{2, \Omega} \\
& \lesssim h\left\|\delta^{\boldsymbol{\sigma}}\right\|_{\mathcal{T}_{h}}\left\|e_{h}^{u}\right\|_{\mathcal{T}_{h}} .
\end{aligned}
$$

Finally, combining all the estimates of $\Pi_{j}(j=1, \cdots, 7)$ indicates the conclusion.
Theorem 4.2. Let $(u, \boldsymbol{\sigma}) \in H^{k+2}\left(\mathcal{T}_{h}\right) \times\left[H^{k+1}\left(\mathcal{T}_{h}\right)\right]^{d}$ be the weak solution to model (1.1) with $k \geq 0$, and let $\left(u_{h}, \widetilde{\lambda}_{h}, \sigma_{h}\right) \in V_{h} \times \widetilde{M}_{h}(g) \times \mathbf{W}_{h}$ be the solution to the EDG scheme (3.6)-(3.8). Then, under the regularity assumption (4.15), it holds

$$
\begin{equation*}
\left\|u-u_{h}\right\| \lesssim h^{k+2}\left(\|\boldsymbol{\sigma}\|_{k+1, \mathcal{T}_{h}}+\|u\|_{k+2, \mathcal{T}_{h}}\right) \tag{4.23}
\end{equation*}
$$

Proof. From the triangle inequality and Lemmas 4.6, 4.4 and 4.1, we have

$$
\left\|u-u_{h}\right\|_{\mathcal{T}_{h}} \leq\left\|e_{h}^{u}\right\|_{\mathcal{T}_{h}}+\left\|\delta^{u}\right\|_{\mathcal{T}_{h}} \lesssim h^{k+2}\left(|\boldsymbol{\sigma}|_{k+1}+|u|_{k+2}\right) .
$$

## 5 Numerical results

In this section, we use a two-dimensional numerical example to verify the theoretical results. We take $\Omega=[0,1] \times[0,1]$, and let the exact solution to (1.1) be $u(x, y)=\sin (\pi x) \sin (\pi y)$ with the coefficient matrix

$$
\boldsymbol{c}=\left[\begin{array}{cc}
1+x^{2} y^{2} & 0  \tag{5.1}\\
0 & 1+x^{2} y^{2}
\end{array}\right]
$$

We consider two types of meshes: uniform triangular meshes and quadrilateral meshes (Figure 1). Numerical results of the flux and potential approximations are listed in Tables 1 and 2 for the proposed EDG methods and the corresponding HDG methods with $k=0,1,2$. We can see that both the HDG and EDG methods converge with the optimal rates.

Table 3 shows the numbers of unknowns of the reduced system (3.16) with $k=0,1,2$ which contains the degrees of freedom of the numerical traces on interelement boundary as the only unknowns. In this example, the EDG method always leads to smaller systems than the corresponding HDG method.

Table 1: Convergence history on triangular meshes

|  |  | EDG |  |  | HDG |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $k$ | Mesh | $\left\\|u-u_{h}\right\\|$ | rate | $\left\\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\\|$ | rate | $\left\\|u-u_{h}\right\\|$ | rate | $\left\\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\\|$ | rate |
| 0 | $4 \times 4$ | $8.076 \mathrm{e}-2$ | - | $3.820 \mathrm{e}-1$ | - | $1.940 \mathrm{e}-1$ | - | $2.775 \mathrm{e}-1$ | - |
|  | $8 \times 8$ | $2.084 \mathrm{e}-2$ | 1.954 | $1.966 \mathrm{e}-1$ | 0.958 | $4.917 \mathrm{e}-2$ | 1.980 | $1.412 \mathrm{e}-1$ | 0.975 |
|  | $16 \times 16$ | $5.274 \mathrm{e}-3$ | 1.982 | $9.907 \mathrm{e}-2$ | 0.989 | $1.233 \mathrm{e}-2$ | 1.996 | $7.089 \mathrm{e}-2$ | 0.994 |
|  | $32 \times 32$ | $1.323 \mathrm{e}-3$ | 1.995 | $4.963 \mathrm{e}-2$ | 0.997 | $3.086 \mathrm{e}-3$ | 1.998 | $3.549 \mathrm{e}-2$ | 0.998 |
|  | $64 \times 64$ | $3.310 \mathrm{e}-4$ | 1.999 | $2.483 \mathrm{e}-2$ | 0.999 | $7.717 \mathrm{e}-4$ | 2.000 | $1.775 \mathrm{e}-2$ | 1.000 |
| 1 | $4 \times 4$ | $2.305 \mathrm{e}-2$ | - | $4.770 \mathrm{e}-2$ | - | $2.410 \mathrm{e}-2$ | - | $4.232 \mathrm{e}-2$ |  |
|  | $8 \times 8$ | $2.883 \mathrm{e}-3$ | 2.999 | $1.312 \mathrm{e}-2$ | 1.862 | $3.031 \mathrm{e}-3$ | 2.991 | $1.085 \mathrm{e}-2$ | 1.964 |
|  | $16 \times 16$ | $3.522 \mathrm{e}-4$ | 3.033 | $3.580 \mathrm{e}-3$ | 1.874 | $3.791 \mathrm{e}-4$ | 2.999 | $2.730 \mathrm{e}-3$ | 1.991 |
|  | $32 \times 32$ | $4.336 \mathrm{e}-5$ | 3.022 | $9.358 \mathrm{e}-4$ | 1.936 | $4.739 \mathrm{e}-5$ | 2.999 | $6.839 \mathrm{e}-4$ | 1.997 |
|  | $64 \times 64$ | $5.387 \mathrm{e}-6$ | 3.009 | $2.377 \mathrm{e}-4$ | 1.977 | $5.923 \mathrm{e}-6$ | 3.000 | $1.711 \mathrm{e}-4$ | 1.999 |
| 2 | $4 \times 4$ | $2.677 \mathrm{e}-3$ | - | $6.482 \mathrm{e}-3$ | - | $2.880 \mathrm{e}-3$ | - | $5.378 \mathrm{e}-3$ |  |
|  | $8 \times 8$ | $1.746 \mathrm{e}-4$ | 3.938 | $8.074 \mathrm{e}-4$ | 3.005 | $1.827 \mathrm{e}-4$ | 3.978 | $6.874 \mathrm{e}-4$ | 2.968 |
|  | $16 \times 16$ | $1.106 \mathrm{e}-5$ | 3.981 | $1.004 \mathrm{e}-4$ | 3.008 | $1.146 \mathrm{e}-5$ | 3.995 | $8.650 \mathrm{e}-5$ | 2.990 |
|  | $32 \times 32$ | $6.938 \mathrm{e}-7$ | 3.994 | $1.251 \mathrm{e}-5$ | 3.004 | $7.166 \mathrm{e}-7$ | 3.999 | $1.084 \mathrm{e}-5$ | 2.997 |
|  | $64 \times 64$ | $4.342 \mathrm{e}-08$ | 3.998 | $1.561 \mathrm{e}-6$ | 3.002 | $4.479 \mathrm{e}-8$ | 4.000 | $1.355 \mathrm{e}-6$ | 2.999 |



Figure 1: Two types of meshes: Left: $4 \times 4 ;$ Right: $8 \times 8$.

Table 2: Convergence history on quadrilateral meshes

|  |  | EDG |  |  | HDG |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $k$ | Mesh | $\left\\|u-u_{h}\right\\|$ | rate | $\left\\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\\|$ | rate | $\left\\|u-u_{h}\right\\|$ | rate | $\left\\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h}\right\\|$ | rate |
| 0 | $4 \times 4$ | $2.623 \mathrm{e}-1$ | - | $3.324 \mathrm{e}-1$ | - | $2.722 \mathrm{e}-1$ | - | $3.362 \mathrm{e}-1$ | - |
|  | $8 \times 8$ | $6.917 \mathrm{e}-2$ | 1.923 | $1.706 \mathrm{e}-1$ | 0.962 | $7.228 \mathrm{e}-2$ | 1.913 | $1.721 \mathrm{e}-1$ | 0.950 |
|  | $16 \times 16$ | $1.752 \mathrm{e}-2$ | 1.981 | $8.583 \mathrm{e}-2$ | 0.991 | $1.845 \mathrm{e}-2$ | 1.970 | $8.621 \mathrm{e}-2$ | 0.997 |
|  | $32 \times 32$ | $4.395 \mathrm{e}-3$ | 1.995 | $4.298 \mathrm{e}-2$ | 0.998 | $4.644 \mathrm{e}-3$ | 1.990 | $4.305 \mathrm{e}-2$ | 1.002 |
|  | $64 \times 64$ | $1.100 \mathrm{e}-3$ | 1.998 | $2.150 \mathrm{e}-2$ | 0.999 | $1.163 \mathrm{e}-3$ | 1.998 | $2.151 \mathrm{e}-2$ | 1.001 |
| 1 | $4 \times 4$ | $4.666 \mathrm{e}-2$ | - | $7.782 \mathrm{e}-2$ | - | $4.498 \mathrm{e}-2$ | - | $8.057 \mathrm{e}-2$ | - |
|  | $8 \times 8$ | $6.023 \mathrm{e}-3$ | 2.960 | $1.973 \mathrm{e}-2$ | 1.982 | $5.408 \mathrm{e}-3$ | 3.056 | $2.058 \mathrm{e}-2$ | 1.919 |
|  | $16 \times 16$ | $7.633 \mathrm{e}-4$ | 2.980 | $4.958 \mathrm{e}-3$ | 1.993 | $6.592 \mathrm{e}-4$ | 3.036 | $5.094 \mathrm{e}-3$ | 2.014 |
|  | $32 \times 32$ | $9.590 \mathrm{e}-5$ | 2.993 | $1.241 \mathrm{e}-3$ | 1.998 | $8.201 \mathrm{e}-5$ | 3.007 | $1.258 \mathrm{e}-3$ | 2.018 |
|  | $64 \times 64$ | $1.201 \mathrm{e}-5$ | 2.997 | $3.104 \mathrm{e}-4$ | 1.999 | $1.026 \mathrm{e}-5$ | 3.000 | $3.123 \mathrm{e}-4$ | 2.010 |
| 2 | $4 \times 4$ | $7.744 \mathrm{e}-3$ | - | $1.200 \mathrm{e}-2$ | - | $7.720 \mathrm{e}-3$ | - | $1.219 \mathrm{e}-2$ | - |
|  | $8 \times 8$ | $4.942 \mathrm{e}-4$ | 3.970 | $1.387 \mathrm{e}-3$ | 3.113 | $4.936 \mathrm{e}-4$ | 3.967 | $1.412 \mathrm{e}-3$ | 3.111 |
|  | $16 \times 16$ | $3.096 \mathrm{e}-5$ | 3.997 | $1.729 \mathrm{e}-4$ | 3.004 | $3.101 \mathrm{e}-5$ | 3.993 | $1.749 \mathrm{e}-4$ | 3.013 |
|  | $32 \times 32$ | $1.933 \mathrm{e}-6$ | 4.002 | $2.139 \mathrm{e}-5$ | 3.015 | $1.940 \mathrm{e}-6$ | 3.999 | $2.152 \mathrm{e}-5$ | 3.023 |
|  | $64 \times 64$ | $1.207 \mathrm{e}-7$ | 4.001 | $2.659 \mathrm{e}-6$ | 3.008 | $1.213 \mathrm{e}-7$ | 4.000 | $2.667 \mathrm{e}-6$ | 3.012 |

Table 3: Comparison of numbers of degrees of freedom

|  |  | simplex | meshes | quadrilateral | meshes |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $k$ | Mesh | EDG | HDG | EDG | HDG |
| 0 | $4 \times 4$ | 9 | 80 | 9 | 48 |
|  | $8 \times 8$ | 49 | 352 | 49 | 224 |
|  | $16 \times 16$ | 225 | 1472 | 225 | 960 |
|  | $32 \times 32$ | 961 | 6016 | 961 | 3968 |
|  | $64 \times 64$ | 3936 | 24320 | 3936 | 16128 |
| 1 | $4 \times 4$ | 49 | 120 | 33 | 72 |
|  | $8 \times 8$ | 225 | 528 | 161 | 336 |
|  | $16 \times 16$ | 961 | 2208 | 705 | 1440 |
|  | $32 \times 32$ | 3936 | 9024 | 2945 | 5952 |
|  | $64 \times 64$ | 16129 | 36480 | 12033 | 24192 |
| 2 | $4 \times 4$ | 89 | 160 | 57 | 96 |
|  | $8 \times 8$ | 401 | 704 | 273 | 448 |
|  | $16 \times 16$ | 1697 | 2944 | 1185 | 1920 |
|  | $32 \times 32$ | 6977 | 12032 | 4929 | 7936 |
|  | $64 \times 64$ | 28289 | 48640 | 20097 | 32256 |

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