

Research Article

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Nonlinear Evolution Equations with Exponentially Decaying Memory: Existence via Time Discretisation, Uniqueness, and Stability

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Abstract: The initial value problem for an evolution equation of type $v' + Av + BKv = f$ is studied, where $A : V_A \rightarrow V'_A$ is a monotone, coercive operator and where $B : V_B \rightarrow V'_B$ induces an inner product. The Banach space V_A is not required to be embedded in V_B or vice versa. The operator K incorporates a Volterra integral operator in time of convolution type with an exponentially decaying kernel. Existence of a global-in-time solution is shown by proving convergence of a suitable time discretisation. Moreover, uniqueness as well as stability results are proved. Appropriate integration-by-parts formulae are a key ingredient for the analysis.

Keywords: Nonlinear Evolution Equation, Monotone Operator, Volterra Operator, Exponentially Decaying Memory, Existence, Uniqueness, Stability, Time Discretisation, Convergence

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Dedicated to Professor Rolf D. Grigorieff on the occasion of his 80th birthday

1 Introduction

1.1 Problem Statement and Main Result

We are concerned with initial value problems of the type

$$v' + Av + BKv = f \quad \text{in } (0, T), \quad (1.1a)$$

$$v(0) = v_0, \quad (1.1b)$$

where

$$(Kv)(t) = u_0 + \int_0^t k(t-s)v(s) ds, \quad k(z) = \lambda e^{-\lambda z}. \quad (1.1c)$$

Here, $T > 0$ is the time under consideration, $\lambda > 0$ is a given parameter, and u_0, v_0, f are given data of the problem.

The operator $A : V_A \rightarrow V'_A$ is a hemicontinuous, monotone, coercive operator satisfying a certain growth condition, where V_A denotes a real, reflexive Banach space. The operator $B : V_B \rightarrow V'_B$ is a linear, symmetric,

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bounded, strongly positive operator on a real Hilbert space V_B . We assume that both V_A and V_B are densely and continuously embedded in a real Hilbert space H , which is identified with its dual. We emphasise that we do not require V_A to be embedded in V_B (or vice versa) but only assume that $V = V_A \cap V_B$ is densely embedded in V_A as well as in V_B . This yields the scale

$$V_A \cap V_B = V \subset V_C \subset H = H' \subset V'_C \subset V' = V'_A + V'_B, \quad C \in \{A, B\}, \quad (1.2)$$

of Banach and Hilbert spaces with dense and continuous embeddings. Moreover, we assume that V is separable.

Since the kernel k in (1.1c) is assumed to be of exponential type, (1.1a) can easily be derived from the system

$$v' + Av + Cu = f, \quad (1.3a)$$

$$(u - u_0)' + \lambda(u - u_0) = \lambda Dv, \quad (1.3b)$$

with suitable linear operators C, D such that $B = CD$ (and with u_0 being replaced by Du_0).

Such systems appear, e.g., in the description of viscoelastic fluid flow (see, e.g., [15, 38]) as well as heat flow in materials with memory (see, e.g., [39, 43]).

Other applications arise in, e.g., non-Fickian diffusion models describing diffusion processes of a penetrant through a viscoelastic material (see, e.g., [17, 18, 46]), where, apart from the usual diffusion rate of the penetrant, the change in the internal structure of the viscoelastic material has to be considered. Non-Fickian diffusion also plays a role in, e.g., mathematical biology (see, e.g., [13, 20, 42]).

Note that all the results obtained in this paper are also valid for kernels of the type $k(z) = ce^{-\lambda z}$ ($c, \lambda > 0$). The proof only differs for the stability with respect to perturbations of the kernel. However, in many applications, kernels of the type given in (1.1c) naturally arise. In such applications, one often deals with a coupled system of the type (1.3), where $\frac{1}{\lambda}$ can be interpreted as a relaxation or averaged delay time. If λ tends to 0, then the system decouples with $u(t) \equiv u_0$. The limit $\lambda \rightarrow \infty$ results in a first-order equation for v without memory. In contrast to this, in the case where $k(z) = ce^{-\lambda z}$, the limit $\lambda \rightarrow 0$ results in an evolution equation for u of second order in time (see [19]), whereas $\lambda \rightarrow \infty$ results again in a first-order evolution equation for v .

Our goal in this paper is twofold: On the one hand, we wish to prove existence of generalised solutions to (1.1). On the other hand, we wish to prove convergence of a simple time discretisation method, which relies on the implicit Euler or Rothe method combined with a product quadrature. Moreover, we prove stability of the solution against perturbations of the problem data (including perturbations of λ), which also implies uniqueness.

1.2 Literature Overview

Most of the results on evolutionary Volterra integral equations available so far are, indeed, more general with respect to the integral kernel (often dealing with memory of positive type). Our results, however, include the case of different domains of definition of the underlying operators involved without assuming that V_A is embedded in V_B (or vice versa). Such a situation occurs, e.g., if A is a spatial differential operator of order lower than that of B . This has, to the best knowledge of the authors, not yet been studied. Dealing with memory of exponentially decaying type can thus be seen as a first step within this more general functional analytic framework.

Well-posedness for linear evolutionary integral equations in Banach or Hilbert spaces has been studied in detail by many authors for a long time. We only refer to the standard monographs [30, 44] and the references cited therein. For the semigroup approach, we also refer to [2]. For the finite-dimensional case (including nonlinear problems), one may also consult [9].

Whereas there are many results available for linear problems, there is less known for the nonlinear case. Classical results for nonlinear problems are due to Clément, Crandall, Dafermos, Desch, Gripenberg, Londen, MacCamy, Nohel, and others. One of the equations that is mostly studied is of the type

$$v' + a * Av = f,$$

where a is a suitable kernel function (e.g., completely monotone or of positive type) and A is a suitable nonlinear operator satisfying certain monotonicity assumptions; see, e.g., [14, 27–29, 33, 50–52].

For example, MacCamy, and Wong [40] consider a class of nonlinear integro-differential equations with completely monotone kernel in a Hilbert space setting. Gajewski, Gröger, and Zacharias [22] study rather general classes of nonlinear evolution equations with Volterra operators, but are restricted to the Hilbert space case $V_A = V_B$. Crandall, Londen, and Nohel [10] consider a doubly nonlinear problem governed by maximal monotone subdifferential operators, where the domain of definition of one of the operators is dense and continuously embedded into the domain of definition of the other operator; see also [11]. An interesting result for a class of doubly nonlinear integro-differential equations governed by an m -dissipative operator can be found in [26]. The nonlocality in time here is of a special form incorporating this m -dissipative operator again. The domain of definition of the principal part of the equations, however, equals the domain of definition of the nonlocality in time. The method of proof relies upon nonlinear semigroup theory. More recent results can be found, e.g., in [1, 3, 6, 23–25, 48, 53, 54]. The above-cited references do not cover the class of problems we consider here.

The question of numerical approximation, mostly for linear or semilinear problems with memory of positive type, has been dealt with by McLean, Sloan, Thomée, and Wahlbin (see, e.g., [31, 37, 41, 47]). The focus is on Galerkin finite element methods combined with suitable time discretisation methods based on the backward Euler scheme (see, e.g., [7, 8]). Time discretisation methods have also been studied by Calvo, Lubich, and Palencia (see, e.g., [5, 12]) and, in particular, convolution quadrature by Lubich [34–36] and Ferreira [21]. Indeed, our proof of existence also relies upon the convergence of a numerical scheme that is based on a convolution quadrature.

1.3 Organisation of the Paper

The paper is organised as follows: The general notation is explained in Section 2. In Section 3, we state the main assumptions on the operators A and B and collect some preliminary results on their properties. The main existence result (Theorem 4.2) is provided in Section 4 by showing (weak) convergence of a suitable time discretisation to (1.1). The crucial point here is an integration-by-parts formula given in Lemma 4.3. In Section 5, we show uniqueness and stability with respect to perturbations of the problem data. In particular, we also consider perturbations of the kernel parameter λ .

2 Notation

For a Banach space X , we denote its norm by $\|\cdot\|_X$, its dual by X' , equipped with the standard norm $\|\cdot\|_{X'}$, and the duality pairing by $\langle \cdot, \cdot \rangle$. We recall that X' is reflexive and separable if X is. For a Hilbert space X , we denote the inner product (inducing the norm $\|\cdot\|_X$) by $(\cdot, \cdot)_X$. The intersection of two Banach spaces X, Y is equipped with the norm $\|\cdot\|_{X \cap Y} = \|\cdot\|_X + \|\cdot\|_Y$, whereas the sum $X + Y$ is equipped with the norm

$$\|g\|_{X+Y} = \inf\{\max(\|g_X\|_X, \|g_Y\|_Y) : g = g_X + g_Y \text{ with } g_X \in X, g_Y \in Y\}.$$

We recall that $(X \cap Y)' = X' + Y'$; see also [22, pp. 12ff.].

For a real, reflexive, separable Banach space X , the Bochner–Lebesgue spaces $L^r(0, T; X)$ ($r \in [1, \infty]$) are defined in the usual way and equipped with the standard norm. Denoting by $r' = \frac{r}{r-1}$ the conjugate of $r \in (1, \infty)$ with $r' = \infty$ if $r = 1$, we have that $(L^r(0, T; X))' = L^{r'}(0, T; X')$ if $r \in [1, \infty)$; the duality pairing is given by

$$\langle g, v \rangle = \int_0^T \langle g(t), v(t) \rangle dt;$$

see, e.g., [16, Theorem 1 on p. 98, Corollary 13 on p. 76, Theorem 1 on p. 79]. Moreover, $L^r(0, T; X)$ is reflexive if $r \in (1, \infty)$ (see [16, Corollary 2 on p. 100]) and $L^1(0, T; X)$ is separable.

By $W^{1,r}(0, T; X)$ ($r \in [1, \infty]$) we denote the Banach space of functions $u \in L^r(0, T; X)$ whose distributional time derivative u' is again in $L^r(0, T; X)$; the space is equipped with the standard norm. Note that if $u \in W^{1,1}(0, T; X)$, then u equals almost everywhere a function that is in $\mathcal{AC}([0, T]; X)$, i.e., a function that is absolutely continuous on $[0, T]$ as a function taking values in X . Moreover, $W^{1,1}(0, T; X)$ is continuously embedded in the Banach space $\mathcal{C}([0, T]; X)$ of functions that are continuous on $[0, T]$ as functions with values in X (see, e.g., [45, Chapter 7] for more details). By $\mathcal{C}_w([0, T]; X)$ we denote the space of functions that are continuous on $[0, T]$ with respect to the weak topology in X .

The space of infinitely many times differentiable real functions with compact support in $(0, T)$ is denoted by $\mathcal{C}_c^\infty(0, T)$. The space of on $[0, T]$ continuously differentiable real functions is denoted by $\mathcal{C}^1([0, T])$. By c we denote a generic positive constant. We set $\sum_{j=1}^0 x_j = 0$ for x_j whatsoever.

3 Main Assumptions and Preliminary Results

Let V_A be a real, reflexive Banach space and let V_B and H be real Hilbert spaces satisfying (1.2). Moreover, we assume that $V = V_A \cap V_B$ is separable.

The structural properties we assume for the operators A and B throughout this paper are as follows.

Assumption A. The operator $A : V_A \rightarrow V'_A$ satisfies for all $u, v, w \in V_A$,

(i) $\theta \mapsto \langle A(u + \theta v), w \rangle \in \mathcal{C}([0, 1])$ (hemicontinuity),

and there exist $p \in (2, \infty)$, $\mu_A, \beta_A > 0$, $c_A \geq 0$ such that for all $v, w \in V_A$,

(ii) $\langle Av - Aw, v - w \rangle \geq 0$ (monotonicity),

(iii) $\langle Av, v \rangle \geq \mu_A \|v\|_{V_A}^p - c_A$ (p -coercivity),

(iv) $\|Av\|_{V'_A} \leq \beta_A (1 + \|v\|_{V_A}^{p-1})$ ($(p-1)$ -growth).

Assumption B. The linear operator $B : V_B \rightarrow V'_B$ is symmetric and there exist $\mu_B, \beta_B > 0$ such that for all $v \in V_B$,

(i) $\langle Bv, v \rangle \geq \mu_B \|v\|_{V_B}^2$ (strong positivity),

(ii) $\|Bv\|_{V'_B} \leq \beta_B \|v\|_{V_B}$ (boundedness).

By $\|\cdot\|_B = \langle B\cdot, \cdot \rangle^{1/2}$ we denote the norm induced by B , which is equivalent to $\|\cdot\|_{V_B}$. Further, by $L^2(0, T; B)$ we denote the space $L^2(0, T; (V_B, \|\cdot\|_B))$.

We shall remark that the operators A and B can be extended, as usual, to operators acting on functions defined on $[0, T]$ and taking values in V_A and V_B , respectively. Since the operator $A : V_A \rightarrow V'_A$ is hemicontinuous and monotone, it is also demicontinuous (see, e.g., [55, Proposition 26.4 on p. 555]). Therefore, the operator A maps, in view of the separability of V'_A and the theorem of Pettis (see, e.g., [16, Theorem 2 on p. 42]), a Bochner measurable function $v : [0, T] \rightarrow V_A$ into a Bochner measurable function $Av : [0, T] \rightarrow V'_A$ with $(Av)(t) = Av(t)$ ($t \in [0, T]$). Because of the growth condition, the operator A then maps $L^p(0, T; V_A)$ into $(L^p(0, T; V_A))' = L^{p'}(0, T; V'_A)$. The linear, symmetric, bounded, strongly positive operator $B : V_B \rightarrow V'_B$ extends, via $(Bu)(t) = Bu(t)$ for a function $u : [0, T] \rightarrow V_B$, to a linear, symmetric, bounded, strongly positive operator mapping $L^2(0, T; V_B)$ into its dual. Indeed, B can also be seen as a linear, bounded operator mapping $L^r(0, T; V_B)$ into $L^r(0, T; V'_B)$ for any $r \in [1, \infty]$.

With respect to the operator K defined by (1.1c), we make the following observations.

Lemma 3.1. *Let k be as in (1.1c) and let $u_0 \in V_B$. Then K is an affine-linear, bounded mapping of $L^2(0, T; V_B)$ into itself with*

$$\|Kv - u_0\|_{L^2(0, T; V_B)} \leq \|k\|_{L^1(0, T)} \|v\|_{L^2(0, T; V_B)}, \quad v \in L^2(0, T; V_B),$$

where $\|k\|_{L^1(0, T)} = 1 - e^{-\lambda T}$. Further, K is an affine-linear, bounded mapping of $L^1(0, T; V_B)$ into $\mathcal{AC}([0, T]; V_B)$ with

$$\|Kv - u_0\|_{\mathcal{C}([0, T]; V_B)} \leq \lambda \|v\|_{L^1(0, T; V_B)}, \quad v \in L^1(0, T; V_B).$$

The standard proof is omitted here.

Note that in the assertions above V_B can be replaced by H . An immediate consequence of the lemma above is that $BK : L^2(0, T; V_B) \rightarrow L^2(0, T; V'_B)$ as well as $BK : L^1(0, T; V_B) \rightarrow L^\infty(0, T; V'_B)$ is affine-linear and bounded if Assumption B is satisfied.

We will often make use of the following relation, which indeed is crucial within this work and reflects the exponential type of the memory kernel. For a function $v \in L^1(0, T; X)$, with X being an arbitrary Banach space, we have that for almost all $t \in (0, T)$,

$$(Kv)'(t) = \lambda(v(t) - ((Kv)(t) - u_0)). \quad (3.1)$$

If X is a Hilbert space and if $v \in L^2(0, T; X)$, then testing this relation by $Kv - u_0$ immediately implies for all $t \in [0, T]$,

$$\int_0^t ((Kv)(s) - u_0, v(s))_X ds = \int_0^t \|(Kv)(s) - u_0\|_X^2 ds + \frac{1}{2\lambda} \|(Kv)(t) - u_0\|_X^2,$$

which shows that the memory term is of positive type.

4 Main Result: Existence via Time Discretisation

In this section, we show existence of generalised solutions to (1.1) by proving weak or weak* convergence of a sequence of approximate solutions constructed from a suitable time discretisation. We commence by studying the corresponding numerical scheme and its properties.

4.1 Time Discretisation

For $N \in \mathbb{N}$, let $\tau = \frac{T}{N}$ and $t_n = n\tau$ ($n = 0, 1, \dots, N$). Let $v^0 \approx v_0$, $u^0 \approx u_0$, and $\{f^n\}_{n=1}^N \approx f$ be given approximations of the problem data v_0 , u_0 , and f , respectively. We look for approximations $v^n \approx v(t_n)$ ($n = 1, 2, \dots, N$).

The numerical scheme we consider combines the implicit Euler method with a convolution or product quadrature for the integral operator K and reads

$$\frac{1}{\tau}(v^n - v^{n-1}) + Av^n + (BK_{u^0}^\tau v)^n = f^n, \quad n = 1, 2, \dots, N, \quad (4.1a)$$

where

$$(K_{u^0}^\tau v)^n := u^0 + \tau \sum_{j=1}^n \gamma_{n-j+1} v^j. \quad (4.1b)$$

To be precise, $K_{u^0}^\tau$ acts on a grid function $\{v^n\}_{n=1}^N$ and, with a slight abuse of notation, the evaluation at n is denoted by $(K_{u^0}^\tau v)^n$. The coefficients γ_i ($i = 1, 2, \dots, N$) are given by

$$\gamma_i = \int_0^1 k((i - \hat{s})\tau) d\hat{s}, \quad (4.1c)$$

and hence

$$\gamma_{n-j+1} = \int_0^1 k((n - j + 1 - \hat{s})\tau) d\hat{s} = \frac{1}{\tau} \int_{t_{j-1}}^{t_j} k(t_n - s) ds.$$

The idea behind is the approximation

$$\int_0^t k(t-s)v(s) ds \approx \sum_{j=1}^n \int_{t_{j-1}}^{t_j} k(t_n - s) ds v(t_j) \quad \text{for } t \in (t_{n-1}, t_n] \quad (n = 1, 2, \dots, N).$$

As we deal with kernels k of exponential type given by (1.1c), we can explicitly calculate

$$\gamma_i = \frac{e^{\lambda\tau} - 1}{\tau} e^{-\lambda t_i}, \quad i = 1, 2, \dots, N,$$

which immediately leads to the properties

$$0 < \gamma_1 = \frac{1 - e^{-\lambda\tau}}{\tau} \rightarrow \lambda = k(0) \quad \text{as } \tau \rightarrow 0, \quad (4.2)$$

and

$$\gamma_{i+1} - \gamma_i = -(e^{\lambda\tau} - 1)\gamma_{i+1}, \quad i = 1, 2, \dots, N-1. \quad (4.3)$$

4.2 Existence, Uniqueness, and A Priori Estimates for the Time Discrete Problem

In what follows, we show existence of solutions for the time discrete equation (4.1) and derive suitable a priori estimates.

Theorem 4.1 (Time Discrete Problem). *Let Assumptions A and B be fulfilled and let $u^0 \in V_B$, $v^0 \in H$, and $\{f^n\}_{n=1}^N = \{f_0^n\}_{n=1}^N + \{f_1^n\}_{n=1}^N \subset V'_A + H$ be given. Then there is a unique solution $\{v^n\}_{n=1}^N \subset V = V_A \cap V_B$ to (4.1). Moreover, there holds for $n = 1, 2, \dots, N$,*

$$\begin{aligned} & \|v^n\|_H^2 + \sum_{j=1}^n \|v^j - v^{j-1}\|_H^2 + \mu_A \tau \sum_{j=1}^n \|v^j\|_{V_A}^p + \frac{T}{e^{\lambda T} - 1} \|(K_{u^0}^\tau v)^n\|_{V_B}^2 \\ & \leq c \left(1 + \|u^0\|_{V_B}^2 + \|v^0\|_H^2 + \tau \sum_{j=1}^N \|f_0^j\|_{V'_A}^{p'} + \left(\tau \sum_{j=1}^N \|f_1^j\|_H \right)^2 \right). \end{aligned} \quad (4.4)$$

Proof. For better readability, we write K^τ instead of $K_{u^0}^\tau$ during this proof. We commence with proving existence and uniqueness of a solution step by step. In the n -th step, (4.1a) is equivalent to determine v^n from $\{v^j\}_{j=1}^{n-1} \subset V$ and the data of the problem by solving

$$\left(\frac{1}{\tau} I + A + \tau \gamma_1 B \right) v^n = f^n + \frac{1}{\tau} v^{n-1} - B u^0 - \tau \sum_{j=1}^{n-1} \gamma_{n-j+1} B v^j.$$

Because of the continuous embeddings (1.2), we know that the right-hand side of the foregoing relation is in V' . The operator $M := \frac{1}{\tau} I + A + \tau \gamma_1 B$ is easily shown to be hemicontinuous, coercive, and strictly monotone as a mapping of V into V' . Here, we make use of the fact that $\gamma_1 > 0$. In particular, we observe that for all $w \in V$,

$$\langle M w, w \rangle \geq \frac{1}{\tau} \|w\|_H^2 + \mu_A \|w\|_{V_A}^p - c_A + \tau \gamma_1 \mu_B \|w\|_{V_B}^2 \geq c \|w\|_V^2 - c.$$

The famous theorem of Browder and Minty (see, e.g., [55, Theorem 26.A on p. 557]) now provides existence of a solution $v^n \in V$; uniqueness immediately follows from the strict monotonicity.

For proving the a priori estimate, we test (4.1a) by $v^n \in V$ ($n = 1, 2, \dots, N$). With

$$(a - b)a = \frac{1}{2}(a^2 - b^2 + (a - b)^2), \quad a, b \in \mathbb{R}, \quad (4.5)$$

the coercivity of A , and Young's inequality, we find for $n = 1, 2, \dots, N$,

$$\begin{aligned} & \frac{1}{2\tau} (\|v^n\|_H^2 - \|v^{n-1}\|_H^2 + \|v^n - v^{n-1}\|_H^2) + \mu_A \|v^n\|_{V_A}^p - c_A + \langle (BK^\tau v)^n, v^n \rangle \\ & \leq \langle f^n, v^n \rangle \\ & \leq \|f_0^n\|_{V'_A} \|v^n\|_{V_A} + \|f_1^n\|_H \|v^n\|_H \\ & \leq \frac{\mu_A}{2} \|v^n\|_{V_A}^p + c \|f_0^n\|_{V'_A}^{p'} + \|f_1^n\|_H \|v^n\|_H. \end{aligned}$$

By summing up, we obtain for $n = 1, 2, \dots, N$,

$$\begin{aligned} & \|v^n\|_H^2 + \sum_{j=1}^n \|v^j - v^{j-1}\|_H^2 + \mu_A \tau \sum_{j=1}^n \|v^n\|_{V_A}^p + 2\tau \sum_{j=1}^n \langle (BK^\tau v)^j, v^j \rangle \\ & \leq c \left(1 + \|v^0\|_H^2 + \tau \sum_{j=1}^n \|f_0^j\|_{V_A}^{p'} \right) + 2\tau \sum_{j=1}^n \|f_1^j\|_H \|v^j\|_H, \end{aligned}$$

where

$$2\tau \sum_{j=1}^n \|f_1^j\|_H \|v^j\|_H \leq 2\tau \sum_{j=1}^n \|f_1^j\|_H \max_{j=1, \dots, n} \|v^j\|_H \leq 2 \left(\tau \sum_{j=1}^n \|f_1^j\|_H \right)^2 + \frac{1}{2} \max_{j=1, \dots, n} \|v^j\|_H^2.$$

We observe that (4.1b) together with (4.2), (4.3) implies for $n = 2, 3, \dots, N$,

$$\frac{1}{\tau} ((K^\tau v)^n - (K^\tau v)^{n-1}) = \gamma_1 v^n + \sum_{j=1}^{n-1} (\gamma_{n-j+1} - \gamma_{n-j}) v^j = \frac{e^{\lambda\tau} - 1}{\tau} (v^n - ((K^\tau v)^n - u^0)), \quad (4.6)$$

which is the discrete analogue of the crucial relation (3.1). Note that $(e^{\lambda\tau} - 1)/\tau \rightarrow \lambda$ as $\tau \rightarrow 0$. By setting $(K^\tau v)^0 := u^0$, relation (4.6) remains true for $n = 1$.

Resolving (4.6) for v^n , together with (4.5) (recall that $B : V_B \rightarrow V'_B$ induces an inner product on V_B), gives

$$\begin{aligned} 2\tau \sum_{j=1}^n \langle (BK^\tau v)^j, v^j \rangle &= \frac{2\tau}{e^{\lambda\tau} - 1} \sum_{j=1}^n \langle (BK^\tau v)^j, (K^\tau v)^j - (K^\tau v)^{j-1} \rangle + 2\tau \sum_{j=1}^n \langle (BK^\tau v)^j, (K^\tau v)^j - u^0 \rangle \\ &\geq \frac{\tau}{e^{\lambda\tau} - 1} (\|(K^\tau v)^n\|_B^2 - \|(K^\tau v)^0\|_B^2) + \tau \sum_{j=1}^n \|(K^\tau v)^j\|_B^2 - T \|u^0\|_B^2, \end{aligned}$$

where

$$\frac{\tau}{e^{\lambda\tau} - 1} \geq \frac{T}{e^{\lambda T} - 1}.$$

Altogether, we find for $n = 1, 2, \dots, N$,

$$\begin{aligned} & \|v^n\|_H^2 + \sum_{j=1}^n \|v^j - v^{j-1}\|_H^2 + \mu_A \tau \sum_{j=1}^n \|v^n\|_{V_A}^p + \frac{T}{e^{\lambda T} - 1} \|(K^\tau v)^n\|_B^2 \\ & \leq c \left(1 + \|u^0\|_{V_B}^2 + \|v^0\|_H^2 + \tau \sum_{j=1}^N \|f_0^j\|_{V_A}^{p'} + \left(\tau \sum_{j=1}^N \|f_1^j\|_H \right)^2 \right) + \frac{1}{2} \max_{j=1, \dots, n} \|v^j\|_H^2. \end{aligned} \quad (4.7)$$

Taking the maximum over all $n \in \{1, \dots, N\}$ first on the right-hand side and then on the left-hand side leads to

$$\max_{j=1, \dots, N} \|v^j\|_H^2 \leq C + \frac{1}{2} \max_{j=1, \dots, N} \|v^j\|_H^2,$$

where $C > 0$ depends on u^0, v^0 , and $\{f^n\}$. This yields

$$\max_{j=1, \dots, N} \|v^j\|_H^2 \leq 2C,$$

which, inserted in (4.7), proves the assertion. \square

4.3 Convergence of Approximate Solutions and Existence of a Generalised Solution

Let $\{N_\ell\}$ be a sequence of positive integers such that $N_\ell \rightarrow \infty$ as $\ell \rightarrow \infty$. We consider the corresponding sequence of time discrete problems (4.1) with step sizes $\tau_\ell = T/N_\ell$, starting values $v_\ell^0 \in H$ with $v_\ell^0 \rightarrow v_0$ in H as well as $u_\ell^0 \in V_B$ with $u_\ell^0 \rightarrow u_0$ in V_B , and right-hand sides $\{f^n\}_{n=1}^{N_\ell} \subset V'_A + H$ given by

$$f^n = f_0^n + f_1^n, \quad f_{0,1}^n := \frac{1}{\tau_\ell} \int_{t_{n-1}}^{t_n} f_{0,1}(t) dt,$$

which is well-defined for $f = f_0 + f_1 \in L^{p'}(0, T; V'_A) + L^1(0, T; H)$. As a slight abuse of notation, in general we do not call the dependence of u^n, v^n, f^n , and of the time instances t_n on ℓ .

Let $\{v^n\}_{n=1}^{N_\ell} \subset V$ denote the solution to (4.1) with step size τ_ℓ . We then consider the piecewise constant functions v_ℓ with $v_\ell(t) = v^n$ for $t \in (t_{n-1}, t_n]$ ($n = 1, 2, \dots, N_\ell$) and $v_\ell(0) = v^1$. Moreover, let \hat{v}_ℓ be the piecewise affine-linear interpolation of the points (t_n, v^n) ($n = 0, 1, \dots, N_\ell$) and let f_ℓ be the piecewise constant function with $f_\ell(t) = f^n$ for $t \in (t_{n-1}, t_n]$ ($n = 1, 2, \dots, N_\ell$) and $f_\ell(0) = f^1$.

Regarding the integral operator K , we define for any integrable function w the piecewise constant function $K_\ell w$ by means of

$$(K_\ell w)(t) = u_\ell^0 + \int_0^{t_n} k(t_n - s)w(s) \, ds \quad \text{if } t \in (t_{n-1}, t_n] \quad (n = 1, \dots, N_\ell),$$

with $(K_\ell w)(0) := u_\ell^0$. As an immediate consequence, with (4.1b) and (4.1c) we obtain

$$(K_\ell v_\ell)(t) = (K_{u_\ell^0}^{\tau_\ell} v)^n \quad \text{if } t \in (t_{n-1}, t_n] \quad (n = 1, \dots, N_\ell). \quad (4.8)$$

We are now able to state the main result.

Theorem 4.2 (Existence via Convergence). *Let Assumptions A and B be fulfilled, and let $u_0 \in V_B, v_0 \in H$, and $f \in L^{p'}(0, T; V'_A) + L^1(0, T; H)$. Then there exists a solution $v \in L^p(0, T; V_A) \cap \mathcal{C}_w([0, T]; H)$ to problem (1.1) with $Kv \in \mathcal{C}_w([0, T]; V_B)$ such that (1.1a) holds in the sense of $L^{p'}(0, T; V'_A) + L^1(0, T; H)$.*

By passing to a subsequence if necessary, both the piecewise constant prolongation v_ℓ and the piecewise affine-linear prolongation \hat{v}_ℓ of the approximate solutions to (4.1) converge weakly in $L^\infty(0, T; H)$ to v as $\ell \rightarrow \infty$. Furthermore, again by passing to a subsequence if necessary and as $\ell \rightarrow \infty$, the piecewise constant prolongation v_ℓ converges weakly to v in $L^p(0, T; V_A)$ and the approximation $K_\ell v_\ell$ of the memory term converges weakly* to Kv in $L^\infty(0, T; V_B)$. The approximation \hat{v}'_ℓ of the time derivative converges to*

$$v' \in L^{p'}(0, T; V'_A) + L^1(0, T; H) + L^\infty(0, T; V'_B)$$

in the sense that $\langle \hat{v}'_\ell, w \rangle \rightarrow \langle v', w \rangle$ for all $w \in L^p(0, T; V_A) \cap L^\infty(0, T; H) \cap L^1(0, T; V_B)$.¹

Note that if $\{\tau_\ell \|v_\ell^0\|_{V_A}^p\}$ is bounded, then also \hat{v}_ℓ converges weakly in $L^p(0, T; V_A)$ to v as $\ell \rightarrow \infty$ (passing to a subsequence if necessary).

The proof of Theorem 4.2 will be prepared by the following integration-by-parts formula.

Lemma 4.3. *Let $u_0 \in V_B$ and $w \in L^p(0, T; V_A) \cap L^\infty(0, T; H)$ such that*

$$Kw \in L^\infty(0, T; V_B) \quad \text{and} \quad w' + BKw \in L^{p'}(0, T; V'_A) + L^1(0, T; H).$$

Then for all $t \in [0, T]$ there holds

$$\begin{aligned} & \int_0^t \langle w'(s) + (BKw)(s), w(s) \rangle \, ds \\ &= \frac{1}{2} \|w(t)\|_H^2 - \frac{1}{2} \|w(0)\|_H^2 + \frac{1}{2\lambda} \|(Kw)(t)\|_B^2 - \frac{1}{2\lambda} \|u_0\|_B^2 - \int_0^t \langle (BKw)(s), u_0 \rangle \, ds + \int_0^t \|(Kw)(s)\|_B^2 \, ds. \end{aligned} \quad (4.9)$$

Proof. The difficulty in proving (4.9) is that neither

$$w' \in L^{p'}(0, T; V'_A) + L^1(0, T; H) \quad \text{nor} \quad BKw \in L^{p'}(0, T; V'_A) + L^1(0, T; H)$$

¹ Here, we slightly abuse the notation of the duality pairing because $L^{p'}(0, T; V'_A) + L^1(0, T; H) + L^\infty(0, T; V'_B)$ is not the dual space of $L^p(0, T; V_A) \cap L^\infty(0, T; H) \cap L^1(0, T; V_B)$. However, we can consider the sum of the duality pairings between $L^{p'}(0, T; V'_A) + L^1(0, T; H)$ and $L^1(0, T; V_B)$ and their respective dual spaces.

can be assumed but only the sum $w' + BKw$ is known to be in $L^p(0, T; V'_A) + L^1(0, T; H)$. Hence it is not possible to split the sum on the left-hand side and to carry out integration by parts separately for both terms.

As $w' + BKw \in L^p(0, T; V'_A) + L^1(0, T; H)$ and $BKw \in L^\infty(0, T; V'_B)$, it is easy to show that

$$w' \in L^p(0, T; V'_A) + L^1(0, T; V'_B) \subset L^1(0, T; V'_A + V'_B),$$

and thus $w \in \mathcal{A}\mathcal{C}([0, T]; V'_A + V'_B)$. In view of [32, Chapter 3, Lemma 8.1], we find $w \in \mathcal{C}_w([0, T]; H)$. Because of (3.1), we also see that $(Kw)' \in L^1(0, T; V_A + V_B)$, and thus $Kw \in \mathcal{A}\mathcal{C}([0, T]; V_A + V_B)$, which implies $Kw \in \mathcal{C}_w([0, T]; V_B)$. Because of (3.1), we also have $(Kw)' \in \mathcal{C}_w([0, T]; H)$. This justifies evaluating $\|w(t)\|_H$, $\|Kw(t)\|_B$, and $\|(Kw)'(t)\|_H$ for fixed t .

Further, we remark that $Kw - u_0 \in L^\infty(0, T; V)$ since $w \in L^p(0, T; V_A) \cap L^\infty(0, T; H)$. Due to (3.1), we thus have $(Kw)' \in L^p(0, T; V_A) \cap L^\infty(0, T; H)$. Differentiating (3.1) yields

$$(Kw)''(t) = \lambda(w'(t) - (Kw)'(t))$$

for almost all $t \in (0, T)$. Therefore, we have

$$\langle w' + BKw, w \rangle = \left\langle \frac{1}{\lambda}(Kw)'' + (Kw)' + BKw, \frac{1}{\lambda}(Kw)' + Kw - u_0 \right\rangle.$$

Since $\frac{1}{\lambda}(Kw)'' + (Kw)' + BKw \in L^p(0, T; V'_A) + L^1(0, T; H)$ and $Kw - u_0 \in L^\infty(0, T; V)$, we are allowed to split the right-hand side into

$$\begin{aligned} & \left\langle \frac{1}{\lambda}(Kw)'' + (Kw)' + BKw, \frac{1}{\lambda}(Kw)' \right\rangle + \left\langle \frac{1}{\lambda}(Kw)'' + (Kw)' + BKw, Kw - u_0 \right\rangle \\ &= \left\langle \frac{1}{\lambda}(Kw)'' + BKw, \frac{1}{\lambda}(Kw)' \right\rangle + \frac{1}{\lambda} \|(Kw)'\|_{L^2(0, T; H)}^2 + \left\langle \frac{1}{\lambda}(Kw)'' + (Kw)' + BKw, Kw - u_0 \right\rangle. \end{aligned} \quad (4.10)$$

For the first term on the right-hand side, we apply the result of Strauss [49, Theorem 4.1] providing the integration-by-parts formula

$$\left\langle \frac{1}{\lambda}(Kw)'' + BKw, \frac{1}{\lambda}(Kw)' \right\rangle = \frac{1}{2\lambda^2} (\|(Kw)'(T)\|_H^2 - \|(Kw)'(0)\|_H^2 + \lambda \|(Kw)(T)\|_B^2 - \lambda \|u_0\|_B^2).$$

Since $(Kw)' \in L^p(0, T; V_A) \cap L^\infty(0, T; H)$, $BKw \in L^\infty(0, T; V'_B)$, and $Kw - u_0 \in L^\infty(0, T; V)$, the third term on the right-hand side of (4.10) yields

$$\begin{aligned} & \left\langle \frac{1}{\lambda}(Kw)'' + (Kw)' + BKw, Kw - u_0 \right\rangle \\ &= \left\langle \frac{1}{\lambda}(Kw)'', Kw - u_0 \right\rangle + \langle (Kw)', Kw - u_0 \rangle + \langle BKw, Kw - u_0 \rangle \\ &= \frac{1}{\lambda} \langle (Kw - u_0)'(T), (Kw - u_0)(T) \rangle - \frac{1}{\lambda} \langle (Kw - u_0)'(0), (Kw - u_0)(0) \rangle - \frac{1}{\lambda} \|(Kw - u_0)'\|_{L^2(0, T; H)}^2 \\ & \quad + \frac{1}{2} \|(Kw)(T) - u_0\|_H^2 - \frac{1}{2} \|(Kw)(0) - u_0\|_H^2 + \|Kw\|_{L^2(0, T; B)}^2 - \langle BKw, u_0 \rangle. \end{aligned}$$

Altogether, we have

$$\begin{aligned} \langle w' + BKw, w \rangle &= \frac{1}{2\lambda^2} (\|(Kw)'(T)\|_H^2 - \|(Kw)'(0)\|_H^2 + \lambda \|(Kw)(T)\|_B^2 - \lambda \|u_0\|_B^2) \\ & \quad + \frac{1}{\lambda} \langle (Kw - u_0)'(T), (Kw - u_0)(T) \rangle + \frac{1}{2} \|(Kw)(T) - u_0\|_H^2 \\ & \quad + \|Kw\|_{L^2(0, T; B)}^2 - \langle BKw, u_0 \rangle. \end{aligned} \quad (4.11)$$

Due to (3.1), there holds

$$\frac{1}{2} \|w(T)\|_H^2 = \frac{1}{2\lambda^2} \|(Kw)'(T)\|_H^2 + \frac{1}{2} \|(Kw)(T) - u_0\|_H^2 + \frac{1}{\lambda} \langle (Kw)'(T), (Kw)(T) - u_0 \rangle$$

as well as

$$\frac{1}{2} \|w(0)\|_H^2 = \frac{1}{2\lambda^2} \|(Kw)'(0)\|_H^2.$$

Inserting this into (4.11) yields the desired integration-by-parts formula for $t = T$. However, everything above remains true for an arbitrary $t \in [0, T]$, which proves the assertion. \square

Proof of Theorem 4.2. For simplicity, we write τ and N instead of τ_ℓ and N_ℓ , respectively, during this proof. Since, by assumption, $v_\ell^0 \rightarrow v_0$ in H , $u_\ell^0 \rightarrow u_0$ in V_B , and since $f_\ell \rightarrow f$ in $L^{p'}(0, T; V'_A) + L^1(0, T; H)$, the right-hand side of the a priori estimate (4.4) stays bounded as $\ell \rightarrow \infty$. Since

$$\begin{aligned} \|v_\ell\|_{L^\infty(0, T; H)} &= \max_{n=1, 2, \dots, N} \|v^n\|_H, \quad \|\hat{v}_\ell\|_{L^\infty(0, T; H)} = \max_{n=0, 1, \dots, N} \|v^n\|_H, \\ \|v_\ell\|_{L^p(0, T; V_A)} &= \left(\tau \sum_{n=1}^N \|v^n\|_{V_A}^p \right)^{1/p}, \end{aligned}$$

the a priori estimate (4.4) implies that the sequence $\{v_\ell\}$ is bounded in $L^\infty(0, T; H) \cap L^p(0, T; V_A)$, the sequence $\{\hat{v}_\ell\}$ is bounded in $L^\infty(0, T; H)$, and – in view of (4.8) – the sequence $\{K_\ell v_\ell\}$ is bounded in $L^\infty(0, T; V_B)$. Due to the growth condition on A , we also conclude that the sequence $\{Av_\ell\}$ is bounded in $L^{p'}(0, T; V'_A)$.

Since $L^\infty(0, T; H)$ and $L^\infty(0, T; V_B)$ are the duals of the separable normed spaces $L^1(0, T; H)$ and $L^1(0, T; V'_B)$, respectively, and since $L^p(0, T; V_A)$ and $L^{p'}(0, T; V'_A)$ are reflexive Banach spaces, there exists a common subsequence, again denoted by ℓ , and elements $v \in L^\infty(0, T; H) \cap L^p(0, T; V_A)$, $\hat{v} \in L^\infty(0, T; H)$, $u \in L^\infty(0, T; V_B)$, and $a \in L^{p'}(0, T; V'_A)$ such that, as $\ell \rightarrow \infty$

$$\begin{aligned} v_\ell \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0, T; H), \quad v_\ell \rightharpoonup v \quad \text{in } L^p(0, T; V_A), \quad \hat{v}_\ell \overset{*}{\rightharpoonup} \hat{v} \quad \text{in } L^\infty(0, T; H), \\ K_\ell v_\ell \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0, T; V_B), \quad Av_\ell \rightharpoonup a \quad \text{in } L^{p'}(0, T; V'_A); \end{aligned}$$

see, e.g., [4, Corollaire III.26, Théorème III.27]. Since $B : V_B \rightarrow V'_B$ is linear and bounded, we also find that $BK_\ell v_\ell \overset{*}{\rightharpoonup} Bu$ in $L^\infty(0, T; V'_B)$.

A short calculation shows that

$$\|v_\ell - \hat{v}_\ell\|_{L^2(0, T; H)}^2 = \frac{\tau}{3} \sum_{j=1}^N \|v^j - v^{j-1}\|_H^2,$$

which tends to 0 as $\ell \rightarrow \infty$ because of the a priori estimate (4.4). Hence $v = \hat{v}$.

Let us show that $u = Kv$, and thereby $Kv \in L^\infty(0, T; V_B)$. We already know that $K_\ell v_\ell \overset{*}{\rightharpoonup} u$ in $L^\infty(0, T; V_B)$, and thus $K_\ell v_\ell \rightharpoonup u$ in $L^2(0, T; H)$. The integral operator \hat{K} defined by $\hat{K}w := Kw - u_0$ can easily be shown to be a linear, bounded and thus weakly-weakly continuous mapping of $L^2(0, T; H)$ into itself (see also Lemma 3.1). Therefore, in view of $v_\ell \overset{*}{\rightharpoonup} v$ in $L^\infty(0, T; H)$, and thus $v_\ell \rightharpoonup v$ in $L^2(0, T; H)$, we find that $Kv_\ell - Kv = \hat{K}v_\ell - \hat{K}v \rightharpoonup 0$ in $L^2(0, T; H)$. Because of

$$u - Kv = u - K_\ell v_\ell + K_\ell v_\ell - Kv_\ell + Kv_\ell - Kv,$$

it thus remains to prove that $K_\ell v_\ell - Kv_\ell$ converges at least weakly in $L^2(0, T; H)$ to 0 in order to prove $u = Kv$. For $w \in L^2(0, T; H)$ and $t \in (t_{n-1}, t_n]$ ($n = 1, 2, \dots, N$), we have

$$(K_\ell w)(t) - (Kw)(t) = u_\ell^0 - u_0 + \int_0^{t_n} (k(t_n - s) - k(t - s))w(s) \, ds + \int_t^{t_n} k(t - s)w(s) \, ds.$$

Since $|k(t - s)| = \lambda e^{-\lambda(t-s)} \leq \lambda e^{\lambda\tau}$ for $s \in (t, t_n) \subset (t_{n-1}, t_n)$ as well as

$$|k(t_n - s) - k(t - s)| \leq \lambda(e^{\lambda\tau} - 1) \leq c\tau \quad \text{for } s \in (0, t_n),$$

the Cauchy–Schwarz inequality yields

$$\|K_\ell w - Kw\|_{L^2(0, T; H)} \leq \sqrt{T} \|u_\ell^0 - u_0\|_H + c\tau \|w\|_{L^2(0, T; H)}.$$

Since $u_\ell^0 \rightarrow u_0$ in V_B and v_ℓ is bounded in $L^2(0, T; H)$, we immediately get $K_\ell v_\ell - Kv_\ell \rightarrow 0$ in $L^2(0, T; H)$ as $\ell \rightarrow \infty$ (and thus $\tau \rightarrow 0$), which is the last step to prove $u = Kv \in L^\infty(0, T; V_B)$.

With the help of the piecewise constant and piecewise linear interpolation of the corresponding grid functions, we can now rewrite the numerical scheme (4.1a) as

$$\hat{v}'_\ell + Av_\ell + BK_\ell v_\ell = f_\ell. \tag{4.12}$$

For any $w \in V$ and $\phi \in \mathcal{C}_c^\infty(0, T)$, we thus obtain

$$-\int_0^T \langle \hat{v}_\ell(t), w \rangle \phi'(t) dt = \int_0^T \langle f_\ell(t) - Av_\ell(t) - (BK_\ell v_\ell)(t), w \rangle \phi(t) dt.$$

Since $f_\ell \rightarrow f$ in $L^{p'}(0, T; V'_A) + L^1(0, T; H)$, and employing the (weak and weak*) convergence just shown, we can pass to the limit in the foregoing equation to come up with

$$-\int_0^T \langle v(t), w \rangle \phi'(t) dt = \int_0^T \langle f(t) - a(t) - (BKv)(t), w \rangle \phi(t) dt.$$

This shows that v is differentiable in the weak sense with

$$v' = f - a - BKv \in L^{p'}(0, T; V'_A) + L^\infty(0, T; V'_B) + L^1(0, T; H). \quad (4.13)$$

Let us summarise the regularity properties of the limits we have proven so far: As

$$v \in L^p(0, T; V_A) \cap L^\infty(0, T; H) \quad \text{and} \quad v' \in L^{p'}(0, T; V'_A) + L^\infty(0, T; V'_B) + L^1(0, T; H) \subset L^1(0, T; V'),$$

we conclude that $v \in \mathcal{S}\mathcal{C}([0, T]; V')$. This implies $v \in \mathcal{C}_w([0, T]; H)$ (see [32, Chapter 3, Lemma 8.1]). Likewise, from

$$Kv \in L^\infty(0, T; V_B) \quad \text{and} \quad (Kv)' = \lambda v - \lambda Kv + \lambda u_0 \in L^p(0, T; V_A) + L^\infty(0, T; V_B) \subset L^1(0, T; V_A + V_B)$$

(see (3.1)), we deduce that $Kv \in \mathcal{S}\mathcal{C}([0, T]; V_A + V_B)$, and thus $Kv \in \mathcal{C}_w([0, T]; V_B)$. We remark that, in addition, one can easily show that $Kv - u_0 \in W^{1,p}(0, T; V_A)$.

By construction, we have $\hat{v}_\ell(0) = v_\ell^0 \rightarrow v_0$ in H and $\hat{v}_\ell(T) = v^N$. In view of the a priori estimate (4.4), the sequence $\{\hat{v}_\ell(T)\}$ is bounded in H , and hence possesses a subsequence, again denoted by ℓ , that converges weakly in H to an element $v_T \in H$. We shall show that $v_T = v(T)$.

For any $\phi \in \mathcal{C}^1([0, T])$, $w \in V$, we test (4.13) with the function $t \mapsto \phi(t)w \in \mathcal{C}^1([0, T]; V)$, which allows us to integrate by parts. Together with (4.12), we find

$$\begin{aligned} (v(T), w)\phi(T) - (v(0), w)\phi(0) &= \int_0^T \langle v'(t), w \rangle \phi(t) dt + \int_0^T \langle v(t), w \rangle \phi'(t) dt \\ &= \int_0^T \langle f(t) - f_\ell(t) + Av_\ell(t) - a(t) + (BK_\ell v_\ell)(t) - (BKv)(t), w \rangle \phi(t) dt \\ &\quad + \int_0^T \langle v(t), w \rangle \phi'(t) dt + \int_0^T \langle \hat{v}'_\ell(t), w \rangle \phi(t) dt \\ &= \int_0^T \langle f(t) - f_\ell(t) + Av_\ell(t) - a(t) + (BK_\ell v_\ell)(t) - (BKv)(t), w \rangle \phi(t) dt \\ &\quad + \int_0^T \langle v(t) - \hat{v}_\ell(t), w \rangle \phi'(t) dt + (\hat{v}_\ell(T), w)\phi(T) - (\hat{v}_\ell(0), w)\phi(0). \end{aligned} \quad (4.14)$$

The first two terms on the right-hand side tend to zero as $\ell \rightarrow \infty$. This yields

$$(v(T), w)\phi(T) - (v(0), w)\phi(0) = (v_T, w)\phi(T) - (v_0, w)\phi(0) \quad \text{for all } w \in V.$$

Choosing ϕ such that $\phi(T) = 0$ and $\phi(0) = 0$, respectively, shows that $v(0) = v_0$ and $v(T) = v_T$, respectively.

By definition, we have $(K_\ell v_\ell)(0) = u_\ell^0 \rightarrow u_0$ in V_B and $(K_\ell v_\ell)(T) = (K_{u_\ell^0}^T v)^N$. Again, the a priori estimate yields the boundedness of $\{(K_\ell v_\ell)(T)\}$ in V_B , and thus the existence of a weakly convergent subsequence, again denoted by ℓ , and an element $u_T \in V_B$ such that $(K_\ell v_\ell)(T) \rightharpoonup u_T$ in V_B .

We shall show that $u_T = (Kv)(T)$. Analogously to (4.14), choosing $\phi(t) = \frac{t}{T}$, $t \in [0, T]$, we have for any $w \in V$,

$$\langle (Kv)(T), w \rangle = \int_0^T \langle (Kv)'(t), w \rangle \frac{t}{T} dt + \int_0^T \langle (Kv)(t), w \rangle \frac{1}{T} dt.$$

Using (3.1), we obtain

$$\begin{aligned} \langle (Kv)(T), w \rangle &= \lambda \int_0^T \langle v(t) - ((Kv)(t) - u_0), w \rangle \frac{t}{T} dt + \int_0^T \langle (Kv)(t), w \rangle \frac{1}{T} dt \\ &= \lim_{\ell \rightarrow \infty} \left(\lambda \int_0^T \langle v_\ell(t) - ((K_\ell v_\ell)(t) - u_\ell^0), w \rangle \frac{t}{T} dt + \int_0^T \langle (K_\ell v_\ell)(t), w \rangle \frac{1}{T} dt \right). \end{aligned} \quad (4.15)$$

For the first integral, (4.6) yields

$$\begin{aligned} \int_0^T \langle v_\ell(t) - ((K_\ell v_\ell)(t) - u_\ell^0), w \rangle \frac{t}{T} dt &= \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \langle v^j - ((K_{u_\ell^0}^\tau v)^j - u_\ell^0), w \rangle \frac{t}{T} dt \\ &= \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \frac{1}{e^{\lambda\tau} - 1} \langle (K_{u_\ell^0}^\tau v)^j - (K_{u_\ell^0}^\tau v)^{j-1}, w \rangle \frac{t}{T} dt. \end{aligned}$$

For better readability, we write K^τ instead of $K_{u_\ell^0}^\tau$ for the rest of this proof. Splitting up the sum and shifting the index in the second sum yields

$$\begin{aligned} &\sum_{j=1}^N \int_{t_{j-1}}^{t_j} \langle (K^\tau v)^j - (K^\tau v)^{j-1}, w \rangle \frac{t}{T} dt \\ &= \langle (K^\tau v)^N, w \rangle \int_{t_{N-1}}^T \frac{t}{T} dt - \langle u_\ell^0, w \rangle \int_0^\tau \frac{t}{T} dt + \sum_{j=1}^{N-1} \langle (K^\tau v)^j, w \rangle \left(\int_{t_{j-1}}^{t_j} \frac{t}{T} dt - \int_{t_j}^{t_{j+1}} \frac{t}{T} dt \right) \\ &= \frac{(2T - \tau)\tau}{2T} \langle (K^\tau v)^N, w \rangle - \frac{\tau^2}{2T} \langle u_\ell^0, w \rangle - \frac{\tau^2}{T} \sum_{j=1}^{N-1} \langle (K^\tau v)^j, w \rangle + \frac{\tau^2}{T} \langle (K^\tau v)^N, w \rangle \\ &= \left(\frac{(2T - \tau)\tau}{2T} + \frac{\tau^2}{T} \right) \langle (K_\ell v_\ell)(T), w \rangle - \frac{\tau^2}{2T} \langle u_\ell^0, w \rangle - \frac{\tau}{T} \int_0^T \langle (K_\ell v_\ell)(t), w \rangle dt. \end{aligned}$$

Inserting this into (4.15) and using that $\{u_\ell^0\}$ is bounded in V_B , $(K_\ell v_\ell)(T) \rightarrow u_T$ in V_B , $K_\ell v_\ell \xrightarrow{*} Kv$ in $L^\infty(0, T; V_B)$ as well as $\lambda\tau(e^{\lambda\tau} - 1)^{-1} \rightarrow 1$ as $\ell \rightarrow \infty$ (and thus $\tau \rightarrow 0$), we end up with

$$\begin{aligned} \langle (Kv)(T), w \rangle &= \lim_{\ell \rightarrow \infty} \left(\frac{\lambda\tau}{e^{\lambda\tau} - 1} \left(\left(\frac{2T - \tau}{2T} + \frac{\tau}{T} \right) \langle (K_\ell v_\ell)(T), w \rangle - \frac{\tau}{2T} \langle u_\ell^0, w \rangle - \frac{1}{T} \int_0^T \langle (K_\ell v_\ell)(t), w \rangle dt \right) \right. \\ &\quad \left. + \int_0^T \langle (K_\ell v_\ell)(t), w \rangle \frac{1}{T} dt \right) \\ &= \langle u_T, w \rangle, \end{aligned}$$

which proves $u_T = (Kv)(T)$.

It remains to show that a equals Av . We recall that $v_\ell \in L^\infty(0, T; V)$. Hence, for arbitrary $w \in L^p(0, T; V_A)$ the monotonicity of A implies

$$\langle Av_\ell, v_\ell \rangle = \langle Av_\ell - Aw, v_\ell - w \rangle + \langle Aw, v_\ell - w \rangle + \langle Av_\ell, w \rangle \geq \langle Aw, v_\ell - w \rangle + \langle Av_\ell, w \rangle.$$

The right-hand side converges to

$$\langle Aw, v - w \rangle + \langle a, w \rangle.$$

This already implies

$$\liminf_{\ell \rightarrow \infty} \langle Av_\ell, v_\ell \rangle \geq \langle Aw, v - w \rangle + \langle a, w \rangle.$$

Below we will show

$$\limsup_{\ell \rightarrow \infty} \langle Av_\ell, v_\ell \rangle \leq \langle a, v \rangle, \quad (4.16)$$

hence altogether proving

$$\langle Aw - a, v - w \rangle \leq 0.$$

Taking $w = v \pm rz$ for any $z \in L^p(0, T; V_A)$, $r > 0$, passing to the limit $r \rightarrow 0$, and Lebesgue's theorem on dominated convergence together with the hemicontinuity and the growth condition of $A: V_A \rightarrow V'_A$ yield

$$\langle Av - a, z \rangle = 0 \quad \text{for all } z \in L^p(0, T; V_A),$$

which proves the assertion.

To finish the proof of existence, it remains to show (4.16). We start with

$$\langle Av_\ell, v_\ell \rangle = \langle f_\ell, v_\ell \rangle - \langle \hat{v}'_\ell, v_\ell \rangle - \langle BK_\ell v_\ell, v_\ell \rangle. \quad (4.17)$$

Since $f_\ell \rightarrow f$ in $L^p(0, T; V'_A) + L^1(0, T; H)$, the first term on the right-hand side tends to $\langle f, v \rangle$ as $\ell \rightarrow \infty$. Using (4.5), we estimate the second term as

$$\langle \hat{v}'_\ell, v_\ell \rangle = \sum_{j=1}^N (v^j - v^{j-1}, v^j) \geq \frac{1}{2} \|v^N\|_H^2 - \frac{1}{2} \|v_\ell^0\|_H^2.$$

As $v_\ell^0 \rightarrow v_0 = v(0)$ and $v^N \rightarrow v_T = v(T)$ in H , the weak lower semicontinuity of the norm provides

$$\liminf_{\ell \rightarrow \infty} \langle \hat{v}'_\ell, v_\ell \rangle \geq \frac{1}{2} \|v(T)\|_H^2 - \frac{1}{2} \|v(0)\|_H^2.$$

Regarding the last term on the right-hand side of equation (4.17), note that only $v_\ell \rightarrow v$ in $L^p(0, T; V_A)$ and $BK_\ell v_\ell \rightarrow BKv$ in $L^2(0, T; V'_B)$. Therefore, a finer examination of that term is necessary. To start with, a short calculation (see (4.6)) shows that

$$v^n = \frac{e^{-\lambda\tau}}{\tau\gamma_1} ((K^\tau v)^n - (K^\tau v)^{n-1}) + \frac{1 - e^{-\lambda\tau}}{\tau\gamma_1} (K^\tau v)^n - \frac{1 - e^{-\lambda\tau}}{\tau\gamma_1} u_\ell^0.$$

We conclude with (4.5) that

$$\begin{aligned} \langle BK_\ell v_\ell, v_\ell \rangle &= \tau \sum_{j=1}^N \langle (BK^\tau v)^j, v^j \rangle \\ &= \frac{e^{-\lambda\tau}}{\gamma_1} \sum_{j=1}^N \langle (BK^\tau v)^j, (K^\tau v)^j - (K^\tau v)^{j-1} \rangle + \frac{1 - e^{-\lambda\tau}}{\gamma_1} \sum_{j=1}^N \langle (BK^\tau v)^j, (K^\tau v)^j \rangle \\ &\quad - \frac{1 - e^{-\lambda\tau}}{\gamma_1} \sum_{j=1}^N \langle (BK^\tau v)^j, u_\ell^0 \rangle \\ &\geq \frac{e^{-\lambda\tau}}{2\gamma_1} (\|(K^\tau v)^N\|_B^2 - \|(K^\tau v)^0\|_B^2) + \frac{1 - e^{-\lambda\tau}}{\tau\gamma_1} \|K_\ell v_\ell\|_{L^2(0, T; B)}^2 - \frac{1 - e^{-\lambda\tau}}{\tau\gamma_1} \int_0^T \langle (BK_\ell v_\ell)(t), u_\ell^0 \rangle dt. \end{aligned}$$

Since $(K^\tau v)^0 = u_\ell^0 \rightarrow u_0 = (Kv)(0)$ in V_B and $(K^\tau v)^N = (K_\ell v_\ell)(T) \rightarrow u_T = (Kv)(T)$ in V_B as well as $K_\ell v_\ell \rightarrow Kv$ in $L^2(0, T; V_B)$, the weak lower semicontinuity of the norms involved shows, by passing to the limit and employing $\gamma_1 \rightarrow \lambda$ (see (4.2)),

$$\liminf_{\ell \rightarrow \infty} \langle BK_\ell v_\ell, v_\ell \rangle \geq \frac{1}{2\lambda} \|(Kv)(T)\|_B^2 - \frac{1}{2\lambda} \|(Kv)(0)\|_B^2 + \|Kv\|_{L^2(0, T; B)}^2 - \int_0^T \langle (BKv)(t), u_0 \rangle dt.$$

Altogether, we find

$$\begin{aligned} \limsup_{\ell \rightarrow \infty} \langle Av_\ell, v_\ell \rangle &\leq \langle f, v \rangle - \frac{1}{2} \|v(T)\|_H^2 + \frac{1}{2} \|v(0)\|_H^2 - \frac{1}{2\lambda} \|(Kv)(T)\|_B^2 + \frac{1}{2\lambda} \|(Kv)(0)\|_B^2 \\ &\quad - \|Kv\|_{L^2(0,T;B)}^2 + \int_0^T \langle (BKv)(t), u_0 \rangle dt. \end{aligned}$$

We replace f by $v' + a + BKv$ and remind that $v \in \mathcal{C}_w([0, T]; H)$ as well as $Kv \in \mathcal{C}_w([0, T]; V_B)$. Thus we are allowed to apply Lemma 4.3, which yields (4.16). \square

5 Uniqueness and Continuous Dependence on the Problem Data

Using again the integration-by-parts formula of Lemma 4.3, we are able to prove a stability and uniqueness result. Further, we are able to prove stability with respect to perturbations of the kernel parameter λ . In order to do so, we again have to prove an integration-by-parts formula using a different approach that requires less structure than the one used in the proof of Lemma 4.3.

Theorem 5.1 (Stability). *Suppose Assumptions A and B are fulfilled and let $u_0, \hat{u}_0 \in V_B$, $v_0, \hat{v}_0 \in H$, and $f, \hat{f} \in L^{p'}(0, T; V'_A) + L^1(0, T; H)$.*

- (i) *Let $f - \hat{f} \in L^1(0, T; H)$ and let v and \hat{v} be solutions to (1.1) with data (u_0, v_0, f) and $(\hat{u}_0, \hat{v}_0, \hat{f})$, respectively. Then the stability estimate*

$$\begin{aligned} \|v(t) - \hat{v}(t)\|_H^2 + \frac{1}{\lambda} \|(K_{u_0}v)(t) - (K_{\hat{u}_0}\hat{v})(t)\|_B^2 + \int_0^t \|(K_{u_0}v)(s) - (K_{\hat{u}_0}\hat{v})(s)\|_B^2 ds \\ \leq c(\|v_0 - \hat{v}_0\|_H^2 + \|u_0 - \hat{u}_0\|_{V_B}^2 + \|f - \hat{f}\|_{L^1(0,T;H)}^2) \end{aligned}$$

holds for all $t \in [0, T]$, where the index in the notations K_{u_0} and $K_{\hat{u}_0}$ denotes the dependence of the integral operator on u_0 and \hat{u}_0 , respectively.

- (ii) *Let $f - \hat{f} \in L^{p'}(0, T; V'_A)$. Assume in addition that A is uniformly monotone in the sense that there is $\mu > 0$ such that*

$$\langle Av - Aw, v - w \rangle \geq \mu \|v - w\|_{V_A}^p$$

for all $v, w \in V_A$. Let v and \hat{v} be solutions to (1.1) with data (u_0, v_0, f) and $(\hat{u}_0, \hat{v}_0, \hat{f})$, respectively. Then the stability estimate

$$\begin{aligned} \|v(t) - \hat{v}(t)\|_H^2 + \mu \int_0^t \|v(s) - \hat{v}(s)\|_{V_A}^p ds + \frac{1}{\lambda} \|(K_{u_0}v)(t) - (K_{\hat{u}_0}\hat{v})(t)\|_B^2 + \int_0^t \|(K_{u_0}v)(s) - (K_{\hat{u}_0}\hat{v})(s)\|_B^2 ds \\ \leq c(\|v_0 - \hat{v}_0\|_H^2 + \|u_0 - \hat{u}_0\|_{V_B}^2 + \|f - \hat{f}\|_{L^{p'}(0,T;V'_A)}^2) \end{aligned}$$

holds for all $t \in [0, T]$.

Proof. Let v and \hat{v} be solutions for (u_0, v_0, f) and $(\hat{u}_0, \hat{v}_0, \hat{f})$, respectively. The difference of both equations (1.1a) then reads

$$(v - \hat{v})' + Av - A\hat{v} + B(K_{u_0}v - K_{\hat{u}_0}\hat{v}) = f - \hat{f}. \quad (5.1)$$

Lemma 4.3 implies for all $t \in [0, T]$,

$$\begin{aligned} \int_0^t \langle (v - \hat{v})'(s) + B((K_{u_0}v)(s) - (K_{\hat{u}_0}\hat{v})(s)), (v - \hat{v})(s) \rangle ds \\ = \frac{1}{2} \|v(t) - \hat{v}(t)\|_H^2 - \frac{1}{2} \|v_0 - \hat{v}_0\|_H^2 + \frac{1}{2\lambda} \|(K_{u_0}v)(t) - (K_{\hat{u}_0}\hat{v})(t)\|_B^2 \\ - \frac{1}{2\lambda} \|u_0 - \hat{u}_0\|_B^2 - \int_0^t \langle B((K_{u_0}v)(s) - (K_{\hat{u}_0}\hat{v})(s)), u_0 - \hat{u}_0 \rangle ds + \int_0^t \|(K_{u_0}v)(s) - (K_{\hat{u}_0}\hat{v})(s)\|_B^2 ds. \end{aligned}$$

We test in equation (5.1) with $v - \hat{v}$. To prove (i), the monotonicity of A and Young's inequality imply

$$\begin{aligned} & \frac{1}{2} \|v(t) - \hat{v}(t)\|_H^2 + \frac{1}{2\lambda} \|(K_{u_0} v)(t) - (K_{\hat{u}_0} \hat{v})(t)\|_B^2 + \int_0^t \|(K_{u_0} v)(s) - (K_{\hat{u}_0} \hat{v})(s)\|_B^2 ds \\ & \leq \frac{1}{2} \|v_0 - \hat{v}_0\|_H^2 + \frac{1}{2\lambda} \|u_0 - \hat{u}_0\|_B^2 + \int_0^t \langle B((K_{u_0} v)(s) - (K_{\hat{u}_0} \hat{v})(s)), u_0 - \hat{u}_0 \rangle ds + \int_0^t \langle (f - \hat{f})(s), (v - \hat{v})(s) \rangle ds \\ & \leq c(\|v_0 - \hat{v}_0\|_H^2 + \|u_0 - \hat{u}_0\|_B^2 + \|f - \hat{f}\|_{L^1(0,T;H)}^2) + \frac{1}{2} \int_0^t \|(K_{u_0} v)(s) - (K_{\hat{u}_0} \hat{v})(s)\|_B^2 ds + \frac{1}{4} \|v - \hat{v}\|_{L^\infty(0,T;H)}^2. \end{aligned}$$

An argument analogous to the one used in the proof of Theorem 4.1 yields the first statement. The second one follows analogously from the uniform monotonicity of A and

$$\int_0^t \langle (f - \hat{f})(s), (v - \hat{v})(s) \rangle ds \leq \|f - \hat{f}\|_{L^{p'}(0,T;V'_A)} \int_0^t \|(v - \hat{v})(s)\|_{V_A}^p ds,$$

which proves the theorem. \square

As usually, the stability estimates directly provide a uniqueness result, which we formulate in the following corollary.

Corollary 5.2 (Uniqueness). *Under the assumptions of Theorem 4.2, the solution to problem (1.1) is unique. Furthermore, the whole sequences $\{v_\ell\}$ and $\{\hat{v}_\ell\}$ of piecewise constant and of piecewise affine-linear prolongations of solutions to the discrete problem (4.1) converge to the solution in the sense stated in Theorem 4.2.*

Proof. We consider $u_0 = \hat{u}_0 \in V_B$, $v_0 = \hat{v}_0 \in H$ and $f = \hat{f} \in L^{p'}(0, T; V'_A) + L^1(0, T; H)$ with corresponding solutions v and \hat{v} . As $f - \hat{f} = 0 \in L^1(0, T; H)$, the first assumption of Theorem 5.1 is fulfilled, which shows

$$\|v(t) - \hat{v}(t)\|_H^2 + \frac{1}{\lambda} \|(Kv)(t) - (K\hat{v})(t)\|_B^2 + \int_0^t \|(Kv)(s) - (K\hat{v})(s)\|_B^2 ds \leq 0$$

for all $t \in [0, T]$. This proves the uniqueness. Convergence of the whole sequences then follows as usual by contradiction. \square

Finally, we aim to derive Lipschitz-type dependence on the kernel parameter $\frac{1}{\lambda}$, that is, Lipschitz dependence on the average relaxation-time.

Theorem 5.3 (Perturbation of λ). *Let the assumptions of Theorem 4.2 be fulfilled and let $v_0 \in V$. For $\lambda > 0$, let K_λ denote the operator related to the kernel $k_\lambda(t) = \lambda e^{-\lambda t}$ and let $v_\lambda \in L^p(0, T; V_A) \cap L^\infty(0, T; H)$ denote the corresponding solution to problem (1.1). Moreover, assume that $v_\lambda \in L^2(0, T; V_B)$. Then for all $\mu > 0$ and corresponding solutions $v_\mu \in L^p(0, T; V_A) \cap L^\infty(0, T; H)$ there holds*

$$\|(v_\lambda - v_\mu)(t)\|_H^2 + \int_0^t \|(K_\lambda v_\lambda - K_\mu v_\mu)(s)\|_B^2 ds \leq \frac{\lambda^2}{2} (1 + \lambda^2 T^2) \left| \frac{1}{\lambda} - \frac{1}{\mu} \right|^2 \|v_\lambda\|_{L^2(0,T;B)}^2$$

for almost all t .

Proof. Taking the difference of the equations (1.1a) for v_λ and v_μ , respectively, and testing with $v_\lambda - v_\mu$ leads to

$$\begin{aligned} & \int_0^t \langle (v_\lambda - v_\mu)'(s) + B((K_\lambda v_\lambda)(s) - (K_\mu v_\mu)(s)), (v_\lambda - v_\mu)(s) \rangle ds \\ & = - \int_0^t \langle Av_\lambda(s) - Av_\mu(s), (v_\lambda - v_\mu)(s) \rangle ds \leq 0. \end{aligned} \quad (5.2)$$

To deal with the integral on the left-hand side, we need to prove an integration-by-parts formula similar to (4.9). Unfortunately, due to the fact that we have two different parameters λ and μ instead of one, we are not able to apply the same method as in the proof of Lemma 4.3. Therefore, we use another method of proof using the centered Steklov average.²

In order to prove the integration-by-parts formula on $(0, t)$, we first have to extend the functions considered to the interval $(-\eta, 0)$ for a fixed $\eta > 0$. We then prove the formula on an arbitrary interval (α, β) with

$$-\eta + h_0 < \alpha < \beta < T - h_0.$$

At the end, instead of a fixed α , we consider a sequence $\{\alpha_k\}$ with $\alpha_k < 0$ and $\alpha_k \rightarrow 0$ as $k \rightarrow \infty$. If we considered a sequence of positive α_k , we would not be able to identify the limit properly.

We fix $h_0 > 0$. The centered Steklov average for a function $z \in L^p(-\eta, T; X)$, with X being an arbitrary Banach space, is defined by

$$(S_h z)(t) := \frac{1}{2h} \int_{t-h}^{t+h} z(s) \, ds,$$

where $0 < h < h_0$ and $t \in [-\eta + h_0, T - h_0]$. It is well known (see [16, Theorem 9, p. 49]) that $(S_h z)(t) \rightarrow z(t)$ in X for almost all $t \in (-\eta + h_0, T - h_0)$. In addition, it is easy to show that $S_h z \in L^p(-\eta + h_0, T - h_0; X)$ as well as, by using Fubini's theorem,

$$\|S_h z\|_{L^p(-\eta+h_0, T-h_0; X)} \leq \|z\|_{L^p(-\eta, T; X)},$$

and thus $S_h z \rightarrow z$ in $L^p(-\eta + h_0, T - h_0; X)$.

First, note that both v_λ and v_μ are in $\mathcal{C}_w([0, T]; H)$, and take the same value v_0 at $t = 0$. Hence, the difference $v_\lambda - v_\mu$ fulfills the initial condition $(v_\lambda - v_\mu)(0) = 0$. We assume both v_λ and v_μ to be extended by $v_0 \in V$ for $t < 0$, and thus the difference to be extended by 0 for $t < 0$. We conclude that $v_\lambda - v_\mu \in L^p(-\eta, T; V_A) \cap \mathcal{C}_w([-\eta, T]; H)$ and similarly $K_\lambda v_\lambda - K_\mu v_\mu \in \mathcal{C}_w([-\eta, T]; V_B)$. Relation (3.1) still holds for both v_λ and v_μ for almost all $t \in (-\eta, 0)$.

As

$$v_\lambda - v_\mu \in L^p(-\eta, T; V_A) \cap \mathcal{C}_w([-\eta, T]; H),$$

it is easy to show that $S_h(v_\lambda - v_\mu)$ is bounded in $L^p(-\eta + h_0, T - h_0; V_A) \cap L^\infty(-\eta + h_0, T - h_0; H)$. Therefore, there exists a subsequence, again denoted by h , such that $S_h(v_\lambda - v_\mu) \xrightarrow{*} v_\lambda - v_\mu$ in

$$L^p(-\eta + h_0, T - h_0; V_A) \cap L^\infty(-\eta + h_0, T - h_0; H).$$

In view of (3.1), we have

$$(S_h(v_\lambda - v_\mu))(t) = \frac{1}{2h} \left(\left(\frac{1}{\lambda} K_\lambda v_\lambda - \frac{1}{\mu} K_\mu v_\mu \right)(t+h) - \left(\frac{1}{\lambda} K_\lambda v_\lambda - \frac{1}{\mu} K_\mu v_\mu \right)(t-h) \right) + (S_h K_\lambda v_\lambda)(t) - (S_h K_\mu v_\mu)(t)$$

such that $K_\lambda v_\lambda, K_\mu v_\mu \in \mathcal{C}_w([-\eta, T]; V_B)$ provide $S_h(v_\lambda - v_\mu) \in \mathcal{C}_w([-\eta + h_0, T - h_0]; V_B)$. Thus, in contrast to the difference $v_\lambda - v_\mu$ itself, the Steklov average $S_h(v_\lambda - v_\mu)$ is an element of

$$L^p(-\eta + h_0, T - h_0; V_A) \cap \mathcal{C}_w([-\eta + h_0, T - h_0]; V_B),$$

which provides the regularity needed.

² We could have also used this method to prove Lemma 4.3, but it only provides the integration-by-parts formula to hold on (α, β) for almost all $\alpha, \beta \in (0, T)$.

For all α, β with $-\eta + h_0 < \alpha < \beta < T - h_0$ we find

$$\begin{aligned}
& \int_{\alpha}^{\beta} \langle (v_{\lambda} - v_{\mu})'(t) + B((K_{\lambda}v_{\lambda})(t) - (K_{\mu}v_{\mu})(t)), (v_{\lambda} - v_{\mu})(t) \rangle dt \\
&= \int_{\alpha}^{\beta} \langle (v_{\lambda} - v_{\mu})'(t) + B((K_{\lambda}v_{\lambda})(t) - (K_{\mu}v_{\mu})(t)), (v_{\lambda} - v_{\mu})(t) - (S_h(v_{\lambda} - v_{\mu}))(t) \rangle dt \\
&\quad + \int_{\alpha}^{\beta} \langle (v_{\lambda} - v_{\mu})'(t) + B((K_{\lambda}v_{\lambda})(t) - (K_{\mu}v_{\mu})(t)) \\
&\quad - (S_h((v_{\lambda} - v_{\mu})' + B(K_{\lambda}v_{\lambda} - K_{\mu}v_{\mu}))(t)), (S_h(v_{\lambda} - v_{\mu}))(t) \rangle dt \\
&\quad + \int_{\alpha}^{\beta} \langle (S_h(v_{\lambda} - v_{\mu})')(t) + B(S_h(K_{\lambda}v_{\lambda} - K_{\mu}v_{\mu}))(t), (S_h(v_{\lambda} - v_{\mu}))(t) \rangle dt. \tag{5.3}
\end{aligned}$$

The first term on the right-hand side tends to zero as $h \rightarrow 0$ since

$$(v_{\lambda} - v_{\mu})' + B(K_{\lambda}v_{\lambda} - K_{\mu}v_{\mu}) = -(Av_{\lambda} - Av_{\mu}) \in L^{p'}(-\eta + h_0, T - h_0; V'_A) + L^1(-\eta + h_0, T - h_0; H)$$

and

$$S_h(v_{\lambda} - v_{\mu}) \xrightarrow{*} v_{\lambda} - v_{\mu}$$

in $L^p(-\eta + h_0, T - h_0; V_A) \cap L^{\infty}(-\eta + h_0, T - h_0; H)$ as $h \rightarrow 0$, the second one since $\{S_h(v_{\lambda} - v_{\mu})\}$ is bounded in $L^p(-\eta + h_0, T - h_0; V_A) \cap L^{\infty}(-\eta + h_0, T - h_0; H)$, and

$$S_h((v_{\lambda} - v_{\mu})' + B(K_{\lambda}v_{\lambda} - K_{\mu}v_{\mu})) \rightarrow (v_{\lambda} - v_{\mu})' + B(K_{\lambda}v_{\lambda} - K_{\mu}v_{\mu})$$

in $L^{p'}(-\eta + h_0, T - h_0; V'_A) + L^1(-\eta + h_0, T - h_0; H)$ as $h \rightarrow 0$. The latter follows from the linearity of S_h .

Due to the regularity properties of $S_h(v_{\lambda} - v_{\mu})$, we are allowed to split the last integrand in (5.3). We use

$$v_{\lambda} - v_{\mu} = \frac{1}{\lambda}(K_{\lambda}v_{\lambda})' - \frac{1}{\mu}(K_{\mu}v_{\mu})' + K_{\lambda}v_{\lambda} - K_{\mu}v_{\mu}$$

as well as $S_h(z') = (S_h z)'$ for any integrable function z on $(\alpha, \beta) \subset (-\eta + h_0, T - h_0)$. We rewrite the third term as

$$\begin{aligned}
& \int_{\alpha}^{\beta} \langle (S_h(v_{\lambda} - v_{\mu})')(t) + B(S_h(K_{\lambda}v_{\lambda} - K_{\mu}v_{\mu}))(t), (S_h(v_{\lambda} - v_{\mu}))(t) \rangle dt \\
&= \int_{\alpha}^{\beta} \langle (S_h(v_{\lambda} - v_{\mu})')(t), (S_h(v_{\lambda} - v_{\mu}))(t) \rangle dt + \int_{\alpha}^{\beta} \langle B(S_h(K_{\lambda}v_{\lambda} - K_{\mu}v_{\mu}))(t), (S_h(K_{\lambda}v_{\lambda} - K_{\mu}v_{\mu}))(t) \rangle dt \\
&\quad + \int_{\alpha}^{\beta} \langle B(S_h(K_{\lambda}v_{\lambda} - K_{\mu}v_{\mu}))(t), \frac{1}{\lambda}(S_h(K_{\lambda}v_{\lambda})')(t) - \frac{1}{\mu}(S_h(K_{\mu}v_{\mu})')(t) \rangle dt \\
&= \frac{1}{2} \|(S_h(v_{\lambda} - v_{\mu}))(\beta)\|_H^2 - \frac{1}{2} \|(S_h(v_{\lambda} - v_{\mu}))(\alpha)\|_H^2 + \int_{\alpha}^{\beta} \|(S_h(K_{\lambda}v_{\lambda} - K_{\mu}v_{\mu}))(t)\|_B^2 dt \\
&\quad + \frac{1}{\mu} \int_{\alpha}^{\beta} \langle B(S_h(K_{\lambda}v_{\lambda} - K_{\mu}v_{\mu}))(t), (S_h(K_{\lambda}v_{\lambda} - K_{\mu}v_{\mu}))'(t) \rangle dt \\
&\quad + \left(\frac{1}{\lambda} - \frac{1}{\mu}\right) \int_{\alpha}^{\beta} \langle B(S_h(K_{\lambda}v_{\lambda} - K_{\mu}v_{\mu}))(t), (S_h(K_{\lambda}v_{\lambda}))'(t) \rangle dt
\end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{2} \|(S_h(v_\lambda - v_\mu))(\beta)\|_H^2 - \frac{1}{2} \|(S_h(v_\lambda - v_\mu))(\alpha)\|_H^2 + \int_\alpha^\beta \|(S_h(K_\lambda v_\lambda - K_\mu v_\mu))(t)\|_B^2 dt \\ &\quad + \frac{1}{2\mu} \|(S_h(K_\lambda v_\lambda - K_\mu v_\mu))(\beta)\|_B^2 - \frac{1}{2\mu} \|(S_h(K_\lambda v_\lambda - K_\mu v_\mu))(\alpha)\|_B^2 \\ &\quad - \left| \frac{1}{\lambda} - \frac{1}{\mu} \right| \int_\alpha^\beta \|(S_h(K_\lambda v_\lambda - K_\mu v_\mu))(t)\|_B \|(S_h(K_\lambda v_\lambda))'(t)\|_B dt. \end{aligned}$$

Taking the limit $h \rightarrow \infty$ and employing (5.2) results in

$$\begin{aligned} &\frac{1}{2} \|(v_\lambda - v_\mu)(\beta)\|_H^2 + \int_\alpha^\beta \|(K_\lambda v_\lambda - K_\mu v_\mu)(t)\|_B^2 dt + \frac{1}{2\mu} \|(K_\lambda v_\lambda - K_\mu v_\mu)(\beta)\|_B^2 \\ &\leq \frac{1}{2} \|(v_\lambda - v_\mu)(\alpha)\|_H^2 + \frac{1}{2\mu} \|(K_\lambda v_\lambda - K_\mu v_\mu)(\alpha)\|_B^2 \\ &\quad + \left| \frac{1}{\lambda} - \frac{1}{\mu} \right| \int_\alpha^\beta \|(K_\lambda v_\lambda - K_\mu v_\mu)(t)\|_B \|(K_\lambda v_\lambda)'(t)\|_B dt \end{aligned} \tag{5.4}$$

for almost all α, β , namely for those that are Lebesgue points of $v_\lambda - v_\mu$ and $K_\lambda v_\lambda - K_\mu v_\mu$. Finally, we consider a sequence $\{\alpha_k\}$ with $\alpha_k < 0$ and $\alpha_k \rightarrow 0$ as $k \rightarrow \infty$. We remark that $v_\lambda - v_\mu \in \mathcal{C}_w([-\eta, T]; H)$ and $K_\lambda v_\lambda - K_\mu v_\mu \in \mathcal{C}_w([-\eta, T]; V_B)$. Due to the choice of the extension for $t < 0$, equation (5.4) shows for $\beta = t$ that

$$\begin{aligned} &\frac{1}{2} \|(v_\lambda - v_\mu)(t)\|_H^2 + \int_0^t \|(K_\lambda v_\lambda - K_\mu v_\mu)(s)\|_B^2 ds + \frac{1}{2\mu} \|(K_\lambda v_\lambda - K_\mu v_\mu)(t)\|_B^2 \\ &\leq \left| \frac{1}{\lambda} - \frac{1}{\mu} \right| \int_0^t \|(K_\lambda v_\lambda - K_\mu v_\mu)(s)\|_B \|(K_\lambda v_\lambda)'(s)\|_B ds \end{aligned}$$

for almost all $t \in (0, T)$, and thus

$$\begin{aligned} &\frac{1}{2} \|(v_\lambda - v_\mu)(t)\|_H^2 + \frac{1}{2} \int_0^t \|(K_\lambda v_\lambda - K_\mu v_\mu)(s)\|_B^2 ds + \frac{1}{2\mu} \|(K_\lambda v_\lambda - K_\mu v_\mu)(t)\|_B^2 \\ &\leq \frac{1}{2} \left| \frac{1}{\lambda} - \frac{1}{\mu} \right|^2 \|(K_\lambda v_\lambda)'\|_{L^2(0, T; B)}^2 \\ &\leq \frac{\lambda^2}{2} \left| \frac{1}{\lambda} - \frac{1}{\mu} \right|^2 (\|v_\lambda\|_{L^2(0, T; B)}^2 + \|K_\lambda v_\lambda - u_0\|_{L^2(0, T; B)}^2). \end{aligned}$$

It is easy to show that $\|K_\lambda v_\lambda - u_0\|_{L^2(0, T; B)}^2 \leq \lambda^2 T^2 \|v_\lambda\|_{L^2(0, T; B)}^2$, which proves the assertion. □

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References

- [1] V. Barbu, P. Colli, G. Gilardi and M. Grasselli, Existence, uniqueness, and longtime behavior for a nonlinear Volterra integrodifferential equation, *Differential Integral Equations* **13** (2000), no. 10–12, 1233–1262.
- [2] A. Bátkai and S. Piazzera, *Semigroups for Delay Equations*, Res. Notes Math. 10, A K Peters, Wellesley, 2005.
- [3] S. Bonaccorsi and G. Desch, Volterra equations in Banach spaces with completely monotone kernels, *NoDEA Nonlinear Differential Equations Appl.* **20** (2013), no. 3, 557–594.

- [4] H. Brézis, *Analyse Fonctionnelle: Théorie et Applications*, Dunod, Paris, 1999.
- [5] M. P. Calvo, E. Cuesta and C. Palencia, Runge–Kutta convolution quadrature methods for well-posed equations with memory, *Numer. Math.* **107** (2007), no. 4, 589–614.
- [6] P. Cannarsa and D. Sforza, Global solutions of abstract semilinear parabolic equations with memory terms, *NoDEA Nonlinear Differential Equations Appl.* **10** (2003), no. 4, 399–430.
- [7] C. Chen and T. Shih, *Finite Element Methods for Integrodifferential Equations*, Ser. Appl. Math. 9, World Scientific, River Edge, 1998.
- [8] C. Chen, V. Thomée and L. B. Wahlbin, Finite element approximation of a parabolic integro-differential equation with a weakly singular kernel, *Math. Comp.* **58** (1992), no. 198, 587–602.
- [9] C. Corduneanu, *Functional Equations with Causal Operators*, Stab. Control Theory Methods Appl. 16, Taylor & Francis, London, 2002.
- [10] M. G. Crandall, S.-O. Londen and J. A. Nohel, An abstract nonlinear Volterra integrodifferential equation, *J. Math. Anal. Appl.* **64** (1978), no. 3, 701–735.
- [11] M. G. Crandall and J. A. Nohel, An abstract functional differential equation and a related nonlinear Volterra equation, *Israel J. Math.* **29** (1978), no. 4, 313–328.
- [12] E. Cuesta and C. Palencia, A numerical method for an integro-differential equation with memory in Banach spaces: Qualitative properties, *SIAM J. Numer. Anal.* **41** (2003), no. 4, 1232–1241.
- [13] J. M. Cushing, *Integrodifferential Equations and Delay Models in Population Dynamics*, Springer, Berlin, 1977.
- [14] C. M. Dafermos and J. A. Nohel, Energy methods for nonlinear hyperbolic Volterra integrodifferential equations, *Comm. Partial Differential Equations* **4** (1979), no. 3, 219–278.
- [15] W. Desch, R. Grimmer and W. Schappacher, Well-posedness and wave propagation for a class of integrodifferential equations in Banach space, *J. Differential Equations* **74** (1988), no. 2, 391–411.
- [16] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, American Mathematical Society, Providence, 1977.
- [17] D. A. Edwards, Non-Fickian diffusion in thin polymer films, *J. Polym. Sci. Part B Polym. Phys.* **34** (1996), 981–997.
- [18] D. A. Edwards and D. S. Cohen, A mathematical model for a dissolving polymer, *AIChE Journal* **41** (1995), no. 11, 2345–2355.
- [19] E. Emmrich and M. Thälhammer, Doubly nonlinear evolution equations of second order: existence and fully discrete approximation, *J. Differential Equations* **251** (2011), no. 1, 82–118.
- [20] S. Fedotov and A. Iomin, Probabilistic approach to a proliferation and migration dichotomy in tumor cell invasion, *Phys. Rev. E* (3) **77** (2008), no. 3, Article ID 031911.
- [21] J. A. Ferreira, E. Gudiño and P. de Oliveira, A second order approximation for quasilinear non-Fickian diffusion models, *Comput. Methods Appl. Math.* **13** (2013), no. 4, 471–493.
- [22] H. Gajewski, K. Gröger and K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Akademie, Berlin, 1974.
- [23] G. Gilardi and U. Stefanelli, Time-discretization and global solution for a doubly nonlinear Volterra equation, *J. Differential Equations* **228** (2006), no. 2, 707–736.
- [24] G. Gilardi and U. Stefanelli, Existence for a doubly nonlinear Volterra equation, *J. Math. Anal. Appl.* **333** (2007), no. 2, 839–862.
- [25] M. Grasselli and A. Lorenzi, Abstract nonlinear Volterra integrodifferential equations with nonsmooth kernels, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **2** (1991), no. 1, 43–53.
- [26] R. Grimmer and M. Zeman, Nonlinear Volterra integro-differential equations in a Banach space, *Israel J. Math.* **42** (1982), no. 1–2, 162–176.
- [27] G. Gripenberg, On a nonlinear Volterra integral equation in a Banach space, *J. Math. Anal. Appl.* **66** (1978), no. 1, 207–219.
- [28] G. Gripenberg, An abstract nonlinear Volterra equation, *Israel J. Math.* **34** (1979), no. 3, 198–212.
- [29] G. Gripenberg, Nonlinear Volterra equations of parabolic type due to singular kernels, *J. Differential Equations* **112** (1994), no. 1, 154–169.
- [30] G. Gripenberg, S.-O. Londen and O. Staffans, *Volterra Integral and Functional Equations*, Encyclopedia Math. Appl. 34, Cambridge University Press, Cambridge, 1990.
- [31] S. Larsson, V. Thomée and L. B. Wahlbin, Numerical solution of parabolic integro-differential equations by the discontinuous Galerkin method, *Math. Comp.* **67** (1998), no. 221, 45–71.
- [32] J.-L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications. Vol. I*, Springer, New York, 1972.
- [33] S.-O. Londen, On an integrodifferential Volterra equation with a maximal monotone mapping, *J. Differential Equations* **27** (1978), no. 3, 405–420.
- [34] C. Lubich, Convolution quadrature and discretized operational calculus. I, *Numer. Math.* **52** (1988), no. 2, 129–145.
- [35] C. Lubich, Convolution quadrature and discretized operational calculus. II, *Numer. Math.* **52** (1988), no. 4, 413–425.
- [36] C. Lubich, Convolution quadrature revisited, *BIT* **44** (2004), no. 3, 503–514.
- [37] C. Lubich, I. H. Sloan and V. Thomée, Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term, *Math. Comp.* **65** (1996), no. 213, 1–17.
- [38] R. C. MacCamy, A model for one-dimensional, nonlinear viscoelasticity, *Quart. Appl. Math.* **35** (1977/78), no. 1, 21–33.

- [39] R. C. MacCamy, An integro-differential equation with application in heat flow, *Quart. Appl. Math.* **35** (1977/78), no. 1, 1–19.
- [40] R. C. MacCamy and J. S. W. Wong, Stability theorems for some functional equations, *Trans. Amer. Math. Soc.* **164** (1972), 1–37.
- [41] W. McLean and V. Thomée, Numerical solution of an evolution equation with a positive-type memory term, *J. Aust. Math. Soc. Ser. B* **35** (1993), no. 1, 23–70.
- [42] A. Mehrabian and Y. Abousleiman, General solutions to poroviscoelastic model of hydrocephalic human brain tissue, *J. Theoret. Biol.* **291** (2011), 105–118.
- [43] R. K. Miller, An integro-differential equation for rigid heat conductors with memory, *J. Math. Anal. Appl.* **66** (1978), no. 2, 313–332.
- [44] J. Prüß, *Evolutionary Integral Equations and Applications*, Birkhäuser, Basel, 1993.
- [45] T. Roubíček, *Nonlinear Partial Differential Equations with Applications*, Birkhäuser, Basel, 2005.
- [46] S. Shaw and J. R. Whiteman, Some partial differential Volterra equation problems arising in viscoelasticity, in: *Proceedings of Equadiff 9*, Masaryk University, Brno (1998), 183–200.
- [47] I. H. Sloan and V. Thomée, Time discretization of an integro-differential equation of parabolic type, *SIAM J. Numer. Anal.* **23** (1986), no. 5, 1052–1061.
- [48] U. Stefanelli, On some nonlocal evolution equations in Banach spaces, *J. Evol. Equ.* **4** (2004), no. 1, 1–26.
- [49] W. A. Strauss, On continuity of functions with values in various Banach spaces, *Pacific J. Math.* **19** (1966), 543–551.
- [50] G. F. Webb, Functional differential equations and nonlinear semigroups in L^p -spaces, *J. Differential Equations* **20** (1976), no. 1, 71–89.
- [51] G. F. Webb, Volterra integral equations and nonlinear semigroups, *Nonlinear Anal.* **1** (1976/77), no. 4, 415–427.
- [52] G. F. Webb, An abstract semilinear Volterra integrodifferential equation, *Proc. Amer. Math. Soc.* **69** (1978), no. 2, 255–260.
- [53] R. Zacher, Boundedness of weak solutions to evolutionary partial integro-differential equations with discontinuous coefficients, *J. Math. Anal. Appl.* **348** (2008), no. 1, 137–149.
- [54] R. Zacher, Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces, *Funkcial. Ekvac.* **52** (2009), no. 1, 1–18.
- [55] E. Zeidler, *Nonlinear Functional Analysis and its Applications. II/B*, Springer, New York, 1990.