## **Research Article**

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# A Priori Analysis of a Symmetric Interior Penalty Discontinuous Galerkin Finite Element Method for a Dynamic Linear Viscoelasticity Model

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**Abstract:** The stress-strain constitutive law for viscoelastic materials such as soft tissues, metals at high temperature, and polymers can be written as a Volterra integral equation of the second kind with a *fading memory* kernel. This integral relationship yields current stress for a given strain history and can be used in the momentum balance law to derive a mathematical model for the resulting deformation. We consider such a dynamic linear viscoelastic model problem resulting from using a *Dirichlet–Prony* series of decaying exponentials to provide the fading memory in the Volterra kernel. We introduce two types of *internal variable* to replace the Volterra integral with a system of auxiliary ordinary differential equations and then use a spatially discontinuous symmetric interior penalty Galerkin (SIPG) finite element method and – in time – a Crank–Nicolson method to formulate the fully discrete problems: one for each type of internal variable. We present a priori stability and error analyses without using Grönwall's inequality and with the result that the constants in our estimates grow linearly with time rather than exponentially. In this sense, the schemes are therefore suited to simulating long time viscoelastic response, and this (to our knowledge) is the first time that such high quality estimates have been presented for SIPG finite element approximation of dynamic viscoelasticity problems. We also carry out a number of numerical experiments using the FEniCS environment (https://fenicsproject.org), describe a simulation using "real" material data, and explain how the codes can be obtained and all of the results reproduced.

**Keywords:** Viscoelasticity, Generalised Maxwell Solid, Symmetric Interior Penalty, Discontinuous Galerkin Finite Element Method, A Priori Analysis, Internal Variables

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# **1** Introduction

The application of a *nonsymmetric interior penalty discontinuous Galerkin* (NIPG) *finite element method* (DGFEM) to a dynamic linear solid viscoelasticity problem with tensor-valued internal variable stress representation was presented by Rivière, Shaw, and Whiteman in [21]. They gave an a priori energy error estimate by using the standard Grönwall inequality to deal with the time accumulation of error, and hence the constants in the stability and error bounds are too large to give confidence in the long time simulation of viscoelastic response. In this paper, we use the symmetric interior penalty Galerkin (SIPG) method and prove stability bounds and a priori error estimates (not only in the energy norm but also in the spatial  $L_2$  norm) without the use of Grönwall's inequality. We therefore obtain non-exponentially increasing bounds for temporal  $L_{\infty}$ -type norms. Further-

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more, we introduce vector-valued internal variables in *displacement form* and *velocity form* (to be defined below). This has the advantage of reducing computer memory requirements in that we need only store vectors of dimension *d*, instead of symmetric second-order tensors of dimension d(d + 1)/2, as in [21]. This can be significant for high fidelity 3D simulations.

We consider a linear homogeneous and isotropic viscoelastic solid material (see e.g. [11]) occupying a bounded polytopic domain, the interior of which is denoted by  $\Omega \subset \mathbb{R}^d$ , and consider the deformation and stress-strain state of this material over times  $t \in [0, T]$ , where T > 0. The deformation u and stress  $\underline{\sigma}$  follow the momentum equation

$$\rho \ddot{\boldsymbol{u}} - \nabla \cdot \underline{\boldsymbol{\sigma}} = \boldsymbol{f} \quad \text{on } \Omega \times (0, T], \tag{1.1}$$

where overdots denote time differentiation so that  $\ddot{u}$  is acceleration,  $\rho$  is the mass density of the material (assumed constant),  $\nabla \cdot \underline{\sigma}$  is the divergence of stress, and f is an external body force (see e.g. [21, 22]). Similarly,  $\dot{u}$  denotes velocity. In addition to this vector-valued governing equation, we assume a mix of essential and natural boundary conditions so that

$$\boldsymbol{u}(t) = 0 \qquad \text{on } \boldsymbol{\Gamma}_D \times [0, T], \tag{1.2}$$

$$\underline{\boldsymbol{\sigma}}(t) \cdot \boldsymbol{n} = \boldsymbol{g}_N(t) \quad \text{on } \Gamma_N \times [0, T], \tag{1.3}$$

where  $\Gamma_D$  is the *Dirichlet* boundary (assumed to have positive surface measure),  $\Gamma_N$  is the *Neumann* boundary given by  $\Gamma_N = \partial \Omega \setminus \Gamma_D$ ,  $\boldsymbol{n}$  is an outward unit normal vector defined a.e. on  $\Gamma_N$ , and  $\boldsymbol{g}_N$  prescribes a surface traction on  $\Gamma_N$ . Furthermore, for initial conditions on the displacement and the velocity, we take

$$u(0) = u_0$$
 and  $\dot{u}(0) = w_0$  (1.4)

for given functions  $\boldsymbol{u}_0$  and  $\boldsymbol{w}_0$ .

To close the problem and solve for displacement, we need a constitutive equation expressing stress in terms of displacement. In the linear viscoelasticity model considered here, this involves a Volterra (or "fading memory") integral with the specific material characterised by stiffness  $\underline{D}$  and a stress relaxation function  $\varphi$ ; see e.g. [8, 11, 14, 22]. The stress is then given by

$$\underline{\boldsymbol{\sigma}}(t) = \underline{\boldsymbol{D}}\varphi(t)\underline{\boldsymbol{\varepsilon}}(0) + \int_{0}^{t} \underline{\boldsymbol{D}}\varphi(t-s)\underline{\dot{\boldsymbol{\varepsilon}}}(s) \, ds, \qquad (1.5)$$

where  $\underline{D}$  is a fourth-order positive definite tensor satisfying the symmetries  $D_{ijkl} = D_{jikl} = D_{ijlk} = D_{klij}$ , and  $\underline{e}$  is the strain defined by

$$\varepsilon_{ij}(\mathbf{v}) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \text{ for } i, j = 1, \dots, d.$$

Note that, in (1.5), we use the shorthand  $\underline{\mathbf{\varepsilon}}(t) = \underline{\mathbf{\varepsilon}}(\mathbf{u}(t))$ . The form of  $\varphi$  depends on which viscoelastic model is invoked. There are several (see e.g. [8, 11] and the references therein), but here we focus on the *generalised Maxwell solid* where

$$\varphi(t) = \varphi_0 + \sum_{q=1}^{N_{\varphi}} \varphi_q e^{-t/\tau_q}$$
(1.6)

with  $N_{\varphi} \in \mathbb{N}$ , strictly positive delay times  $\{\tau_q\}_{q=1}^{N_{\varphi}}$ , and coefficients  $\{\varphi_q\}_{q=0}^{N_{\varphi}}$ , the latter of which are normalised so that  $\varphi(0) = 1$ . The positivity requirement excludes the case  $\varphi_0 = 0$  (a fluid in the sense used by Golden and Graham in [11]): this is an important assumption in the arguments developed below. In particular, in moving from (4.4) to (4.5), we will use the fact that

$$1 - \sum_{q=1}^{N_{\varphi}} \varphi_q = \varphi(0) - \sum_{q=1}^{N_{\varphi}} \varphi_q = \varphi_0 > 0.$$
(1.7)

This paper is arranged as follows. In Section 2, we give our notations and the preliminary background for DGFEM. In Section 3, we introduce two forms of internal variables, each of which are used to represent the

Volterra (or "history") integral, and formulate a variational problem for each form. We then state and prove (without using Grönwall's inequality) stability bounds in Section 4 and error bounds in Section 5, carry out some illustrative numerical experiments in Section 6, using FEniCS (see [1] and https://fenicsproject.org), and then end with some concluding remarks in Section 7.

# 2 Preliminary

We use standard notation so that  $L_p(\Omega)$ ,  $H^s(\Omega)$ , and  $W_p^s(\Omega)$  (with *s* and *p* non-negative) denote the usual Lebesgue, Hilbert, and Sobolev spaces. For any normed space X,  $\|\cdot\|_X$  is the *X* norm which, for inner product spaces, is always the norm induced by the inner product. For example,  $\|\cdot\|_{L_2(\Omega)}$  is the  $L_2(\Omega)$  norm, as induced by the  $L_2(\Omega)$  inner product denoted – for brevity – by  $(\cdot, \cdot)$ , but for  $S \subset \overline{\Omega}$ , we use  $(\cdot, \cdot)_{L_2(S)}$  for the  $L_2(S)$  inner product. For time dependent functions, we expand this notation so that, for *X* a normed target space,  $f \in L_p(0, T; X)$  denotes the space of  $L_p(0, T) \to X$  functions with norm

$$\|f\|_{L_p(0,T;X)} = \left(\int_0^T \|f(t)\|_X^p \, dt\right)^{1/p}$$

for  $1 \le p < \infty$ . When  $p = \infty$ , this becomes the *essential supremum* norm

$$||f||_{L_{\infty}(0,T;X)} = \underset{0 \le t \le T}{\operatorname{ess \, sup}} ||f(t)||_{X}.$$

When convenient, we shall often replace the upper limit T in these expressions by some other value  $t \in [0, T]$ .

For inner products of vector-valued and tensor-valued functions, we use the same notation as for the scalar cases. For instance, we have

$$(\mathbf{v}, \mathbf{w}) = \int_{\Omega} \mathbf{v} \cdot \mathbf{w} \, d\Omega, \quad (\underline{\mathbf{v}}, \underline{\mathbf{w}}) = \int_{\Omega} \underline{\mathbf{v}} : \underline{\mathbf{w}} \, d\Omega = \sum_{i,j=1}^{d} \int_{\Omega} v_{ij} w_{ij} \, d\Omega$$

for vector-valued functions v and w, and second-order tensors v and w.

## Meshes

We refer to [19] for a detailed explanation of the framework of the DGFEM and here just summarise the main points. Assume that the closure of  $\Omega$  is subdivided into closed elements *E*, where *E* is a triangle in 2D or a tetrahedron in 3D, and the intersection of any pair of elements is either a vertex, an edge, a face, or empty. The diameter of *E* is defined by  $h_E := \sup_{x,y \in E} ||x - y||$ , where  $|| \cdot ||$  is the Euclidean norm, and |E| denotes the measure (area/volume) of *E*. In a similar way, let *e* be an edge of *E* and use |e| to denote its measure (length/area). Let *h* be the maximum of the diameters  $h_E$  over all the elements *E*, and define the set  $\mathcal{E}_h$  of all of those elements. Then  $|e| \leq h_E^{d-1} \leq h^{d-1}$  for all  $e \subset \partial E$  for each  $E \in \mathcal{E}_h$ . We further suppose that the subdivision is quasi-uniform, which means that there exists a positive constant *C* such that  $h \leq Ch_E$  for all  $E \in \mathcal{E}_h$ .

Next, let  $\Gamma_h$  be the set of interior edges (in 2D) or faces (in 3D) contained in the subdivision  $\mathcal{E}_h$ . Then, for each edge or face element e, we can define a unit normal vector  $\mathbf{n}_e$ . If  $e \in \partial \Omega$ , then  $\mathbf{n}_e$  is the outward unit normal vector. For an interior edge e such that  $e \in E_i \cap E_j$  with i < j, the normal vector  $\mathbf{n}_e$  is oriented from  $E_i$  to  $E_j$ .

## **Test Spaces**

We introduce the broken Sobolev space  $H^{s}(\mathcal{E}_{h}) = \{v \in L_{2}(\Omega) \mid \text{ for all } E \in \mathcal{E}_{h}, v|_{E} \in H^{s}(E)\}$  and endow it with the broken Sobolev norm  $\|\cdot\|_{H^{s}(\mathcal{E}_{h})}$  defined by

$$\|\|\boldsymbol{v}\|\|_{H^{s}(\mathcal{E}_{h})} = \left(\sum_{E \in \mathcal{E}_{h}} \|\boldsymbol{v}\|_{H^{s}(E)}^{2}\right)^{1/2}$$

We note the following facts:  $H^{s}(\Omega) \subset H^{s}(\mathcal{E}_{h})$  and  $H^{s+1}(\mathcal{E}_{h}) \subset H^{s}(\mathcal{E}_{h})$ . These definitions and notations are extended in an obvious way to the vector field analogue  $H^{s}(\mathcal{E}_{h})$ .

We define the space of polynomials of degree less than or equal to k on E, for  $E \subset \mathbb{R}^d$ , by

$$\mathcal{P}_k(E) = \operatorname{span}\left\{ x_1^{i_1} \cdots x_d^{i_d} \mid \sum_{m=1}^d i_m \leq k, \ x \in E, \ i_m \in \mathbb{N} \cup \{0\} \text{ for each } m \right\}$$

and then define our DG finite element space as

$$\mathcal{D}_k(\mathcal{E}_h) = \{ v \in H^1(\mathcal{E}_h) \mid v|_E \in \mathcal{P}_k(E) \text{ for each } E \in \mathcal{E}_h \}.$$

The analogous vector field is given by  $\mathcal{D}_k(\mathcal{E}_h) := [\mathcal{D}_k(\mathcal{E}_h)]^d$ .

## Average and Jump

Suppose two elements  $E_i^e$  and  $E_j^e$  share the common edge e with i < j and that there is a vector-valued function v and a second-order tensor  $\underline{v}$  on  $E_i^e$  and  $E_j^e$ . Then we define an average and a jump for v and  $\underline{v}$  by

$$\{\mathbf{v}\} = \frac{(\mathbf{v}|_{E_i^e})|_e + (\mathbf{v}|_{E_j^e})|_e}{2}, \qquad \{\underline{v}\} = \frac{(\underline{v}|_{E_i^e})|_e + (\underline{v}|_{E_j^e})|_e}{2}, \\ [\mathbf{v}] = (\mathbf{v}|_{E_i^e})|_e - (\mathbf{v}|_{E_j^e})|_e, \qquad [\mathbf{v} \otimes \mathbf{n}_e] = (\mathbf{v}|_{E_i^e})|_e \otimes \mathbf{n}_e - (\mathbf{v}|_{E_j^e})|_e \otimes \mathbf{n}_e$$

where the normal vector  $\mathbf{n}_e$  is oriented from  $E_i^e$  to  $E_j^e$  and  $\otimes$  is the outer product defined, for vectors  $\mathbf{a}$  and  $\mathbf{b}$ , by  $(\mathbf{a} \otimes \mathbf{b})_{mn} = a_m b_n$  for m, n = 1, ..., d. On the other hand, if  $e \in \partial \Omega$  and  $e \in \partial E$ ,

 $\{\boldsymbol{\nu}\} = \boldsymbol{\nu}|_e, \quad \{\underline{\nu}\} = \underline{\nu}|_e, \quad [\boldsymbol{\nu}] = \boldsymbol{\nu}|_e \cdot \boldsymbol{n}_e, \text{ and } [\boldsymbol{\nu} \otimes \boldsymbol{n}_e] = \boldsymbol{\nu}|_e \otimes \boldsymbol{n}_e.$ 

We can now introduce the jump penalty operator

$$J_0^{\alpha_0,\beta_0}(\boldsymbol{v},\boldsymbol{w}) = \sum_{e \in \Gamma_h \cup \Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \int_e [\boldsymbol{v}] \cdot [\boldsymbol{w}] \, de,$$

where  $\alpha_0$  and  $\beta_0$  are positive constants.

#### **Useful Inequalities**

We now recall the following inequalities for use later in the a priori analysis.

• Inverse polynomial trace inequalities [23]: For any  $v \in \mathcal{P}_k(E)$ , for all  $e \subset \partial E$ ,

$$\begin{cases} \|v\|_{L_{2}(e)} \leq C|e|^{1/2}|E|^{-1/2}\|v\|_{L_{2}(E)}, \\ \|v\|_{L_{2}(e)} \leq Ch_{E}^{-1/2}\|v\|_{L_{2}(E)}, \\ \|\nabla v \cdot \boldsymbol{n}_{e}\|_{L_{2}(e)} \leq C|e|^{1/2}|E|^{-1/2}\|\nabla v\|_{L_{2}(E)}, \\ \|\nabla v \cdot \boldsymbol{n}_{e}\|_{L_{2}(e)} \leq Ch_{E}^{-1/2}\|\nabla v\|_{L_{2}(E)}, \end{cases}$$

$$(2.1)$$

where C is a positive constant and is independent of  $h_E$  but depends on the polynomial degree k.

• Poincaré's inequality [3, 19]: If  $\beta_0(d-1) \ge 1$  and  $|e| \le 1$  for every  $e \in \Gamma_h \cup \Gamma_D$ , then

$$\|v\|_{L_{2}(\Omega)} \leq C \Big( \|\nabla v\|_{H^{0}(\mathcal{E}_{h})}^{2} + \sum_{e \in \Gamma_{h} \cup \Gamma_{D}} \frac{1}{|e|^{\beta_{0}}} \|[v]\|_{L_{2}(e)}^{2} \Big)^{1/2}$$

$$(2.2)$$

for any  $v \in H^1(\mathcal{E}_h)$ .

Inverse inequality (or Markov inequality) [17, 19]: For any  $E \in \mathcal{E}_h$ , there is a positive constant C such that

for all 
$$v \in \mathcal{P}_k(E)$$
,  $\|\nabla^j v\|_{L_2(E)} \leq C h_E^{-j} \|v\|_{L_2(E)}$  for all  $j \in \{0, 1, \dots, k\}$ , (2.3)

where

$$\nabla^{j} v = \begin{cases} \nabla \cdot \nabla^{j-1} v & \text{for even } j, \\ \nabla (\nabla^{j-1} v) & \text{for odd } j, \end{cases} \quad \text{and} \quad \nabla^{0} v = v$$

Note that these three inequalities can also be applied in the obvious way to vector-valued functions.

#### **DG Bilinear Form**

We define a DG bilinear form  $a: H^{s}(\mathcal{E}_{h}) \times H^{s}(\mathcal{E}_{h}) \mapsto \mathbb{R}$  for s > 3/2 by

$$a(\mathbf{v}, \mathbf{w}) = \sum_{E \in \mathcal{E}_h} \int_E \underline{D} \underline{\mathbf{\varepsilon}}(\mathbf{v}) : \underline{\mathbf{\varepsilon}}(\mathbf{w}) \, dE - \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\underline{D} \underline{\mathbf{\varepsilon}}(\mathbf{v})\} : [\mathbf{w} \otimes \mathbf{n}_e] \, de$$
$$- \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\underline{D} \underline{\mathbf{\varepsilon}}(\mathbf{w})\} : [\mathbf{v} \otimes \mathbf{n}_e] \, de + J_0^{\alpha_0, \beta_0}(\mathbf{v}, \mathbf{w})$$
(2.4)

for any  $v, w \in H^{s}(\mathcal{E}_{h})$ . We also define our DG energy norm by

$$\|\boldsymbol{\nu}\|_{V} = \left(\sum_{E \in \mathcal{E}_{h}} \int_{E} \underline{\boldsymbol{D}} \underline{\boldsymbol{\varepsilon}}(\boldsymbol{\nu}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{\nu}) \, dE + J_{0}^{\alpha_{0},\beta_{0}}(\boldsymbol{\nu},\boldsymbol{\nu})\right)^{1/2} \quad \text{for } \boldsymbol{\nu} \in \boldsymbol{H}^{s}(\mathcal{E}_{h}).$$
(2.5)

Comparing these, we can observe that

$$a(\boldsymbol{\nu},\boldsymbol{\nu}) = \|\boldsymbol{\nu}\|_{V}^{2} - 2 \sum_{e \in \Gamma_{h} \cup \Gamma_{D}} \int_{e} \{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{\nu})\} : [\boldsymbol{\nu} \otimes \boldsymbol{n}_{e}] de.$$
(2.6)

**Remark 2.1.** In the DG bilinear form, the third term is called the *"interior penalty"* term and the last one is called the *"jump penalty"*. Depending on the sign of the interior penalty, the bilinear form is either symmetric or nonsymmetric. In this article, we consider only the symmetric DG method and refer to [15, 21] for an application of the nonsymmetric method for viscoelasticity. The reason why we employ SIPG is that it requires only the standard penalisation  $\beta_0(d-1) \ge 1$  for optimal spatial error estimates, while NIPG needs the super-penalisation  $\beta_0(d-1) \ge 3$ . Since the use of the super-penalisation enforces the linear system to be more ill-conditioned, we may encounter some difficulty in solving the system with iterative solvers. For more details, we refer to [15].

**Remark 2.2** (Korn's Inequality for Piecewise  $H^1$  Vector Fields [4, 19]). If we have  $\beta_0(d-1) \ge 1$ , then since  $\underline{D}$  is symmetric positive definite and the jump penalty is defined not only on the interior edges but also on the positive measured Dirichlet boundary, Korn's inequality yields, for any  $\mathbf{v} \in H^1(\mathcal{E}_h)$ ,

$$\sum_{E \in \mathcal{E}_h} \|\nabla \boldsymbol{\nu}\|_{L_2(E)}^2 \leq C \|\boldsymbol{\nu}\|_V^2$$
(2.7)

for some positive *C* independent of v.

In CGFEM, the squared energy norm  $\|v\|_V^2$  is usually given by the energy inner product a(v, v). Here, in DGFEM, we see from (2.6) that this is nearly true but there is an extra term. The following lemma allows us to deal with that term.

**Lemma 2.1.** Suppose  $\alpha_0 > 0$  and  $\beta_0(d-1) \ge 1$ . For any  $v, w \in \mathcal{D}_k(\mathcal{E}_h)$  and for any pair  $E_1, E_2 \in \mathcal{E}_h$ , we have

$$\int_{e} \{\underline{D}\underline{\varepsilon}(\mathbf{v})\} : [\mathbf{w} \otimes \mathbf{n}_{e}] de \bigg| \leq \frac{C}{\sqrt{\alpha_{0}}} \bigg( \|\underline{D}\underline{\varepsilon}(\mathbf{v})\|_{L_{2}(E_{1})}^{2} + \|\underline{D}\underline{\varepsilon}(\mathbf{v})\|_{L_{2}(E_{2})}^{2} + \frac{\alpha_{0}}{|e|^{\beta_{0}}} \|[\mathbf{w}]\|_{L_{2}(e)}^{2} \bigg),$$
(2.8)

where e is the shared edge of elements  $E_1$  and  $E_2$ , and C is a positive constant independent of v and w but dependent on the inverse polynomial trace inequality's constants and the domain. When  $e \in \partial \Omega$ , (2.8) holds with  $E_1 = E_2$ .

*Proof.* Let  $e \in \Gamma_h$  and  $e \in E_1 \cap E_2$ , where  $E_1, E_2 \in \mathcal{E}_h$ . Recalling the definitions of the average and jump, and using the Cauchy–Schwarz inequality, we have

$$\left|\int\limits_{e} \{\underline{D}\underline{\varepsilon}(\mathbf{v})\} : [\mathbf{w} \otimes \mathbf{n}_{e}] de\right| \leq \frac{1}{2} (\|\underline{D}\underline{\varepsilon}(\mathbf{v}|_{E_{1}})\|_{L_{2}(e)} + \|\underline{D}\underline{\varepsilon}(\mathbf{v}|_{E_{2}})\|_{L_{2}(e)}) \frac{|e|^{\beta_{0}/2}}{|e|^{\beta_{0}/2}} \|[\mathbf{w}]\|_{L_{2}(e)},$$

after noting that  $[w \otimes n_e] = [w] \otimes n_e$  since  $n_e|_{E_1} = -n_e|_{E_2}$ . The inverse polynomial trace inequality (2.1) implies that

$$\left|\int_{e} \{\underline{D}\underline{\varepsilon}(\mathbf{v})\} : [\mathbf{w} \otimes \mathbf{n}_{e}] de\right| \leq C(|e|^{\beta_{0}/2} h_{E_{1}}^{-1/2} \|\underline{D}\underline{\varepsilon}(\mathbf{v})\|_{L_{2}(E_{1})} + |e|^{\beta_{0}/2} h_{E_{2}}^{-1/2} \|\underline{D}\underline{\varepsilon}(\mathbf{v})\|_{L_{2}(E_{2})}) \frac{\|[\mathbf{w}]\|_{L_{2}(e)}}{|e|^{\beta_{0}/2}} de^{-1/2} \|\underline{D}\underline{\varepsilon}(\mathbf{v})\|_{L_{2}(E_{2})} de^{-1/2} \|\underline{D}\underline{$$

for a positive constant *C*. Since  $|e| \leq h^{d-1}$ , the discrete Cauchy–Schwarz inequality yields

$$\begin{aligned} \left| \int_{e} \{\underline{D}\underline{\varepsilon}(\mathbf{v})\} : [\mathbf{w} \otimes \mathbf{n}_{e}] \, de \right| &\leq C(h_{E_{1}}^{\beta_{0}(d-1)-1} + h_{E_{2}}^{\beta_{0}(d-1)-1})^{1/2} \big( \|\underline{D}\underline{\varepsilon}(\mathbf{v})\|_{L_{2}(E_{1})}^{2} + \|\underline{D}\underline{\varepsilon}(\mathbf{v})\|_{L_{2}(E_{2})}^{2} \big)^{1/2} \frac{\|[\mathbf{w}]\|_{L_{2}(E_{2})}}{|e|^{\beta_{0}/2}} \\ &\leq C \big( \|\underline{D}\underline{\varepsilon}(\mathbf{v})\|_{L_{2}(E_{1})}^{2} + \|\underline{D}\underline{\varepsilon}(\mathbf{v})\|_{L_{2}(E_{2})}^{2} \big)^{1/2} \frac{\|[\mathbf{w}]\|_{L_{2}(E_{2})}}{|e|^{\beta_{0}/2}} \end{aligned}$$

because  $h_{E_1} \leq \text{diam}(\Omega)$ ,  $h_{E_2} \leq \text{diam}(\Omega)$ , and  $\beta_0(d-1) \geq 1$ . Using Young's inequality, we then obtain

$$\left|\int_{e} \{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{\nu})\} : [\boldsymbol{w} \otimes \boldsymbol{n}_{e}] de\right| \leq C \left(\frac{\epsilon}{2} \left(\|\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{\nu})\|_{L_{2}(E_{1})}^{2} + \|\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{\nu})\|_{L_{2}(E_{2})}^{2}\right) + \frac{1}{2\epsilon} \frac{\|[\boldsymbol{w}]\|_{L_{2}(e)}^{2}}{|e|^{\beta_{0}}}\right)$$

for any positive  $\epsilon$ . Taking  $\epsilon = 1/\sqrt{\alpha_0}$  then completes the proof.

Corollary 2.1. By (2.8), we can also derive

$$\sum_{e \in \Gamma_h \cup \Gamma_D} \int_e |\{\underline{\underline{D}}\underline{\underline{\varepsilon}}(v)\} : [w \otimes n_e] de| \leq \frac{C}{\sqrt{\alpha_0}} (\|v\|_V^2 + J_0^{\alpha_0, \beta_0}(w, w)),$$
(2.9)

where *C* is a positive constant independent of  $v, w \in \mathcal{D}_k(\mathcal{E}_h)$  but dependent on the inverse polynomial trace inequality's constant in (2.1).

**Theorem 2.1** (Coercivity and Continuity). Suppose  $\alpha_0 > 0$  is sufficiently large and  $\beta_0(d-1) \ge 1$ . Then there exist positive constants  $\kappa$  and K such that

$$\kappa \|\mathbf{v}\|_{V}^{2} \leq a(\mathbf{v}, \mathbf{v}), \quad |a(\mathbf{v}, \mathbf{w})| \leq K \|\mathbf{v}\|_{V} \|\mathbf{w}\|_{V} \quad \text{for all } \mathbf{v}, \mathbf{w} \in \mathcal{D}_{k}(\mathcal{E}_{h}),$$

where  $\kappa$  and K are independent of v and w.

*Proof.* The proof follows the same arguments as in [15, Theorems 4.5 and 4.6]. For example, the use of (2.8) and (2.9) leads us to show the coercivity and the continuity. For details, please see [15].

**Remark 2.3.** Due to the use of inverse polynomial trace inequality, the DG bilinear form will not be coercive and continuous on the broken Sobolev space. In other words, Theorem 2.1 holds only on the finite element space. For the choice of the penalty parameter  $\alpha_0$ , we refer to [12, 24]. For instance, we will take  $\alpha_0 \in [10, 100]$  in the numerical experiments Section 6.

### **DG Elliptic Projection**

The DG elliptic projector **R** is defined for  $u \in H^{s}(\mathcal{E}_{h})$  and s > 3/2 by

$$R: H^{s}(\mathcal{E}_{h}) \mapsto \mathcal{D}_{k}(\mathcal{E}_{h})$$
 such that  $a(\boldsymbol{u}, \boldsymbol{v}) = a(\boldsymbol{R}\boldsymbol{u}, \boldsymbol{v})$  for all  $\boldsymbol{v} \in \mathcal{D}_{k}(\mathcal{E}_{h})$ 

. .. . . .

Referring to [13, 19, 20, 24], for example, we recall the following elliptic-error estimates:

$$\|\boldsymbol{u} - \boldsymbol{R}\boldsymbol{u}\|_{V} \leq Ch^{\min(k+1,s)-1} \|\|\boldsymbol{u}\|_{H^{s}(\mathcal{E}_{h})},$$
(2.10)

$$\|\boldsymbol{u} - \boldsymbol{R}\boldsymbol{u}\|_{L_{2}(\Omega)} \leq Ch^{\min(k+1,s)} \|\|\boldsymbol{u}\|_{H^{s}(\mathcal{E}_{h})}$$
(2.11)

for  $u \in H^{s}(\mathcal{E}_{h})$  with s > 3/2 and for sufficiently large penalty parameters  $\alpha_{0}$  and  $\beta_{0} \ge (d-1)^{-1}$ . Here, the positive constant *C* is independent of *u* but dependent on the domain, its boundary, and the polynomial degree *k*.

We now move on to describe the model problem.

# 3 Model Problem

We recall our primal equations (1.1)–(1.6) and hereafter assume the data terms are bounded and smooth so that  $f \in C(0, T; L_2(\Omega)), g_N \in C^1(0, T; L_2(\Gamma_N)), u_0 \in H^s(\Omega)$ , and  $w_0 \in L_2(\Omega)$ . We first of all set up the internal variable representations of viscoelasticity and then give the variational formulations.

## 3.1 Internal Variables

Recalling the constitutive equation (1.5) and the stress relaxation function (1.6), we define internal variables as

$$\psi_{q}(t) = \int_{0}^{t} \frac{\varphi_{q}}{\tau_{q}} e^{-(t-t')/\tau_{q}} \boldsymbol{u}(t') dt' \quad \text{and} \quad \boldsymbol{\zeta}_{q}(t) = \int_{0}^{t} \varphi_{q} e^{-(t-t')/\tau_{q}} \dot{\boldsymbol{u}}(t') dt'$$
(3.1)

for  $q = 1, ..., N_{\varphi}$  and note that these are zero at t = 0. Using (1.6) in (1.5), and the Leibniz integral rule leads us to two alternative forms of the constitutive equation,

$$\underline{\boldsymbol{\sigma}}(\boldsymbol{u}(t)) = \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}\left(\varphi_{0}\boldsymbol{u}(t) + \sum_{q=1}^{N_{\varphi}}\boldsymbol{\zeta}_{q}(t)\right) + \sum_{q=1}^{N_{\varphi}}\varphi_{q}\boldsymbol{e}^{-t/\tau_{q}}\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{0})$$
(3.2)

or 
$$\underline{\boldsymbol{\sigma}}(\boldsymbol{u}(t)) = \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}\left(\boldsymbol{u}(t) - \sum_{q=1}^{N_{\varphi}} \boldsymbol{\psi}_{q}(t)\right),$$
 (3.3)

where we note that  $\zeta_q(t) + \varphi_q e^{-t/\tau_q} u(0) = \varphi_q u(t) - \psi_q(t)$ . We call  $\psi_q$  and  $\zeta_q$  the *displacement form* and *velocity form*, respectively, of the internal variable. It is easy to check that (3.3) and (3.2) are equivalent by integration by parts and remembering that  $\sum_{q=1}^{N_{\varphi}} \varphi_q = 1 - \varphi_0$ . Using these constitutive relationships, we can derive dynamic linear viscoelasticity model problems in two forms as follows.

#### **Displacement Form**

We consider (1.1) in the following form: find  $\boldsymbol{u}, \boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots$  such that

$$\rho \ddot{\boldsymbol{u}} - \nabla \cdot \underline{\boldsymbol{D}} \underline{\boldsymbol{\varepsilon}} \left( \boldsymbol{u} - \sum_{q=1}^{N_{\varphi}} \psi_{q} \right) = \boldsymbol{f}, \qquad (3.4)$$

$$\tau_q \dot{\psi}_q + \psi_q = \varphi_q \boldsymbol{u}, \quad \text{with } \psi(0) = \boldsymbol{0}, \quad \text{for } q = 1, \dots, N_{\varphi}.$$
(3.5)

#### **Velocity Form**

We consider (1.1) in the form

$$\rho \ddot{\boldsymbol{u}} - \nabla \cdot \underline{\boldsymbol{D}} \underline{\boldsymbol{\varepsilon}} \left( \varphi_0 \boldsymbol{u} + \sum_{q=1}^{N_{\varphi}} \boldsymbol{\zeta}_q \right) = \boldsymbol{f} + \nabla \cdot \left( \sum_{q=1}^{N_{\varphi}} \varphi_q e^{-t/\tau_q} \underline{\boldsymbol{D}} \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_0) \right),$$
(3.6)

$$\tau_q \dot{\boldsymbol{\zeta}}_q + \boldsymbol{\zeta}_q = \tau_q \varphi_q \dot{\boldsymbol{u}} \quad \text{with } \boldsymbol{\zeta}(0) = \boldsymbol{0}, \quad \text{for } q = 1, \dots, N_{\varphi}.$$
(3.7)

In these, we note that the auxiliary equations (3.5) and (3.7) are derived by time-differentiation of each of (3.1).

**Remark 3.1.** The (well-known) idea here is to replace the integral form in the constitutive hereditary laws by ODEs for a set of internal variables. With these, we use a time stepper rather than numerical integration to compute the discrete solution. Furthermore, while [21] employed internal variables as second-order tensors, we have defined vector-valued internal variables. This reduces the computer memory requirement by a factor of *d*.

## 3.2 Variational Formulation

We begin by multiplying each of (3.4) and (3.6) by test functions  $v \in H^s(\mathcal{E}_h)$  for s > 3/2, integrating by parts over each *E*, summing over every *E*, and then collecting up terms to form the edge average and jump terms.

We then assume sufficient spatial continuity of the continuous solution to (3.4) and (3.7) so that we can add the terms involving the edge jumps of the exact solution: terms three and four on the right of (2.4). This produces the DG bilinear form on the left-hand side and gives us a weak form for each choice of the form of the internal variables. The procedure is standard in the DGFEM literature, and for more details in this specific setting, we refer to [15] (where it may be useful to note that  $\underline{\sigma} \cdot \mathbf{n} \cdot \mathbf{v} = \underline{\sigma} : (\mathbf{v} \otimes \mathbf{n})$ ).

#### Weak Forms

Let us define linear forms by

$$\begin{split} F_d(t; \boldsymbol{v}) &= (\boldsymbol{f}(t), \boldsymbol{v}) + (\boldsymbol{g}_N(t), \boldsymbol{v})_{L_2(\Gamma_N)}, \\ F_v(t; \boldsymbol{v}) &= (\boldsymbol{f}(t), \boldsymbol{v}) + (\boldsymbol{g}_N(t), \boldsymbol{v})_{L_2(\Gamma_N)} - \sum_{q=1}^{N_\varphi} \varphi_q e^{-t/\tau_q} a(\boldsymbol{u}_0, \boldsymbol{v}). \end{split}$$

Recalling (2.4), we now use these linear forms to formulate variational problems for the displacement form and the velocity form. First for the displacement form of the problem.

**Displacement Form Problem (D).** Find  $\boldsymbol{u}$  and  $\{\boldsymbol{\psi}_q\}_{q=1}^{N_{\varphi}}$  such that, for all  $\boldsymbol{\nu} \in \boldsymbol{H}^s(\mathcal{E}_h)$  with s > 3/2, we have

$$(\rho \ddot{u}(t), \mathbf{v}) + a(u(t), \mathbf{v}) - \sum_{q=1}^{N_{\varphi}} a(\psi_q(t), \mathbf{v}) + J_0^{a_0, \beta_0}(\dot{u}(t), \mathbf{v}) = F_d(t; \mathbf{v}),$$
(3.8)

$$a(\tau_q \dot{\psi}_q(t) + \psi_q(t), \mathbf{v}) = a(\varphi_q \mathbf{u}(t), \mathbf{v}) \quad \text{for each } q, \tag{3.9}$$

where  $\boldsymbol{u}(0) = \boldsymbol{u}_0$ ,  $\dot{\boldsymbol{u}}(0) = \boldsymbol{w}_0$  and  $\boldsymbol{\psi}_a(0) = \boldsymbol{0}$ . And, secondly, for the velocity form.

**Velocity Form Problem (V).** Find u and  $\{\zeta_q\}_{q=1}^{N_{\varphi}}$  such that, for all  $v \in H^s(\mathcal{E}_h)$  with s > 3/2, we have

$$(\rho \ddot{\boldsymbol{u}}(t), \boldsymbol{v}) + \varphi_0 a(\boldsymbol{u}(t), \boldsymbol{v}) + \sum_{q=1}^{N_{\varphi}} a(\boldsymbol{\zeta}_q(t), \boldsymbol{v}) + J_0^{a_0, \beta_0}(\dot{\boldsymbol{u}}(t), \boldsymbol{v}) = F_{\boldsymbol{v}}(t; \boldsymbol{v}),$$
(3.10)

$$a(\tau_q \dot{\boldsymbol{\zeta}}_q(t) + \boldsymbol{\zeta}_q(t), \boldsymbol{v}) = a(\tau_q \varphi_q \dot{\boldsymbol{u}}(t), \boldsymbol{v}) \quad \text{for each } q, \qquad (3.11)$$

where  $u(0) = u_0$ ,  $\dot{u}(0) = w_0$  and  $\zeta_q(0) = 0$ .

Note that, in (3.9) and (3.11), the energy bilinear form, rather than the  $L_2(\Omega)$  inner product, is used to enforce the evolution equations for the internal variables. This is because, in the testing procedures used below for the stability and error analyses, we want to take linear combinations of (3.8) and (3.9), and of (3.10) and (3.11), and their numerical analogues, to cancel certain terms out. We will return to this in the proof of Theorem 4.1 below.

#### **Fully Discrete Formulations**

For the time discretisation, we employ a *Crank–Nicolson* type finite difference scheme [7]. Let  $t_n = n \Delta t$  with  $\Delta t = T/N$  for some  $N \in \mathbb{N}$ , and denote the fully discrete approximations to  $\boldsymbol{u}$  and  $\dot{\boldsymbol{u}}$  by  $\boldsymbol{u}(t_n) = \boldsymbol{u}^n \approx \boldsymbol{U}_h^n \in \mathcal{D}_k(\mathcal{E}_h)$  and  $\dot{\boldsymbol{u}}(t_n) = \dot{\boldsymbol{u}}^n \approx \boldsymbol{W}_h^n \in \mathcal{D}_k(\mathcal{E}_h)$ . Similarly, for the internal variables, we introduce  $\psi_q(t_n) \approx \Psi_{hq}^n \in \mathcal{D}_k(\mathcal{E}_h)$  and  $\zeta_q(t_n) \approx S_{hq}^n \in \mathcal{D}_k(\mathcal{E}_h)$  for each q. Furthermore, in the schemes that follow, we will use the following approximations:

$$\frac{\ddot{\boldsymbol{u}}(t_{n+1})+\ddot{\boldsymbol{u}}(t_n)}{2}\approx\frac{\boldsymbol{W}_h^{n+1}-\boldsymbol{W}_h^n}{\Delta t}\quad\text{and}\quad\frac{\boldsymbol{u}(t_{n+1})+\boldsymbol{u}(t_n)}{2}\approx\frac{\boldsymbol{U}_h^{n+1}+\boldsymbol{U}_h^n}{2},$$

and we will impose the relation

$$\frac{\boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}}{2} = \frac{\boldsymbol{U}_{h}^{n+1} - \boldsymbol{U}_{h}^{n}}{\Delta t}.$$
(3.12)

We can now give the fully discrete schemes.

**Displacement Form Problem (D)**<sup>*h*</sup>. Find  $W_h^n, U_h^n, \Psi_{h1}^n, \dots, \Psi_{hN_{\varphi}}^n \in \mathcal{D}_k(\mathcal{E}_h)$  for  $n = 0, \dots, N$  such that, for all  $v \in \mathcal{D}_k(\mathcal{E}_h)$ , we have for  $n = 0, 1, \dots, N - 1$  firstly that

$$\left(\rho \frac{W_{h}^{n+1} - W_{h}^{n}}{\Delta t}, \nu\right) + a\left(\frac{U_{h}^{n+1} + U_{h}^{n}}{2}, \nu\right) - \sum_{q=1}^{N_{\varphi}} a\left(\frac{\Psi_{hq}^{n+1} + \Psi_{hq}^{n}}{2}, \nu\right) + J_{0}^{a_{0},\beta_{0}}\left(\frac{W_{h}^{n+1} + W_{h}^{n}}{2}, \nu\right)$$
$$= \frac{1}{2}\left(F_{d}^{n+1}(\nu) + F_{d}^{n}(\nu)\right)$$
(3.13)

and secondly that, for  $q = 1, 2, ..., N_{\varphi}$ ,

$$a\left(\tau_{q}\frac{\Psi_{hq}^{n+1}-\Psi_{hq}^{n}}{\Delta t}+\frac{\Psi_{hq}^{n+1}+\Psi_{hq}^{n}}{2},\nu\right)=a\left(\varphi_{q}\frac{U_{h}^{n+1}+U_{h}^{n}}{2},\nu\right),$$
(3.14)

with initial data

$$a(\boldsymbol{U}_{h}^{0},\boldsymbol{\nu}) = a(\boldsymbol{u}_{0},\boldsymbol{\nu}), \qquad (3.15)$$

$$(W_{h}^{0}, v) = (w_{0}, v),$$
 (3.16)

$$\Psi_{hq}^{0} = \mathbf{0} \quad \text{for all } q = 1, \dots, N_{\varphi}, \tag{3.17}$$

and where  $F_d^n(\mathbf{v}) = F_d(t_n; \mathbf{v})$ .

**Velocity Form Problem (V)**<sup>*h*</sup>. Find  $W_h^n, U_h^n, S_{h1}^n, \ldots, S_{hN_{\varphi}}^n \in \mathcal{D}_k(\mathcal{E}_h)$  for  $n = 0, \ldots, N$  such that, for all  $v \in \mathcal{D}_k(\mathcal{E}_h)$ , we have for  $n = 0, 1, \ldots, N - 1$  firstly that

$$\left(\rho \frac{W_{h}^{n+1} - W_{h}^{n}}{\Delta t}, \nu\right) + \varphi_{0} a \left(\frac{U_{h}^{n+1} + U_{h}^{n}}{2}, \nu\right) + \sum_{q=1}^{N_{\varphi}} a \left(\frac{S_{hq}^{n+1} + S_{hq}^{n}}{2}, \nu\right) + J_{0}^{a_{0},\beta_{0}} \left(\frac{W_{h}^{n+1} + W_{h}^{n}}{2}, \nu\right) \\ = \frac{1}{2} \left(F_{\nu}^{n+1}(\nu) + F_{\nu}^{n}(\nu)\right)$$
(3.18)

and secondly that, for  $q = 1, 2, ..., N_{\varphi}$ ,

$$a\left(\tau_q \frac{\mathcal{S}_{hq}^{n+1} - \mathcal{S}_{hq}^n}{\Delta t} + \frac{\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n}{2}, \boldsymbol{\nu}\right) = a\left(\tau_q \varphi_q \frac{\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n}{2}, \boldsymbol{\nu}\right),\tag{3.19}$$

with initial data (3.15), (3.16),

$$S_{hq}^0 = \mathbf{0}$$
 for all  $q = 1, \ldots, N_{\varphi}$ ,

and where  $F_{v}^{n}(v) = F_{v}(t_{n}; v)$ .

The jump term penalisation of the velocities in (3.8) and (3.10) is included in these formulations to facilitate the error analysis later: see (5.10) in particular. This term is not required for the stability estimates that follow.

# **4** Stability Analysis

In this section, we derive stability bounds for the solutions to the fully discrete displacement and velocity forms of the problem. As the problems are linear, these stability bounds can also be used to conclude the existence and uniqueness of the numerical solutions and, hence, the well-posedness of the discrete problems. At first, we present bounds that are non-optimal in that they contain a factor of  $h^{-1} \| g_N \|_{H^1(0,T;L_2(\Gamma_N))}^2$  on the right-hand side. This arises from use of the trace inequality on the traction term. This unwelcome appearance of  $h^{-1}$  is not unusual (see for example [19, 21]) and is a technical issue stemming from the straightforward use of (4.3) below.

The  $h^{-1}$  factor is not observed in practical computations, suggesting that it can be removed. This can be seen by using the deeper and non-trivial arguments given by Buffa and Ortner in [5]. There, under some quite mild mesh assumptions, the trace bound follows without the factor of  $h^{-1}$  appearing. We give more details on this below the non-optimal results that follow, and there, we set the improved results forward as a corollary.

**Theorem 4.1** (Stability Bound for (D)<sup>*h*</sup>). If  $\beta_0(d-1) \ge 1$  and  $\alpha_0$  is large enough, and if  $W_h^n$ ,  $U_h^n$  and  $\{\Psi_{hq}^n\}_{q=1}^{N_{\phi}}$  in  $\mathcal{D}_k(\mathcal{E}_h)$  satisfy the fully discrete formulation (D)<sup>*h*</sup>, then there exists a positive constant *C* such that

$$\begin{split} \max_{0 \le n \le N} \|\boldsymbol{W}_{h}^{n}\|_{L_{2}(\Omega)}^{2} + \max_{0 \le n \le N} \|\boldsymbol{U}_{h}^{n}\|_{V}^{2} + \sum_{q=1}^{N_{\varphi}} \max_{0 \le n \le N} \|\boldsymbol{\Psi}_{hq}^{n}\|_{V}^{2} + \sum_{n=0}^{N-1} \sum_{q=1}^{N_{\varphi}} \frac{1}{\Delta t} \|\boldsymbol{\Psi}_{hq}^{n+1} - \boldsymbol{\Psi}_{hq}^{n}\|_{V}^{2} \\ + \Delta t \sum_{n=0}^{N-1} J_{0}^{\alpha_{0},\beta_{0}} (\boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}, \boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}) \\ \le CT^{2} (\|\boldsymbol{w}_{0}\|_{L_{2}(\Omega)}^{2} + \|\boldsymbol{u}_{0}\|_{V}^{2} + \|\boldsymbol{f}\|_{L_{\infty}(0,T;L_{2}(\Omega))}^{2} + h^{-1} \|\boldsymbol{g}_{N}\|_{H^{1}(0,T;L_{2}(\Gamma_{N}))}^{2}). \end{split}$$

Here, *C* is independent of the discrete solutions,  $\Delta t$ , and *h*, but dependent on the polynomial degree *k*, the domain  $\Omega$ , its boundary, and the material properties.

*Proof.* Choose  $m \in \mathbb{N}$  so that m < N. Take  $v = \Delta t(W_h^{n+1} + W_h^n)$  in (3.13) and  $v = 2(\Psi_{hq}^{n+1} - \Psi_{hq}^n)$  in (3.14) for each q and for n = 0, ..., m - 1, and then add the results and sum over n to get

$$\rho \| \boldsymbol{W}_{h}^{m} \|_{L_{2}(\Omega)}^{2} + a(\boldsymbol{U}_{h}^{m}, \boldsymbol{U}_{h}^{m}) + \sum_{q=1}^{N_{\varphi}} \frac{1}{\varphi_{q}} a(\boldsymbol{\Psi}_{hq}^{m}, \boldsymbol{\Psi}_{hq}^{m}) \\
+ \sum_{n=0}^{m-1} \sum_{q=1}^{N_{\varphi}} \frac{2\tau_{q}}{\Delta t \varphi_{q}} a(\boldsymbol{\Psi}_{hq}^{n+1} - \boldsymbol{\Psi}_{hq}^{n}, \boldsymbol{\Psi}_{hq}^{n+1} - \boldsymbol{\Psi}_{hq}^{n}) \\
+ \frac{\Delta t}{2} \sum_{n=0}^{m-1} \int_{0}^{\alpha_{0},\beta_{0}} (\boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}, \boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}) \\
= \rho \| \boldsymbol{W}_{h}^{0} \|_{L_{2}(\Omega)}^{2} + a(\boldsymbol{U}_{h}^{0}, \boldsymbol{U}_{h}^{0}) + 2 \sum_{q=1}^{N_{\varphi}} a(\boldsymbol{\Psi}_{hq}^{m}, \boldsymbol{U}_{h}^{m}) \\
+ \frac{\Delta t}{2} \sum_{n=0}^{m-1} (F_{d}^{n+1}(\boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}) + F_{d}^{n}(\boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}))$$
(4.1)

since, by (3.12),  $\Delta t(\boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}) = 2(\boldsymbol{U}_{h}^{n+1} - \boldsymbol{U}_{h}^{n})$  and, by (3.17),  $\boldsymbol{\Psi}_{hq}^{0} = \boldsymbol{0}$ , and also where we note that

$$a(\Psi_{hq}^{n+1} + \Psi_{hq}^{n}, U_{h}^{n+1} - U_{h}^{n}) = 2a(\Psi_{hq}^{n+1}, U_{h}^{n+1}) - 2a(\Psi_{hq}^{n}, U_{h}^{n}) - a(\Psi_{hq}^{n+1} - \Psi_{hq}^{n}, U_{h}^{n+1} + U_{h}^{n})$$

which, when used with (3.14), gives us

$$\begin{aligned} a(\Psi_{hq}^{n+1} + \Psi_{hq}^{n}, U_{h}^{n+1} - U_{h}^{n}) &= 2a(\Psi_{hq}^{n+1}, U_{h}^{n+1}) - 2a(\Psi_{hq}^{n}, U_{h}^{n}) \\ &- \frac{2\tau_{q}}{\Delta t\varphi_{q}}a(\Psi_{hq}^{n+1} - \Psi_{hq}^{n}, \Psi_{hq}^{n+1} - \Psi_{hq}^{n}) \\ &- \frac{1}{\varphi_{q}} \left( a(\Psi_{hq}^{n+1}, \Psi_{hq}^{n+1}) - a(\Psi_{hq}^{n}, \Psi_{hq}^{n}) \right) \end{aligned}$$

for each n and q. The cancellation of terms in this step illustrates the importance of enforcing the internal variable ODEs in the energy bilinear form in (3.9) and (3.11). Using (2.6), (4.1), and Theorem 2.1 in (4.1), we obtain

$$\rho \| \boldsymbol{W}_{h}^{m} \|_{L_{2}(\Omega)}^{2} + \| \boldsymbol{U}_{h}^{m} \|_{V}^{2} + \sum_{q=1}^{N_{\varphi}} \frac{1}{\varphi_{q}} \| \boldsymbol{\Psi}_{hq}^{m} \|_{V}^{2} + \sum_{n=0}^{m-1} \sum_{q=1}^{N_{\varphi}} \frac{2\kappa\tau_{q}}{\Delta t\varphi_{q}} \| \boldsymbol{\Psi}_{hq}^{n+1} - \boldsymbol{\Psi}_{hq}^{n} \|_{V}^{2} \\
+ \frac{\Delta t}{2} \sum_{n=0}^{m-1} \int_{0}^{a_{0},\beta_{0}} (\boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}, \boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}) \\
\leqslant \left| \rho \| \boldsymbol{W}_{h}^{0} \|_{L_{2}(\Omega)}^{2} + a(\boldsymbol{U}_{h}^{0}, \boldsymbol{U}_{h}^{0}) + 2 \sum_{q=1}^{N_{\varphi}} a(\boldsymbol{\Psi}_{hq}^{m}, \boldsymbol{U}_{h}^{m}) + \frac{\Delta t}{2} \sum_{n=0}^{m-1} (F_{d}^{n+1}(\boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}) + F_{d}^{n}(\boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n})) \\
+ 2 \sum_{e \in \Gamma_{h} \cup \Gamma_{D}} \int_{e} \{ \underline{D} \underline{\varepsilon} (\boldsymbol{U}_{h}^{m}) \} : [\boldsymbol{U}_{h}^{m} \otimes \boldsymbol{n}_{e}] de + \sum_{q=1}^{N_{\varphi}} \frac{2}{\varphi_{q}} \sum_{e \in \Gamma_{h} \cup \Gamma_{D}} \int_{e} \{ \underline{D} \underline{\varepsilon} (\boldsymbol{\Psi}_{hq}^{m}) \} : [\boldsymbol{\Psi}_{hq}^{m} \otimes \boldsymbol{n}_{e}] de \right|. \tag{4.2}$$

Now we show an upper bound for the right-hand side of (4.2).

•  $\|\boldsymbol{W}_{h}^{0}\|_{L_{2}(\Omega)}^{2}$ : By the Cauchy–Schwarz inequality, (3.16) yields

$$\|\boldsymbol{W}_{h}^{0}\|_{L_{2}(\Omega)}^{2} = (\boldsymbol{W}_{h}^{0}, \boldsymbol{W}_{h}^{0}) = (\boldsymbol{W}_{h}^{0}, \boldsymbol{w}_{0}) \leq \|\boldsymbol{W}_{h}^{0}\|_{L_{2}(\Omega)}\|\boldsymbol{w}_{0}\|_{L_{2}(\Omega)},$$

hence  $\|\boldsymbol{W}_{h}^{0}\|_{L_{2}(\Omega)}^{2} \leq \|\boldsymbol{w}_{0}\|_{L_{2}(\Omega)}^{2}$ .

 $|a(U_h^0, U_h^0)|$ : Combining the coercivity and the continuity, we can derive •

$$\kappa \|\boldsymbol{U}_{h}^{0}\|_{V}^{2} \leq a(\boldsymbol{U}_{h}^{0}, \boldsymbol{U}_{h}^{0}) = a(\boldsymbol{U}_{h}^{0}, \boldsymbol{u}_{0}) \leq K \|\boldsymbol{U}_{h}^{0}\|_{V} \|\boldsymbol{u}_{0}\|_{V}$$

by (3.15). This implies  $\|\boldsymbol{U}_{h}^{0}\|_{V} \leq K/\kappa \|\boldsymbol{u}_{0}\|_{V}$ , and so we have

$$|a(\boldsymbol{U}_{h}^{0},\boldsymbol{U}_{h}^{0})| \leq K \|\boldsymbol{U}_{h}^{0}\|_{V}^{2} \leq \bar{K} \|\boldsymbol{u}_{0}\|_{V}^{2}$$
 for  $\bar{K} = \frac{K^{3}}{\kappa^{2}}$ .

 $\left|2\sum_{q=1}^{N_{\varphi}}a(\Psi_{hq}^{m},U_{h}^{m})\right|$ : The Cauchy–Schwarz inequality, Young's inequality and (2.9) yield •

$$\left|2\sum_{q=1}^{N_{\varphi}}a(\boldsymbol{\Psi}_{hq}^{m},\boldsymbol{U}_{h}^{m})\right| \leq \sum_{q=1}^{N_{\varphi}}\left(\epsilon_{q}+\frac{C}{\sqrt{\alpha_{0}}}\right)\|\boldsymbol{U}_{h}^{m}\|_{V}^{2}+\sum_{q=1}^{N_{\varphi}}\left(\frac{1}{\epsilon_{q}}+\frac{C}{\sqrt{\alpha_{0}}}\right)\|\boldsymbol{\Psi}_{hq}^{m}\|_{V}^{2}$$

for any positive  $\epsilon_q$  for all q.  $\left|\frac{\Delta t}{2}\sum_{n=0}^{m-1}(F_d^{n+1}(W_h^{n+1}+W_h^n)+F_d^n(W_h^{n+1}+W_h^n))\right|$ : Note that summation by parts and the fundamental theo-• rem of calculus give

$$\sum_{n=0}^{m-1} (\boldsymbol{g}_{N}(t_{n+1}) + \boldsymbol{g}_{N}(t_{n}), \boldsymbol{U}_{h}^{n+1} - \boldsymbol{U}_{h}^{n})_{L_{2}(e)}$$
  
=  $2(\boldsymbol{g}_{N}(t_{m}), \boldsymbol{U}_{h}^{m})_{L_{2}(e)} - 2(\boldsymbol{g}_{N}(t_{0}), \boldsymbol{U}_{h}^{0})_{L_{2}(e)}$   
 $- \sum_{n=0}^{m-1} \int_{t_{n}}^{t_{n+1}} (\dot{\boldsymbol{g}}_{N}(t'), \boldsymbol{U}_{h}^{n+1} + \boldsymbol{U}_{h}^{n})_{L_{2}(e)} dt' \text{ for all } e \in \Gamma_{N}.$ 

By (3.12) and the Cauchy–Schwarz inequality, we therefore have

$$\begin{split} \left| \frac{\Delta t}{2} \sum_{n=0}^{m-1} (F_d^{n+1}(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n) + F_d^n(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n)) \right| \\ &\leqslant \frac{\Delta t}{2} \sum_{n=0}^{m-1} (\|\boldsymbol{f}^{n+1}\|_{L_2(\Omega)} + \|\boldsymbol{f}^n\|_{L_2(\Omega)}) \|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\|_{L_2(\Omega)} \\ &+ 2 \sum_{e \in \Gamma_N} \|\boldsymbol{g}_N^m\|_{L_2(e)} \|\boldsymbol{U}_h^m\|_{L_2(e)} + 2 \sum_{e \in \Gamma_N} \|\boldsymbol{g}_N^0\|_{L_2(e)} \|\boldsymbol{U}_h^0\|_{L_2(e)} \\ &+ \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} \sum_{e \in \Gamma_N} \|\dot{\boldsymbol{g}}_N(t')\|_{L_2(e)} \|\boldsymbol{U}_h^{n+1} + \boldsymbol{U}_h^n\|_{L_2(e)} dt'. \end{split}$$

Since the inverse polynomial trace inequality (2.1) and Poincaré's inequality (2.2) imply that

$$\sum_{e \in \partial \Omega} \|\boldsymbol{\nu}\|_{L_2(e)}^2 \leq Ch^{-1} \sum_{E \in \mathcal{E}_h} \|\boldsymbol{\nu}\|_{L_2(E)}^2 \leq Ch^{-1} \|\boldsymbol{\nu}\|_V^2 \quad \text{for all } \boldsymbol{\nu} \in \mathcal{D}_k(\mathcal{E}_h),$$
(4.3)

the triangle inequality and Young's inequality lead us to

$$\begin{split} \left| \frac{\Delta t}{2} \sum_{n=0}^{m-1} (F_d^{n+1}(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n) + F_d^n(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n)) \right| \\ &\leqslant \frac{\Delta t}{4\epsilon_a} \sum_{n=0}^{m-1} (\|\boldsymbol{f}^{n+1}\|_{L_2(\Omega)}^2 + \|\boldsymbol{f}^n\|_{L_2(\Omega)}^2) + 2\Delta t\epsilon_a \sum_{n=0}^{m-1} (\|\boldsymbol{W}_h^{n+1}\|_{L_2(\Omega)}^2 + \|\boldsymbol{W}_h^n\|_{L_2(\Omega)}^2) \\ &+ \frac{1}{\epsilon_b} \|\boldsymbol{g}_N^m\|_{L_2(\Gamma_N)}^2 + \frac{C\epsilon_b}{h} \|\boldsymbol{U}_h^m\|_V^2 + \frac{1}{h} \|\boldsymbol{g}_N^0\|_{L_2(\Gamma_N)}^2 + C\|\boldsymbol{U}_h^0\|_V^2 \\ &+ \frac{1}{\epsilon_b} \int_0^{t_m} \|\dot{\boldsymbol{g}}_N(t')\|_{L_2(\Gamma_N)}^2 dt' + \frac{C\Delta t\epsilon_b}{2h} \sum_{n=0}^{m-1} (\|\boldsymbol{U}_h^{n+1}\|_V^2 + \|\boldsymbol{U}_h^n\|_V^2) \end{split}$$

for any positive  $\epsilon_a$  and  $\epsilon_b$ . Maximising over time, we then arrive at

$$\begin{split} \left| \frac{\Delta t}{2} \sum_{n=0}^{m-1} (F_d^{n+1}(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n) + F_d^n(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n)) \right| \\ &\leq \frac{T}{2\epsilon_a} \|\boldsymbol{f}\|_{L_{\infty}(0,T;L_2(\Omega))}^2 + 4T\epsilon_a \max_{0 \leq n \leq N} \|\boldsymbol{W}_h^n\|_{L_2(\Omega)}^2 + \left(\frac{1}{\epsilon_b} + \frac{1}{h}\right) \|\boldsymbol{g}_N\|_{L_{\infty}(0,T;L_2(\Gamma_N))}^2 \\ &+ C \|\boldsymbol{U}_h^0\|_V^2 + \frac{1}{\epsilon_b} \|\boldsymbol{\dot{g}}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2 + \frac{C(T+1)\epsilon_b}{h} \max_{0 \leq n \leq N} \|\boldsymbol{U}_h^n\|_V^2. \end{split}$$

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•  $\left| 2 \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{ \underline{D} \underline{\varepsilon} (U_h^m) \} : [U_h^m \otimes n_e] de \right|$ : (2.9) implies that

$$2\sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{U}_h^m)\} : [\boldsymbol{U}_h^m \otimes \boldsymbol{n}_e] de \leqslant \frac{C}{\sqrt{\alpha_0}} \|\boldsymbol{U}_h^m\|_V^2.$$

•  $\left|\sum_{q=1}^{N_{\varphi}} \frac{2}{\varphi_{q}} \sum_{e \in \Gamma_{h} \cup \Gamma_{D}} \int_{e} \{\underline{D}\underline{\varepsilon}(\Psi_{hq}^{m})\} : [\Psi_{hq}^{m} \otimes n_{e}] de\right|$ : In the same manner, (2.9) yields

$$\left|\sum_{q=1}^{N_{\varphi}} \frac{2}{\varphi_{q}} \sum_{e \in \Gamma_{h} \cup \Gamma_{D}} \int_{e} \{\underline{D}\underline{e}(\Psi_{hq}^{m})\} : [\Psi_{hq}^{m} \otimes \boldsymbol{n}_{e}] de\right| \leq \sum_{q=1}^{N_{\varphi}} \frac{C}{\varphi_{q} \sqrt{\alpha_{0}}} \|\Psi_{hq}^{m}\|_{V}^{2}$$

Collecting these results then gives

$$\rho \| \boldsymbol{W}_{h}^{m} \|_{L_{2}(\Omega)}^{2} + \left(1 - \sum_{q=1}^{N_{\varphi}} \epsilon_{q} - \frac{CN_{\varphi}}{\sqrt{\alpha_{0}}}\right) \| \boldsymbol{U}_{h}^{m} \|_{V}^{2} + \sum_{q=1}^{N_{\varphi}} \left(\frac{1}{\varphi_{q}} - \frac{1}{\epsilon_{q}} - \frac{C}{\sqrt{\alpha_{0}}}\right) \| \boldsymbol{\Psi}_{hq}^{m} \|_{V}^{2} \\
+ \sum_{n=0}^{m-1} \sum_{q=1}^{N_{\varphi}} \frac{2\kappa\tau_{q}}{\Delta t\varphi_{q}} \| \boldsymbol{\Psi}_{hq}^{n+1} - \boldsymbol{\Psi}_{hq}^{n} \|_{V}^{2} + \frac{\Delta t}{2} \sum_{n=0}^{m-1} J_{0}^{\alpha_{0},\beta_{0}} (\boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}, \boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}) \\
\leq \rho \| \boldsymbol{w}_{0} \|_{L_{2}(\Omega)}^{2} + \left(\bar{K} + \frac{CK}{\kappa}\right) \| \boldsymbol{u}_{0} \|_{V}^{2} + \frac{T}{2\epsilon_{a}} \| \boldsymbol{f} \|_{L_{\infty}(0,T;L_{2}(\Omega))}^{2} + \left(\frac{1}{\epsilon_{b}} + \frac{1}{h}\right) \| \boldsymbol{g}_{N} \|_{L_{\infty}(0,T;L_{2}(\Gamma_{N}))}^{2} \\
+ \frac{1}{\epsilon_{b}} \| \dot{\boldsymbol{g}}_{N} \|_{L_{2}(0,T;L_{2}(\Gamma_{N}))}^{2} + 4T\epsilon_{a} \max_{0 \leq n \leq N} \| \boldsymbol{W}_{h}^{n} \|_{L_{2}(\Omega)}^{2} + \frac{C(T+1)\epsilon_{b}}{h} \max_{0 \leq n \leq N} \| \boldsymbol{U}_{h}^{n} \|_{V}^{2}.$$
(4.4)

When we set  $\epsilon_q = \varphi_q + \varphi_0/(2N_{\varphi}) > 0$  for each q and use (1.7) to obtain

$$1 - \sum_{q=1}^{N_{\varphi}} \epsilon_q = 1 - \sum_{q=1}^{N_{\varphi}} \varphi_q - \sum_{q=1}^{N_{\varphi}} \frac{\varphi_0}{2N_{\varphi}} = \varphi_0 - \frac{\varphi_0}{2} = \frac{\varphi_0}{2} > 0,$$

inequality (4.4) gives

$$\rho \| \boldsymbol{W}_{h}^{m} \|_{L_{2}(\Omega)}^{2} + \left( \frac{\varphi_{0}}{2} - \frac{CN_{\varphi}}{\sqrt{\alpha_{0}}} \right) \| \boldsymbol{U}_{h}^{m} \|_{V}^{2} + \sum_{q=1}^{N_{\varphi}} \left( \frac{\varphi_{0}}{2N_{\varphi}\varphi_{q}^{2} + \varphi_{0}\varphi_{q}} - \frac{C}{\sqrt{\alpha_{0}}} \right) \| \boldsymbol{\Psi}_{hq}^{m} \|_{V}^{2} \\
+ \sum_{n=0}^{m-1} \sum_{q=1}^{N_{\varphi}} \frac{2\kappa\tau_{q}}{\Delta t\varphi_{q}} \| \boldsymbol{\Psi}_{hq}^{n+1} - \boldsymbol{\Psi}_{hq}^{n} \|_{V}^{2} + \frac{\Delta t}{2} \sum_{n=0}^{m-1} J_{0}^{\alpha_{0},\beta_{0}} (\boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}, \boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}) \\
\leq \rho \| \boldsymbol{w}_{0} \|_{L_{2}(\Omega)}^{2} + \left( \bar{K} + \frac{CK}{\kappa} \right) \| \boldsymbol{u}_{0} \|_{V}^{2} + \frac{T}{2\epsilon_{a}} \| \boldsymbol{f} \|_{L_{\infty}(0,T;L_{2}(\Omega))}^{2} + \left( \frac{1}{\epsilon_{b}} + \frac{1}{h} \right) \| \boldsymbol{g}_{N} \|_{L_{\infty}(0,T;L_{2}(\Gamma_{N}))}^{2} \\
+ \frac{1}{\epsilon_{b}} \| \dot{\boldsymbol{g}}_{N} \|_{L_{2}(0,T;L_{2}(\Gamma_{N}))}^{2} + 4T\epsilon_{a} \max_{0 \leq n \leq N} \| \boldsymbol{W}_{h}^{n} \|_{L_{2}(\Omega)}^{2} + \frac{C(T+1)\epsilon_{b}}{h} \max_{0 \leq n \leq N} \| \boldsymbol{U}_{h}^{n} \|_{V}^{2}.$$
(4.5)

After noting that the right-hand side of (4.5) is independent of *m* and that *m* is arbitrary, we can obtain

$$\begin{split} \frac{\rho}{2} \max_{0 \leq n \leq N} \|\boldsymbol{W}_{h}^{n}\|_{L_{2}(\Omega)}^{2} + \left(\frac{\varphi_{0}}{4} - \frac{CN_{\varphi}}{\sqrt{a_{0}}}\right) \max_{0 \leq n \leq N} \|\boldsymbol{U}_{h}^{n}\|_{V}^{2} + \sum_{q=1}^{N_{\varphi}} \left(\frac{\varphi_{0}}{2N_{\varphi}\varphi_{q}^{2} + \varphi_{0}\varphi_{q}} - \frac{C}{\sqrt{a_{0}}}\right) \max_{0 \leq n \leq N} \|\boldsymbol{\Psi}_{hq}^{n}\|_{V}^{2} \\ &+ \sum_{n=0}^{N-1} \sum_{q=1}^{N_{\varphi}} \frac{2\kappa\tau_{q}}{\Delta t\varphi_{q}} \|\boldsymbol{\Psi}_{hq}^{n+1} - \boldsymbol{\Psi}_{hq}^{n}\|_{V}^{2} + \frac{\Delta t}{2} \sum_{n=0}^{N-1} \int_{0}^{a_{0},\beta_{0}} (\boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}, \boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}) \\ &\leq \rho \|\boldsymbol{w}_{0}\|_{L_{2}(\Omega)}^{2} + \left(\bar{K} + \frac{CK}{\kappa}\right) \|\boldsymbol{u}_{0}\|_{V}^{2} + \frac{4T^{2}}{\rho} \|\boldsymbol{f}\|_{L_{\infty}(0,T;L_{2}(\Omega))}^{2} \\ &+ \frac{1}{h} \Big(\frac{4C(T+1)}{\varphi_{0}} + 1\Big) \|\boldsymbol{g}_{N}\|_{L_{\infty}(0,T;L_{2}(\Gamma_{N}))}^{2} + \frac{4C(T+1)}{\varphi_{0}h} \|\boldsymbol{\dot{g}}_{N}\|_{L_{2}(0,T;L_{2}(\Gamma_{N}))}^{2}, \end{split}$$

where  $\epsilon_a = \rho/(8T)$  and  $\epsilon_b = \varphi_0 h/(4C(T+1))$ . Taking  $\alpha_0$  sufficiently large then completes the proof.

**Theorem 4.2** (Stability Bound for (V)<sup>*h*</sup>). If  $\beta_0(d-1) \ge 1$  and  $\alpha_0$  is large enough, and if  $W_h^n$ ,  $U_h^n$  and  $\{S_{hq}^n\}_{q=1}^{N_{\varphi}}$  satisfy the fully discrete formulation (V)<sup>*h*</sup>, then there exists a positive constant *C* such that

$$\begin{split} \max_{0 \leq n \leq N} \|\boldsymbol{W}_{h}^{n}\|_{L_{2}(\Omega)}^{2} + \max_{0 \leq n \leq N} \|\boldsymbol{S}_{h}^{n}\|_{V}^{2} + \sum_{q=1}^{N_{\varphi}} \max_{0 \leq n \leq N} \|\boldsymbol{\Psi}_{hq}^{n}\|_{V}^{2} + \sum_{n=0}^{N-1} \sum_{q=1}^{N_{\varphi}} \Delta t \|\boldsymbol{S}_{hq}^{n+1} - \boldsymbol{S}_{hq}^{n}\|_{V}^{2} \\ + \Delta t \sum_{n=0}^{N-1} J_{0}^{\alpha_{0},\beta_{0}} (\boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}, \boldsymbol{W}_{h}^{n+1} + \boldsymbol{W}_{h}^{n}) \\ \leq CT^{2} (\|\boldsymbol{w}_{0}\|_{L_{2}(\Omega)}^{2} + \|\boldsymbol{u}_{0}\|_{V}^{2} + \|\boldsymbol{f}\|_{L_{\infty}(0,T;L_{2}(\Omega))}^{2} + h^{-1}\|\boldsymbol{g}_{N}\|_{H^{1}(0,T;L_{2}(\Gamma_{N}))}^{2}). \end{split}$$

*Here, C is independent of discrete solutions,*  $\Delta t$ *, and h, but dependent on the polynomial degree k, the domain*  $\Omega$ *, its boundary, and the material properties.* 

*Proof.* The proof follows the same arguments as in Theorem 4.1. We take  $v = \Delta t(W_h^{n+1} + W_h^n)$  in (3.18) and  $v = \Delta t(S_h^{n+1} + S_h^{n+1})$  in (3.19) for each q and add the results. We then use (2.2), the inverse polynomial trace inequality, (2.6), (2.9), the Cauchy–Schwarz inequality and Young's inequality, continuity and coercivity of the DG bilinear form, and maximise over discrete time just as before. There is this additional term in  $F_v(\cdot)$ ,

$$\sum_{n=0}^{m-1} \sum_{q=1}^{N_{\varphi}} \varphi_{q} (e^{-t_{n+1}/\tau_{q}} + e^{-t_{n}/\tau_{q}}) a(\boldsymbol{u}_{0}, \boldsymbol{U}_{h}^{n+1} - \boldsymbol{U}_{h}^{n})$$

which can be dealt with by using the fundamental theorem of calculus, summation by parts, and continuity. Specifically,

$$\begin{aligned} \left|\sum_{n=0}^{m-1} \sum_{q=1}^{N_{\varphi}} \varphi_{q}(e^{-t_{n+1}/\tau_{q}} + e^{-t_{n}/\tau_{q}})a(\boldsymbol{u}_{0}, \boldsymbol{U}_{h}^{n+1} - \boldsymbol{U}_{h}^{n})\right| \\ &= \left|2\sum_{q=1}^{N_{\varphi}} \varphi_{q}e^{-t_{m}/\tau_{q}}a(\boldsymbol{u}_{0}, \boldsymbol{U}_{h}^{m}) - 2\sum_{q=1}^{N_{\varphi}} \varphi_{q}e^{-t_{0}/\tau_{q}}a(\boldsymbol{u}_{0}, \boldsymbol{U}_{h}^{0}) \right. \\ &\left. - \sum_{n=0}^{m-1} \sum_{q=1}^{N_{\varphi}} \varphi_{q}(e^{-t_{n+1}/\tau_{q}} - e^{-t_{n}/\tau_{q}})a(\boldsymbol{u}_{0}, \boldsymbol{U}_{h}^{n+1} + \boldsymbol{U}_{h}^{n})\right| \\ &\leqslant 2|a(\boldsymbol{u}_{0}, \boldsymbol{U}_{h}^{m})| + 2|a(\boldsymbol{u}_{0}, \boldsymbol{U}_{h}^{0})| + 2K\sum_{q=1}^{N_{\varphi}} \varphi_{q}\sum_{n=0}^{m-1} \int_{t_{n}}^{t_{n+1}} \frac{e^{-t'/\tau_{q}}}{\tau_{q}} \|\boldsymbol{u}_{0}\|_{V} \|\boldsymbol{U}_{h}^{n}\|_{V} dt' \end{aligned}$$

since  $|e^{-t/\tau_q}| \le 1$  for all  $t \ge 0$ , for each q, and  $\sum_{q=1}^{N_{\phi}} \varphi_q \le 1$ . Young's inequality, continuity, and maximising over time now yield an *m*-independent upper bound. The proof is now completed in a similar way (appropriate choice of Young's inequality constants and large enough  $\alpha_0$ ) to the proof of Theorem 4.1.

As we have already touched upon above, the  $h^{-1}$  factor appearing in Theorems 4.1 and 4.2 can be removed by using the results in [5], under [5, Assumption 2.1] on the quality of the family of meshes. These assumptions require shape regularity and contact regularity. The former is well understood, and the latter says, essentially, that, in  $\mathbb{R}^d$  for d = 2 (d = 3), the length (area) of an edge (face) can be bounded below by  $Ch^{d-1}$ , where h is the local mesh size. The consequence is that  $h_{edge} \sim h_{element}$ . There is a third assumption regarding the existence of a sub-mesh. This is described as a purely technical assumption which allows the proofs to go through with a high degree of generality. For d = 2, 3, they point to the related earlier results of Brenner in [3]. We refer back to [5, Assumption 2.1] for the full details and here just outline the relevant result and give the implications for the improvements in Theorems 4.1 and 4.2.

For a constant  $C_{BT}$ , independent of h, [5, equation (4.6)] gives

$$\|\boldsymbol{\nu}\|_{L_q(\partial\Omega)} \leq C_{\mathrm{BT}}\Big(\|\boldsymbol{\nu}\|_{L_1(\Omega)} + \|\nabla\boldsymbol{\nu}\|_{L_p(\Omega)} + \alpha\Big(\int_{\Gamma_p} h^{1-p} |[\boldsymbol{\nu}]|^p \, ds\Big)^{1/p}\Big),$$

where  $\alpha > 0$  and we can take p = 2 with  $q = \infty$  (q = 4) for d = 2 (d = 3). Using (2.2) and (2.5), with the positivity of the material constants, we can then introduce the DG norm  $\|\cdot\|_V$  into the right-hand side above.

**Corollary 4.1** (to Theorems 4.1 and 4.2). With the techniques and result from [5] just discussed, in Theorems 4.1 and 4.2, the terms  $h^{-1} \|\boldsymbol{g}_N\|_{H^1(0,T;L_2(\Gamma_N))}^2$  can be replaced with  $\|\boldsymbol{g}_N\|_{H^1(0,T;L_2(\Gamma_N))}^2$ .

*Proof.* By  $\beta_0(d-1) = 1$  in (2.5), recalling (2.2), we get, for a generic constant *C*,

$$\|\boldsymbol{\nu}\|_{L_{q}(\partial\Omega)} \leq C \Big(\|\boldsymbol{\nu}\|_{L_{1}(\Omega)} + \|\nabla\boldsymbol{\nu}\|_{L_{2}(\Omega)} + \alpha \Big(\int_{\Gamma_{h}} h^{-1} |[\boldsymbol{\nu}]|^{2} ds \Big)^{1/2} + \alpha \Big(\int_{\Gamma_{D}} h^{-1} |[\boldsymbol{\nu}]|^{2} ds \Big)^{1/2} \Big) \leq C \|\boldsymbol{\nu}\|_{V}$$

for all  $v \in \mathcal{D}_k(\mathcal{E}_h)$ . Therefore, for q = 4 or  $q = \infty$ , we have  $\|v\|_{L_2(\partial\Omega)} \leq \|v\|_{L_q(\partial\Omega)} \leq \|v\|_V$ , which can be used in place of (4.3) above with no need to introduce  $h^{-1}$ .

Again, we emphasise that obtaining these improved results requires lengthy and non-trivial arguments which it would not be appropriate to try and replicate here. The full details are of course in [5].

# **5** Error Analysis

In order to carry out an a priori error analysis, we take the usual approach of splitting the error into "spatial" and "temporal" components by using the elliptic projection. To this end, define

$$\boldsymbol{\theta}(t) \coloneqq \boldsymbol{u}(t) - \boldsymbol{R}\boldsymbol{u}(t), \quad \boldsymbol{\vartheta}_{q}(t) \coloneqq \boldsymbol{\psi}_{q}(t) - \boldsymbol{R}\boldsymbol{\psi}_{q}(t), \quad \boldsymbol{v}_{q}(t) \coloneqq \boldsymbol{\zeta}_{q}(t) - \boldsymbol{R}\boldsymbol{\zeta}_{q}(t),$$
$$\boldsymbol{\chi}^{n} \coloneqq \boldsymbol{U}_{h}^{n} - \boldsymbol{R}\boldsymbol{u}^{n}, \quad \boldsymbol{\varpi}^{n} \coloneqq \boldsymbol{W}_{h}^{n} - \boldsymbol{R}\dot{\boldsymbol{u}}^{n}, \quad \boldsymbol{\zeta}_{q}^{n} \coloneqq \boldsymbol{\Psi}_{hq}^{n} - \boldsymbol{R}\boldsymbol{\psi}_{q}^{n},$$
$$\boldsymbol{\Upsilon}_{q}^{n} \coloneqq \boldsymbol{S}_{hq}^{n} - \boldsymbol{R}\boldsymbol{\zeta}_{q}^{n}, \quad \text{and} \quad \boldsymbol{\varepsilon}_{1}(t) \coloneqq \frac{\ddot{\boldsymbol{u}}(t + \Delta t) + \ddot{\boldsymbol{u}}(t)}{2} - \frac{\dot{\boldsymbol{u}}(t + \Delta t) - \dot{\boldsymbol{u}}(t)}{\Delta t}$$

for  $t \in [0, T]$ ,  $q = 1, ..., N_{\varphi}$ , and n = 0, ..., N, and note that (3.12) implies that

$$\frac{\boldsymbol{\chi}^{n+1} - \boldsymbol{\chi}^n}{\Delta t} = \frac{\boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^n}{2} - \boldsymbol{\mathcal{E}}_2^n - \boldsymbol{\mathcal{E}}_3^n$$
(5.1)

for  $n = 0, \ldots, N - 1$ , where

$$\mathcal{E}_2(t) := \frac{\dot{\boldsymbol{\theta}}(t+\Delta t)+\dot{\boldsymbol{\theta}}(t)}{2} - \frac{\boldsymbol{\theta}(t+\Delta t)-\boldsymbol{\theta}(t)}{\Delta t}, \quad \mathcal{E}_3(t) := \frac{\boldsymbol{u}(t+\Delta t)-\boldsymbol{u}(t)}{\Delta t} - \frac{\dot{\boldsymbol{u}}(t+\Delta t)+\dot{\boldsymbol{u}}(t)}{2}.$$

Also, for a three-times time-differentiable function v(t), with  $v^{(3)}$  denoting the third time derivative, we have

$$\frac{\dot{\mathbf{v}}(t_{n+1}) + \dot{\mathbf{v}}(t_n)}{2} - \frac{\mathbf{v}(t_{n+1}) - \mathbf{v}(t_n)}{\Delta t} = \frac{1}{2\Delta t} \int_{t_n}^{t_{n+1}} \mathbf{v}^{(3)}(t)(t_{n+1} - t)(t - t_n) dt$$

Hence, if  $v^{(3)} \in L_2(t_n, t_{n+1}; X)$ , the Cauchy–Schwarz inequality gives

$$\left\|\frac{\dot{\boldsymbol{\nu}}(t_{n+1}) + \dot{\boldsymbol{\nu}}(t_n)}{2} - \frac{\boldsymbol{\nu}(t_{n+1}) - \boldsymbol{\nu}(t_n)}{\Delta t}\right\|_X^2 \leqslant \frac{\Delta t^3}{4} \|\boldsymbol{\nu}^{(3)}\|_{L_2(t_n, t_{n+1}; X)}^2.$$
(5.2)

As is well known, the usual path to an error bound is to use the triangle inequality to split the error using the spatial and temporal components introduced above. The spatial errors are bounded using standard results for elliptic problems, and so it is the temporal error components that demand the bulk of the effort. For our problems, this effort is contained in the next two lemmas: the first for  $(D)^h$  and the second for  $(V)^h$ .

**Lemma 5.1.** Suppose  $\boldsymbol{u} \in H^2(0, T; \boldsymbol{C}^2(\Omega)) \cap W^1_{\infty}(0, T; \boldsymbol{H}^s(\mathcal{E}_h)) \cap H^4(0, T; \boldsymbol{H}^s(\mathcal{E}_h))$  and  $\beta_0(d-1) \ge 1$  for s > 3/2. If, for n = 0, ..., N, the solution to  $(D)^h$  is  $\boldsymbol{U}^n_h, \boldsymbol{W}^n_h, \boldsymbol{\Psi}^n_{h1}, ..., \boldsymbol{\Psi}^n_{hN_{\varphi}}$ , then, for large enough  $\alpha_0$ , there exists a positive constant C such that

$$\max_{1 \le n \le N} \|\boldsymbol{\varpi}^n\|_{L_2(\Omega)} + \max_{1 \le n \le N} \|\boldsymbol{\chi}^n\|_V \le CT \|\boldsymbol{u}\|_{H^4(0,T;H^s(\mathcal{E}_h))} (h^r + \Delta t^2),$$

where  $r = \min(k + 1, s)$  and C are independent of h,  $\Delta t$ , T, and the numerical solution.

*Proof.* The proof will follow similar arguments in the stability analysis, but we additionally use Galerkin orthogonality, the elliptic error estimates (2.10) and (2.11), and the time discretisation error (5.2). Averaging over  $t_{n+1}$  and  $t_n$  and subtracting (3.8) from (3.13) give

$$\frac{\rho}{\Delta t}(\boldsymbol{\varpi}^{n+1} - \boldsymbol{\varpi}^n, \boldsymbol{v}) + \frac{1}{2}a(\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n, \boldsymbol{v}) - \frac{1}{2}\sum_{q=1}^{N_{\varphi}}a(\boldsymbol{\varsigma}_q^{n+1} + \boldsymbol{\varsigma}_q^n, \boldsymbol{v}) + \frac{1}{2}J_0^{\alpha_0,\beta_0}(\boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^n, \boldsymbol{v}) \\
= \frac{\rho}{\Delta t}(\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^n, \boldsymbol{v}) + \rho(\mathcal{E}_1^n, \boldsymbol{v})$$
(5.3)

for all  $v \in \mathcal{D}_k(\mathcal{E}_h)$ , for  $0 \le n \le N - 1$ . In this manner, a subtraction of (3.14) from (3.9) gives

$$\frac{\tau_q}{\Delta t}a(\boldsymbol{\varsigma}_q^{n+1} - \boldsymbol{\varsigma}_q^n, \boldsymbol{\nu}) + \frac{1}{2}a(\boldsymbol{\varsigma}_q^{n+1} + \boldsymbol{\varsigma}_q^n, \boldsymbol{\nu}) - \frac{\varphi_q}{2}a(\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n, \boldsymbol{\nu}) = \tau_q a(\boldsymbol{E}_q^n, \boldsymbol{\nu})$$
(5.4)

by Galerkin orthogonality, for any  $v \in \mathcal{D}_k(\mathcal{E}_h)$ , where

$$E_q(t) = \frac{\dot{\psi}_q(t + \Delta t) + \dot{\psi}_q(t)}{2} - \frac{\psi_q(t + \Delta t) - \psi_q(t)}{\Delta t} \quad \text{for each } q.$$

Inserting  $v = 2(\chi^{n+1} - \chi^n)$  into (5.3), with equation (5.1), and  $v = \varsigma_q^{n+1} - \varsigma_q^n$  into (5.4) for each q, summing over n = 0, ..., m - 1 for  $m \leq N$  gives

$$\begin{split} \rho \| \boldsymbol{\varpi}^{m} \|_{L_{2}(\Omega)}^{2} + a(\boldsymbol{\chi}^{m}, \boldsymbol{\chi}^{m}) + \sum_{q=1}^{N_{\varphi}} \frac{1}{\varphi_{q}} a(\boldsymbol{\varsigma}_{q}^{m}, \boldsymbol{\varsigma}_{q}^{m}) + \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_{\varphi}} \frac{2\tau_{q}}{\varphi_{q}} a\left(\frac{\boldsymbol{\varsigma}_{q}^{n+1} - \boldsymbol{\varsigma}_{q}^{n}}{\Delta t}, \frac{\boldsymbol{\varsigma}_{q}^{n+1} - \boldsymbol{\varsigma}_{q}^{n}}{\Delta t}\right) \\ & + \frac{\Delta t}{2} \sum_{n=0}^{m-1} \int_{0}^{a_{0},\beta_{0}} (\boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^{n}, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^{n}) \\ & = \rho \| \boldsymbol{\varpi}^{0} \|_{L_{2}(\Omega)}^{2} + a(\boldsymbol{\chi}^{0}, \boldsymbol{\chi}^{0}) + \rho \sum_{n=0}^{m-1} (\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^{n}, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^{n}) - 2\rho \sum_{n=0}^{m-1} (\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^{n}, \boldsymbol{\varepsilon}_{2}^{n}) \\ & - 2\rho \sum_{n=0}^{m-1} (\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^{n}, \boldsymbol{\varepsilon}_{3}^{n}) + 2\rho \Delta t \sum_{n=0}^{m-1} (\boldsymbol{\varepsilon}_{1}^{n}, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^{n}) - \rho \Delta t \sum_{n=0}^{m-1} (\boldsymbol{\varepsilon}_{1}^{n}, \boldsymbol{\varepsilon}_{2}^{n}) \\ & - 2\rho \Delta t \sum_{n=0}^{m-1} (\boldsymbol{\varepsilon}_{1}^{n}, \boldsymbol{\varepsilon}_{3}^{n}) + 2\rho \sum_{n=0}^{m-1} (\boldsymbol{\varpi}^{n+1} - \boldsymbol{\varpi}^{n}, \boldsymbol{\varepsilon}_{2}^{n}) + 2\rho \sum_{n=0}^{m-1} (\boldsymbol{\varpi}^{n+1} - \boldsymbol{\varpi}^{n}, \boldsymbol{\varepsilon}_{3}^{n}) \\ & + 2 \sum_{q=1}^{N_{\varphi}} a(\boldsymbol{\chi}^{m}, \boldsymbol{\varsigma}_{q}^{m}) + \sum_{q=1}^{N_{\varphi}} \frac{2\tau_{q}}{\varphi_{q}} a(\boldsymbol{E}_{q}^{m-1}, \boldsymbol{\varsigma}_{q}^{m}) - \sum_{n=0}^{m-1} \sum_{q=1}^{N_{\varphi}} \frac{2\tau_{q}}{\varphi_{q}} a(\boldsymbol{E}_{q}^{n+1} - \boldsymbol{E}_{q}^{n}, \boldsymbol{\varsigma}_{q}^{n+1}) \end{split}$$

when we apply summation by parts to the summation terms from (5.4) with the fact  $\boldsymbol{\varsigma}_q^0 = \boldsymbol{0}$  for all q. Hence the definition of DG bilinear form and its coercivity imply that

$$\rho \|\boldsymbol{\varpi}^{m}\|_{L_{2}(\Omega)}^{2} + \|\boldsymbol{\chi}^{m}\|_{V}^{2} + \sum_{q=1}^{N_{\varphi}} \frac{1}{\varphi_{q}} \|\boldsymbol{\varsigma}_{q}^{m}\|_{V}^{2} + \Delta t \sum_{n=0}^{N_{\varphi}} \sum_{q=1}^{N_{\varphi}} \frac{2\kappa\tau_{q}}{\varphi_{q}} \|\frac{\boldsymbol{\varsigma}_{q}^{n+1} - \boldsymbol{\varsigma}_{q}^{n}}{\Delta t}\|^{2} + \frac{\Delta t}{2} \sum_{n=0}^{m-1} f_{0}^{a_{0},\beta_{0}}(\boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^{n}, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^{n}) \\ \leq \left|\rho\|\boldsymbol{\varpi}^{0}\|_{L_{2}(\Omega)}^{2} + \|\boldsymbol{\chi}^{0}\|_{V}^{2} + \rho \sum_{n=0}^{m-1} (\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^{n}, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^{n}) - 2\rho \sum_{n=0}^{m-1} (\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^{n}, \boldsymbol{\varepsilon}_{2}^{n}) \\ - 2\rho \sum_{n=0}^{m-1} (\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^{n}, \boldsymbol{\varepsilon}_{3}^{n}) + \rho \Delta t \sum_{n=0}^{m-1} (\boldsymbol{\varepsilon}_{1}^{n}, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^{n}) - 2\rho \Delta t \sum_{n=0}^{m-1} (\boldsymbol{\varepsilon}_{1}^{n}, \boldsymbol{\varepsilon}_{2}^{n}) \\ - 2\rho \Delta t \sum_{n=0}^{m-1} (\boldsymbol{\varepsilon}_{1}^{n}, \boldsymbol{\varepsilon}_{3}^{n}) + 2\rho \sum_{n=0}^{m-1} (\boldsymbol{\varpi}^{n+1} - \boldsymbol{\varpi}^{n}, \boldsymbol{\varepsilon}_{2}^{n}) + 2\rho \sum_{n=0}^{m-1} (\boldsymbol{\varpi}^{n+1} - \boldsymbol{\varpi}^{n}, \boldsymbol{\varepsilon}_{3}^{n}) \\ + 2 \sum_{q=1}^{N_{\varphi}} a(\boldsymbol{\chi}^{m}, \boldsymbol{\varsigma}_{q}^{m}) + \sum_{q=1}^{N_{\varphi}} \frac{2\tau_{q}}{\varphi_{q}} a(\boldsymbol{E}_{q}^{m-1}, \boldsymbol{\varsigma}_{q}^{m}) - \sum_{n=0}^{m-1} \sum_{q=1}^{N_{\varphi}} \frac{2\tau_{q}}{\varphi_{q}} a(\boldsymbol{E}_{q}^{n+1} - \boldsymbol{E}_{q}^{n}, \boldsymbol{\varsigma}_{q}^{n+1}) \\ + 2 \sum_{e \in \Gamma_{h} \cup \Gamma_{D}} \sum_{e} \{\underline{D}\underline{e}(\boldsymbol{\chi}^{m})\} : [\boldsymbol{\chi}^{m} \otimes \boldsymbol{n}_{e}] de + \sum_{q=1}^{N_{\varphi}} \frac{2}{\varphi_{q}} \sum_{e \in \Gamma_{h} \cup \Gamma_{D}} \sum_{e} \{\underline{D}\underline{e}(\boldsymbol{\varsigma}_{q}^{m})\} : [\boldsymbol{\varsigma}_{q}^{m} \otimes \boldsymbol{n}_{e}] de \right|. \tag{5.5}$$

(5.6)

Using the elliptic projection, (3.15), (3.16), and initial conditions, we have  $\|\boldsymbol{\sigma}^0\|_{L_2(\Omega)} \leq \|\dot{\boldsymbol{\theta}}^0\|_{L_2(\Omega)}$  and  $\|\boldsymbol{\chi}^0\|_V = 0$ . Further, we note that the fundamental theorem of calculus allows us to deal with the time differences with expressions like  $\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^n = \int_{t_n}^{t_{n+1}} \ddot{\boldsymbol{\theta}}(t') dt'$ , and then using (5.2), we can bound  $\mathcal{E}_1^n$ ,  $\mathcal{E}_2^n$ ,  $\mathcal{E}_3^n$ , and  $E_q^n$  for all n and for each q in an optimal way. For example,

$$\left|\Delta t \sum_{n=0}^{m-1} (\mathcal{E}_1^n, \mathcal{E}_2^n)\right| \leq \frac{\Delta t}{2} \sum_{n=0}^{N-1} \|\mathcal{E}_1^n\|_{L_2(\Omega)}^2 + \frac{\Delta t}{2} \sum_{n=0}^{N-1} \|\mathcal{E}_2^n\|_{L_2(\Omega)}^2 \leq \frac{\Delta t^4}{8} \|\boldsymbol{u}^{(4)}\|_{L_2(0,T;L_2(\Omega))}^2 + \frac{\Delta t^4}{8} \|\boldsymbol{\theta}^{(3)}\|_{L_2(0,T;L_2(\Omega))}^2$$

by the Cauchy–Schwarz inequality, Young's inequality, and (5.2). On the other hand, the elliptic error estimates such as (2.11) provide spatial error estimates for  $\theta(t)$  and its time derivatives, and noting that the internal variables are analogues of displacement u, we can employ elliptic error estimates for the internal variables as well. Therefore, in the same way as for the stability estimate, using the Cauchy–Schwarz inequality and Young's inequality, (2.9), summation by parts, maximising in time, and choosing large enough penalty parameters, we eventually arrive at

$$\begin{aligned} \max_{1 \le n \le N} \|\boldsymbol{\varpi}^{n}\|_{L_{2}(\Omega)} + \max_{1 \le n \le N} \|\boldsymbol{\chi}^{n}\|_{V} + \sum_{q=1}^{N_{\varphi}} \max_{1 \le n \le N} \|\boldsymbol{\varsigma}^{n}_{q}\|_{V} + \left(\Delta t \sum_{n=0}^{N-1} \sum_{q=1}^{N_{\varphi}} \frac{\tau_{q}}{\varphi_{q}} \left\|\frac{\boldsymbol{\varsigma}^{n+1}_{q} - \boldsymbol{\varsigma}^{n}_{q}}{\Delta t}\right\|^{2}\right)^{1/2} \\ + \left(\Delta t \sum_{n=0}^{N-1} J_{0}^{\alpha_{0},\beta_{0}}(\boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^{n}, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^{n})\right)^{1/2} \le CT \|\boldsymbol{u}\|_{H^{4}(0,T;\boldsymbol{H}^{s}(\mathcal{E}_{h}))}(h^{r} + \Delta t^{2}) \end{aligned}$$

Here, *C* is independent of *h*,  $\Delta t$ , and the solutions, but depends on  $\rho$ ,  $\Omega$ ,  $\partial \Omega$ , and the material properties. For full technical details for the proof, we refer to [15].

Next, the analogous estimate for  $(V)^h$ .

**Lemma 5.2.** Suppose  $\mathbf{u} \in H^2(0, T; \mathbf{C}^2(\Omega)) \cap W^1_{\infty}(0, T; \mathbf{H}^s(\mathcal{E}_h)) \cap H^4(0, T; \mathbf{H}^s(\mathcal{E}_h))$  and  $\beta_0(d-1) \ge 1$  for s > 3/2. If, for n = 0, 1, ..., N, the solution to  $(\nabla)^h$  is  $U^n_h, W^n_h, S^n_{h1}, ..., S^n_{hN_{\varphi}}$ , then, for large enough  $\alpha_0$ , there exists a positive constant C such that

$$\max_{1 \leq n \leq N} \|\boldsymbol{\varpi}^n\|_{L_2(\Omega)} + \max_{1 \leq n \leq N} \|\boldsymbol{\chi}^n\|_V \leq CT \|\boldsymbol{u}\|_{H^4(0,T;H^s(\mathcal{E}_h))} (h^r + \Delta t^2),$$

where  $r = \min(k + 1, s)$  and C are independent of h,  $\Delta t$ , T, and the numerical solution.

*Proof.* We follow similar steps to the proof of Lemma 5.1 for the displacement form to show this claim. By averaging at  $t = t_{n+1}$  and  $t = t_n$ , subtracting (3.10) from (3.18) with  $v = 2(\chi^{n+1} - \chi^n)$ , and subtracting (3.11) from (3.19), with  $v = 2(\chi^{n+1} + \chi^n_q)$  for each n and q, summing over n = 0, ..., m - 1 for  $m \leq N$ , employing summation by parts, recalling the coercivity of  $a(\cdot, \cdot)$ , and using Galerkin orthogonality, we eventually get

$$\begin{split} \rho \|\boldsymbol{\varpi}^{m}\|_{L_{2}(\Omega)}^{2} + \varphi_{0}\|\boldsymbol{\chi}^{m}\|_{V}^{2} + \sum_{q=1}^{N_{\varphi}} \frac{\kappa}{\varphi_{q}} \|\boldsymbol{Y}_{q}^{m}\|_{V}^{2} + \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_{\varphi}} \frac{\kappa}{\tau_{q} \varphi_{q}} \|\boldsymbol{Y}_{q}^{n+1} + \boldsymbol{Y}_{q}^{n}\|_{V}^{2} \\ &+ \frac{\Delta t}{2} \sum_{n=0}^{m-1} J_{0}^{\alpha_{0},\beta_{0}}(\boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^{n}, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^{n}) \\ \leqslant \left| \rho \|\boldsymbol{\varpi}^{0} \|_{L_{2}(\Omega)}^{2} + \varphi_{0} \|\boldsymbol{\chi}^{0}\|_{V}^{2} + \rho \sum_{n=0}^{m-1} (\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^{n}, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^{n}) + \rho \Delta t \sum_{n=0}^{m-1} (\mathcal{E}_{1}^{n}, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^{n}) \right. \\ &- 2\rho \sum_{n=0}^{m-1} (\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^{n}, \mathcal{E}_{2}^{n}) - 2\rho \sum_{n=0}^{m-1} (\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^{n}, \mathcal{E}_{3}^{n}) \\ &- 2\rho \Delta t \sum_{n=0}^{m-1} (\mathcal{E}_{1}^{n}, \mathcal{E}_{2}^{n}) - 2\rho \Delta t \sum_{n=0}^{m-1} (\mathcal{E}_{1}^{n}, \mathcal{E}_{3}^{n}) \\ &+ \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_{\varphi}} \frac{1}{\varphi_{q}} a(\mathcal{E}_{q}^{n}, \boldsymbol{\Upsilon}_{q}^{n+1} + \boldsymbol{\Upsilon}_{q}^{n}) - \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_{\varphi}} a(\mathcal{E}_{3}^{n}, \boldsymbol{\Upsilon}_{q}^{n+1} + \boldsymbol{\Upsilon}_{q}^{n}) \\ &+ 2\varphi_{0} \sum_{e \in \Gamma_{h} \cup \Gamma_{D}} \int_{e} \left\{ \underline{\mathcal{D}} \underline{\varepsilon}(\boldsymbol{\chi}^{m}) \right\} : [\boldsymbol{\chi}^{m} \otimes \boldsymbol{n}_{e}] de \bigg|, \end{split}$$

where

$$\boldsymbol{E}_q(t) \coloneqq \frac{\boldsymbol{\zeta}_q(t+\Delta t) + \boldsymbol{\zeta}_q(t)}{2} - \frac{\boldsymbol{\zeta}_q(t+\Delta t) - \boldsymbol{\zeta}_q(t)}{\Delta t} \quad \text{for each } q.$$

We complete the proof by using the same techniques and results as used earlier: the Cauchy–Schwarz inequality and Young's inequality, integration and summation by parts, (2.9), (2.11), and (5.2).

We can now use Lemmas 5.1 and 5.2 to prove the following a priori error estimates.

**Theorem 5.1** (Error Analysis). Suppose the discrete solutions in  $\mathcal{D}_k(\mathcal{E}_h)$  satisfy either  $(\mathbb{D})^h$  or  $(\mathbb{V})^h$  for integer  $s \ge 2$ , and also that  $\mathbf{u} \in H^2(0, T; \mathbf{C}^2(\Omega)) \cap W^1_{\infty}(0, T; \mathbf{H}^s(\mathcal{E}_h)) \cap H^4(0, T; \mathbf{H}^s(\mathcal{E}_h))$ . If Lemmas 5.1 and 5.2 hold, then we have

$$\begin{split} & \max_{1 \le n \le N} \| \boldsymbol{u}(t_n) - \boldsymbol{U}_h^n \|_V \le CT \| \boldsymbol{u} \|_{H^4(0,T;H^s(\mathcal{E}_h))} (h^{r-1} + \Delta t^2), \\ & \max_{1 \le n \le N} \| \boldsymbol{u}(t_n) - \boldsymbol{U}_h^n \|_{L_2(\Omega)} \le CT \| \boldsymbol{u} \|_{H^4(0,T;H^s(\mathcal{E}_h))} (h^r + \Delta t^2), \\ & \max_{1 \le n \le N} \| \boldsymbol{\dot{u}}(t_n) - \boldsymbol{W}_h^n \|_V \le CT \| \boldsymbol{u} \|_{H^4(0,T;H^s(\mathcal{E}_h))} (h^{r-1} + \Delta t^2), \\ & \max_{1 \le n \le N} \| \boldsymbol{\dot{u}}(t_n) - \boldsymbol{W}_h^n \|_{L_2(\Omega)} \le CT \| \boldsymbol{u} \|_{H^4(0,T;H^s(\mathcal{E}_h))} (h^r + \Delta t^2), \end{split}$$

where  $r = \min(s + 1, k)$  and C is a positive constant that is independent of h,  $\Delta t$ , T, and the exact and discrete solutions, but depends on the polynomial degree k,  $\Omega$ ,  $\partial \Omega$ , and the material coefficients. Moreover, the initial discrete errors are given by

$$\|\boldsymbol{u}_0 - \boldsymbol{U}_h^0\|_V \leq Ch^{r-1} \|\|\boldsymbol{u}_0\|_{H^s(\mathcal{E}_h)}$$
 and  $\|\boldsymbol{w}_0 - \boldsymbol{W}_h^0\|_{L_2(\Omega)} \leq Ch^r \|\|\boldsymbol{w}_0\|_{H^s(\mathcal{E}_h)}$ 

*Proof.* Using the triangle inequality with Lemmas 5.1 and 5.2, and the DG elliptic error estimates, we get for any  $n \in \{1, ..., N\}$  that

$$\|\boldsymbol{u}(t_n) - \boldsymbol{U}_h^n\|_V \leq \|\boldsymbol{\theta}(t_n)\|_V + \|\boldsymbol{\chi}^n\|_V \leq CT \|\boldsymbol{u}\|_{H^4(0,T;H^s(\mathcal{E}_h))}(h^{r-1} + \Delta t^2).$$
(5.7)

Poincaré's inequality (2.2) indicates that  $\|v\|_{L_2(\Omega)} \leq C \|v\|_V$  for any  $v \in \mathcal{D}_k(\mathcal{E}_h)$ . Hence we have

$$\|\boldsymbol{u}(t_n) - \boldsymbol{U}_h^n\|_{L_2(\Omega)} \leq \|\boldsymbol{\theta}(t_n)\|_{L_2(\Omega)} + C\|\boldsymbol{\chi}^n\|_V \leq CT\|\boldsymbol{u}\|_{H^4(0,T;H^s(\mathcal{E}_h))}(h^{r-1} + \Delta t^2).$$
(5.8)

In a similar way, we have

$$\|\dot{\boldsymbol{u}}(t_n) - \boldsymbol{W}_h^n\|_{L_2(\Omega)} \le \|\dot{\boldsymbol{\theta}}(t_n)\|_{L_2(\Omega)} + \|\boldsymbol{\varpi}^n\|_{L_2(\Omega)} \le CT \|\boldsymbol{u}\|_{H^4(0,T;H^s(\mathcal{E}_h))}(h^r + \Delta t^2).$$
(5.9)

On the other hand, as seen in (5.5) and (5.6), the jump penalty term of  $\boldsymbol{\varpi}^n$  is also bounded by the same upper bound. In other words, we have

$$\left(J_0^{\alpha_0,\beta_0}(\boldsymbol{\varpi}^n,\boldsymbol{\varpi}^n)\right)^{1/2} = O(h^r + \Delta t^2).$$
(5.10)

Using this argument with the inverse inequality (2.3), we can derive energy norm error estimates for the velocity vector field and get

$$\begin{aligned} \|\dot{\boldsymbol{u}}(t_{n}) - \boldsymbol{W}_{h}^{n}\|_{V} &\leq \|\dot{\boldsymbol{\theta}}(t_{n})\|_{V} + \|\boldsymbol{\varpi}^{n}\|_{V} \leq \|\dot{\boldsymbol{\theta}}(t_{n})\|_{V} + Ch^{-1}\|\boldsymbol{\varpi}^{n}\|_{L_{2}(\Omega)} + \left(J_{0}^{\alpha_{0},\beta_{0}}(\boldsymbol{\varpi}^{n},\boldsymbol{\varpi}^{n})\right)^{1/2} \\ &\leq CT\|\boldsymbol{u}\|_{H^{4}(0,T;H^{s}(\mathcal{E}_{h}))}(h^{r-1} + \Delta t^{2}). \end{aligned}$$
(5.11)

Since *n* is arbitrary and the right-hand sides of (5.7)–(5.11) are independent of *n*, the proof is completed. Furthermore, to show the discrete errors for n = 0, we want to use the triangular inequalities, DG elliptic error estimates, and initial (DG elliptic/ $L_2$ ) projections, e.g. (3.15) and (3.16). Hence we have

$$\|\boldsymbol{u}_{0} - \boldsymbol{U}_{h}^{0}\|_{V} = \|\boldsymbol{\theta}(0) - \boldsymbol{\chi}^{0}\|_{V} \leq \|\boldsymbol{\theta}(0)\|_{V} + \|\boldsymbol{\chi}^{0}\|_{V} = \|\boldsymbol{\theta}(0)\|_{V} \leq Ch^{r-1}\|\|\boldsymbol{u}_{0}\|_{H^{s}(\mathcal{E}_{h})}$$

by the fact  $\|\chi^0\|_V = 0$  and (2.10). Also, since  $\|\varpi^0\|_{L_2(\Omega)} \leq \|\dot{\theta}(0)\|_{L_2(\Omega)}$ , (2.11) leads to

$$\|\boldsymbol{w}_{0} - \boldsymbol{W}_{h}^{0}\|_{L_{2}(\Omega)} = \|\dot{\boldsymbol{\theta}}(0) - \boldsymbol{\varpi}^{0}\|_{L_{2}(\Omega)} \leq \|\dot{\boldsymbol{\theta}}(0)\|_{L_{2}(\Omega)} + \|\boldsymbol{\varpi}^{0}\|_{L_{2}(\Omega)} \leq 2\|\dot{\boldsymbol{\theta}}(0)\|_{L_{2}(\Omega)} \leq Ch^{r}\|\|\boldsymbol{w}_{0}\|_{H^{s}(\mathcal{E}_{h})}.$$

# **6** Numerical Experiments

In this section, we use the FEniCS environment (see [1] and https://fenicsproject.org) to give the results of some numerical experiments that were carried out to demonstrate the convergence rates proven above: we present tables of numerical errors, as well as convergence rates. The Python code to reproduce these results is available at Jang's GitHub (https://github.com/Yongseok7717/Visco\_Int\_CG) and can be used to reproduce the results that are given below. Alternatively, these results can also be reproduced by using Docker. In this case, the commands to run are

docker pull jangyour/fenics\_fem
docker run -ti jangyour/fenics\_fem
cd visco/Int\_DG; . main.sh

## **An Exact Solution**

As is common, we use an artificial, or *manufactured*, exact solution in order to calculate the errors. Let this manufactured strong solution be

$$\boldsymbol{u}(x,y,t) = \begin{bmatrix} xye^{1-t} \\ \cos(t)\sin(xy) \end{bmatrix} \in C^{\infty}(0,T; \boldsymbol{C}^{\infty}(\Omega)),$$

where  $\Omega = (0, 1)^2$  and T = 1. We use two internal variables with coefficients and time delays  $\varphi_0 = 0.5$ ,  $\varphi_1 = 0.1$ ,  $\varphi_2 = 0.4$ ,  $\tau_1 = 0.5$ ,  $\tau_2 = 1.5$ , and (because we are not constrained by the physical reality here) we assume for simplicity an identity fourth-order tensor as our  $\underline{D}$  so that  $\underline{D}\underline{\varepsilon} = \underline{\varepsilon}$ . We define the Dirichlet boundary as

$$\Gamma_D = \{ (x, y) \in \partial \Omega \mid x = 0 \text{ or } y = 0 \},\$$

and then  $\boldsymbol{u} = \boldsymbol{0}$  on  $\Gamma_D$ , and the other data are readily computed.

## **Numerical Results**

We present numerical errors to demonstrate the convergence rates given by the theorems, and for this, we define  $e^n = u(t_n) - U_h^n$  and  $\tilde{e}^n = w(t_n) - W_h^n$  for n = 0, ..., N. According to Theorem 5.1, we have the  $L_2$  norm error as well as DG energy norm error such that

$$\|\boldsymbol{e}^{n}\|_{V} = O(h^{k} + \Delta t^{2})$$
 and  $\|\boldsymbol{e}^{n}\|_{L_{2}(\Omega)} = O(h^{k+1} + \Delta t^{2})$ 

since  $s = \infty$ . Also, we have the same rates of convergence for the velocity error. By recalling Korn's inequality (2.7), we note that the broken  $H^1$  norm is bounded by the energy norm. Therefore, we have broken  $H^1$  error estimates such that

$$\|\|\boldsymbol{e}^{n}\|\|_{H^{1}(\mathcal{E}_{h})} + \|\|\tilde{\boldsymbol{e}}^{n}\|\|_{H^{1}(\mathcal{E}_{h})} \leq C(h^{k} + \Delta t^{2})$$

for some positive constant *C*. Since the DG energy norm depends on the penalty parameters  $\alpha_0$  and  $\beta_0$ , we hereafter consider the errors in the broken  $H^1$  norm instead.

Firstly, we set  $\Delta t = h$  and define the numerical convergence rate  $d_c$  by

$$d_c = \frac{\log(\text{error of } h_1) - \log(\text{error of } h_2)}{\log(h_1) - \log(h_2)}$$

We can see in Figure 1 that  $d_c$  is 1 for both the displacement and velocity form errors when k = 1, and that this suggests quadratic convergence orders for  $L_2$  norm error or when k = 2, regardless of which internal variable form is used.



Figure 1: Numerical convergence order: *linear* (top) and *quadratic* (bottom) polynomial basis

k	h	Displacement form		Velocity form	
		$   e^{N}   _{H^{1}(\mathcal{E}_{h})}$	$\ \! \!  \tilde{\pmb{e}}^{\pmb{N}} \ \! _{H^1(\mathcal{E}_h)}$	<b>∥</b> <i>e<sup>N</sup></i> <b>∥</b> <i>H</i> <sup>1</sup> ( <i>ε</i> <sub><i>h</i></sub> )	$\  \tilde{\boldsymbol{e}}^{\boldsymbol{N}} \  _{\boldsymbol{H}^{1}(\mathcal{E}_{h})}$
1	1/4	1.298e-01	1.951e-01	1.298e-01	1.951e-01
	1/8	6.177e-02 (1.07)	8.741e-02 (1.16)	6.177e-02 (1.07)	8.741e-02 (1.16)
	1/16	2.993e-02 (1.05)	4.130e-02 (1.08)	2.993e-02 (1.05)	4.130e-02 (1.08)
	1/32	1.473e-02 (1.02)	2.001e-02 (1.04)	1.473e-02 (1.02)	2.001e-02 (1.04)
2	1/4	3.168e-03	4.996e-03	3.168e-03	4.996e-03
	1/8	8.030e-04 (1.98)	1.284e-03 (1.96)	8.030e-04 (1.98)	1.284e-03 (1.96)
	1/16	2.008e-04 (2.00)	3.256e-04 (1.98)	2.008e-04 (2.00)	3.256e-04 (1.98)
	1/32	5.010e-05 (2.00)	8.206e-05 (1.99)	5.010e-05 (2.00)	8.206e-05 (1.99)

**Table 1:**  $H^1$  norm of numerical errors (orders) when  $\Delta t = 1/2048$ 

k	h	Displacement form		Velocity form	
		$\ \boldsymbol{e}^{N}\ _{L_{2}(\Omega)}$	$\ \tilde{\boldsymbol{e}}^{\boldsymbol{N}}\ _{L_2(\Omega)}$	$\ \boldsymbol{e}^{N}\ _{L_{2}(\Omega)}$	$\ \tilde{\boldsymbol{e}}^{\boldsymbol{N}}\ _{L_2(\Omega)}$
1	1/4	1.067e-02	2.293e-02	1.067e-02	2.293e-02
	1/8	2.808e-03 (1.93)	6.691e-03 (1.78)	2.808e-03 (1.93)	6.691e-03 (1.78)
	1/16	7.094e-04 (1.98)	1.182e-03 (1.88)	7.094e-04 (1.98)	1.182e-03 (1.88)
	1/32	1.781e-04 (1.99)	4.686e-04 (1.95)	1.781e-04 (1.99)	4.686e-04 (1.95)
2	1/4	8.362e-05	1.496e-04	8.362e-05	1.496e-04
	1/8	1.011e-05 (3.05)	1.861e-05 (3.01)	1.011e-05 (3.05)	1.861e-05 (3.01)
	1/16	1.231e-06 (3.04)	2.315e-06 (3.01)	1.231e-06 (3.04)	2.315e-06 (3.01)
	1/32	1.515e-07 (3.02)	2.906e-07 (3.00)	1.514e-07 (3.02)	2.902e-07 (3.00)

**Table 2:**  $L_2$  norm of numerical errors (orders) when  $\Delta t = 1/2048$ 

Secondly, if we take  $\Delta t \ll h$ , we can render the "time error" negligible in comparison to the "space error" and isolate the spatial convergence rates. Some example results for this are shown in Tables 1 and 2, where we see that the numerical rates follow the spatial convergence order of  $d_c = k$  and  $d_c = k + 1$  in the  $H^1$  and  $L_2$  norms respectively. We can also see that there is no significant difference between the numerical schemes (D)<sup>h</sup> and (V)<sup>h</sup>, in terms of convergence rates as well as the size of the errors.

Thirdly, in Table 3, we give some results for when the spatial error became negligibly small to the temporal error, to observe second-order accuracy in time regardless of spatial norms, displacement/velocity fields, and internal variable forms. All of these results are consistent with the claims in the theorems.

Norm	Δt	Displacement form		Velocity form	
		e <sup>N</sup>	ẽ <sup>ℕ</sup>	e <sup>N</sup>	ẽ <sup>ℕ</sup>
$H^1$	1/2	2.696e-02	9.579e-02	1.766e-02	7.348e-02
	1/4	7.445e-03 (1.86)	2.461e-02 (1.96)	4.879e-03 (1.86)	1.880e-02 (1.97)
	1/8	1.894e-03 (1.97)	6.169e-03 (2.00)	1.2429e-03 (1.97)	4.712e-03 (2.00)
	1/16	4.747e-04 (2.00)	1.549e-04 (2.00)	3.117e-04 (2.00)	1.181e-03 (2.00)
L <sub>2</sub>	1/2	8.121e-02	3.222e-02	5.256e-03	2.586e-02
	1/4	2.410e-03 (1.75)	8.221e-03 (1.97)	1.534e-03 (1.78)	6.601e-03 (1.97)
	1/8	6.252e-04 (1.95)	2.067e-03 (1.99)	3.974e-04 (1.95)	1.659e-03 (1.99)
	1/16	1.577e-04 (1.99)	5.178e-04 (2.00)	1.001e-04 (1.99)	4.155e-04 (2.00)

**Table 3:** Time convergence errors (orders) for fixed h = 1/128 and k = 2

	Displacement form			Velocity form		
h	1/2	1/4	1/8	1/2	1/4	1/8
Error	2.262	9.527e+03	1.262e+14	2.286	9.364e+03	1.266e+14

**Table 4:**  $||e^{N}||_{L_{2}(\Omega)}$  for  $a_{0} = 0.1$  where  $\Delta t = h$  and k = 1

#### **Penalty Parameters**

For well-posedness as well as optimal convergence, we need to have "large enough" penalty parameters, and following studies of elasticity problems [13, 24], we took  $a_0 = 10$  and  $\beta_0 = 1$  for the computations above. If  $a_0$  is small, our numerical schemes will lose stability and convergence. This is illustrated in Table 4, where we see that the numerical errors diverged when  $a_0 = 0.1$ . On the other hand, in order to observe the requirements of the stability and error analyses above, we need  $\beta_0(d-1) \ge 1$ , but a too large  $\beta_0$  will result in the assembled global matrix being ill-conditioned. Therefore, taking  $\beta_0 = 1/(d-1)$  seems to be the most reasonable choice. In fact, the condition number of the assembled system matrix in the interior penalty method is of order  $O(h^{-(\beta_0+1)})$  (see [15, 19]), and hence we may encounter difficulty in solving linear systems for large  $\beta_0$  if we use iterative solvers. For more details, we refer to [15, 19].

#### An Example with Real Material Data

The results above are based on artificial choices of the material data – the point was merely to show convergence and illustrate the estimates. In this closing section, we illustrate the displacement formulation using real material data for PMMA (polymethyl methacrylate), taken from [18, Table 1]. The demonstration given below shows clearly the effect of viscoelastic damping on the longer time response.

First, the Prony series relaxation data from [18, Table 1] are converted from dyne/cm<sup>2</sup> to N/m<sup>2</sup> by dividing by ten, and then the coefficients are normalised (and nondimensionalised) to fit our assumptions for (1.6). The results of this are in Table 5. The stiffness modulus resulting from this normalisation turns out to be  $E = 2.23947 \text{ GN/m}^2$  and is absorbed into the tensor  $\underline{D}$  in (1.5). We assume a compressible isotropic homogeneous medium, and so this tensor, in two space dimensions, has a matrix representation as

$$\underline{\underline{D}} = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \quad \text{with} \quad \underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} \quad \text{and} \quad \underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix}$$

for Lamé parameters  $\lambda = \nu E/((1 + \nu)(1 - 2\nu))$  and  $\mu = \frac{1}{2}E/(1 + \nu)$  (see e.g. [6, 9–11, 16]). We assume from, for example, [2] a Poisson's ratio of  $\nu = 0.35$  and a mass density of  $\rho = 1190 \text{ kg/m}^3$ .

We take  $\Omega = (0, 2) \times (0, 1)$  with zero initial data and solve over (0, T) = (0, 0.3). There are homogeneous boundary conditions of Dirichlet type on the left side and homogeneous Neumann conditions on the top and

q	$\tau_q$ (s)	dyne/cm <sup>2</sup>	$oldsymbol{arphi}_q$
0		$2.24 \times 10^{7}$	0.001000236663139
1	0.02	$1.94  imes 10^{9}$	0.086627639575435
2	0.20	$2.83 \times 10^{9}$	0.126369185566228
3	2	$5.54 imes10^9$	0.247379960437068
4	20	$6.02 \times 10^{9}$	0.268813603218619
5	200	$3.88 \times 10^{9}$	0.173255279150871
6	2000	1.56 × 10 <sup>9</sup>	0.069659339040041
7	20000	$4.10 \times 10^{8}$	0.018307903209241
8	200000	$1.38 \times 10^{8}$	0.006162172299696
9	2000000	$3.68 \times 10^{7}$	0.001643245946586
10	2000000	$7.90  imes 10^{6}$	0.000352762037446
11	20000000	$9.60  imes 10^{6}$	0.000428672855631

**Table 5:** Prony series data for PMMA taken from [18, Table 1]. The second and third columns are from the source, one tenth of the sum of the third column (because 10 dyne/cm<sup>2</sup> = 1 N/m<sup>2</sup>) gives 2.23947 GN/m<sup>2</sup>, and the fourth column is the normalisation of one tenth of the third column (to ensure that  $\varphi(0) = 1$  in (1.6)).

bottom edges. The body force **f** in (1.1) is zero. On the right-hand edge of the domain, we impose the traction

$$\boldsymbol{g}(\boldsymbol{x},t) = \begin{pmatrix} g_1(\boldsymbol{x},t) \\ 0 \end{pmatrix}$$
 N/m<sup>2</sup>, where  $g_1(\boldsymbol{x},t) = A\mathcal{H}(0.01)$ 

for  $A = 50 \text{ MN/m}^2$  and where  $\mathcal{H}$  denotes the Heaviside step. The finite element space is piecewise quadratic with an  $N_x \times N_y$  uniform spatial mesh of right-angled triangles (diagonals from north-west to south-east), and we use  $N_t$  time steps of size  $\Delta T = T/N_t$ . We took  $\alpha = 10$  and  $\beta = 1$ , and scaled the entire discrete system by dividing through by  $10^9$ : this was needed to prevent runtime precision errors being thrown by the linear solvers.

We give two different solutions to compare elastic and viscoelastic short time response to the traction. The elastic solution is generated by setting  $\varphi_q = 0$  for q = 1, ..., 11, and setting  $\varphi_0 = 1$ . The results for the  $u_1$  displacement are shown as surface plots in Figure 2 for t = 0.06 s and t = 0.3 s. The similarity at the shorter time as well as the viscoelastic decay at the longer time is evident. To see this more clearly, the pointwise displacement  $u_1(2, 0.5)$  is shown in Figure 3.

The code that produced these results can be obtained from https://github.com/variationalform/SIPG\_ PMMA\_demo\_2022 with

git clone git@github.com:variationalform/SIPG\_PMMA\_demo\_2022.git

Or one can generate the results in a Docker container from https://hub.docker.com/r/variationalform/fem with

```
docker pull variationalform/fem:SIPG_PMMA_demo_2022
docker run -ti variationalform/fem:SIPG_PMMA_demo_2022
# in the container
cd runtime/le
./longrun_le.sh | tee ./longrun_le_out.txt
```

for the linear elastic runs. Alter 1e to ve to run with the viscoelastic data. This Docker image uses FEniCS with DOLFIN version 2017.2.0.

# 7 Conclusion and Discussion

We have presented an a priori stability and error analysis of the DGFEM for linear dynamic viscoelasticity with two types of internal variables used to replace the fading memory Volterra integral. We have given illustrative numerical results, explained how these results can be reproduced, and we have observed that both of the schemes (for each type of internal variable) behave similarly. The numerics are consistent with the predicted convergence rates.



**Figure 2:** Snapshots at t = 0.06 s and t = 0.3 s of the discrete solution  $u_1$  for the linear elasticity problem on the left and the viscoelastic problem on the right. Here,  $N_x \times N_y = 120 \times 60$  with  $N_t = 24000$ .



**Figure 3:** Plots of the discrete solution  $u_1(2, 0.5)$  for the linear elasticity problem on the left and the viscoelastic problem on the right. Here,  $N_x \times N_y = 120 \times 60$  with  $N_t = 24000$ .

The main achievement of this work is that we have been able to give the stability and error bounds without recourse to Grönwall's inequality, with the result that the stability and error bounds depend explicitly on the final time linearly, and not exponentially (along with the time dependence in the exact solution norms). This is a significant improvement over the estimates given in [21]. Moreover, the schemes presented here use displacement or velocity internal variables rather than stresses, and so – compared to [21] – for the schemes presented here, there is a modest reduction in the computer memory requirements.

An inspection of Tables 1, 2, and 3 indicates that there is no particular reason to prefer one formulation over the other. This is perhaps not surprising, but we are not aware of any previous work that has demonstrated it both in theoretical estimates and in numerical experiments. We return to this point below. We assumed throughout that  $\varphi_0 > 0$  in (1.6), which is reasonable for materials with long term stiffness ("solids" as defined by Golden and Graham in [11]). However, for "fluids", under the specific condition  $\varphi_0 = 0$ , the velocity form will yield an interesting alternative model. In particular, with  $u_0 = 0$  for simplicity, we can rewrite (3.6) as

$$\rho \dot{\boldsymbol{w}} - \sum_{q=1}^{N_{\varphi}} \nabla \cdot \left( \underline{\boldsymbol{D}} \underline{\boldsymbol{\varepsilon}}(\boldsymbol{\zeta}_{q}) \right) = \boldsymbol{f},$$

where we have set  $\dot{u} = w$ . The internal variable still evolves according to the ODE (3.7), which holds at every point in space, and so, in the numerical scheme, there is no need to introduce displacement. This results in a reduced memory requirement because, in this case, we would certainly prefer the velocity formulation. We plan to explore this interesting variant in a future study.

Finally, the results in Figures 2 and 3 demonstrate that this method can be applied to real materials.

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