

Cyclic rewriting and conjugacy problems

Volker Diekert and Andrew J. Duncan and Alexei Myasnikov

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Abstract

Cyclic words are equivalence classes of cyclic permutations of ordinary words. When a group is given by a rewriting relation, a rewriting system on cyclic words is induced, which is used to construct algorithms to find minimal length elements of conjugacy classes in the group. These techniques are applied to the universal groups of Stallings pregroups and in particular to free products with amalgamation, HNN-extensions and virtually free groups, to yield simple and intuitive algorithms and proofs of conjugacy criteria.

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1 Introduction

Rewriting systems are used in the theory of groups and monoids to specify presentations together with conditions under which certain algorithmic problems may be solved. Typically, presentations given by convergent rewriting systems are sought as these give rise to algorithms for the word and geodesics problems. Recently, less stringent conditions on rewriting systems which still allow the word problem and/or the geodesics problem to be decided, have also been investigated: for example geodesic or geodesically perfect systems[10, 7]. In contrast to the classical case, geodesically perfect systems are confluent, but not necessarily convergent or finite, and are designed to seek geodesics in a group, rather than normal forms of elements. In any case, all these systems depend on rewriting of strings of letters, or words, from the free monoid on the generating set of a group or monoid.

In this paper we consider applications of rewriting systems to the conjugacy problem in groups. To this end we apply rewriting to cyclic words rather than ordinary words. Cyclic words can be viewed as sets of all cyclic permutations of standard words or, equivalently, as graphs, which are directed labelled cycles. This allows us to construct algorithms for finding representatives of minimal length in the conjugacy classes of elements in groups.

We describe analogues of Knuth-Bendix completion for rewriting systems on cyclic words and consider how to realise these procedures in particular situations. Our approach to the completion processes on cyclic words is rather different from the one developed by Chouraqui in [6], where cyclic rewriting systems are used to construct algorithmic solutions to the conjugacy and transposition problems in monoids, under suitable conditions. One significant difference is that we introduce certain new rewriting rules, which are specific to the cyclic rewriting. These rules are absolutely essential but are not “induced” by standard string rewriting. Furthermore, in [6] the rewriting systems considered are all finite, whereas here we allow infinite systems. As in the case of geodesically perfect string rewrite systems we do not require our systems to be convergent. This allows us to construct confluent cyclic rewriting systems which are particularly suitable for working with the

conjugacy problem in groups.

We apply these techniques to the conjugacy problem in universal groups of Stallings pregroups and fundamental groups of graphs of groups. As a warm-up we give short intuitive proofs of the conjugacy criteria of free products with amalgamation [21] and in HNN-extensions (Collin's Lemma) [20]. Moreover we are able to describe a linear time algorithm for the conjugacy problem in finitely generated virtually free groups. (Epstein and Holt [8] have constructed a linear time algorithm for the conjugacy problem in arbitrary hyperbolic groups. However, for this special case we give a very simple construction based on the underlying finite rewriting system.)

Canonical examples of pregroups and their universal groups arise from free products with amalgamation, HNN-extensions and, more generally, fundamental groups of graphs of groups. The conjugacy problem may behave badly with respect to these constructions: for example in [18] an HNN extension $G = \text{HNN}(H, t; t^{-1}at = b)$ is constructed, where the base group H has solvable conjugacy problem and the elements a and b of H are infinite cyclic, but G has unsolvable conjugacy problem. Several authors have studied conditions under which amalgams, HNN-extensions and graphs of groups do have solvable conjugacy problem, see for example [12, 13, 14, 19] and the references therein. Our results show that the obstruction to deciding the conjugacy problem in such groups arises only from the determination of conjugacy of elements of length one, with respect to the corresponding pregroup. Thus, if the conjugacy problem in the group is undecidable our systems do not provide a computable rewriting, but they do indicate where the difficulties are. This also gives a different view-point on the results of papers [4, 3, 5, 9] where efficient generic algorithms for the conjugacy problem in free products with amalgamation and HNN-extensions were constructed. These algorithms are fast correct partial algorithms that give the answer on most ("generic") inputs, and do not give an answer only on a negligible set of inputs.

The structure of the paper is as follows. In Section 2 we outline the transposition and conjugacy problems for monoids and groups and give a brief introduction to string rewriting systems. Section 2.4 contains the definitions of cyclic words, cyclic rewriting systems and the appropriate notions of geodesic and geodesically perfect cyclic systems, needed later in the paper. In Section 2.5 we consider how a semi-Thue system may be "completed" to give a larger, semi-Thue, system which is confluent on cyclic words. This is possible under a weak termination condition, but the price is that, in general, length increasing rules may be introduced. This leads in 2.7, 2.8 and 2.9 to consideration of analogues of Knuth-Bendix completion processes in which we add context sensitive rules, that rewrite transposed words directly to each

other; when they are of some globally bounded length.

In Section 3 we describe Stallings pregroups, their universal groups and the rewriting systems to which they are naturally associated. Section 4 contains the main results on conjugacy in the universal groups of pregroups, namely Theorem 4.4, Corollary 4.5 and Theorem 4.6. These results are applied to free products with amalgamation, HNN-extensions and virtually free groups in Sections 5 and 6.

2 Preliminaries

2.1 Transposition, conjugacy and involution

Let M be a monoid and $f, g \in M$. Then f and g are said to be *transpose*, if there exist elements $r, s \in M$ such that $f = rs$ and $g = sr$. We write $f \sim g$ to denote transposition. The elements f and g are called *conjugate*, if there exists an element $z \in M$ such that $fz = zg$.

In general these definitions describe different relations. Indeed, conjugacy is transitive, but not necessarily symmetric, while the transposition relation is reflexive and symmetric, but not in general transitive. All transpose elements are conjugate. If the monoid M is a group, then conjugacy is an equivalence relation and f and g are conjugate if and only if there exists an element $z \in M$ such that $f = zgz^{-1}$.

Throughout Γ denotes an alphabet, which simply means it is a set, which might be finite or infinite in this paper. An element $a \in \Gamma$ is called a *letter* and an element u in the free monoid Γ^* is called a *word*. A non-empty word can be written as $u = a_1 \cdots a_n$, where $a_i \in \Gamma$ and $n \geq 0$. The number n is then called the *length* of u and denoted $|u|$. The empty word has *length* 0 and is denoted 1, as is customary for the neutral element in monoids or groups.

A crucial, but elementary fact for free monoids is that transposition is equal to conjugacy. More precisely, in free monoids $fz = zg$ implies that $f = rs$, $g = sr$, and $z = r(sr)^m$ for some $m \geq 0$. Essentially this implies a straightforward algorithm for the conjugacy problem in free groups: on input elements f and g of a free group first do cyclic reductions, to cyclically reduce f and g . This costs only linear time. Then check whether the cyclically reduced words f and g are transpose by searching for the word f as a factor of the word g^2 . This is possible in linear time by a well-known pattern matching algorithm, usually attributed to Knuth-Morris-Pratt [17], although it was described earlier by Matiyasevich [22].

Frequently, sets and monoids come with an involution. An *involution* on a set X is a permutation $a \mapsto \bar{a}$ such that $\overline{\bar{a}} = a$. An involution of a monoid

satisfies in addition $\overline{xy} = \overline{y} \overline{x}$. If the monoid is a group G then we always assume that the involution is given by the inverse, thus $\overline{g} = g^{-1}$ for group elements. If the alphabet Γ has an involution, then it is extended to Γ^* by defining $\overline{a_1 \cdots a_n} = \overline{a_n} \cdots \overline{a_1}$ for $a_i \in \Gamma$ and $n \geq 0$. From now on we always assume that Γ is equipped with an involution $\bar{\cdot} : \Gamma \rightarrow \Gamma$. Since the identity id_Γ is an involution, this is no restriction.

2.2 Rewriting systems

Monoids and groups can be defined through a set of monoid generators Γ and a set of defining relations $S \subseteq \Gamma^* \times \Gamma^*$. A subset $S \subseteq \Gamma^* \times \Gamma^*$ is called a *semi-Thue system*, or a *string rewriting system*. Given S , we define a relation \Longrightarrow_S , called a *one-step rewriting relation*, on Γ^* by $u \Longrightarrow_S v$ if and only if $u = p\ell q$ and $v = prq$ for some $(\ell, r) \in S$.

Let X be any set and $\Longrightarrow \subseteq X \times X$ be a relation. The iteration of at most k steps of \Longrightarrow is denoted by $\Longrightarrow^{\leq k}$ while the reflexive and transitive closure of \Longrightarrow is denoted by \Longrightarrow^* . We also write $x \Leftarrow y$ and $x \Leftarrow^* y$ to denote $y \Longrightarrow x$ and $y \Longrightarrow^* x$, respectively. The reflexive, symmetric and transitive closure of \Longrightarrow is denoted by \Leftarrow^* . Elements $x \in X$ such that there is no y with $x \Longrightarrow y$ are called *irreducible*. The relation \Longrightarrow is called:

1. *strongly confluent*, if $y \Leftarrow x \Longrightarrow z$ implies $y \Longrightarrow^{\leq 1} w \Leftarrow^{\leq 1} z$ for some w ;
2. *confluent*, if $y \Leftarrow^* x \Longrightarrow^* z$ implies $y \Longrightarrow^* w \Leftarrow^* z$ for some w and
3. *Church-Rosser*, if $y \Leftarrow^* z$ implies $y \Longrightarrow^* w \Leftarrow^* z$ for some w .

The following facts are well-known and easy to prove, see e.g. [2, 15].

1. Strong confluence implies confluence.
2. Confluence is equivalent to Church-Rosser.

A relation $\Longrightarrow \subseteq X \times X$ is called *terminating* (or *Noetherian*), if there is no infinite chain

$$x_0 \Longrightarrow x_1 \Longrightarrow \cdots x_{i-1} \Longrightarrow x_i \Longrightarrow \cdots$$

For a semi-Thue system S the equivalence relation $\overset{*}{\longleftrightarrow}_S$ is a congruence, hence the equivalence classes form a monoid which is denoted by Γ^*/S . This is the quotient of the monoid Γ^* when S is viewed as a set of defining relations. We also say that S is confluent, terminating etc., whenever $\overset{*}{\implies}_S$ has the corresponding property.

The main interest in a terminating and confluent system S stems from the fact that these properties (together with some other natural condition on the computability of the one-step rewriting process) yield a procedure to solve the word problem in the quotient monoid Γ^*/S . If Γ is finite, then decidability of the word problem is equivalent to the ability to compute shortlex normal forms: first we endow the alphabet Γ with a linear order \leq . The *shortlex normal form* for an element g in a quotient monoid Γ^*/S is then the lexicographically first word among all geodesic words $u \in \Gamma^*$ representing $g \in M$. Recall, that a word $u \in \Gamma^*$ is called a *geodesic*, if u has minimal length among all words representing the same element as u in Γ^*/S .

Example 2.1. *If the involution $\bar{}$ on Γ is without fixed points, then we can write Γ as a disjoint union $\Gamma = \Sigma \dot{\cup} \bar{\Sigma}$. Then the rewriting system $S = \{ a\bar{a} \rightarrow 1 \mid a \in \Gamma \}$ is strongly confluent and terminating; and the quotient monoid Γ^*/S defines the free group $F(\Sigma)$. In this case geodesics are unique.*

2.3 Thue systems

A semi-Thue system S is called a *Thue system*, if S does not contain any length increasing rules and all length preserving rules are symmetric. This means $(\ell, r) \in S$ implies $|\ell| \geq |r|$ and that $|\ell| = |r|$ implies $(r, \ell) \in S$, too. The set S of a Thue system splits naturally into two parts $S = R \dot{\cup} T$, where R contains the length reducing rules and T contains the symmetric length preserving rules. In particular, $R \cap R^{-1} = \emptyset$ and $T = T^{-1}$, where, as usual, $P^{-1} = \{ (y, x) \mid (x, y) \in P \}$ for any relation P .

A Thue system S is called *geodesic*, if starting from any word u and applying only length decreasing rules we eventually obtain a *geodesic* word v (a shortest word in the set $\{v \mid u \overset{*}{\underset{S}{\longrightarrow}} v\}$). Thus, we have $u \overset{*}{\underset{R}{\longrightarrow}} v$ for some geodesic word v .

A confluent, geodesic, Thue system is called *geodesically perfect*. This means whenever $u \overset{*}{\underset{S}{\longleftrightarrow}} v$, then we can first compute geodesics $u \overset{*}{\underset{R}{\longrightarrow}} \widehat{u}$ and $v \overset{*}{\underset{R}{\longrightarrow}} \widehat{v}$, by applying length reducing rules, and then we can transform \widehat{u} into \widehat{v} by symmetric rules from T , that is $\widehat{u} \overset{*}{\underset{T}{\longleftrightarrow}} \widehat{v}$ (which in turn is equivalent

to $\widehat{u} \xrightarrow[T]{*} \widehat{v}$). Thus the following statements are equivalent for geodesically perfect systems.

1. $u \xleftrightarrow[S]{*} v$.
2. $\exists \widehat{u}, \widehat{v}: u \xrightarrow[R]{*} \widehat{u} \xrightarrow[T]{*} \widehat{v} \xleftarrow[R]{*} v$.

2.4 Cyclic words and cyclic rewriting

There are two principal ways of introducing cyclic words over an alphabet Γ . The first one is based on combinatorics of words: in this case one defines a *cyclic word* as an equivalence class of the transposition relation on Γ^* . Thus, if $w \in \Gamma^*$ then the cyclic word represented by w is the set $w_\sim = \{vu \in \Gamma^* \mid uv = w\}$. The second one, defines the cyclic word represented by w to be the directed, Γ -labelled, cycle graph C_w , such that the label of the cycle, when read with orientation, starting at an appropriate vertex, is w . More precisely, if $w = a_1 \dots a_n, n > 0$, then C_w is a directed graph with vertices v_1, \dots, v_n and directed edges $e_1 = (v_1 \rightarrow v_2), \dots, e_{n-1} = (v_{n-1} \rightarrow v_n), e_n = (v_n \rightarrow v_1)$ where each edge e_i is labelled by a_i . In the graph-theoretic version, an ordinary word $w \in \Gamma^*$ can be viewed as a directed Γ -labelled path-graph P_w : with vertices v_1, \dots, v_{n+1} and edges $e_1 = (v_1 \rightarrow v_2), \dots, e_n = (v_n \rightarrow v_{n+1})$ with labels a_1, \dots, a_n , respectively. If w is the empty word 1 then P_w and C_w consist of a single vertex. We regard the combinatorial and graph theoretic views of words and cyclic words as different aspects of the same objects and pass from one to the other without further comment.

Graph rewriting (or transformation) is a well-established technique of computing with graphs. We refer to the book [25] for details. In general, a graph rewriting system consists of a set of graph rewriting rules of the form (L, R) , where L and R are graphs. To apply such a rule to a given graph G one finds a subgraph of G isomorphic to L and replaces it by R according to some prescribed procedure.

In our case the graphs G are cycles C_w , where $w \in \Gamma^*$ and the rewriting rules are of the following two types:

- 1) (P_ℓ, P_r) for some $\ell, r \in \Gamma^*$;
- 2) (C_ℓ, C_r) for some $\ell, r \in \Gamma^*, \ell \neq 1$.

Application of a rule (P_ℓ, P_r) to a graph C_w involves replacing some path subgraph P_ℓ of C_w by the path P_r . This can be clearly visualised as in Figure 1. Application of the rule (C_ℓ, C_r) to C_w is straightforward: if C_w

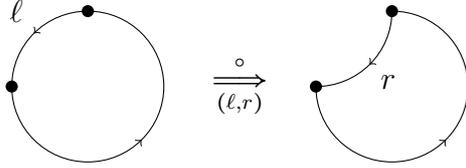


Figure 1: Cyclic rewriting when $(\ell, r) \in S$ and ℓ appears on the cycle.

is isomorphic to C_ℓ (as a directed, labelled graph) then replace C_w by C_ℓ . Otherwise the rule does not apply. Clearly, the result of applying one of these rules to cyclic word is a cyclic word. A *rewriting system on cyclic words* is a set T of rules of the type 1) and 2). We write $C_u \xRightarrow{T} C_v$ if C_v can be obtained from C_u by one of the rules from T . In this case we may also write $u_\sim \xRightarrow{T} v_\sim$ or $u \xRightarrow{T} v$. The definitions of Section 2.2 apply to an arbitrary binary relation on a set X , and in particular to the relation \xRightarrow{T} on the set of cyclic words over Γ^* . Hence, we can talk about confluent, strongly confluent, terminating, etc. rewriting systems on cyclic words.

The subsystem of T consisting of the rules of type 1) corresponds to a string rewriting semi-Thue system $S = \{(\ell, r) \mid (P_\ell, P_r) \in T\}$. On the other hand, let $S \subseteq \Gamma^* \times \Gamma^*$ be a semi-Thue system. Then S composed on the right and left with the relation \sim defines a one-step relation \xRightarrow{S} on cyclic words. That is, we have $u_\sim \xRightarrow{S} v_\sim$, if and only if there are words u' and v' such that $u \sim u'$, $u' \xRightarrow{S} v'$, and $v' \sim v$. Obviously, if a rule $(\ell, r) \in S$ is applied to u_\sim , then the rewriting step $u_\sim \xRightarrow{(\ell, r)} v_\sim$ may be understood as applying the rule (P_ℓ, P_r) to the graph P_u , as in Figure 1.

By analogy with string rewriting, we denote by $\xRightarrow{S^\otimes}$ the reflexive and transitive closure of \xRightarrow{S} ; write $u \xleftarrow{S} v$ and $u \xleftarrow{S^\otimes} v$ for $v \xRightarrow{S} u$ and $v \xleftarrow{S^\otimes} u$, respectively; and denote by $\xleftrightarrow{S^\otimes}$ the reflexive, symmetric, transitive closure of \xRightarrow{S} .

Neither confluence nor termination transfers from S (defined on words) to \xRightarrow{S} (defined on cyclic words).

Example 2.2. 1. Let $\Gamma = \{a, b, c, d\}$ and let S consist of the following four rules

$$abc \longrightarrow bac, \quad cda \longrightarrow dca, \quad bad \longrightarrow abd, \quad dc b \longrightarrow cbd.$$

To see that $\xrightarrow[S]{\circ}$ is confluent it is necessary to check all four cases where the left-hand sides of rules overlap. For example the left-hand side of $abc \rightarrow bac$ overlaps with the left-hand side of $cda \rightarrow dca$. Therefore we can rewrite $abcd_a$ in two ways:

$$bacda \xleftarrow[S]{} abcd_a \xrightarrow[S]{} abdca.$$

However

$$bacda \xrightarrow[S]{} badca \xrightarrow[S]{} abdca,$$

so either way results in the same reduced word. The other three cases are similar and so $\xrightarrow[S]{\circ}$ is confluent. However, the cyclic rewriting system defined by S is not confluent. In fact $abcd_{\sim} \xrightarrow[S]{\circ} bacd_{\sim}$ and $abcd_{\sim} = bcda_{\sim} \xrightarrow[S]{\circ} bdca_{\sim}$. Both $bacd_{\sim}$ and $bdca_{\sim}$ are irreducible and they are not equal.

2. Let $\Gamma = \{a, b\}$ and $S = \{ba \rightarrow ab^2\}$. It is not difficult to see that $\xrightarrow[S]{\circ}$ is terminating. However the relation $\xrightarrow[S]{\circ}$ on cyclic words is non-terminating as

$$ba_{\sim} \xrightarrow[S]{\circ} b^2a_{\sim} \xrightarrow[S]{\circ} b^3a_{\sim} \xrightarrow[S]{\circ} b^4a_{\sim} \xrightarrow[S]{\circ} \dots$$

A semi-Thue system S is called *C-confluent*, if $\xrightarrow[S]{\circ}$ is confluent on cyclic words. If W is subset of cyclic words, then we also say that S is *C-confluent on W* , if $\xrightarrow[S]{\circ}$ is confluent on all cyclic words in W .

In the rest of the section we consider some general methods of transforming confluent semi-Thue systems into C-confluent systems.

2.5 From confluence to cyclic confluence

Let $S \subseteq \Gamma^* \times \Gamma^*$ be a *confluent* semi-Thue system such that $G = \Gamma^*/S$ is a *group*. In this section we consider the general question (in the spirit of a Knuth-Bendix or Shirshov-Gröbner completion) of how to enlarge the system S by adding new rules in order to obtain another system \widehat{S} such that the following hold:

1. $S \subseteq \widehat{S}$ and $\Gamma^*/S = \Gamma^*/\widehat{S}$ (i.e., \widehat{S} is a *conservative extension* of S);
2. $\xrightarrow[\widehat{S}]{\circ}$ is confluent on cyclic words (i.e., \widehat{S} is C-confluent).

Usually, we refer to \widehat{S} satisfying 1 as an *extension* of S (omitting conservative). \widehat{S} satisfying 1 and 2 is termed a *C-extension* of S . Condition 1 ensures that \widehat{S} is still confluent (since S , and hence \widehat{S} , is Church-Rosser).

Now we fix a confluent semi-Thue system $S \subseteq \Gamma^* \times \Gamma^*$ such that $G = \Gamma^*/S$ is a group. For each letter $a \in \Gamma$ we can choose some fixed word $\tilde{a} \in \Gamma^*$ such that $a\tilde{a} = 1$ in G . We extend this definition (in a unique way) to all words of Γ^* as follows. Define $\tilde{1} = 1$ and assume that \tilde{u} has been defined for all words u of length at most n . Let $u = va$ be a word of length $n+1$, with $a \in \Gamma, v \in \Gamma^*$. Then define $\tilde{u} = \tilde{a}\tilde{v}$. Clearly, $\tilde{w} = w$ in G for all words $w \in \Gamma^*$.

For $x, y \in \Gamma^*$ write $x > y$, if $x \xrightarrow[S]{*} pyq$ with $pq \neq 1$. Then $>$ is a partial order on Γ^* . Since $x \xrightarrow[S]{*} x = x1$ (here 1 is the empty word) then $x > 1$ for every non-empty word x . We call the system S *weakly-terminating* if the partial order $>$ is well-founded, i.e., there are no infinite chains $x_1 > x_2 > x_3 > \dots$. Clearly, if the system S is terminating, then it is weakly-terminating. Moreover, every semi-Thue system without length increasing rules is weakly-terminating. Note that the empty word 1 is irreducible in every weakly-terminating system. In particular, such a system does not have rules of the type $1 \rightarrow x\tilde{x}$ or $1 \rightarrow \tilde{x}x$, but $x\tilde{x} \xrightarrow[S]{*} 1$ and $\tilde{x}x \xrightarrow[S]{*} 1$ for any $x \in \Gamma^*$, since S is Church-Rosser and 1 is irreducible.

For a system S define a semi-Thue system \widehat{S} by the following rules $u \xrightarrow[\widehat{S}]{} u'$ where:

1. $u \xrightarrow[S]{} u'$ (*original rule*).
2. $u = qv$ and $u' = \tilde{p}rv$, if exists $pq \rightarrow r \in S, p \neq 1 \neq q$ (*prefix rule*).
3. $u = vp$ and $u' = vr\tilde{q}$, if exists $pq \rightarrow r \in S, p \neq 1 \neq q$ (*suffix rule*).
4. $u' = \tilde{p}r\tilde{q}$, if exists $puq \rightarrow r \in S, p \neq 1 \neq q$ (*infix rule*).

It is clear that \widehat{S} satisfies $S \subseteq \widehat{S}$ and $\Gamma^*/S = \Gamma^*/\widehat{S}$. As before $\xrightarrow[\widehat{S}]{\circledast}$ denotes the reflexive and transitive closure of $\xrightarrow[\widehat{S}]{\circ}$.

Theorem 2.3. *Let $S \subseteq \Gamma^* \times \Gamma^*$ be a confluent weakly-terminating semi-Thue system such that $G = \Gamma^*/S$ is a group. Then the following hold:*

- 1) u and v are conjugate in G if and only if $u \xrightarrow[\widehat{S}]{\circledast} t \xleftarrow[\widehat{S}]{\circledast} v$ for some (cyclic) word t .
- 2) the rewrite system $\xrightarrow[\widehat{S}]{\circ}$ is confluent on (cyclic) words.

Proof. To prove 1) observe first (by inspection of all the rules in \widehat{S}) that

$$u \xrightarrow[\widehat{S}]{\otimes} t \xleftarrow[\widehat{S}]{\otimes} v \quad (1)$$

(in fact, even $u \xrightarrow[\widehat{S}]{\otimes} v$) for some word $t \in \Gamma^*$ implies that u and v are conjugate in G .

Assume now that $u, v \in \Gamma^*$ define conjugate elements in G , i.e., $xu\tilde{x} \xrightarrow[S]{*} v$ for some $x \in \Gamma^*$. We claim that in this case there exists $t \in \Gamma^*$ for which (1) holds. We proceed by Noetherian induction on x , i.e., by induction on the number of predecessors of x relative to $>$.

Since S is Church-Rosser there exists $w \in \Gamma^*$ such that $xu\tilde{x} \xrightarrow[S]{*} w \xleftarrow[S]{*} v$. If x has no predecessors then $x = 1$ and the claim is obvious (in this case $t = w$). Thus, we may assume that the claim holds for all $y < x$.

In the reduction $xu\tilde{x} \xrightarrow[S]{*} w$ the following cases may occur.

Case 1 (no overlap). Suppose one can factorise $w = x'u'x''$ in such a way that $x \xrightarrow[S]{*} x'$, $u \xrightarrow[S]{*} u'$, and $\tilde{x} \xrightarrow[S]{*} x''$, then we are done since $x''x' = 1$ in G , so $x''x' \xrightarrow[S]{*} 1$ (1 is S -irreducible), and hence:

$$u \xrightarrow[S]{*} u' \xleftarrow[S]{*} u'x''x' \sim x'u'x'' = w,$$

which proves the claim. Thus we may assume that there is no such factorisation.

Case 2 (overlap). Assume now that $x \xrightarrow[S]{*} yp$, $u \xrightarrow[S]{*} qv$ such that $p \neq 1, q \neq 1$ and $pq \rightarrow r$ is a rule of S . Then one has $xu\tilde{x} = y(rv\tilde{p})\tilde{y} = v$ in G and $y < x$. Hence, by induction, $rv\tilde{p} \xrightarrow[\widehat{S}]{\otimes} t \xleftarrow[\widehat{S}]{\otimes} v$ for some word $t \in \Gamma^*$. Notice, that we can apply a prefix rule to qv and after a transposition obtain $u \xrightarrow[\widehat{S}]{\otimes} rv\tilde{p}$. Therefore, $u \xrightarrow[\widehat{S}]{\otimes} t \xleftarrow[\widehat{S}]{\otimes} v$ and the claim holds.

The argument for the other possible overlap, when $\tilde{x} \xrightarrow[S]{*} qy$ and $u \xrightarrow[S]{*} vp$, is similar and we omit it.

Case 3 (nesting). We are left to consider the following situation: $x \xrightarrow[S]{*} yp$, $u \xrightarrow[S]{*} s$, and $\tilde{x} \xrightarrow[S]{*} qz$ where $p \neq 1 \neq q$ and $psq \rightarrow r$ is a rule of S . Again $xu\tilde{x} = y(r\tilde{q}\tilde{p})\tilde{y}$ in G and $y < x$. Hence, by induction, $r\tilde{q}\tilde{p} \xrightarrow[\widehat{S}]{\otimes} t \xleftarrow[\widehat{S}]{\otimes} v$, for some word t . Applying an infix rule to s and a transposition yields $u \xrightarrow[\widehat{S}]{\otimes} r\tilde{q}\tilde{p}$. The claim follows.

This finishes the proof of 1). Statement 2) follows from 1) since, as mentioned above, $u \xleftrightarrow[\widehat{S}]{\otimes} v$ implies that u and v are conjugate in G . \square

Now we show that an extension S° (defined below) of S which is, in this context, extremely natural is also a C -extension of S . Define S° to be the extension of S obtained by adding the rules $1 \rightarrow a\tilde{a}$ and $1 \rightarrow \tilde{a}a$, for every $a \in \Gamma$. Thus

$$S^\circ = S \cup \{1 \rightarrow a\tilde{a}, 1 \rightarrow \tilde{a}a \mid a \in \Gamma\}$$

and, since $G = \Gamma^*/S$ is a group, S° is indeed a C -conservative extension of S .

Theorem 2.4. *Let $S \subseteq \Gamma^* \times \Gamma^*$ be a confluent weakly-terminating semi-Thue system such that $G = \Gamma^*/S$ is a group. Then the following hold:*

- 1) u and v are conjugate in G if and only if $u \xrightarrow[S^\circ]{\otimes} t \xleftarrow[S^\circ]{\otimes} v$ for some (cyclic) word t .
- 2) S° is a C -extension of S .

Proof. If $u \xrightarrow[S^\circ]{\otimes} t \xleftarrow[S^\circ]{\otimes} v$ for some word t then u and v are obviously conjugate in G . Conversely, if u and v are conjugate in G , then by Theorem 2.3 $u \xrightarrow[\widehat{S}]{\otimes} t \xleftarrow[\widehat{S}]{\otimes} v$ for some word t . Observe, that application of a prefix, suffix, or infix rule from \widehat{S} is equivalent to a sequence of rule applications from S° , so $u \xrightarrow[\widehat{S}]{\otimes} t \xleftarrow[\widehat{S}]{\otimes} v$ implies $u \xrightarrow[S^\circ]{\otimes} t \xleftarrow[S^\circ]{\otimes} v$. Now the result follows. \square

2.6 From strong to cyclic confluence in groups

The transformation of a semi-Thue system S into the larger system S° described in Section 2.5 leads to length increasing rules. This is in some sense unavoidable. Indeed, assume we have $ab = c$ and $ba = d$ in the quotient $M = \Gamma^*/S$ where $c, d \in \Gamma$ are letters. In general we cannot expect that $c = d \in M$. But c and d are transpose, so we need cyclic rewriting rules to pass from c to d or vice versa. If we wish to do this by string rewriting and transpositions, then we are forced to pass from c to d via cyclic words of length at least 2. This is what happens in building \widehat{S} and S° .

Another idea is to introduce special rules which directly rewrite short cyclic words into each other, if they represent distinct conjugate elements. In this case we have rules that rewrite cyclic words, but these rules are not induced by any string rewriting rules in the system (via equivalence relation \sim). We now make this precise.

We start with a semi-Thue system $S \subseteq \Gamma^* \times \Gamma^*$, which we allow to be infinite. Define

$$m(S) = \sup \{ |\ell| \mid (\ell, r) \in S \}.$$

We say that S is *left-bounded* if $m(S) < \infty$. From now on we assume that the empty word is S -irreducible and, to exclude trivial cases, that $2 \leq m(S)$. To this end, we say that S is a *standard* semi-Thue system, if it satisfies the two conditions above: that is

1. $(1, r) \notin S$, for all non-empty words $r \in \Gamma^*$, and
2. $2 \leq m(S) < \infty$.

A cyclic word w_\sim is called *S -short* if $|w| \leq 2m(S) - 2$, and it is called *strictly S -short* if $|w| < 2m(S) - 2$. When S is fixed we refer to such words simply as short or strictly short.

In the following let $C(S)$ denote any relation defined on the set of cyclic words which satisfies the following two conditions:

C1 If $u_\sim \xrightarrow[S]{\circ} v_\sim$, then $(u_\sim, v_\sim) \in C(S)$, i.e., $\xrightarrow[S]{\circ} \subseteq C(S)$.

C2 If $(u_\sim, v_\sim) \in C(S)$, then u and v are conjugate in Γ^*/S .

Later, we discuss the possibility of constructing relations $C(S)$ with these properties. We write $u \xrightarrow[C(S)]{\circ} v$ and $v \xleftarrow[C(S)]{\circ} u$ if $(u_\sim, v_\sim) \in C(S)$ or if $u_\sim = v_\sim$. (Thus, both $\xrightarrow[C(S)]{\circ}$ and $\xleftarrow[C(S)]{\circ}$ are reflexive.) Moreover, we use $\xrightarrow[C(S)]{\circledast}$ and $\xleftarrow[C(S)]{\circledast}$ again, for the transitive, and for the symmetric and transitive closure, respectively, of $\xrightarrow[C(S)]{\circ}$. As $C(S)$ is a relation on cyclic words, when we say $C(S)$ is confluent, or strongly confluent, unless we explicitly specify an alternative, we mean confluent or strongly confluent on the set of all cyclic words.

Theorem 2.5. *Let $S \subseteq \Gamma^* \times \Gamma^*$ be a standard strongly confluent semi-Thue system such that $C(S)$ satisfies the two conditions C1 and C2 above. Then the following assertions are equivalent:*

- 1.) *The system $C(S)$ is confluent.*
- 2.) *The system $C(S)$ is confluent on all short cyclic words w . (That is if w is short and $u \xleftarrow[C(S)]{\circledast} w \xrightarrow[C(S)]{\circledast} v$ then there exists t such that $u \xrightarrow[C(S)]{\circledast} t \xleftarrow[C(S)]{\circledast} v$.)*

Proof. We have to show only that if $C(S)$ is confluent on all short cyclic words, then $C(S)$ is confluent.

First consider $u \xrightarrow[C(S)]{\circ} w \xrightarrow[C(S)]{\circ} v$ where $|w| \geq 2m(S) - 1$. Then the two rules applied to the cyclic word w_{\sim} are inherited from the semi-Thue system S . Since w is long enough the corresponding left-hand sides overlap in the cyclic word w at most once. Since S is strongly confluent, we see that there is some cyclic word t such that

$$u \xrightarrow[C(S)]{\circ} t \xleftarrow[C(S)]{\circ} v.$$

Next, consider

$$u = w_k \xleftarrow[C(S)]{\circ} \cdots \xleftarrow[C(S)]{\circ} w_0 \xrightarrow[C(S)]{\circ} v_1 \xrightarrow[C(S)]{\circ} \cdots \xrightarrow[C(S)]{\circ} v_m = v.$$

We may assume that $m \geq k \geq 1$. We perform an induction on (k, m) in the lexicographical order.

If none of w_0, \dots, w_{k-1} is short, then by strong confluence of S we have the following situation.

$$u \xrightarrow[C(S)]{\circ} w'_k \xleftarrow[C(S)]{\circ} \cdots \xleftarrow[C(S)]{\circ} w'_1 \xleftarrow[C(S)]{\circ} v_1 \xrightarrow[C(S)]{\circ} v.$$

Thus, we are done by induction on m . Therefore let w_ℓ be a short cyclic word where $\ell \leq k - 1$. By induction on k we see that there exists

$$w_\ell \xrightarrow[C(S)]{\circ} t \xleftarrow[C(S)]{\circ} v.$$

Moreover, $u \xleftarrow[C(S)]{\circ} w_\ell$ and $C(S)$ is confluent on w_ℓ because w_ℓ is a short cyclic word. Hence we find

$$u \xrightarrow[C(S)]{\circ} t' \xleftarrow[C(S)]{\circ} t \xleftarrow[C(S)]{\circ} v.$$

□

Corollary 2.6. *Let $S \subseteq \Gamma^* \times \Gamma^*$ be a standard strongly confluent semi-Thue system such that Γ^*/S is a group and such that first, $C(S)$ is confluent on all short cyclic words and second, it satisfies the two conditions C1 and C2 above. Then two words u and v are conjugate in Γ^*/S if and only if there exists a cyclic word t such that*

$$u_{\sim} \xrightarrow[C(S)]{\circ} t \xleftarrow[C(S)]{\circ} v_{\sim}.$$

Proof. Clearly, $u_{\sim} \xleftrightarrow[C(S)]{\circledast} v_{\sim}$ implies conjugacy. Now, if u and v are conjugate, then there is some x such that $xux^{-1} \xleftrightarrow[S]{*} v$. This implies $xux^{-1} \xleftrightarrow[C(S)]{\circledast} v_{\sim}$. We have $xx^{-1} \xleftrightarrow[S]{*} 1$, because S is standard and confluent, hence $xux^{-1} \xleftrightarrow[C(S)]{\circledast} u_{\sim}$. We conclude $u_{\sim} \xleftrightarrow[C(S)]{\circledast} v_{\sim}$. The result follows by Theorem 2.5. \square

2.7 A Knuth-Bendix-like procedure on cyclic words

If a system $C(S)$, satisfying C1 and C2 above, is large enough to ensure $u_{\sim} \xleftrightarrow[C(S)]{\circledast} v_{\sim}$ whenever u_{\sim} and v_{\sim} are conjugate in Γ^*/S with u short, then we can apply Theorem 2.5; and we can use the system $C(S)$ for solving conjugacy in Γ^*/S . In order to construct such a system we may use a form of Knuth-Bendix completion. This can be done in a very general way; which is fairly standard but technical, if we work out all details. Here we wish to restrict an analogue of Knuth-Bendix completion to short words; for which we need some additional hypotheses.

We assume throughout this section that the alphabet Γ is well-ordered by $<$. We extend this well-order to the *shortlex* order $<$ on Γ^* as usual: we write $u < v$ if either $|u| < |v|$ or $|u| = |v|$ and $u = pax$, $v = pby$ with $a, b \in \Gamma$ such that $a < b$. Moreover, we extend the well-order to cyclic words by representing a cyclic word w_{\sim} by the minimal shortlex word in its class $w_{\sim} = \{uv \mid vu = w\}$. Hence, there is well-order on the set of cyclic words. For any relation $R \subseteq \Gamma^* \times \Gamma^*$ we define the *descending part* of R to be

$$\tilde{R} = \{(l, r) \in R : l > r \text{ in the shortlex ordering}\}.$$

The new restriction we put on S is that we assume that, for all $(\ell, r) \in S$, we have $|r| \leq |\ell|$. In particular, if w is short and $w \xleftrightarrow[S]{\circledast} v$, then v is short, too. Now let $C(S)$ satisfy C1 and C2 above. We say that $(u_{\sim}, v_{\sim}) \in C(S)$ is a *short critical pair*, if $u_{\sim} > v_{\sim}$ (in the shortlex ordering) and for some S -short word w we have:

$$u_{\sim} \xleftrightarrow[C(S)]{\circ} w_{\sim} \xrightarrow[C(S)]{\circ} v_{\sim} \quad (2)$$

We say that the critical pair (u_{\sim}, v_{\sim}) in (2) is *shortlex resolved*, if

$$u_{\sim} \xrightarrow[\tilde{C}(S)]{\circledast} t_{\sim} \xleftarrow[\tilde{C}(S)]{\circledast} v_{\sim},$$

for some t with $v \geq t$ (where $\tilde{C}(S)$ is the descending part of $C(S)$). By *resolving* (u_{\sim}, v_{\sim}) we mean adding the rule (u_{\sim}, v_{\sim}) to $C(S)$. (Note that, by definition, $u_{\sim} > v_{\sim}$.) Hence by resolving we force (u_{\sim}, v_{\sim}) to be resolved.

If we begin by taking $C(S)$ equal to $\xrightarrow[S]{\circ}$ then, by resolving short pairs, we may form new systems which still satisfy C1 and C2. If the alphabet Γ is finite, then this procedure of adding more rules terminates because there are only finitely many short words. In general, there exists a limit system $C^*(S)$, satisfying C1 and C2 and such that all short critical pairs are shortlex resolved, but if Γ is infinite then we may only have a semi-procedure for its construction.

Theorem 2.7. *Let $S \subseteq \Gamma^* \times \Gamma^*$ be a standard strongly confluent semi-Thue system such that Γ^*/S is a group and such that, for all $(\ell, r) \in S$, we have $|r| \leq |\ell|$. Let $C^*(S)$ be constructed as above by resolving short critical pairs. Then the following two assertions hold:*

1. *The system $C^*(S)$ is standard and confluent.*
2. *Two words u and v are conjugate in Γ^*/S if and only if there exists a cyclic word t_{\sim} such that*

$$u_{\sim} \xrightarrow[C^*(S)]{\otimes} t_{\sim} \xleftarrow[C^*(S)]{\otimes} v_{\sim}.$$

Proof. By construction $C^*(S)$ is standard. Having shortlex resolved all short critical pairs, the descending part $\tilde{C} = \tilde{C}^*(S)$ of $C^*(S)$ is terminating and contains all new rules (u_{\sim}, v_{\sim}) added to the system $\xrightarrow[S]{\circ}$. Therefore \tilde{C} is locally confluent on short words. Moreover, if $u_{\sim} \xrightarrow[S]{\circ} v_{\sim}$ then, since $<$ is a total order and $|l| \geq |r|$ for all $(l, r) \in S$, either (u_{\sim}, v_{\sim}) or (v_{\sim}, u_{\sim}) belongs to \tilde{C} . Hence $u_{\sim} \xleftrightarrow[C^*(S)]{\otimes} v_{\sim}$ if and only if $u_{\sim} \xleftrightarrow[\tilde{C}]{\otimes} v_{\sim}$. (Note that we don't claim that \tilde{C} satisfies C1 or C2. We don't even have $S \subseteq \tilde{C}$, in general.)

We are now ready to show that $C^*(S)$ is confluent on short words. Consider the following situation where w is short:

$$u_{\sim} \xleftarrow[C^*(S)]{\otimes} w_{\sim} \xrightarrow[C^*(S)]{\otimes} v_{\sim}.$$

Since $|r| \leq |\ell|$ for $(\ell, r) \in S$ (and hence for all $(\ell, r) \in C^*(S)$) we see that u and v are short, and moreover $u_{\sim} \xleftrightarrow[\tilde{C}]{\otimes} v_{\sim}$. Note that the path $u_{\sim} \xleftrightarrow[\tilde{C}]{\otimes} u'_{\sim} \xleftrightarrow[\tilde{C}]{\otimes} \dots \xleftrightarrow[\tilde{C}]{\otimes} v_{\sim}$ (via w_{\sim}) never leaves the set of short words. Being terminating and locally confluent, the system \tilde{C} is confluent on short words. Hence, since $u_{\sim} \xleftrightarrow[\tilde{C}]{\otimes} v_{\sim}$, there exists a cyclic word t_{\sim} such that

$$u_{\sim} \xrightarrow[\tilde{C}]{\otimes} t_{\sim} \xleftarrow[\tilde{C}]{\otimes} v_{\sim}.$$

As $\tilde{C} \subseteq C^*(S)$, we see that $C^*(S)$ is confluent on short words, as claimed.

Finally, $C^*(S)$ satisfies conditions C1 and C2 above; and so Corollary 2.6 applies, to give the result. \square

2.8 Strongly confluent Thue systems

If the system S is Thue (c.f. Section 2.3) then we may construct $C^*(S)$ in finitely many steps as follows. We start with $C_0 = C_0(S) = \overset{\circ}{\underset{S}{\rightrightarrows}}$. This is a relation defined on the set of cyclic words where all rules are either length decreasing or length preserving and then symmetric. We call any such relation on cyclic words *Thue*.

At each step let us define a Thue relation C_i satisfying conditions C1 and C2 above. We let U_i be the set of “unresolved short critical pairs” (u_\sim, v_\sim) , which are defined in the Thue case as follows:

$$u_\sim \overset{\circ}{\underset{C_i}{\longleftarrow}} w_\sim \overset{\circledast}{\underset{C_i}{\longrightarrow}} w'_\sim \overset{\circ}{\underset{C_i}{\longrightarrow}} v_\sim$$

where w is S -short, $|w| = |w'| \geq |u| \geq |v| \geq 1$, and neither $u_\sim \overset{\circledast}{\underset{C_i}{\longrightarrow}} v_\sim$ nor $u_\sim \overset{\circledast}{\underset{C_i}{\longleftarrow}} v_\sim$.

Note that, since $|w| = |w'|$ we have $u_\sim \overset{\circ}{\underset{C_i}{\longleftarrow}} w_\sim \overset{\circledast}{\underset{C_i}{\longleftarrow}} w'_\sim \overset{\circ}{\underset{C_i}{\longrightarrow}} v_\sim$, too. Thus, for unresolved pairs we must have $|w| > |u| \geq |v| \geq 1$. (Because if, say $|w| = |v|$, then $u_\sim \overset{\circledast}{\underset{C_i}{\longleftarrow}} w'_\sim \overset{\circ}{\underset{C_i}{\longleftarrow}} v_\sim$.)

At the next step we let C_{i+1} be the relation obtained from C_i by adding a pair (u_\sim, v_\sim) to C_i , for all $(u_\sim, v_\sim) \in U_i$, and, in addition, by adding (v_\sim, u_\sim) whenever $|u_\sim| = |v_\sim|$. This keeps C_{i+1} Thue. Finally, we let

$$C^*(S) = \bigcup \{ C_i(S) \mid i \in \mathbb{N} \}. \quad (3)$$

Theorem 2.8. *Let S be a standard, strongly confluent, Thue system, let $m = m(S)$ and let $C^*(S)$ be the system defined in (3) above. Then $C^*(S) = C_{2m-2}$, and $C^*(S)$ is a confluent, Thue system, satisfying conditions C1 and C2.*

Proof. By definition C_i are Thue for all $i \geq 0$; and short words have length at most $2m - 2$. When considering $u_\sim \overset{\circ}{\underset{C_i}{\longleftarrow}} w_\sim \overset{\circledast}{\underset{C_i}{\longrightarrow}} w'_\sim \overset{\circ}{\underset{C_i}{\longrightarrow}} v_\sim$ we may assume that $|u| < |w|$ (see above) and that $u_\sim \overset{\circ}{\underset{C_i \setminus C_{i-1}}{\longleftarrow}} w_\sim$ (or $w'_\sim \overset{\circ}{\underset{C_i \setminus C_{i-1}}{\longrightarrow}} v_\sim$). Thus, at every step the words w under consideration get shorter. We conclude $C^*(S) = C_{2m-2}(S)$, as claimed.

Next, we show that $C^*(S)$ is confluent on short cyclic words. To this end we define an equivalence relation \equiv on cyclic words by $u \equiv v$ if $u \xrightarrow[C^*(S)]{\circledast} v$ and $u \xleftarrow[C^*(S)]{\circledast} v$. Thus, if $u \equiv v$ then $u \xleftrightarrow[C^*(S)]{\circledast} v$ and $|u| = |v|$. We can view $C^*(S)$ as a terminating rewriting system on equivalence classes $[u] = \{v \mid v \equiv u\}$. By construction, $C^*(S)$ is locally confluent on classes $[w]$, where w is short. But together with termination, we see that $C^*(S)$ is actually confluent on these classes $[w]$. But this implies that $C^*(S)$ is confluent on short cyclic words, because it is Thue. Finally, $C^*(S)$ satisfies the two conditions C1 and C2 above. Since S is also a standard, strongly confluent semi-Thue-system, we may apply Theorem 2.5. \square

2.9 Cyclic geodesically perfect systems

In this section we consider an analogue for cyclic rewriting systems of geodesically perfect string rewriting systems; and adapt our Knuth-Bendix completion process to these systems. Let $S \subseteq \Gamma^* \times \Gamma^*$ be a standard semi-Thue system such that Γ^*/S is a group. A cyclic word w_\sim is called *geodesic* (w.r.t. S), if w is a shortest word in its conjugacy class. That is

$$|w| = \min \{ |u| \mid u \in \Gamma^* \text{ and } \exists x : xux^{-1} = w \in \Gamma^*/S \}.$$

A cyclic word w_\sim is called *quasi-geodesic* (w.r.t. S), if it is either geodesic or it is strictly S -short, but it is not equal to the neutral element in Γ^*/S . Note that all non-trivial geodesic cyclic words are quasi-geodesic and more importantly in 2-monadic systems every quasi-geodesic cyclic word is actually geodesic.

Now, a Thue relation $C(S)$ on cyclic words, satisfying C1 and C2 above, is called *quasi-geodesic*, if by applying a sequence of length reducing rules from $C(S)$ to a cyclic word w_\sim we eventually derive a quasi-geodesic cyclic word u_\sim . In order to be geodesically perfect $C(S)$ must satisfy stronger conditions: $C(S)$ is called *geodesically perfect* if, by applying a sequence of length reducing rules from $C(S)$ to a cyclic word w_\sim , we eventually derive a geodesic cyclic word u_\sim . Moreover, if two geodesics u_\sim and v_\sim can both be derived from w_\sim , then it must be possible to rewrite u_\sim into v_\sim using only length preserving rules from $C(S)$. Note that every geodesically perfect Thue system on cyclic words is confluent.

Now, if $S \subseteq \Gamma^* \times \Gamma^*$ is a Thue system then we say that S is *C-quasi-geodesic* if the system $\xrightarrow[S]{\circ}$, on cyclic words, is quasi-geodesic. The following result shows that a geodesic Thue system is innately C-quasi-geodesic.

Theorem 2.9. *Let $S \subseteq \Gamma^* \times \Gamma^*$ be a standard, geodesic, Thue system. Then S is C -quasi-geodesic.*

Proof. We have to show the following: if $u_\sim \overset{\circ}{\leftarrow}_S w_\sim$ and $|u| < |w|$, then either a length reducing rule applies to the cyclic word w_\sim or w_\sim is strictly S -short. To begin with let $u_\sim \overset{\circ}{\leftarrow}_S w_\sim$. Then there is a sequence $u = w_0, \dots, w_\ell = w$ such that w_{i-1} and w_i are related in one of the following three ways:

$$w_{i-1} \xrightarrow{S} w_i \quad \text{or} \quad w_{i-1} \xleftarrow{S} w_i \quad \text{or} \quad w_{i-1} \sim w_i.$$

First, we claim that there exist $m \in \mathbb{N}$ and $u_1, u_2 \in \Gamma^*$ such that $u_1 u^{k-m} u_2 \overset{*}{\leftarrow}_S w^k$, for all $k > m$.

This is true for $\ell = 0$ with $m = 0$. For $\ell \geq 1$ the result holds by induction for $v = w_1, \dots, w_\ell = w$ with some $m \in \mathbb{N}$ and $v_1, v_2 \in \Gamma^*$. Now, if $u \xrightarrow{S} v$, then we have $v_1 u^{k-m} v_2 \xrightarrow{S} v_1 v^{k-m} v_2 \overset{*}{\leftarrow}_S w^k$, for all $k > m$. Similarly, if $u \xleftarrow{S} v$, then we have $v_1 u^{k-m} v_2 \overset{*}{\leftarrow}_S v_1 v^{k-m} v_2 \overset{*}{\leftarrow}_S w^k$, for all $k > m$. Now, let $u = u_2 u_1$ and $v = u_1 u_2$. Define $m' = m + 1$. We have:

$$v_1 u_1 u^{k-m-1} u_2 v_2 \overset{*}{\leftarrow}_S v_1 v^{k-m} v_2 \overset{*}{\leftarrow}_S w^k, \quad \text{for all } k > m.$$

Replacing m , u_1 and u_2 with m' , $v_1 u_1$, and $u_2 v_2$, respectively, we see that the claim holds.

Next, assume that we have $|u| < |w|$ and choose m , u_1 and u_2 as above. Take k large enough to make $|w^k| > |u_1 u^{k-m} u_2|$. Since $u_1 u^{k-m} u_2 \overset{*}{\leftarrow}_S w^k$ and S is geodesic, a length reducing rule $(\ell, r) \in S$ applies to w^k . If $|\ell| \leq |w|$, then the same rule applies to the cyclic word w_\sim , and we are done. In the other case, w is strictly S -short, and we are done, too. \square

In the next section of the paper we shall be concerned with standard, geodesically perfect, Thue string rewriting systems S , which are 2-monadic: that is $m(S) = 2$. For the rewriting system $\overset{\circ}{\Rightarrow}_S$ induced by such S , there is a particularly simple form of Knuth-Bendix completion. In this case we consider an short critical pair (u_\sim, v_\sim) to be “unresolved” if it arises from the situation

$$u_\sim \overset{\circ}{\leftarrow}_S w_\sim \overset{\circ}{\rightarrow}_S v_\sim, \tag{4}$$

where w is short and $|w| > |u| \geq |v| \geq 1$. We *resolve* the short critical pair of (4) by adding the rules (u_\sim, v_\sim) and (v_\sim, u_\sim) . Let $C^\dagger(S)$ be the system obtained from $\overset{\circ}{\Rightarrow}_S$ by resolving all short critical pairs of the form (4). Note

that if (u_\sim, v_\sim) is a short critical pair then both u and v are strictly short and non-trivial so, S being 2-monadic, we have $|u| = |v| = 1$.

Corollary 2.10. *Let $S \subseteq \Gamma^* \times \Gamma^*$ be a standard, 2-monadic, geodesically perfect, Thue system, such that Γ^*/S is a group, and $C^\dagger(S)$ is confluent. Then $C^\dagger(S)$ satisfies C1 and C2 and is geodesically perfect. Moreover two cyclic words u_\sim and v_\sim are conjugate in Γ^*/S if and only if there exists a cyclic word t_\sim such that*

$$u_\sim \xrightarrow[C^\dagger(S)]{\otimes} t_\sim \xleftarrow[C^\dagger(S)]{\otimes} .$$

Proof. By construction $C^\dagger = C^\dagger(S)$ satisfies C1 and C2. Two elements $u, v \in \Gamma^*$ are conjugate if and only if $u_\sim \xleftrightarrow[C^\dagger]{\otimes} v_\sim$; so the final statement holds if C^\dagger is confluent. Therefore it is sufficient to prove that C^\dagger is geodesically perfect.

Consider $w \xrightarrow[C^\dagger]{\otimes} v$ such that v has minimal length with this property (so is geodesic) and let $w \xrightarrow[C^\dagger]{\otimes} u$ be some maximal derivation using only length reducing rules from the cyclic rewriting system C^\dagger . Clearly, $|u| \geq |v|$; and Theorem 2.9 implies that S is C-quasi-geodesic so either $|u| = |v|$ or u is strictly S -short. We have to show that we can transform u into v by length preserving rules from C^\dagger . This is clear, if v is not strictly S -short, because then $|u| = |v|$, and C^\dagger is confluent and Thue. For $m(S) = 2$, a strictly S -short word v is either a letter or the empty word 1. But if $v = 1$ we have $w \xrightarrow[S]{*} v$ because S is a confluent semi-Thue system and 1 is irreducible; and it follows from the definitions of $\xrightarrow[C^\dagger]{\otimes}$ and u that $u = 1$ as well. There remains the case $v \in \Gamma$. Since S is C-quasi-geodesic we have $|u| = 1$, too. As C^\dagger is confluent and Thue we can transform the letter u into v , by applying length preserving rules of C^\dagger . \square

3 Stallings' pregroups and their universal groups

We now turn to the notion of pregroup in the sense of Stallings, [27], [28]. A *pregroup* is a set P with a distinguished element ε , equipped with a partial multiplication $(a, b) \mapsto ab$ which is defined for $(a, b) \in D$, where $D \subseteq P \times P$, and an involution $a \mapsto \bar{a}$, satisfying the following axioms, for all $a, b, c, d \in P$. (By “ ab is defined” we mean that $(a, b) \in D$.)

(P1) $a\varepsilon$ and εa are defined and $a\varepsilon = \varepsilon a = a$;

(P2) $\bar{\bar{a}}a$ and $a\bar{\bar{a}}$ are defined and $\bar{\bar{a}}a = a\bar{\bar{a}} = \varepsilon$;

(P3) if ab is defined, then so is $\overline{b\bar{a}}$, and $\overline{ab} = \overline{b\bar{a}}$;

(P4) if ab and bc are defined, then $(ab)c$ is defined if and only if $a(bc)$ is defined, in which case

$$(ab)c = a(bc);$$

(P5) if ab, bc , and cd are all defined then either abc or bcd is defined.

It is shown in [11] that (P3) follows from (P1), (P2), and (P4), hence can be omitted.

For $a, b \in P$ we write $ab \in P$, to mean that ab is defined. Also we use the notation $[ab]$ to indicate that $ab \in P$ and, under the partial multiplication, $(a, b) \mapsto [ab]$. This notation is extended recursively to products of more than two elements of P : if $w \in P^*$, where the notation has been established for words shorter than w , and w has a factorisation $w = uv$, such that $u, v \in P$ and $[u][v]$ is defined, we write $w \in P$ and use $[w]$ to denote the product $[u][v] \in P$. (Note though that, for example, $[abc]$ means only that one of $[ab]c$ or $a[bc]$ belongs to P . (cf. Lemma 3.2.))

The set P can be considered as a possibly infinite alphabet. The axioms above lead to the following definitions of Thue systems S_ε , $S(P)$ and the universal group $U(P)$.

Definition 3.1. 1. The system $S_\varepsilon \subseteq P^* \times P^*$ is defined by the following rules:

$$\begin{aligned} \varepsilon &\longrightarrow 1 && (= \text{the empty word}) \\ ab &\longrightarrow [ab] && \text{if } (a, b) \in D \\ ab &\longleftrightarrow [ac][\bar{c}b] && \text{if } (a, c), (\bar{c}, b) \in D \end{aligned}$$

2. Let $\Gamma = P \setminus \{\varepsilon\}$. The system $S(P) \subseteq \Gamma^* \times \Gamma^*$ is defined as follows:

$$\begin{aligned} ab &\longrightarrow 1 && \text{if } (a, b) \in D \text{ and } [ab] = \varepsilon. \\ ab &\longrightarrow [ab] && \text{if } (a, b) \in D \text{ and } [ab] \neq \varepsilon. \\ ab &\longleftrightarrow [ac][\bar{b}] && \text{if } (a, c), (\bar{c}, b) \in D, \text{ and } (a, b) \notin D. \end{aligned}$$

We say that $S(P)$ is the Thue system associated with P .

3. The universal group $U(P)$ of a pregroup P is the group

$$U(P) = \Gamma^* / \{ \ell = r \mid (\ell, r) \in S(P) \}.$$

Tietze transformations may be applied to the presentation P^*/S_ε to give the presentation $\Gamma^*/S(P)$; so $U(P) \cong P^*/S_\varepsilon$.

A *reduced word* is an element $p_1 \cdots p_n$ of P^* such that all $p_i \in \Gamma$ and $[p_i p_{i+1}] \notin P$, for i from 1 to $n - 1$.

The relationships between a pregroup, these rewriting systems and the universal group rest on several key lemmas, the most important of which we restate here for completeness.

Lemma 3.2 ([27]). *Let $a, b, c, d, g, h \in P$.*

- 1.) *If $ab \in P$ then $[ab]\bar{b} \in P$ and $[ab\bar{b}] = a$.*
- 2.) *If $ab \notin P$ but ac and $\bar{c}b \in P$ then $[ac][\bar{c}b] \notin P$.*
- 3.) *If abc is a reduced word and $a\bar{d}, db \in P$ then $[a\bar{d}][db]c$ is a reduced word.*
- 4.) *If $ab \notin P$ but $ac, \bar{c}b, bd \in P$ then $\bar{c}bd \in P$. (That is $[\bar{c}b]d \in P$ from which it follows that $[\bar{c}[bd]] = [[\bar{c}b]d]$.)*
- 5.) *If $gb, \bar{b}h, gbc, \bar{c}bh \in P$, but $gh \notin P$ then $bc \in P$.*

Proof. 1.) Apply (P4) to the triple a, b, \bar{b} .

2.) Use 1.) and apply (P4) to the triple $[ac], \bar{c}$ and b .

3.) From the above $[a\bar{d}][db]$ is reduced and $\bar{d}db \in P$. If $dbc \in P$ then consider the four element product $a\bar{d}[db]c$. From (P5), either $ab \in P$ or $bc \in P$, a contradiction.

4.) Consider the four elements $[ac], \bar{c}, b$ and d , of P . The product of each adjacent pair is defined, so (P5) implies either $ab = [ac][\bar{c}b] \in P$, or $\bar{c}bd \in P$.

5.) Consider the product of four elements $\bar{g}[gb]c[\bar{c}bh]$. By hypothesis we have $gbc, \bar{b}h \in P$. Moreover, $[gb]c[\bar{c}bh] = gh \notin P$. Hence, by (P5) we conclude $\bar{g}[gb]c = [bc] \in P$.

□

As a consequence of Lemma 3.2.3.) and 4.) the set of reduced words coincides with the set of $S(P)$ -geodesic and the set of S_ε -geodesic words.

The length preserving rule \longleftrightarrow of S_ε is the length 2 case of Stallings' *interleaving* relation \approx defined on words in P^* as follows. If $a_i, c_i \in P$, for $i = 1, \dots, n$, and $\bar{c}_{i-1}a_i, a_i c_i$ and $\bar{c}_{i-1}a_i c_i \in P$ with $c_0 = c_n = \varepsilon$, then

$$a_1 \cdots a_n \approx b_1 \cdots b_n,$$

where $b_i = [\bar{c}_{i-1}a_i c_i]$. Stallings used Lemma 3.2 to show that interleaving is an equivalence relation on reduced words and this equivalence relation is central to the proof of Theorem 3.4.1) in [27]. Another approach is taken in [7], based on the following lemma, which is again proved using Lemma 3.2.

Lemma 3.3. *The Thue system S_ε is strongly confluent.*

Parts 1) and 2) of the following theorem are from Stallings [27]. Part 3) is from [7].

Theorem 3.4 ([27],[7]). *Let P be a pregroup. Then the following hold.*

- 1) P embeds into $U(P)$.
- 2) If g and h are reduced words P^* then $g =_{U(P)} h$ if and only if h is an interleaving of g .
- 3) $S(P)$ is a geodesically perfect Thue system.

Proof. 1) is a direct consequence of Lemma 3.3 and the remark following the proof of Proposition 3.6. 2) follows from 3) and Lemma 3.2. The proof of 3) is given in [7]: however, for completeness we give a proof. Consider a word $u = a_1 \cdots a_n$ with $a_i \in \Gamma$ such that $a_i a_{i+1}$ is not defined in P for $1 \leq i < n$. Assume that after a sequence of applications of symmetric rules, we can apply a length reducing one. We have to show that some length reducing rule applies to u . We may assume that the sequence of applications of symmetric rules is not empty, but as short as possible. The corresponding word contains a factor $abcd$ with $a, b, c, d \in \Gamma$ and neither ab , bc nor cd defined in P . Applying the last symmetric rule yields $a[b\bar{x}][xc]d$. The length reducing rule cannot then apply to $[b\bar{x}][xc]$, since this is not defined, by Lemma 3.2.2.), and so must apply to $a[b\bar{x}]$ or $[xc]d$. In both cases we have a contradiction to Lemma 3.2.3.). \square

Remark 3.5. *Every group G is the universal pregroup of some pregroup P . Indeed, $G = U(G)$. Moreover, Theorem 3.4 tells us that every pregroup P can be defined as a subset $P \subseteq G$ inside a group G such that $1 \in P$, $a \in P$ implies $a^{-1} \in P$, and P satisfies the axiom (P5). Having such a subset the domain D becomes $D = \{ (a, b) \in P \times P \mid ab \in P \}$.*

3.1 Amalgamated products and HNN-extensions

The guiding example of an universal group in the sense of Stallings is the amalgamated product $G = A *_H B$ of two groups over a common subgroup $H = A \cap B$. In this case $P = A \cup B$ forms a pregroup with $U(P) = G$. In this case, for $a, b \in P$, the product ab is defined in P if and only if $a, b \in A$ or $a, b \in B$. The verification of (P5) is straightforward.

The other obvious example of an universal group is the case where $G = \text{HNN}(H, t; t^{-1}At = B)$ is an HNN-extension over two isomorphic subgroups

A, B in some base group H . (That is there is an isomorphism $\varphi : A \rightarrow B$ and “ $t^{-1}At = B$ ” denotes the set of relations of the form $t^{-1}at = a\varphi$, for all $a \in A$.) In this case we can choose $P = H \cup Ht^{-1}H \cup HtH$. Again, the verification of (P5) is straightforward.

3.2 Fundamental groups of graph of groups

The notion of the fundamental group of a graph of groups generalises amalgamated product and HNN-extension to a much broader class. The concept of a *graph of groups* is due to Serre and the development of Bass-Serre theory has been a major achievement in modern group theory. We refer to the books [26], [1], and to [24] for the background.

A *virtually free group* is a group G having a free subgroup of finite index. They are related to graphs of groups as follows.

Proposition 3.6. *Let G be a finitely generated group. The following conditions are equivalent.*

1. G is the fundamental group of a finite connected graph of groups where all vertex groups are finite.
2. G is the universal group of some finite pregroup.
3. G can be presented by some finite geodesic system.
4. G is virtually free.

Proposition 3.6 is taken from [7, Cor. 8.7] and combines several results from the literature. It follows from [24], [7], [16], and [23].

4 Conjugacy in universal groups

We shall apply Theorem 2.5 and Corollary 2.10 to the universal group of a pregroup and in particular to the conjugacy problem. For this we fix a pregroup P , we let $U(P)$ be its universal group; and denote by S_ε and $S = S(P)$ the Thue systems of Definition 3.1. Let $C^\dagger(S_\varepsilon)$ and $C^\dagger(S)$ be the cyclic rewriting systems defined by resolving short critical pairs in the sense of Section 2.9.

A *cyclically reduced word* is a cyclic word over Γ^* which is geodesic with respect to the rewriting system $C^\dagger(S)$. We also refer to words $w \in w_\sim$ as cyclically reduced if w_\sim is cyclically reduced. In particular all elements of Γ are cyclically reduced.

Lemma 4.1. *Let $g \in P^*$ be a cyclically reduced word and let $h \in P^*$ be a word such that h_{\sim} is obtained from g_{\sim} by applying a sequence of length preserving rules of $C^\dagger(S_\varepsilon)$. Then h_{\sim} is cyclically reduced and $|h| = |g|$.*

Proof. By induction it is enough to prove the case where h_{\sim} is obtained from g_{\sim} by applying a single rule. If $g \in P$ then $g \neq \varepsilon$, as ε is not cyclically reduced, so $g \in \Gamma$, and the result follows.

If $|g| = n \geq 2$ then there exists a word $g_1 \cdots g_n \in g_{\sim}$ and an element $c \in P$ such that $g_i g_{i+1} \notin P$ for all i (subscripts modulo n), $g_1 c \in P$, $\bar{c} g_2 \in P$ and $f = [g_1 c][\bar{c} g_2] \cdots g_n \in h_{\sim}$. As $g_1 \cdots g_n g_1 \cdots g_n$ is reduced, it follows from Lemma 3.2, 2.) & 3.) that f^2 is reduced. Therefore f , and so also h , is cyclically reduced, as required. \square

This lemma suggests that cyclically reduced cyclic words under cyclic rewriting should play the role of reduced words under standard rewriting. This works as expected, with the exception of the behaviour of words of length 1. From Theorem 3.4, two elements of Γ are equivalent under S only if they are equal in Γ . However this is not true of cyclic words of length 1 and the system $C^\dagger(S)$, and we often have to treat words of length one separately in what follows.

Let $u = a_1 \cdots a_n \in \Gamma^*$ with $a_i \in \Gamma$. A *cyclic permutation* of u is any element of u_{\sim} . Thus, a cyclic permutation is the same as a transposition in Γ^* . Let $n \geq 2$. If for $i = 1, \dots, n$, there are elements $b_i, c_i \in P$ such that $\bar{c}_{i-1} a_i, a_i c_i$ are in P , and $b_i = [\bar{c}_{i-1} a_i c_i]$ (subscripts modulo n), then any element of v_{\sim} , where $v = b_1 \cdots b_n$ is called a *cyclic interleaving* of u ; and u_{\sim} is also called a cyclic interleaving of v_{\sim} . A *preconjugation* of u by $c \in P$ (when $n \geq 2$) is the cyclic interleaving $v = [\bar{c} a_1] a_2 \cdots a_{n-1} [a_n c]$.

For $u \in \Gamma$ (i.e., $n = 1$) a *cyclic interleaving* of u by $c \in P$ is defined as $v = [c u \bar{c}]$ in case that $c u \bar{c} \in P$ is defined. A *preconjugation* is defined to be a cyclic interleaving in this case.

In all cases every cyclic interleaving of u may be obtained by a cyclic permutation, followed by an interleaving, followed by a preconjugation. Moreover every cyclic interleaving of u is conjugate to u in $U(P)$. The following lemma describes more precisely how these definitions are related.

Lemma 4.2. *Let g and h be cyclically reduced words over Γ^* . If $|g| = 1$ then h is a cyclic interleaving of g if and only if h_{\sim} is obtained from g_{\sim} by applying a length preserving rule from $C^\dagger(S)$. If $|g| \geq 2$, then the following are equivalent.*

- 1.) h_{\sim} is obtained from g_{\sim} by the application of a finite sequence of length preserving rules from $C^\dagger(S)$.

2.) There exists a word f , obtained from g by a cyclic permutation followed by a single preconjugation, such that $h =_{U(P)} f$.

3.) h is a cyclic interleaving of g .

Proof. First consider the case $n = 1$. Then h is a cyclic interleaving of g if only if there exists $b \in P$ such that either $\bar{b}g$ or $gb \in P$ and $[\bar{b}gb] = h \in P$. On the other hand, there is a symmetric rule in $C^\dagger(S)$ transforming g_\sim to h_\sim if and only if there exists $b \in P$ such that either $\bar{b}g \in P$ and $[\bar{b}g]b \xrightarrow[S]{} h$ (in which case $b[\bar{b}g] \xrightarrow[S]{} g$); or $gb \in P$ and $\bar{b}[gb] \xrightarrow[S]{} h$.

Now suppose $n \geq 2$. We show first that 3.) implies 1.). If 3.) holds then there exist $g_i, a_i \in P$ such that $\bar{a}_{i-1}g_i, g_i a_i$ and $\bar{a}_{i-1}g_i a_i \in P$ and h is a cyclic permutation of $h_1 \cdots h_n$, where $h_i = [\bar{a}_{i-1}g_i a_i]$. Therefore, we may successively apply symmetric rules of S to g_\sim to obtain $(h_1 \cdots h_n)_\sim = h_\sim$ as required.

Next we show that 1.) implies 2.). If 1.) holds then there exist words g_0, \dots, g_n in Γ^* such that $g_0 = g$, $h \in g_n \sim$ and $g_{i+1} \sim$ is obtained by applying a symmetric rule of $C^\dagger(S)$ to $g_i \sim$. If $n = 0$ then h is a cyclic permutation of g and there is nothing further to do. Assume then that $n > 0$. From Lemma 4.1 g_i is cyclically reduced for all i . By definition there exists a word $g_0 = a_1 \cdots a_n \in g_\sim$ and an element $c \in P$ such that $a_1 c \in P$, $\bar{c}a_2 \in P$ and $g_1 = b_1 \cdots b_n$, where $b_1 = [a_1 c]$, $b_2 = [\bar{c}a_2]$ and $b_i = a_i$, for $i \geq 3$. By induction, there exists a word f_1 , obtained from g_1 by a cyclic permutation followed by a single preconjugation, such that $h =_{U(P)} f_1$. There are several cases to consider, depending on which cyclic permutation of g_1 is taken. Assume f_1 is a preconjugation of a cyclic permutation $b_{i+1} \cdots b_i$ of g_1 , where $0 \leq i \leq n-1$. That is, there exists $d \in P$ such that $\bar{d}b_{i+1} \in P$, $b_i d \in P$ and $f_1 = c_{i+1} \cdots c_i$, where $c_{i+1} = [\bar{d}b_{i+1}]$, $c_i = [b_i d]$ and $c_j = b_j$, if $j \neq i, i+1$. Thus

$$f_1 = \begin{cases} c_1 c_2 = [\bar{d}[a_1 c]][[\bar{c}a_2]d] & \text{if } i = 0 \text{ and } n = 2 \\ c_1 c_2 \cdots c_n = [\bar{d}[a_1 c]][\bar{c}a_2] \cdots [a_n d] & \text{if } i = 0 \text{ and } n \geq 3 \\ c_2 c_3 \cdots c_n c_1 = [\bar{d}[\bar{c}a_2]]a_3 \cdots a_n [[a_1 c]d] & \text{if } i = 1 \\ c_3 \cdots c_n c_1 c_2 = [\bar{d}a_3] \cdots a_n [a_1 c][[\bar{c}a_2]d] & \text{if } i = 2 \\ c_{i+1} \cdots c_n c_1 c_2 \cdots c_i = [\bar{d}a_{i+1}] \cdots a_n [a_1 c][\bar{c}a_2] \cdots [a_i d] & \text{if } i \geq 3 \end{cases}$$

If $i \geq 3$ then $n \geq 3$ and, as $i+1 \leq n$, we have $c_1 c_2 =_{U(P)} a_1 a_2$ so

$$h =_{U(P)} f_1 =_{U(P)} [da_{i+1}] \cdots a_1 a_2 \cdots [a_i d],$$

a preconjugation of the cyclic permutation $a_{i+1} \cdots a_i$ of g .

If $i = 2$ then Lemma 3.2.4.) applied to the four elements $[a_1 c]$, $[\bar{c}a_2]$, \bar{c} and d , shows that $[c\bar{c}a_2 d] = [a_2 d] \in P$. Therefore $c_1 c_2 =_{U(P)} [a_1 c][\bar{c}a_2 d] =_{U(P)}$

$a_1[a_2d]$ and

$$f_1 =_{U(P)} [\bar{d}a_3] \cdots a_n a_1 [a_2d],$$

as required.

If $i = 1$ then from Lemma 3.2.5.) it follows that $cd \in P$ so

$$f_1 =_{U(P)} [(\bar{cd})a_2] a_3 \cdots a_n [a_1(cd)],$$

as required.

If $i = 0$ and $n \geq 3$ then the result follows, by symmetry, from the case $i = 2$, leaving the case $i = 0$ and $n = 2$. As g is cyclically reduced, $a_1 a_2 a_1 a_2$ is a reduced word and therefore so is $[a_1c][\bar{c}a_2][a_1c][\bar{c}a_2]$. Hence $[a_1c][\bar{c}a_2]$ is cyclically reduced. Applying Lemma 3.2.4.) to $[\bar{c}a_2]$, d , $[a_1c]$ and \bar{c} , gives $\bar{d}a_1 \in P$. Similarly $a_2d \in P$ and the result follows as before.

Finally, to show that 2.) implies 3.), suppose that $h =_{U(P)} f$, where

$$f = [\bar{b}g_1]g_2 \cdots g_{n-1}[g_n b],$$

and

$$g = g_{i+1} \cdots g_n g_1 \cdots g_i,$$

for some i . Then, from Lemma 4.1, f is cyclically reduced and hence, from Theorem 3.4, h is an interleaving of f . From Lemma 3.2, it follows that h is a cyclic interleaving of g . \square

Lemma 4.3. *The system $C^\dagger(S_\varepsilon)$ is confluent.*

Proof. The system S_ε is standard and it is strongly confluent by Lemma 3.3. Thus, by Theorem 2.5 it is enough to show that $C^\dagger(S_\varepsilon)$ is confluent on all short cyclic words. Thus we have to consider the situation:

$$d_\sim \xleftrightarrow[C^\dagger(S_\varepsilon)]{\otimes} w_\sim \xleftrightarrow[C^\dagger(S_\varepsilon)]{\otimes} e_\sim, \quad (5)$$

where w is short. We must show that

$$d_\sim \xleftrightarrow[C^\dagger(S_\varepsilon)]{\otimes} t_\sim \xleftrightarrow[C^\dagger(S_\varepsilon)]{\otimes} e_\sim,$$

for some t_\sim . As w_\sim is a short cyclic word we have $|w_\sim| \leq 2$. If $w = 1$ in $U(P)$, then $u \xrightarrow[S_\varepsilon]{*} 1$, for all $u \xleftrightarrow[C^\dagger(S_\varepsilon)]{\otimes} w$, and we may take $t_\sim = 1$. Thus, we may assume $1 \leq |w_\sim|$ and $w \neq 1 \in U(P)$. If $|w_\sim| = 1$ then w_\sim is cyclically reduced, since $w \neq \varepsilon$. Hence all rules involved in (5) are symmetric and we may take $t_\sim = w_\sim$. Thus, from now on in the proof we may assume $|w_\sim| = 2$. Since all

length preserving rules in $\xrightarrow[\circ]{C^\dagger(S_\varepsilon)}$ are symmetric, we are done if $d \notin \Gamma$ or $e \notin \Gamma$.

Thus, as suggested by the notation we have $d, e \in \Gamma$. Again, since length preserving rules are symmetric, we may assume that the situation is

$$d_\sim \xleftarrow[\circ]{C^\dagger(S_\varepsilon)} w_{0\sim} \xleftrightarrow[\circ]{C^\dagger(S_\varepsilon)} \cdots \xleftrightarrow[\circ]{C^\dagger(S_\varepsilon)} w_{k\sim} \xrightarrow[\circ]{C^\dagger(S_\varepsilon)} e_\sim,$$

where all w_i have length 2. As $w_{0\sim}$ is not cyclically reduced, Lemma 4.1 implies that no $w_{i\sim}$ is cyclically reduced. Hence, for all i there exists $e_i \in \Gamma$ such that $w_{i\sim} \xrightarrow[\circ]{C^\dagger(S_\varepsilon)} e_i \in \Gamma$. It therefore suffices to show that if

$$d_\sim \xleftarrow[\circ]{C^\dagger(S_\varepsilon)} u_\sim \xleftrightarrow[\circ]{C^\dagger(S_\varepsilon)} v_\sim \xrightarrow[\circ]{C^\dagger(S_\varepsilon)} e_\sim,$$

where $|u| = |v| = 2$ and $|d| = |e| = 1$, then

$$d_\sim \xleftrightarrow[\circ]{C_1} e_\sim,$$

where C_1 is the length preserving part of $C^\dagger(S_\varepsilon)$. We may assume that $u_\sim = (ab)_\sim$ with $a, b \in \Gamma$, $[ab] = d \in \Gamma$, and that there exists $c \in \Gamma$ such that either $v_\sim = ([ac][\bar{c}b])_\sim$ or $v_\sim = ([\bar{c}a][bc])_\sim$. If $v_\sim = ([ac][\bar{c}b])_\sim$ then

$$d_\sim \xleftarrow[\circ]{C^\dagger(S_\varepsilon)} v_\sim \xrightarrow[\circ]{C^\dagger(S_\varepsilon)} e_\sim,$$

so $d_\sim \xleftrightarrow[\circ]{C_1} e_\sim$ and we are done.

Assume then that $v_\sim = ([\bar{c}a][bc])_\sim$. If $ba \in P$ then

$$d_\sim \xleftarrow[\circ]{C^\dagger(S_\varepsilon)} u_\sim \xrightarrow[\circ]{C^\dagger(S_\varepsilon)} [ba]_\sim,$$

and so $d_\sim \xleftrightarrow[\circ]{C_1} [ba]_\sim$. Also

$$e_\sim \xleftarrow[\circ]{C^\dagger(S_\varepsilon)} v_\sim \xrightarrow[\circ]{C^\dagger(S_\varepsilon)} [ba]_\sim,$$

so $e_\sim \xleftrightarrow[\circ]{C_1} [ba]_\sim \xleftrightarrow[\circ]{C_1} d_\sim$, as required.

Therefore we may assume that $ba \notin P$. Applying (P5) to the elements \bar{c} , a , b and c we have $\bar{c}ab$ or $abc \in P$. Also, from Lemma 3.2.3.), $[bc][\bar{c}a] \notin P$. As v_\sim is not cyclically reduced it follows that $[\bar{c}a][bc] \in P$, so we have

$$d_\sim \xleftarrow[\circ]{C^\dagger(S_\varepsilon)} ([\bar{c}ab]c)_\sim \xrightarrow[\circ]{C^\dagger(S_\varepsilon)} [\bar{c}abc]_\sim \quad \text{or} \quad d_\sim \xleftarrow[\circ]{C^\dagger(S_\varepsilon)} (\bar{c}[abc])_\sim \xrightarrow[\circ]{C^\dagger(S_\varepsilon)} [\bar{c}abc]_\sim,$$

and in both cases

$$d_{\sim} \xleftrightarrow[C_1]{\circ} [\bar{c}abc]_{\sim}.$$

Moreover,

$$e_{\sim} \xleftrightarrow[C^{\dagger}(S_{\varepsilon})]{\circ} v_{\sim} \xrightarrow[C^{\dagger}(S_{\varepsilon})]{\circ} [\bar{c}abc]_{\sim},$$

so $d_{\sim} \xleftrightarrow[C_1]{\circ} [\bar{c}abc]_{\sim} \xleftrightarrow[C_1]{\circ} e$, as required. \square

Having established the confluence of $C^{\dagger}(S_{\varepsilon})$ we may get rid of the letter ε and the rule $\varepsilon \rightarrow 1$. That is: we switch back to the system $S = S(P)$.

Theorem 4.4. *Let $S \subseteq \Gamma^* \times \Gamma^*$ be the Thue system associated with P , c.f. Definition 3.1. Then $C^{\dagger}(S)$ is geodesically perfect.*

Proof. The system $S = S(P)$ is a standard 2-monadic Thue system. The confluence of $C^{\dagger}(S)$ follows from Lemma 4.3. By Theorem 3.4 the semi-Thue system S is geodesically perfect. The result follows by Corollary 2.10. \square

Corollary 4.5. *Cyclically reduced elements are minimal length representatives of their conjugacy class in $U(P)$. Let g and f be cyclically reduced elements of Γ^* such that g is conjugate to f in $U(P)$. Then the following hold.*

1. g and f have the same length.
2. If $g \notin P$, i.e., $|g| \geq 2$, then we can transform the cyclic word g_{\sim} into the cyclic word f_{\sim} by a sequence of at most $|g|$ length preserving rules from $C^{\dagger}(S)$.
3. If $g \in P$, i.e., $|g| = 1$, then we can transform g into f by a sequence of preconjugations.

Proof. Immediate by the confluence of $C^{\dagger}(S)$ and Lemma 4.2. \square

The following theorem is the main result in this section. It makes statement 2 of Corollary 4.5 much more precise.

Theorem 4.6. *Let g and f be a cyclically reduced elements of Γ^* such that g is conjugate to f in $U(P)$. Let $g = g_1 \cdots g_n$ with $g_i \in P$ and $n = |g| \geq 2$. Then, we may obtain f , as an element in $U(P)$, by a single cyclic permutation followed by a preconjugation. More precisely, we have*

$$f = [bg_i] \cdots g_n g_1 \cdots [g_{i-1} \bar{b}] \in U(P),$$

where $b \in P$ and $bg_i, g_{i-1} \bar{b} \in P$.

Proof. This follows directly from Corollary 4.5. \square

We may strengthen the statement of Theorem 4.6 for pregroups which satisfy certain extra conditions. First, in any pregroup P we can define a canonical subgroup by

$$G_P = \{ x \in P \mid (x, y), (y, x) \in D, \forall y \in P \}.$$

We say that P satisfies the extra axiom (P6) if the following is true.

$$(f, g) \notin D \wedge (f, \bar{b}) \in D \wedge (b, g) \in D \implies b \in G_P. \quad (\text{P6})$$

Axiom (P6) holds for the standard pregroups defining amalgamated products or HNN-extensions (as in Section 3.1), but it does not hold in general for the pregroup defining the fundamental group of a graph of groups, as given in [24].

Remark 4.7. *If P satisfies the axiom (P6) then the element $b \in P$ in the statement of Theorem 4.6 is necessarily in the canonical subgroup G_P .*

We say that P satisfies the extra axiom (P7) if the following is true.

$$(y, z) \in D \wedge (x, [yz]) \in D \wedge [yz] \notin G_P \wedge s \in \{x, \bar{x}\} \wedge t \in \{y, z\} \\ \implies \{(s, t), (t, s)\} \subset D. \quad (\text{P7})$$

First note that (P7) implies axiom (P6). To see this, suppose that b, f and g satisfy the hypotheses of (P6). Let $z = \bar{g}$, $y = [bg]$ and $x = [f\bar{b}]$. Then $(y, z) \in D$, $[yz] = b$ and $(x, [yz]) = (x, b) = ([f\bar{b}], b) \in D$. If $[yz] \notin G_P$ then (P7) gives $(x, y) \in D$ and this implies that $(f\bar{b}, bg) \in D$, from which we infer $(f, g) \in D$, a contradiction. Thus we must have $[yz] \in G_P$ and (P6) holds.

Axiom (P7) holds for the standard pregroup defining an amalgamated product, but not, in general, for the standard pregroup defining an HNN-extension (as in Section 3.1). In contrast the following axiom (P8) holds for the standard pregroup of an HNN-extension, but not for that of an amalgamated product.

$$(a, b) \in D \wedge [ab] = c \implies a \in G_P \vee b \in G_P \vee c \in G_P. \quad (\text{P8})$$

Again, axiom (P8) implies axiom (P6). Indeed, consider the condition $(b, g) \in D$. If we have $[bg] \in G_P$, then $(f, \bar{b}) \in D$ implies $(f, g) \in D$, contrary to the hypothesis of (P6). Given $(f, g) \notin D$, we can exclude $g \in G_P$. Thus, (P8) yields the implication of (P6).

In pregroups in which axiom (P7) holds, elements of P behave well with respect to preconjugation.

Lemma 4.8. *Let P be a pregroup satisfying axiom (P7), let $H = G_P$ be its canonical subgroup and let $a, b \in P$. If $c \in P$ is a preconjugate of both a and b then either $c \in H$ or b is a preconjugate of a .*

Proof. Let $c = [\bar{u}au] = [\bar{v}bv]$, for some $u, v \in P$. Then either $\bar{u}a \in P$ or $au \in P$. Assume $\bar{u}a \in P$. We have $b = [vc\bar{v}] \in P$, so either $vc \in P$ or $c\bar{v} \in P$. Assume $c = [[\bar{u}a]u] \notin H$. Then $vc \in P$ together with (P7) implies $u\bar{v} \in P$. Similarly $c\bar{v} \in P$ implies $u\bar{v} \in P$. By symmetry, if $au \in P$ and $c \notin H$ then again $u\bar{v} \in P$. Therefore, either $c \in H$ or $b = [(v\bar{u})a(u\bar{v})]$, a preconjugate of a . \square

In pregroups in which axiom (P8) holds, elements of $P \setminus G_P$ behave well with respect to preconjugation.

Lemma 4.9. *Let P be a pregroup satisfying axiom (P8) and $H = G_P$ its canonical subgroup. Let $a \in P \setminus H$ and $b \in P$.*

- 1.) *If b is a preconjugate of a then $b = [\bar{h}ah]$, for some element $h \in H$.*
- 2.) *If b is conjugate to a then b is a preconjugate of a .*

Proof. 1. If $b = [\bar{c}ac]$, where $c \in P$, then either $\bar{c}a \in P$ or $ac \in P$. By symmetry, assume $\bar{c}a \in P$. If $c \notin H$ then (P8) implies $[\bar{c}a] = h \in H$, so $c = a\bar{h}$. Thus $b = [\bar{c}ac] = [ha\bar{h}]$, as required.

2. From Corollary 4.5.3 there exist a sequence of elements $a = b_0, \dots, b_n = b$, of P , such that b_{i+1} is a preconjugation of b_i , for all i . As each preconjugation is by an element of H it follows that b is in fact a preconjugate of a .

\square

5 The conjugacy problem in amalgamated products and HNN-extensions

5.1 Conjugacy in amalgamated products

As in Section 3.1, the defining pregroup for the group $G = A *_H B$ can be chosen to be $P = A \cup B$; and the common subgroup H is then equal to the canonical subgroup G_P . Therefore P satisfies (P6). Conjugacy of elements of a free product with amalgamation is described in [21], which now follows easily from Corollary 4.5 and Theorem 4.6 as we show below. First we state the theorem.

Theorem 5.1 ([21], Thm. 4.6). *Let $G = A *_H B$. Every element of G is conjugate to a cyclically reduced element of G . (That is an element g which can be written as $g = g_1 \cdots g_n$ with $g_i \in A \cup B$ and either $n = 1$ or g_{i-1} and g_i do not lie in the same factor for all $i \in \mathbb{Z}/n\mathbb{Z}$.) If g is a cyclically reduced element of G then the following hold.*

1. *If g is conjugate to $h \in H$ then $g \in A \cup B$ and there exists a sequence h, h_1, \dots, h_ℓ, g where $h_i \in H$ and consecutive terms are conjugate in some factor.*
2. *If 1 does not hold and g is conjugate to an element $f \in A \cup B$, then g and f belong to the same factor, A or B , and they are conjugate in that factor.*
3. *If $n = |g| \geq 2$, then 1 and 2 do not hold. If g is conjugate to a cyclically reduced element f , then f can be written as $f = h^{-1}g_i \cdots g_n g_1 \cdots g_{i-1}h$, for some $h \in H$ and i with $1 \leq i \leq n$.*

Proof. Assertion 3 is a trivial consequence of Theorem 4.6. Indeed, for $n \geq 2$ Theorem 4.6 says that $f = b^{-1}g_i \cdots g_n g_1 \cdots g_{i-1}b$ where $b, b^{-1}g_i, g_{i-1}b \in A \cup B$. However, P satisfies (P6), hence $b \in H$ by Remark 4.7. Moreover, 1 or 2 implies $n = 1$ by Corollary 4.5, 1.

Thus, let $n = 1$ and $g, p \in A \cup B$ be conjugate to each other. Applying Corollary 4.5, 3, there is a sequence $p = p_0, p_1, \dots, p_\ell = g$ where consecutive terms are pre-conjugate, i.e., consecutive terms are conjugate in some factor. From Lemma 4.8, either every p_i is in H or the sequence may be shortened. Thus, if g is not conjugate to any $h \in H$, we may assume $g \in A \setminus H$ and $\ell = 1$, so p is a pre-conjugate of g ; that is of the form $a^{-1}ga$ for some $a \in A$. Hence $p = a^{-1}ga \in A \setminus H$, giving 2.

Otherwise every p_i is in H and 1 holds. □

5.2 Conjugacy in HNN-extensions

As in Section 3.1, for $G = \text{HNN}(H, t; t^{-1}At = B)$ the defining pregroup can be chosen as $P = H \cup Ht^{-1}H \cup HtH$; and the base group H is then equal to the canonical subgroup G_P . Therefore P satisfies (P8).

The word problem in G can be solved, if we can effectively perform Britton reductions, see e.g. in [20]: we read non-trivial elements in G as words over $H \setminus \{1\}$ and in $t^{\pm 1}$. Whenever we see a factor in $t^{-1}At$, then we replace it by the corresponding factor in B . Similarly, whenever we see a factor in tBt^{-1} , then we replace it by the corresponding factor in A . This leads to a normal

form where each g becomes an element in H : that is, for some uniquely defined t -sequence of minimal length $n \geq 1$, the element g has the form

$$g = h_0 t^{\varepsilon_1} h_1 \cdots t^{\varepsilon_{n-1}} h_{n-1} t^{\varepsilon_n} h_n.$$

In order to perform a *cyclic reduction* we remove $h_0 t^{\varepsilon_1} h_1$ from the left and put it at the right. We continue with Britton and cyclic reductions for as long as possible and eventually reach a (Britton) cyclically reduced form. Clearly every cyclically reduced form g , with non-trivial t -sequence, is conjugate to an element of the form

$$t^{\varepsilon_1} z_1 \cdots t^{\varepsilon_{n-1}} z_{n-1} t^{\varepsilon_n} z_n, \quad (6)$$

where $t^{\varepsilon_1}, \dots, t^{\varepsilon_n}$ is the t -sequence of g , $z_i \in H$ and, for all i we have

$$t^{\varepsilon_i} z_i t^{\varepsilon_{i+1}} \notin t^{-1} A t \cup t B t^{-1},$$

(subscripts modulo n). Cyclically reduced elements which either belong to H or are written in the form of (6) are called *standard* cyclically reduced elements of G . In terms of the pregroup P , every pregroup cyclically reduced word can be written as a preconjugate, by an element of H , of a standard cyclically reduced word; and conversely, every standard cyclically reduced word is cyclically reduced with respect to P .

The conjugacy theorem for HNN-extensions, Collins' Lemma, can be found in [20, Chapter IV, Theorem 2.5], and is stated for standard cyclically reduced words. In analogy to amalgamated products we restate it as follows.

Theorem 5.2 (D.J. Collins (1969)). *Let $G = \text{HNN}(H, t; t^{-1} A t = B)$ be an HNN-extension over two isomorphic subgroups A, B in some base group H . Every element of G is conjugate to a standard cyclically reduced element. Let g and f be conjugate, standard cyclically reduced elements of G . Then the following (mutually exclusive) statements hold.*

1. *If $f \in A \cup B$ then there exists a sequence $f = c_0, c_1, \dots, c_\ell = g$ of elements of $A \cup B$, such that, for $i = 1, \dots, \ell$, we have $c_i = k_i^{-1} t^{-\delta_i} c_{i-1} t^{\delta_i} k_i$, with $k_i \in H$, $\delta_i = \pm 1$ and $t^{-\delta_i} c_{i-1} t^{\delta_i} \in t^{-1} A t \cup t B t^{-1}$.*
2. *If g is not conjugate to an element of $A \cup B$ and $f \in H$ then g and f are conjugate by an element of the base group H .*
3. *If f is not in H then there exist $n \geq 1$, $z_i \in H$, $\varepsilon_i = \pm 1$, $1 \leq j \leq n$ and $c \in A \cup B$, such that $g = t^{\varepsilon_1} z_1 \cdots t^{\varepsilon_n} z_n$ and f has t -sequence of length n and is equal in G to*

$$c^{-1} t^{\varepsilon_j} z_j \cdots t^{\varepsilon_n} z_n t^{\varepsilon_1} z_1 \cdots t^{\varepsilon_{j-1}} z_{j-1} c,$$

with $c \in A$, if $\varepsilon_j = -1$; and $c \in B$, if $\varepsilon_j = 1$.

Proof. As in the proof of Theorem 5.1, it follows from Theorem 4.6 that if $g = t^{\varepsilon_1} z_1 \cdots t^{\varepsilon_n} z_n$, where $n \geq 2$, then f is equal in G to $c^{-1} t^{\varepsilon_j} z_j \cdots t^{\varepsilon_n} z_n t^{\varepsilon_1} z_1 \cdots t^{\varepsilon_{j-1}} z_{j-1} c$, for some $c \in P$, and as (P6) holds we have $c \in H$.

The pregroup P for G satisfies (P8) and H is the canonical subgroup. Therefore $P \setminus H = HtH \cup Ht^{-1}H$. Hence, if $f = t^\varepsilon z$, for some $z \in H$ and $\varepsilon = \pm 1$, then from Corollary 4.5 and Lemma 4.9 every cyclically reduced conjugate of f has the form $c^{-1} t^\varepsilon z c$, for some $c \in H$. Since g and f are standard, statement 3 holds in both these cases.

This leaves the case where $f \in H$. As in the proof of Theorem 5.1, applying Corollary 4.5, 3, there is a sequence $f = p_0, p_1, \dots, p_\ell = g$ of elements of P , where consecutive terms are preconjugate, say $p_i = q_i^{-1} p_{i-1} q_i$, with $q_i \in P$. If $p_i \in P \setminus H$ then, from Lemma 4.9, the sequence may be shortened, at least while $\ell > 1$. Then, since $p_{\ell-1} \in H$ we have, from (P8), $g \in H$. Hence we may assume that either $p_i \in H$, for $i = 0, \dots, \ell$. Again, if $q_i \in H$ and $\ell > 1$ then the sequence may be shortened, so we may assume that either $q_i \in P \setminus H$, for $i = 1, \dots, \ell$; or that $\ell = 1$.

Applying (P8), $q_i^{-1} p_{i-1} q_i = p_i \in H$, with $q_i \notin H$, implies that p_i is conjugate to an element of $A \cup B$. If g is not conjugate to an element of $A \cup B$ it follows that $\ell = 1$, and $q_1 \in H$, as required in statement 2. Otherwise 1 holds. \square

6 The conjugacy problem in virtually free groups

We consider only the case of finitely generated virtually free groups. Virtually free groups are hyperbolic, and it has been shown by Epstein and Holt [8] that the conjugacy problem for hyperbolic groups can be solved in linear time. Hence, the following is a special case of [8]. However, our algorithm is much simpler and more direct. It can be implemented in a straightforward way using finite pregroups.

Proposition 6.1. *The conjugacy problem in finitely generated virtually free groups can be solved in linear time.*

Proof. A finitely generated virtually free group G is the universal group $U(P)$ of some finite pregroup P , see Proposition 3.6. As above let $\Gamma = P \setminus \{\varepsilon\}$.

By a standard procedure involving Theorem 3.4 we can compute cyclically reduced elements in linear time. Thus, we may assume that our input words are given as $g = g_1 \cdots g_n$ and $f = f_1 \cdots f_n$ with $g_i, f_i \in \Gamma$ such that both sequences are cyclically reduced. For $n = 1$ we can use table look-up. Hence we may assume $n \geq 2$ henceforth.

Now, let us put a linear order on Γ . Then the shortlex normal form of g begins with a letter $[g_1 \bar{a}_1]$ such that the geodesic length of $a_1 g_2 \cdots g_n$ is $n - 1$.

But this implies $a_1g_2 \in P$. Thus, working from left to right we may compute the shortlex normal form of g , in linear time; and we may assume that this is $g_1 \cdots g_n$

We know $f = [bg_i] \cdots g_n g_1 \cdots [g_{i-1} \bar{b}] \in U(P)$ by Theorem 4.6. Thus, $\bar{b}fb = g_i \cdots g_n g_1 \cdots g_{i-1}$ and the number of all $\bar{b}fb$ is bounded by a constant dependent only on the order of P . Hence, we may assume that $f = g_i \cdots g_n g_1 \cdots g_{i-1}$. Since the word problem in finitely generated virtually free groups can be solved in linear time (e.g., using the system $S(P)$ or by computing the shortlex normal form), we may assume $2 < i < n$. (We also see that the conjugacy problem can be solved in quadratic time: but our goal is linear time.)

Now, the shortlex normal form of g^2 can be written as

$$g_1 \cdots g_{n-1} [g_n \bar{a}_1] [a_1 g_1 \bar{a}_2] \cdots [a_n g_n],$$

for appropriate $a_i \in P$. As a consequence,

$$f \bar{a}_i = g_i \cdots g_{n-1} [g_n \bar{a}_1] [a_1 g_1 \bar{a}_2] \cdots [a_{i-1} g_{i-1} \bar{a}_i].$$

However, the word $g_i \cdots g_{n-1} [g_n \bar{a}_1] [a_1 g_1 \bar{a}_2] \cdots [a_{i-1} g_{i-1} \bar{a}_i]$ is in shortlex normal form. Therefore the shortlex normal form of f is $f' [a_{i-1} g_{i-1}]$, where

$$f' = g_i \cdots g_{n-1} [g_n \bar{a}_1] [a_1 g_1 \bar{a}_2] \cdots [a_{i-2} g_{i-2} \bar{a}_{i-1}].$$

Thus, it is enough to compute the shortlex normal form $\widehat{f} = f'p$, of f . Erasing the last letter p yields f' . We can run the pattern matching algorithm of Knuth-Morris-Pratt, in linear time, in order to obtain a list (i_1, \dots, i_k) with $2 < i_j < n$ where the pattern f' appears as $g_{i_j} \cdots g_{n-1} [g_n \bar{a}_1] [a_1 g_1 \bar{a}_2] \cdots [a_{i_j-2} g_{i_j-2} \bar{a}_{i_j-1}]$. All that remains is to verify whether or not $[p \bar{a}_{i_j}] = [a_{i_j-1} g_{i_j-1} \bar{a}_{i_j}]$, for one index in the list.

□

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