# On the Covering Number of Small Symmetric Groups and Some Sporadic Simple Groups 

Luise-Charlotte Kappe, Daniela Nikolova-Popova, and Eric Swartz


#### Abstract

A set of proper subgroups is a covering for a group if its union is the whole group. The minimal number of subgroups needed to cover $G$ is called the covering number of $G$, denoted by $\sigma(G)$. Determining $\sigma(G)$ is an open problem for many non-solvable groups. For symmetric groups $S_{n}$, Maróti determined $\sigma\left(S_{n}\right)$ for odd $n$ with the exception of $n=9$ and gave estimates for $n$ even. In this paper we determine $\sigma\left(S_{n}\right)$ for $n=8,9,10$ and 12. In addition we find the covering number for the Mathieu group $M_{12}$ and improve an estimate given by Holmes for the Janko group $J_{1}$.


## 1. Introduction

Let $G$ be a group and $\mathcal{A}=\left\{A_{i} \mid 1 \leq i \leq n\right\}$ a collection of proper subgroups of $G$. If $G=\bigcup_{i=1}^{n} A_{i}$, then $\mathcal{A}$ is called a cover of $G$. A cover is called irredundant if after the removal of any subgroup, the remaining subgroups do not cover the group. A cover of size $n$ is said to be minimal if no cover of $G$ has fewer than $n$ members. According to J.H.E. Cohn [6], the size of a minimal covering of $G$ is called the covering number, denoted by $\sigma(G)$. By a result of B.H. Neumann [20], a group is the union of finitely many proper subgroups if and only if it has a finite noncyclic homomorphic image. Thus it suffices to restrict our attention to finite groups when determining covering numbers of groups.

Determining the invariant $\sigma(G)$ of a group $G$ and finding the positive integers which can be covering numbers is the topic of ongoing research. It even predates Cohn's 1994 publication [6]. It is a simple exercise to show that no group is the union of two proper subgroups. Already in 1926, Scorza [21] proved that $\sigma(G)=3$ if and only if $G$ has a homomorphic image isomorphic to the Klein-Four group, a result many times rediscovered over the years. In [11], Greco characterizes groups with $\sigma(G)=4$ and in [12] and [13] gives a partial characterization of groups with $\sigma(G)=5$. For further details we refer to the survey article by Serena [22], and for recent applications of this research see for instance [3] and [4].

In [6], Cohn conjectured that the covering number of any solvable group has the form $p^{\alpha}+1$, where $p$ is a prime and $\alpha$ a positive integer, and for every integer of the form $p^{\alpha}+1$ he determined a solvable group with this covering number. In [23], Tomkinson proves Cohn's conjecture and suggests that it might be of interest to investigate minimal covers of non-solvable and in particular simple groups. Bryce, Fedri and Serena [5] started this investigation by determining the covering number for some linear groups such as $\operatorname{PSL}(2, q), \operatorname{PGL}(2, q)$ or $\operatorname{GL}(2, q)$ after Cohn [6] had already shown that $\sigma\left(A_{5}\right)=10$ and $\sigma\left(S_{5}\right)=16$. In [18], Lucido investigates Suzuki groups and determines their covering numbers. For the sporadic groups, such as $M_{11}$,

[^0]$M_{22}, M_{23}, L y$ and $O^{\prime} N$, the covering numbers are established in [15] by Holmes and she gives estimates for those of $J_{1}$ and $M^{c} L$. Some of the results in [15] are established with the help of GAP [10], a first in this context.

The covering numbers of symmetric and alternating groups were investigated by Maróti in [19]. For $n \neq 7,9$, he shows that for the alternating group $\sigma\left(A_{n}\right) \geq 2^{n-2}$ with equality if and only if $n$ is even but not divisible by 4 . For $n=7$ and 9 Maróti establishes $\sigma\left(A_{7}\right) \leq 31$ and $\sigma\left(A_{9}\right) \geq 80$. For the symmetric groups he proves that $\sigma\left(S_{n}\right)=2^{n-1}$ if $n$ is odd unless $n=9$ and $\sigma\left(S_{n}\right) \leq 2^{n-2}$ if $n$ is even. It is a natural question to ask what are the exact covering numbers for alternating and symmetric groups for those values of $n$ where Maróti only gives estimates. In [17] and [8] this was done for alternating groups in case of small values of $n$. As mentioned earlier, Cohn [6] already established $\sigma\left(A_{5}\right)=10$. In [17] it is shown that $\sigma\left(A_{7}\right)=31$ and $\sigma\left(A_{8}\right)=71$. Furthermore, Maróti's bound for $A_{9}$ is improved by establishing that $127 \leq \sigma\left(A_{9}\right) \leq 157$. Recently, it was shown in [8] that $\sigma\left(A_{9}\right)=157$.

The topic of this paper is to determine the covering numbers for symmetric groups of small degree and some sporadic simple groups. We determine the covering numbers for $S_{n}$ in cases when $n=8,9,10$, and 12 . In particular, we show $\sigma\left(S_{9}\right)=256$, establishing that Maróti's result that $\sigma\left(S_{n}\right)=2^{n-1}$ for odd $n$ holds without exceptions. For $n=8,10$ and 12 we have $\sigma\left(S_{8}\right)=64, \sigma\left(S_{10}\right)=221$, and $\sigma\left(S_{12}\right)=761$, respectively. We observe that Maróti [19] gave already 761 as an upper bound for $\sigma\left(S_{12}\right)$. Since we can use the same methods, we establish in addition that the Mathieu group $M_{12}$ has covering number 208 and improve the estimate given for the Janko group $J_{1}$ in [15].

Observing that $\sigma\left(S_{4}\right)=4$ and $\sigma\left(S_{6}\right)=13$ by [1], we know now the covering numbers of $S_{n}$ for all even $n \leq 12$ and observe that in this range $\sigma\left(S_{n}\right)=2^{n-2}$, Maróti's upper bound, is only taken if $n$ is a 2-power. In the remaining cases we have $\sigma\left(S_{n}\right)<2^{n-2}$ and $\sigma\left(S_{n}\right) \sim \frac{1}{2}\binom{n}{n / 2}$. This suggests that perhaps the value for $\sigma\left(S_{n}\right)$ is less than Maróti's bound in case $n$ is not a 2-power. Our current methods rely on explicit tables for the symmetric groups in question and computer calculation to carry out certain optimizations. There are limits to the size of the group on how far these methods can carry us and statements for general values of $n$ are extremely difficult and require entirely different methods than those used for small values of $n$. This will become clearer when we discuss our methods in the following.

The methods employed here are an extension of those used in [17]. In determining a minimal covering of a group we can restrict ourselves to finding a minimal covering by maximal subgroups. The conjugacy classes of subgroups for the groups in question can be found in GAP [10]. To determine a minimal covering by maximal subgroups, it suffices to find a minimal covering of the conjugacy classes of maximal cyclic subgroups by such subgroups of the group. Already in [15] this method is used to determine the covering numbers of sporadic groups. Here this method is adapted to the case of symmetric groups where the generators of maximal cyclic subgroups can easily be identified by their cycle structure.

The following notation is used for the disjoint cycle decomposition of a nontrivial permutation. Let $m_{1}, m_{2}, \ldots, m_{t} \in \mathbb{N}$ with $1<m_{1}<m_{2}<\ldots<m_{t}$ and $k_{1}, \ldots, k_{t} \in \mathbb{N}$. If $\alpha$ is a permutation with disjoint cycle decomposition of $k_{i}$ cycles of length $m_{i}, i=1, \ldots, t$, then we denote the class of $\alpha$ by $\left(m_{1}^{k_{1}}, \ldots, m_{t}^{k_{t}}\right)$. If $k_{i}=1$, we just write $m_{i}$ instead of $m_{i}^{1}$. As is customary, we suppress 1 -cycles and the identity permutation is denoted by (1). For example, the permutation with disjoint cycle decomposition (12)(34)(5678) belongs to the class $\left(2^{2}, 4\right)$. In the case of symmetric groups all elements of a given cycle structure are contained in the subgroups of a conjugacy class of maximal subgroups and the elements with the respective cycle structure are either partitioned into these subgroups or there exists an intersection between some of the subgroups of the conjugacy class.

For the groups $S_{8}, S_{9}, S_{10}$, and $M_{12}$, we provide two tables which are obtained with the help of GAP [10]. (For the group $S_{12}$, we provide only a list of maximal subgroup conjugacy classes and refer to previous work in [19]. For the group $J_{1}$, we refer to previous work in [15].) The first table gives the information on the conjugacy classes of maximal subgroups of the group, such as the isomorphism type and order of the class representative and the size of each class. The second table lists the order and cycle structure of each permutation generating a maximal cyclic subgroup as well as the total of such elements in the group together with the distribution of these elements over the various conjugacy classes. For each conjugacy class we list how many of these elements are contained in a class representative. If elements are partitioned over the representatives, we indicate this with $P$, and if each element is contained in $k$ class representatives and each representative contains $s$ such elements, we indicate this with $s_{k}$. For some of the groups it suffices to give the second table in abbreviated form.

For finding the covering number, the goal is to determine an irredundant covering and show that it is minimal. If the elements of a certain cycle structure are partitioned into the subgroups of a particular conjugacy class, it is not hard to find a minimal covering for such elements. The difficulty arises if the elements in question occur in several class representatives. In this case we interpret the subgroups and group elements as an incidence structure with the subgroup representatives as the sets and the group elements with the specific cycle structure as elements. This leads to a problem in linear optimization. Here are some of the details.

Given two finite collections of objects, call them $U$ and $V$. Call the objects in $V$ elements and the objects in $U$ sets. Given an incidence structure between $U$ and $V$, that is for every $v$ in $V$ and every $u$ in $U$ we have either $v$ incident with $u$ or $v$ not incident with $u, v \in u$ or $v \notin u$ for short. This relation can be represented by a matrix $A=\left(a_{i j}\right)$, the incidence matrix of $(V, U)$. We label the columns of $A$ by the sets in $U$ and the rows by the elements in $V$. For $1 \leq i \leq|V|$ and $1 \leq j \leq|U|$ we set

$$
a_{i j}= \begin{cases}1 & \text { if } v_{i} \in u_{j} \\ 0 & \text { if } v_{i} \notin u_{j}\end{cases}
$$

Let $W$ be a subcollection of $U$. We define a column vector $x(W)=\left(x_{1}, \ldots, x_{|U|}\right)^{T}$ as follows

$$
x_{j}= \begin{cases}1 & \text { if } u_{j} \in W \\ 0 & \text { if } u_{j} \notin W\end{cases}
$$

Let $A x(W)=y(W)=\left(y_{1}, \ldots, y_{|V|}\right)^{T}$, a column vector of length $|V|$ with coordinates $y_{i} \geq 0$. If $y_{i}=0$, then $v_{i} \notin \bigcup_{u \in W} u$ and if $y_{i}>0$, then $v_{i} \in \bigcup_{u \in W} u$, specifically $v_{i}$ is contained in exactly $y_{i}$ members of $W$. If $y_{i}>0$ for $i=1, \ldots,|V|$, every $v_{i} \in V$ is contained in at least one member of $W$ and we say $W$ covers $V$.

In our interpretation the objects in $U$ are representatives of a certain conjugacy class of maximal subgroups and the objects in $V$ are permutations with a certain cycle structure. The goal is to find the minimal size of $|W|$ such that $W$ covers $V$. If the objects in $U$ and $V$ can be suitably labeled, we can use a combinatorial argument to find the optimal solution, e.g., using the Erdős-Ko-Rado Theorem [9] as in the case of $S_{10}$. Otherwise we have to resort to the help of computers to find optimal solutions, e.g., in the case of $S_{9}, M_{12}$ and $J_{1}$. Roughly speaking, a system of linear inequalities with binary variables is prepared by GAP [10] and the optimal solution is found with the help of Gurobi [14]. Naturally, this approach puts a limit on how large our groups can be. In addition, the structure of the cover heavily depends on the arithmetic nature of $n$.

## 2. The Symmetric Group $S_{8}$

The smallest symmetric group for which the covering number is not known is $S_{8}$. Here we determine $\sigma\left(S_{8}\right)$ and show that it equals the upper bound given by Maróti in [19].

THEOREM 2.1. The covering number of $S_{8}$ is 64 .

Proof. First we will show that there exists an irredundant covering of $S_{8}$ by 64 subgroups. As can be seen from Table 2.2, all odd permutations of the group generating maximal cyclic subgroups are contained either in $M S 3$ or $M S 6$. Thus the union of $M S 3$ and $M S 6$ contains all odd permutations in question. We observe that this union does not contain all even permutations generating maximal cyclic subgroups, e.g., the permutation with cycle structure $(3,5)$ is only contained in $M S 1$ and $M S 2$. Thus $M S 1, M S 3$, and $M S 6$ cover all of $S_{8}$, and

$$
\sigma\left(S_{8}\right) \leq|M S 1|+|M S 3|+|M S 6|=64 .
$$

Let $\mathcal{C}$ be the union of $M S 1, M S 3$, and $M S 6$, and define $\Pi$ to be the union of all elements with cycle structure (8), $(3,5)$, or $\left(2,3^{2}\right)$. The elements of $\Pi$ are partitioned among the 64 groups of $\mathcal{C}$, so $\mathcal{C}$ is an irredundant covering.

It remains to be shown that $\mathcal{C}$ is a minimal covering. Assume to the contrary that there exists a cover $\mathcal{B}$ of $S_{8}$ such that $\mathcal{B}$ contains fewer subgroups than $\mathcal{C}$. Since $\mathcal{B}$ covers all the elements of $S_{8}$, it must cover all the elements of $\Pi$. Moreover, $\mathcal{B}$ contains fewer subgroups than $\mathcal{C}$, so we may assume that $\mathcal{C}=(\mathcal{B} \cap \mathcal{C}) \cup \mathcal{C}^{\prime}$ and $\mathcal{B}=(\mathcal{B} \cap \mathcal{C}) \cup \mathcal{B}^{\prime}$, where $\mathcal{C}^{\prime}$ is the set of subgroups in $\mathcal{C}$ but not in $\mathcal{B}$ and $\mathcal{B}^{\prime}$ is the set of subgroups in $\mathcal{B}$ that are not in $\mathcal{C}$. Since $|\mathcal{B}|<|\mathcal{C}|$, it must be that $\left|\mathcal{B}^{\prime}\right|<\left|\mathcal{C}^{\prime}\right|$. This means that $\mathcal{B}^{\prime}$ must cover some subset of elements of $\Pi$ more efficiently than does $\mathcal{C}^{\prime}$.

We will now show that $A_{8}$ is in $\mathcal{B} \cap \mathcal{C}$. If $A_{8} \notin \mathcal{B} \cap \mathcal{C}$, then the only other way to cover the elements of cycle structure $(3,5)$ is by the 56 subgroups of $M S 2$. Since the most efficient way to cover the 8 -cycles is by the 35 subgroups of $M S 6$ and no maximal subgroup contains both elements of cycle structure $(3,5)$ and 8 -cycles, we have $|\mathcal{B}| \geq 56+35>|\mathcal{C}|$, a contradiction. We conclude $A_{8} \in \mathcal{B} \cap \mathcal{C}$.

Define $\mathcal{C}_{1}^{\prime}$ to be the set of subgroups of $\mathcal{C}^{\prime}$ that are in $M S 3$ and $\mathcal{C}_{2}^{\prime}$ to be the set of subgroups of $\mathcal{C}^{\prime}$ in MS6, and let $\Pi_{1}$ and $\Pi_{2}$, respectively, be the elements of $\Pi$ that are in subgroups of $\mathcal{C}_{1}^{\prime}$ and $\mathcal{C}_{2}^{\prime}$, respectively. Note that $\Pi_{1}$ and $\Pi_{2}$ are disjoint since the elements of $\Pi$ are partitioned among the subgroups of $\mathcal{C}$. As can be seen from examining Table 2.2, the maximal subgroups of $S_{8}$ that are not in $\mathcal{C}$ (i.e., those isomorphic to $S_{3} \times S_{5}, S_{7}, \mathrm{PGL}(2,7)$, or $S_{2} \mathrm{wr} S_{4}$ ) contain at most 80 elements of $\Pi_{1} \cup \Pi_{2}$, whereas a subgroup isomorphic to $S_{4} \mathrm{wr} S_{2}$ in $M S 6$ contains 144 elements of this set. Since $\mathcal{C}$ partitions $\Pi_{1} \cup \Pi_{2}$, this means that if $\mathcal{C}_{2}^{\prime}$ contains $n$ subgroups, then $\mathcal{B}^{\prime}$ must contain at least $n+1$ subgroups to cover the elements in $\Pi_{2}$. Since $\left|\mathcal{B}^{\prime}\right|<\left|\mathcal{C}^{\prime}\right|$, this means both that $\mathcal{C}_{1}^{\prime}$ is nonempty and that some collection $\mathcal{B}_{1}^{\prime}$ of $\mathcal{B}^{\prime}$ covers the elements of $\Pi_{1}$ with fewer subgroups than $\mathcal{C}_{1}^{\prime}$. However, $\Pi_{1}$ consists only of elements with cycle structure $\left(2,3^{2}\right)$, and each subgroup of $\mathcal{C}_{1}^{\prime}$ contains exactly 40 such elements. For $\mathcal{B}_{1}^{\prime}$ to be smaller than $\mathcal{C}_{1}^{\prime}$, some subgroup of $S_{8}$ would have to contain more than 40 elements with cycle structure $\left(2,3^{2}\right)$. None does, which is a contradiction. Therefore it follows that no such cover $\mathcal{B}$ can exist and $\mathcal{C}$ is a minimal covering of $S_{8}$. We conclude $\sigma\left(S_{8}\right)=64$, as desired.

| Label | Isomorphism Type | Group Order | Class Size |
| :---: | :---: | :---: | :---: |
| $M S 1$ | $A_{8}$ | 20160 | 1 |
| $M S 2$ | $S_{3} \times S_{5}$ | 720 | 56 |
| $M S 3$ | $S_{2} \times S_{6}$ | 1440 | 28 |
| $M S 4$ | $S_{7}$ | 5040 | 8 |
| $M S 5$ | $S_{2} \mathrm{wr} S_{4}$ | 384 | 105 |
| $M S 6$ | $S_{4} \mathrm{wr} S_{2}$ | 1152 | 35 |
| $M S 7$ | $\mathrm{PGL}(2,7)$ | 336 | 120 |

Table 2.1. Conjugacy classes of maximal subgroups of $S_{8}$.

| Order | C.S. | Size | $M S 1$ <br> (1) | $\begin{gathered} M S 2 \\ (56) \end{gathered}$ | $M S 3$ (28) | MS4 <br> (8) | $\begin{aligned} & M S 5 \\ & (105) \end{aligned}$ | MS6 <br> (35) | $\begin{aligned} & M S 7 \\ & (120) \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| ODD |  |  |  |  |  |  |  |  |  |
| 4 | $\left(2^{2}, 4\right)$ | 1260 | 0 | 0 | $90_{2}$ | 0 | $36_{3}$ | 1805 | 0 |
| 6 | $(2,3)$ | 1120 | 0 | 1005 | 1604 | 4203 | 0 | 963 | 0 |
| 6 | (2, $3^{2}$ ) | 1120 | 0 | $40_{2}$ | 40, P | 0 | 323 | 0 | 0 |
| 6 | 6 | 13360 | 0 | 0 | 120, $P$ | 8402 | 32, $P$ | 0 | $56_{2}$ |
| 8 | 8 | 5040 | 0 | 0 | 0 | 0 | 48, $P$ | 144, P | $84_{2}$ |
| 10 | $(2,5)$ | 4032 | 0 | $72, P$ | 144, $P$ | 504, P | 0 | 0 | 0 |
| 12 | $(3,4)$ | 3360 | 0 | 60, P | 0 | 420, $P$ | 0 | 96, $P$ | 0 |
| EVEN |  |  |  |  |  |  |  |  |  |
| 4 | $(2,4)$ | 2520 | $P$ | 90, P | 1802 | 6302 | 24, P | $72, P$ | 0 |
| 6 | $(2,6)$ | 3360 | $P$ | 0 | 120, P | 0 | 32, P | 192 | 0 |
| 7 | 7 | 5760 | $P$ | 0 | 0 | 720, $P$ | 0 | 0 | 48, $P$ |
| 15 | $(3,5)$ | 2688 | $P$ | 48, $P$ | 0 | 0 | 0 | 0 | 0 |

Table 2.2. Inventory of elements generating maximal cyclic subgroups in $S_{8}$ across conjugacy classes of maximal subgroups.

## 3. The Symmetric Group $S_{9}$

In this section we will determine the exact covering number of $S_{9}$, the case missing in [19], where the covering numbers for $S_{n}$ with $n$ odd were determined with the exception of $n=9$.

Theorem 3.1. The covering number of $S_{9}$ is 256.
This together with Theorem 1.1 in [19] yields the following corollary.
Corollary 3.2. Let $n \geq 3$ be an odd integer. Then $\sigma\left(S_{n}\right)=2^{n-1}$.
To prove the main result of this section, we need the following proposition.
PROPOSITION 3.3. The 84 subgroups of MS3 form a minimal covering of the elements with cycle structure $(3,6)$ in $S_{9}$.

Proof. We prove this computationally with the help of the software GAP [10] and Gurobi [14]. Using the GAP program as given in Function 8.1 for $G=S_{9}$ and the conjugacy classes $M S 3, M S 6$, and $M S 7$ of maximal subgroups, we are setting up the equations readable by Gurobi for the elements of type $(3,6)$. The Gurobi output shows that a minimal covering of these elements consists of 84 subgroups from $M S 3, M S 6$, and $M S 7$. Since the elements with cycle structure $(3,6)$ are partitioned into the subgroups of MS3, these 84 subgroups constitute a minimal covering of these elements.

We note that the GAP output addressed in the above proposition as well as an abbreviated Gurobi output of these calculations is given at the end of Section 8. For further details we refer to http://www.math.binghamton.edu/menger/coverings/. Now we are ready to prove our theorem.

Proof of Theorem 3.1. We will show first that there exists a covering of $S_{9}$ by 256 subgroups. As can be seen with the help of GAP [10], the 9 -cycles in $S_{9}$ are only contained in $A_{9}$, the only subgroup in $M S 1$. Thus it suffices to show that the odd permutations generating maximal cyclic subgroups can be covered by 255 subgroups.

As can be seen from Table 3.2, listing the odd permutations generating maximal cyclic subgroups in $S_{9}$, the elements with cycle structure $(4,5)$ and $(2,7)$ are only contained in the subgroups of $M S 2$ and $M S 4$, respectively. Since these elements are partitioned into the subgroups of the respective classes, the full classes have to be added to the covering. As one can see from Table 3.2, the odd permutations generating maximal cyclic subgroups not covered by the subgroups of $M S 2$ and $M S 4$ are the 8 -cycles and the elements with cycle structure $(3,6)$. Thus adding the subgroups of $M S 3$ and $M S 5$ to those of $M S 1, M S 2$ and $M S 4$ provides a covering of $S_{9}$. We conclude

$$
\sigma\left(S_{9}\right) \leq|M S 1|+|M S 2|+|M S 3|+|M S 4|+|M S 5|=256 .
$$

It remains to be shown that any covering of $S_{9}$ contains at least 256 subgroups. As pointed out earlier, none of the subgroups of $M S 1, M S 2$ and $M S 4$ can be omitted since the respective elements are partitioned into these subgroups. The 8 -cycles are partitioned into the nine subgroups of $M S 5$ with 5040 elements in each subgroup. On the other hand, each such element is contained in two subgroups of MS7 with 1088 -cycles in each subgroup. Obviously, replacing subgroups from $M S 5$ by those from $M S 7$ increases the number of subgroups needed for covering these elements. Hence the nine subgroups of MS5 constitute a minimal covering of the 8 -cycles.

| Label | Isomorphism Type | Group Order | Class Size |
| :---: | :---: | :---: | :---: |
| $M S 1$ | $A_{9}$ | 18140 | 1 |
| $M S 2$ | $S_{4} \times S_{5}$ | 2880 | 126 |
| $M S 3$ | $S_{3} \times S_{6}$ | 4320 | 84 |
| $M S 4$ | $S_{2} \times C_{7}$ | 10080 | 36 |
| $M S 5$ | $S_{8}$ | 40320 | 9 |
| $M S 6$ | $S_{3} w r S_{3}$ | 1296 | 280 |
| $M S 7$ | $\mathrm{AGL}(2,3)$ | 432 | 840 |

Table 3.1. Conjugacy classes of maximal subgroups of $S_{9}$.

| Order | C.S | Size | $M S 2$ | $M S 3$ | $M S 4$ | $M S 5$ | $M S 6$ | $M S 7$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | $\left(2^{2}, 4\right)$ | 11340 | $180_{2}$ | $270_{2}$ | $630_{2}$ | $1260, P$ | $162_{4}$ | 0 |
| 6 | $(2,3)$ | 2520 | $220_{11}$ | $270_{9}$ | $490_{7}$ | $1120_{4}$ | $36_{4}$ | 0 |
| 6 | $\left(2,3^{2}\right)$ | 10080 | $160_{2}$ | $360_{3}$ | $280, P$ | $1120, P$ | $36, P$ | 0 |
| 6 | 6 | 10080 | 0 | $120, P$ | $840_{3}$ | $3360_{3}$ | $36, P$ | $56_{2}$ |
| 6 | $\left(2^{3}, 3\right)$ | 2520 | $60_{3}$ | $30, P$ | $210_{3}$ | 0 | $36_{4}$ | 0 |
| 6 | $(3,6)$ | 20160 | 0 | $240, P$ | 0 | 0 | $288_{4}$ | $72_{3}$ |
| 8 | 8 | 45360 | 0 | 0 | 0 | $5040, P$ | 0 | $108_{2}$ |
| 10 | $(2,5)$ | 18144 | $144, P$ | $432_{2}$ | $1008_{2}$ | $4032_{2}$ | 0 | 0 |
| 12 | $(3,4)$ | 15120 | 360 | $180, P$ | $420, P$ | $3360_{2}$ | 0 | 0 |
| 14 | $(2,7)$ | 25920 | 0 | 0 | $720, P$ | 0 | 0 | 0 |
| 20 | $(4,5)$ | 18144 | $144, P$ | 0 | 0 | 0 | 0 | 0 |

Table 3.2 Inventory of odd permutations generating maximal cyclic subgroups in $S_{9}$ across conjugacy classes of maximal subgroups.

On the other hand, the elements with cycle structure $(3,6)$ are partitioned into the 84 subgroups of $M S 3$ with 240 elements in each subgroup and each such element is contained in four subgroups of MS6 with 288 elements in each subgroup. Thus potentially there could be an arrangement that the elements with cycle structure $(3,6)$ in six subgroups of $M S 3$ can be covered by five subgroups from MS6. However, as shown in Proposition 3.3, this is not the case, and the 84 subgroups of $M S 3$ constitute a minimal covering of these elements. Moreover, the only class of subgroups containing both 8 -cycles and elements with cycle structure $(3,6)$ is $M S 7$. However, each subgroup of $M S 7$ contains a combined total of 1808 -cycles and elements with cycle structure $(3,6)$, and so cannot possibly be a better cover than using $M S 3$ and $M S 5$, in each of which a subgroup covers at least 240 such elements. We conclude $\sigma\left(S_{9}\right)>255$ and thus $\sigma\left(S_{9}\right)=256$.

## 4. The Symmetric Group $S_{10}$

In this section we determine the covering number of $S_{10}$. It turns out to be less than the upper bound of $2^{10-2}$ given by Maróti in [19].

Theorem 4.1. The covering number of $S_{10}$ is 221.
Before we can prove Theorem 4.1, we have to establish some preparatory results involving combinatorics and incidence matrices leading to an application of a result due to Erdős, Ko and Rado [9] (see also Theorem 5.1.2 in [2]).

Theorem 4.2. [9] Let $A_{1}, \ldots, A_{m}$ be $m k$-subsets of an $n$-set $S, k \leq \frac{1}{2} n$, which are pairwise nondisjoint. Then $m \leq\binom{ n-1}{k-1}$. The upper bound for $m$ is best possible. It is attained when the $A_{i}$ are precisely the $k$-subsets of $S$ which contain a chosen fixed element of $S$.

In our application we consider the following incidence structure. We let

$$
\left.U=\left\{\left(k_{1}, k_{2}, k_{3}\right): k_{1}, k_{2}, k_{3} \in\{0,1, \ldots, 9\} \text { and } k_{1}<k_{2}<k_{3}\right)\right\}
$$

and

$$
V=\left\{\left(u, u^{\prime}\right): u, u^{\prime} \in U \text { with } u \cap u^{\prime}=\emptyset\right\} .
$$

We define an incidence relation between $U$ and $V$ as follows. For $v=\left(u, u^{\prime}\right) \in V$ we say $v \in u_{j}$ if $u=u_{j}$ or $u^{\prime}=u_{j}$, and $v \notin u_{j}$ otherwise. For this choice of $U$ and $V$ we make the following claim.

Proposition 4.3. Let $U, V$ and the incidence relation between them defined as above. Then there exists a subcollection $W^{*}$ of $U$ with $\left|W^{*}\right|=84$ which covers $V$ and every subcollection $W$ of $U$ with $|W|<\left|W^{*}\right|$ does not cover $V$. Specifically, $W^{*}$ can be chosen as $U-D$, where

$$
D=\left\{\left(0, k_{2}, k_{3}\right): k_{2}, k_{3} \in\{1,2, \ldots, 9\}, k_{2}<k_{3}\right\} .
$$

Proof. We have $|U|=120$ and $|V|=2100$. Thus the incidence matrix $A$ of $U$ and $V$ is a $2100 \times 120$ matrix with exactly two entries equal to 1 in each row, since $u_{j}=\left(\left(k_{1}, k_{2}, k_{3}\right),\left(k_{1}^{\prime}, k_{2}^{\prime}, k_{3}^{\prime}\right)\right) \in v_{i}$ if and only if $\left(k_{1}, k_{2}, k_{3}\right)=u_{j}$ or $\left(k_{1}^{\prime}, k_{2}^{\prime}, k_{3}^{\prime}\right)=u_{j}$. With $x(U)=(1, \ldots, 1)^{T}$ we have $A x(U)=(2, \ldots, 2)^{T}$. Let $u, u^{\prime} \in U$ with $u \cap u^{\prime}=\emptyset$ and let $X=U-\left\{u, u^{\prime}\right\}$. Then $y(X)$ contains a zero entry and $v=\left(u, u^{\prime}\right)$ is not covered by $X$. On the other hand, removing any subset $\left\{u_{1}, \ldots, u_{t}\right\}$ of $U$ with pairwise non-trivial intersection, i.e. $u_{i} \cap u_{j} \neq \emptyset$, then for $X=U-\left\{u_{1}, \ldots, u_{t}\right\}$ the vector $y(X)$ has all non-zero entries. The largest number of sets we can remove from $U$ has the cardinality of a maximal set with pairwise non-trivial intersection. Applying Theorem 4.2 with $n=10$ and $k=3$, we obtain $\binom{9}{2}=36$ for the cardinality of such a set. Specifically, $D=\left\{\left(0, k_{1}, k_{2}\right): k_{1}<k_{2}, k_{1}, k_{2} \in\{1, \ldots, 9\}\right\}$ is such a set. Let $W^{*}=U-D$. Then $y\left(W^{*}\right)$ has all entries $>0$. On the other hand, for any set $W$ with $|W|<\left|W^{*}\right|$ there exist $u, u^{\prime} \in \bar{W}$, the complement of $W$ in $U$, such that $u \cap u^{\prime}=\emptyset$ and thus $y(W)$ has at least one zero entry.

The following corollary establishes a minimal covering of the elements of type $\left(3^{2}, 4\right)$ by certain subgroups from MS3 (see Table 4.2). Since these subgroups are isomorphic to $S_{3} \times S_{7}$, we can label them by the letters fixed by the respective $S_{7}$. Hence we have

$$
M S 3=\left\{H\left(k_{1}, k_{2}, k_{3}\right): k_{1}, k_{2}, k_{3} \in\{0,1, \ldots, 9\}, k_{1}<k_{2}<k_{3}\right\} .
$$

Corollary 4.4. Let $\mathcal{D}=\left\{H\left(0, k_{2}, k_{3}\right): k_{2}, k_{3} \in\{1,2, \ldots, 9\}, k_{2}<k_{3}\right\}$. Then $\overline{\mathcal{D}}=$ $M S 3-\mathcal{D}$, the complement of $\mathcal{D}$ in $M S 3$, is a minimal covering of the elements of type $\left(3^{2}, 4\right)$ in $S_{10}$.

Proof. By Table 4.2, there are 50400 elements of type $\left(3^{2}, 4\right)$ in $S_{10}$. Each $H\left(k_{1}, k_{2}, k_{3}\right) \in$ $M S 3$ contains 840 such elements and each element of type $\left(3^{2}, 4\right)$ is in exactly two subgroups of $M S 3$. There are exactly six cyclic subgroups generated by elements of type $\left(3^{2}, 4\right)$ in the intersection of $H(u)$ and $H\left(u^{\prime}\right)$ with $u=\left(k_{1}, k_{2}, k_{3}\right)$ and $u^{\prime}=\left(k_{1}^{\prime}, k_{2}^{\prime}, k_{3}^{\prime}\right)$ with each such cyclic subgroup of order 12 containing four elements of type $\left(3^{2}, 4\right)$. Thus any two members $H(u)$ and $H\left(u^{\prime}\right)$ of $M S 3$ with $u \cap u^{\prime}=\emptyset$ share exactly 24 elements of type $\left(3^{2}, 4\right)$. The six cyclic subgroups of order 12 can be represented as $\left\langle t \cdot c_{4 i}\right\rangle, i=1,2,3$, where $t=u \cdot u^{\prime}$ or $u^{-1} \cdot u^{\prime}$ and $c_{4 i}$ is a 4 -cycle in $\left\{j_{1}, j_{2}, j_{3}, j_{4}\right\}$, the complement of $\left\{k_{1}, k_{2}, k_{3}, k_{1}^{\prime}, k_{2}^{\prime}, k_{3}^{\prime}\right\}$ in $\{0,1, \ldots, 9\}$, specifically $c_{41}=\left(j_{1}, j_{2}, j_{3}, j_{4}\right), c_{42}=\left(j_{1}, j_{3}, j_{2}, j_{4}\right)$ and $c_{43}=\left(j_{1}, j_{2}, j_{4}, j_{3}\right)$. For $u \cap u^{\prime}=\emptyset$ we consider

$$
T\left(u, u^{\prime}\right)=\left\{g \in\left\langle t \cdot c_{4 i}\right\rangle: t=u \cdot u^{\prime} \text { or } u^{-1} u^{\prime} ; i=1,2,3 ;|g|=12\right\} .
$$

We have $\left|T\left(u, u^{\prime}\right)\right|=24$. The 50400 elements of type $\left(3^{2}, 4\right)$ are partitioned into the 2100 equivalence classes $T\left(u, u^{\prime}\right)$. Identifying $M S 3$ with $U$ of Proposition 4.3 and setting $V=$ $\left\{T\left(u, u^{\prime}\right) ; u \cap u^{\prime}=\emptyset\right\}$, we have the same incidence structure as in Proposition 4.3 and the conclusion of the corollary follows immediately.

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. By Table 4.2 and Corollary 4.4 we see that $M S 1 \cup M S 5 \cup M S 7 \cup \overline{\mathcal{D}}=S_{10}$, since all permutations generating maximal cyclic subgroups in $S_{10}$ are contained in the union of these subgroups. Hence

$$
\sigma\left(S_{10}\right) \leq|M S 1 \cup M S 5 \cup M S 7 \cup \overline{\mathcal{D}}|=1+10+126+84=221 .
$$

It remains to be shown that the covering obtained by the 221 subgroups is minimal, i.e. $\sigma\left(S_{10}\right) \geq 221$. First we will show that the 126 subgroups of $M S 7$ and any nine subgroups of $M S 5$ constitute a minimal covering of the odd permutations generating maximal cyclic subgroups and not involved in $M S 3$. We observe that the 10 -cycles are partitioned into the three conjugacy classes $M S 6, M S 7$ and $M S 8$. Since $M S 7$ contains only 126 subgroups versus the 945 and 2520 subgroups, respectively, of the other two classes, the 126 subgroups of MS7 constitute a minimal covering of the 10 -cycles in $S_{10}$. Next we will show that any nine subgroups of MS5 are a minimal covering of the 8 -cycles in $S_{10}$. It is obvious that a minimal covering of the 8 -cycles cannot be obtained from using subgroups from $M S 4, M S 6$ or $M S 8$. Let $M S 5=\left\{S_{9}^{(i)} ; i=0, \ldots, 9\right\}$ with $S_{9}^{(i)} \cong S_{9}$ and fixed point $i$. Any 8-cycle in $S_{10}$ has two fixpoints, say $i_{1}$ and $i_{2}$. After removing $S_{9}^{(i)}$, all 8-cycles in $S_{10}$ are still covered by the remaining subgroups in $M S 5$. Removing an additional subgroup from $M S 5$, say $S_{9}^{\left(i_{2}\right)}$, leaves those 8 -cycles with fixed points $i_{1}$ and $i_{2}$ uncovered. Thus any nine subgroups of $M S 5$ constitute a minimal covering of the 8 -cycles in $S_{10}$. It can be seen now from Table 4.2 that all odd permutations generating maximal cyclic subgroups and not involved in $M S 3$ are covered by the subgroups of $M S 7$ and any nine subgroups of $M S 5$.

By Corollary 4.4 , the 84 subgroups of $\overline{\mathcal{D}}$ constitute a minimal covering of the elements of type $\left(3^{2}, 4\right)$. We observe now that the 219 subgroups of $M S 7 \cup \overline{\mathcal{D}} \cup \mathcal{C}_{0}$, where $\mathcal{C}_{0}=\left\{S_{9}^{(i)} ; i=\right.$ $1,2, \ldots, 9\}$ constitute a minimal covering of the elements of type $\left(3^{2}, 4\right)$, the 10 -cycles and the 8 -cycles, since these elements are mutually not contained in the respective subgroups covering the other types of elements. However, not all odd permutations generating maximal cyclic subgroups are contained in $M S 7 \cup \overline{\mathcal{D}} \cup \mathcal{C}_{0}$, specifically the elements of type $(2,7)$ and $(3,6)$ with fixpoint 0 . Adding $S_{9}^{(10)}$ to the covering, we obtain that the 220 subgroups of $M S 7 \cup M S 5 \cup \overline{\mathcal{D}}$ minimally cover the odd permutations of $S_{10}$ generating maximal cyclic subgroups. A look at Table 4.2 shows that the only even permutations generating maximal cyclic subgroups and not contained in $M S 5$ and $M S 7$ are the elements of type ( 3,7 ). They are partitioned into $M S 3$ and $M S 1$. Adding the single subgroup of $M S 1$, which is isomorphic to $A_{10}$, to the cover yields $\sigma\left(S_{10}\right) \geq|M S 1 \cup M S 5 \cup M S 7 \cup \overline{\mathcal{D}}|=221$. This together with the above leads to $\sigma\left(S_{10}\right)=221$.

| Label | Isomorphism Type | Group Order | Class Size |
| :---: | :---: | :---: | :---: |
| $M S 1$ | $A_{10}$ | 1814400 | 1 |
| $M S 2$ | $S_{4} \times S_{6}$ | 17280 | 210 |
| $M S 3$ | $S_{3} \times S_{7}$ | 30240 | 120 |
| $M S 4$ | $S_{2} \times S_{8}$ | 80640 | 45 |
| $M S 5$ | $S_{9}$ | 362880 | 10 |
| $M S 6$ | $S_{2} \mathrm{wr} S_{5}$ | 3840 | 945 |
| $M S 7$ | $S_{5} \mathrm{wr} S_{2}$ | 28800 | 126 |
| $M S 8$ | $\operatorname{P\Gamma L}(2,9)$ | 1440 | 2520 |

Table 4.1. Conjugacy classes of maximal subgroups of $S_{10}$.

| Order | C.S. | Size | $M S 1$ | $M S 2$ | $M S 3$ | $M S 4$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| ODD |  |  |  |  |  |  |
| 4 | $\left(2^{2}, 4\right)$ | 56700 | 0 | $1080_{4}$ | $1890_{4}$ | $3780_{3}$ |
| 4 | $\left(2,4^{2}\right)$ | 56700 | 0 | $540_{2}$ | 0 | $1260, P$ |
| 6 | $\left(2^{3}, 3\right)$ | 25200 | 0 | $480_{4}$ | $840_{4}$ | $1680_{3}$ |
| 6 | $\left(2,3^{2}\right)$ | 50400 | 0 | $1200_{5}$ | $1680_{4}$ | $2240_{2}$ |
| 6 | $\left(2^{2}, 6\right)$ | 75600 | 0 | $360, P$ | 0 | $3360_{2}$ |
| 6 | $(3,6)$ | 201600 | 0 | $960, P$ | $1680, P$ | 0 |
| 8 | 8 | 226800 | 0 | 0 | 0 | $5040, P$ |
| 10 | 10 | 362880 | 0 | 0 | 0 | 0 |
| 12 | $\left(3^{2}, 4\right)$ | 50400 | 0 | $240, P$ | $840_{2}$ | 0 |
| 14 | $(2,7)$ | 259200 | 0 | 0 | $2160, P$ | $5760, P$ |
| 20 | $(4,5)$ | 181440 | 0 | $964, P$ | 0 | 0 |
| 30 | $(2,3,5)$ | 120960 | 0 | 0 | $1008, P$ | $2688, P$ |
| EVEN |  |  |  |  |  |  |
| 6 | $(2,6)$ | 151200 | $P$ | $720, P$ | $72520_{2}$ | $6720_{2}$ |
| 8 | $(8,2)$ | 226800 | $P$ | 0 | 0 | $5040, P$ |
| 9 | 9 | 403200 | $P$ | 0 | 0 | 0 |
| 12 | $(4,6)$ | 151200 | $P$ | $720, P$ | 0 | 0 |
| 12 | $(2,3,4)$ | 151200 | $P$ | $1440_{2}$ | $2520_{2}$ | $3360, P$ |
| 21 | $(3,7)$ | 172800 | $P$ | 0 | $1440, P$ | 0 |
|  |  |  |  |  |  |  |


| Order | C.S. | Size | $M S 5$ | $M S 6$ | $M S 7$ | $M S 8$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| ODD |  |  |  |  |  |  |
| 4 | $\left(2^{2}, 4\right)$ | 56700 | $11340_{2}$ | $180_{3}$ | $900_{2}$ | 0 |
| 4 | $\left(2,4^{2}\right)$ | 56700 | 0 | $300_{5}$ | $1800_{4}$ | $90_{4}$ |
| 6 | $\left(2^{3}, 3\right)$ | 25200 | $2520, P$ | 0 | $600_{3}$ | 0 |
| 6 | $\left(2,3^{2}\right)$ | 50400 | $10080_{2}$ | $160_{3}$ | $800_{2}$ | 0 |
| 6 | $\left(2^{2}, 6\right)$ | 75600 | 0 | $240_{3}$ | $2400_{4}$ | 0 |
| 6 | $(3,6)$ | 201600 | $20160, P$ | 0 | 0 | $240_{3}$ |
| 8 | 8 | 226800 | 45360 | $240, P$ | 0 | $180_{2}$ |
| 10 | 10 | 362880 | 0 | $384, P$ | $2880, P$ | $144, P$ |
| 12 | $\left(3^{2}, 4\right)$ | 50400 | 0 | $160_{3}$ | 0 | 0 |
| 14 | $(2,7)$ | 259200 | $25920, P$ | 0 | 0 | 0 |
| 20 | $(4,5)$ | 181440 | $18144, P$ | 0 | $1440, P$ | 0 |
| 30 | $(2,3,5)$ | 120960 | 0 | 0 | $960, P$ | 0 |
| EVEN |  |  |  |  |  |  |
| 6 | $(2,6)$ | 151200 | $30240_{2}$ | $160, P$ | 0 | 0 |
| 8 | $(8,2)$ | 226800 | 0 | $240, P$ | $3600_{2}$ | $180_{2}$ |
| 9 | 9 | 403200 | $40320, P$ | 0 | 0 | 0 |
| 12 | $(4,6)$ | 151200 | 0 | $160, P$ | $2400_{2}$ | 0 |
| 12 | $(2,3,4)$ | 151200 | $15120, P$ | 0 | $1200, P$ | 0 |
| 21 | $(3,7)$ | 172800 | 0 | 0 | 0 | 0 |

Table 4.2. Inventory of elements generating maximal cyclic subgroups
in $S_{10}$ across conjugacy classes of maximal subgroups.

## 5. The Symmetric Group $S_{12}$

In [19], Maróti gives an upper bound for the covering number of $S_{12}$, which is lower than the general upper bound given there. We will show here that this bound is indeed the covering number of $S_{12}$.

Theorem 5.1. The covering number of $S_{12}$ is 761.

Proof. As noted by Maróti [19, p. 104], the covering number of $S_{12}$ is at most 761, since $S_{12}$ may be written as the union of all subgroups conjugate to $S_{6} \mathrm{wr} S_{2}, S_{11} \times S_{1}, S_{10} \times S_{2}, S_{9} \times$ $S_{3}$, and $A_{12}$, which correspond to the classes $M S 5, M S 2, M S 3, M S 4$, and $M S 1$, respectively, of Table 5.1. Indeed, we will show that this is in fact a minimal cover of $S_{12}$ by demonstrating that there is a particular class of maximal cyclic subgroups that is minimally covered by one of these five classes.

First, we examine the elements with cycle structure (12), i.e., the 12 -cycles of $S_{12}$. It is not hard to see that the classes of maximal subgroups containing 12 -cycles are all imprimitive subgroups in the classes $M S 5, M S 8, M S 9$, and $M S 10$ (a 12-cycle preserves such an imprimitive decomposition of twelve elements), and also the subgroups of class MS11. Moreover, it is easy to see that the 12 -cycles must be partitioned in each of the classes $M S 5, M S 8, M S 9$, and $M S 10$, respectively, since a 12 -cycle stabilizes a unique imprimitive decomposition of twelve elements. Since the 12 -cycles are partitioned among the subgroups in classes MS5, MS8, $M S 9$, and $M S 10$, respectively, and $M S 5$ has the fewest number of subgroups, removing $n$ subgroups from $M S 5$ from the cover would require at least $n+1$ replacements from the other classes. On the other hand, the 12 -cycles are not partitioned in $M S 11$, and simple computation using GAP [10] shows that each such subgroup contains 220 different 12-cycles. Removing even one subgroup from $M S 5$, which contains 86400 different 12 -cycles, would require at least $\lceil 86400 / 220\rceil=393$ different subgroups from MS11 to replace it. Since there are only 462 total subgroups in MS5, it is easy to see that the unique minimal covering of the maximal cyclic subgroups generated by 12 -cycles uses the 462 subgroups from $M S 5$.

Next, we examine the elements with cycle structure $(3,4,5)$. These elements are only contained in the classes $M S 4, M S 6$, and $M S 7$. Since elements with this cycle structure preserve a unique intransitive partition of twelve elements into one set of size nine (by the 5 -cycle and the 4 -cycle) and one set of size three (by the 3 -cycle), the elements with cycle structure $(3,4,5)$ are partitioned among the subgroups of MS4. Similar reasoning shows that these elements are also partitioned in MS6 and MS7, respectively. Arguing as we did for the 12-cycles above, we see that the unique minimal covering of these elements uses the 220 subgroups from the class MS4.

We now examine the elements with cycle structure $\left(2,5^{2}\right)$. These elements are only contained in the classes $M S 3$ and $M S 7$. While these elements are partitioned in class $M S 3$, they are not partitioned in class $M S 7$. On the other hand, each subgroup of $M S 3$ contains 72576 elements with cycle structure $\left(2,5^{2}\right)$, whereas each subgroup in class $M S 7$ contains 12096 elements with this cycle structure. Hence removing any collection of subgroups from MS3 requires at least $72576 / 12096=6$ times as many subgroups from $M S 7$, and the unique minimal covering of these elements uses the 66 subgroups from MS3.

Looking at the elements with cycle structure $(4,7)$, we see that these are contained only in subgroups of the classes $M S 2, M S 6$, and $M S 7$. As with elements examined above, these are partitioned among these three classes, so we see that the unique minimal covering of these elements uses the 12 subgroups from $M S 2$.

Finally, we examine the elements with cycle structure $(5,7)$. These elements are contained in the subgroups of the classes $M S 1$ and $M S 7$, and they are partitioned among the subgroups of $M S 7$. Since $M S 1$ only contains one subgroup (the alternating group $A_{12}$ ), the unique minimal cover of the elements with cycle structure $(5,7)$ uses the single subgroup from $M S 1$.

It only remains to be shown now that no collection of subgroups from $M S 7$ is a more efficient cover of some elements with cycle structure $(3,4,5),\left(2,5^{2}\right),(4,7)$, and $(5,7)$ collectively than those listed above. First, in order to cover all the elements with cycle structure $(5,7)$ which are contained in $A_{12}$, the single subgroup in $M S 1$, we would need all 792 subgroups of $M S 7$, which is larger than our bound of 761 . To cover the elements that are lost when a single subgroup of $M S 2$ isomorphic to $S_{11}$ is removed, 330 subgroups of $M S 7$ are required. However,
the 462 subgroups of MS5 are still needed, so this is a total of 792 subgroups, more than our current bound of 761 . Hence we need only consider the elements with cycle structure $(3,4,5)$ and $\left(2,5^{2}\right)$. However, one subgroup of $M S 7$ contains 10080 elements with cycle structure $(3,4,5)$ and 12096 elements with cycle structure $\left(2,5^{2}\right)$ for a total of 22176 elements of one of these two types, whereas one subgroup of $M S 3$ contains 72576 elements with cycle structure $\left(2,5^{2}\right)$, and one subgroup of $M S 4$ contains 36288 elements with cycle structure ( $3,4,5$ ). Since the elements are partitioned across $M S 3$ and $M S 4$, this shows that no collection of subgroups of $M S 7$ can possibly be a more efficient cover.

Putting this all together, we see that each of the classes MS1, MS2, MS3, MS4, and MS5 is necessary in a minimal cover; on the other hand, these five classes together form a cover. Therefore, these five classes together form the unique minimal cover of the elements of $S_{12}$, and the covering number of $S_{12}$ is 761 .

| Label | Isomorphism Type | Group Order | Class Size |
| :---: | :---: | :---: | :---: |
| $M S 1$ | $A_{12}$ | 239500800 | 1 |
| $M S 2$ | $S_{11}\left(\times S_{1}\right)$ | 39916800 | 12 |
| $M S 3$ | $S_{10} \times S_{2}$ | 7257600 | 66 |
| $M S 4$ | $S_{9} \times S_{3}$ | 2177280 | 220 |
| $M S 5$ | $S_{6} \mathrm{wr} S_{2}$ | 1036800 | 462 |
| $M S 6$ | $S_{8} \times S_{4}$ | 967680 | 495 |
| $M S 7$ | $S_{7} \times S_{5}$ | 604800 | 792 |
| $M S 8$ | $S_{4} \mathrm{wr} S_{3}$ | 82944 | 5775 |
| $M S 9$ | $S_{2} \mathrm{wr} S_{6}$ | 46080 | 10395 |
| $M S 10$ | $S_{3} \mathrm{wr} S_{4}$ | 31104 | 15400 |
| $M S 11$ | $\mathrm{PGL}(2,11)$ | 1320 | 362880 |

Table 5.1. Conjugacy classes of maximal subgroups of $S_{12}$.

## 6. The Mathieu Group $M_{12}$

Only as recently as 2010, it was shown by Holmes and Maróti in [16] that for the Mathieu group $M_{12}$ we have $131 \leq \sigma\left(M_{12}\right) \leq 222$. Here we will determine the exact covering number of $M_{12}$.

Theorem 6.1. The covering number of $M_{12}$ is 208.
Before we can prove this theorem, we need a proposition which gives a minimal covering for the elements with cycle structure $(6,6)$. (We note that $M_{12}$ is represented here as a permutation group embedded into $S_{12}$.) In fact, the minimal cover found contains subgroups from three different conjugacy classes of subgroups. This seems to be a first in this context and explains why the covering number for the group $M_{12}$ was not determined any earlier despite its relatively small order. The use of GAP and Gurobi led to this breakthrough.

Proposition 6.2. There exists a covering of the elements with cycle structure $(6,6)$ in $M_{12}$ by 130 subgroups, and this covering is minimal. This covering is made up of 120 subgroups isomorphic to $\operatorname{PSL}(2,11)$ from MS5, eight subgroups isomorphic to $C_{2} \times S_{5}$ from MS8, and two subgroups isomorphic to $\left(C_{4} \times C_{4}\right): D_{6}$ in MS10.

Proof. Using the GAP [10] program listed in Function 8.1 for the elements with cycle structure $(6,6)$ in $M_{12}$ and the appropriate maximal subgroups of $M_{12}$, Gurobi [14] finds that
there exists a covering of the elements with cycle structure $(6,6)$ by 130 subgroups in MS5, $M S 8, M S 10$, and that this covering is minimal.

A list of generators for the subgroups of $M_{12}$ contained in this can be found on line at http://www.math.binghamton.edu/menger/coverings/. Now we are ready to prove our theorem.

| Label | Isomorphism Type | Group Order | Class Size |
| :---: | :---: | :---: | :---: |
| $M S 1$ | $M_{11}$ | 7920 | 12 |
| $M S 2$ | $M_{11}$ | 7920 | 12 |
| $M S 3$ | $\operatorname{P\Gamma L}(2,9)$ | 1440 | 66 |
| $M S 4$ | $\operatorname{P\Gamma L}(2,9)$ | 1440 | 66 |
| $M S 5$ | $\operatorname{PSL}(2,11)$ | 660 | 144 |
| $M S 6$ | $\left(C_{3} \times C_{3}\right):\left(C_{2} \times S_{4}\right)$ | 432 | 220 |
| $M S 7$ | $\left(C_{3} \times C_{3}\right):\left(C_{2} \times S_{4}\right)$ | 432 | 220 |
| $M S 8$ | $S_{5} \times C_{2}$ | 240 | 396 |
| $M S 9$ | $2^{1+4}: S_{3}$ | 192 | 495 |
| $M S 10$ | $\left(C_{4} \times C_{4}\right): D_{12}$ | 192 | 495 |
| $M S 11$ | $A_{4} \times S_{3}$ | 72 | 1320 |

Table 6.1. Conjugacy classes of maximal subgroups of $M_{12}$.

| Order | C.S. | Size | $M S 1$ <br> $(12)$ | $M S 2$ <br> $(12)$ | $M S 3$ <br> $(66)$ | $M S 4$ <br> $(66)$ | $M S 5$ <br> $(144)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 6 | $(2,3,6)$ | 15840 | $1320, P$ | $1320, P$ | $240, P$ | $240, P$ | 0 |
| 6 | $(6,6)$ | 7920 | 0 | 0 | 0 | 0 | $110_{2}$ |
| 8 | $(8,2)$ | 11880 | 0 | 1980 | 0 | 360 | 0 |
| 8 | $(4,8)$ | 11880 | 1980 | 0 | 360 | 0 | 0 |
| 10 | $(2,10)$ | 9504 | 0 | 0 | $144, P$ | $144, P$ | 0 |
| 11 | $(11)$ | 17280 | $1440, P$ | $1440, P$ | 0 | 0 | $120, P$ |


| Order | C.S. | Size | $M S 6$ <br> $(220)$ | $M S 7$ <br> $(220)$ | $M S 8$ <br> $(396)$ | $M S 9$ <br> $(495)$ | $M S 10$ <br> $(495)$ | $M S 11$ <br> $(1320)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 6 | $(2,3,6)$ | 15840 | 144 | 144 | 0 | $32, P$ | 0 | 24 |
| 6 | $(6,6)$ | 7920 | 0 | 0 | 60 | 0 | 32 | $6, P$ |
| 8 | $(8,2)$ | 11880 | 0 | 108 | 0 | $24, P$ | $24, P$ | 0 |
| 8 | $(4,8)$ | 11880 | 108 | 0 | 0 | $24, P$ | $24, P$ | 0 |
| 10 | $(2,10)$ | 9504 | 0 | 0 | $24, P$ | 0 | 0 | 0 |
| 11 | $(11)$ | 17280 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 6.2. Inventory of elements generating maximal cyclic subgroups in $M_{12}$ across conjugacy classes of maximal subgroups.

Proof of Theorem 6.1. It can be easily seen from Table 6.2 that the subgroups in MS1 and MS4 cover all elements in $M_{12}$ generating maximal cyclic subgroups with the exception of elements of cycle structure $(6,6)$. By Proposition 6.2 there exists a covering of the elements of cycle structure $(6,6)$ by 130 subgroups in $M S 5, M S 8$, and $M S 10$. Thus

$$
\sigma\left(M_{12}\right) \leq|M S 1|+|M S 4|+130=12+66+130=208
$$

It remains to be shown that any covering of $M_{12}$ contains at least 208 subgroups. As can be seen from Table 6.2, a covering of the 11-cycles needs to contain at least 12 subgroups of MS1 or $M S 2$. Similarly, a covering of the elements of cycle structure $(2,10)$ needs to contain the 66 subgroups of $M S 3$ or $M S 4$. Since the covering of the elements with cycle structure $(6,6)$ by the 130 subgroups from $M S 5, M S 8$, and $M S 10$ is minimal by Proposition 6.2, it follows that $\sigma\left(M_{12}\right) \geq 12+66+130=208$. We conclude $\sigma\left(M_{12}\right)=208$.

## 7. The Janko Group $J_{1}$

In [15] it was shown by Holmes that $5165 \leq \sigma\left(J_{1}\right) \leq 5415$ for the Janko group $J_{1}$. Using similar methods employed in this paper for $S_{9}$ and $M_{12}$, we were able to improve these bounds. It should be noted here that longer computation times on more powerful machines would likely improve these bounds.

To better utilize the results from [15], we will follow Holmes and use notation from the Atlas [7] rather than representing the groups as a permutation group as done in the previous cases. Recall that conjugacy classes of elements are named by the orders of their elements and a capital letter. They are written in descending order of centralizer size. Here is our improved estimate for $\sigma\left(J_{1}\right)$.

Theorem 7.1. For the covering number of the Janko group $J_{1}$ we have $5281 \leq \sigma\left(J_{1}\right) \leq$ 5414.

Proof. In [15] it is determined that all 1540 maximal subgroups isomorphic to $C_{19}: C_{6}$ and all 2926 maximal subgroups isomorphic to $S_{3} \times D_{10}$ are needed in a minimal covering. The only remaining elements generating maximal cyclic subgroups that need to be covered are those of type $11 A$ and $7 A$. Holmes shows in [15] that only maximal subgroups isomorphic to $\operatorname{PSL}(2,11)$ are needed to cover all elements of type $11 A$, and also only maximal subgroups isomorphic to $C_{2}^{3}: C_{7}: C_{3}$ are needed to cover elements of type $7 A$. Using the GAP program [10] as given in Function 8.1 for $G=J_{1}$ and the maximal subgroups isomorphic to $\operatorname{PSL}(2,11)$, we are setting up the equations readable by Gurobi [14] for the elements of type $11 A$. The Gurobi output then tells us that a minimal covering of the elements of this type consists of at least 186 and at most 196 subgroups isomorphic to $\operatorname{PSL}(2,11)$. Similarly, preparing the linear equations for Gurobi using Function 8.1 for $G=J_{1}$ and the maximal subgroups isomorphic to $C_{2}^{3}: C_{7}: C_{3}$ for the elements of type $7 A$, the Gurobi output shows that the number of subgroups of this type needed to cover the respective elements is between 629 and 752 . (We have included the files produced by GAP which are read by Gurobi on http://www.math.binghamton.edu/menger/coverings/.) Therefore, we find that the subgroup covering number of $J_{1}$ is between $1540+2926+629+186=5281$ and $1540+2926+752+196=5414$.

## 8. GAP Code

In this section, we start with the code used in GAP [10] to create the output files read by Gurobi [14]. Any solution to the system of equations encoded in the output corresponds to a subgroup cover of the elements, and any time the "best objective" and the "best bound" found by Gurobi are identical, Gurobi has found a minimal subgroup cover. In short, GAP is used to
create a system of linear inequalities, the optimal solution to which corresponds to a minimal cover. Gurobi then performs a linear optimization on this system of linear inequalities.

For the case of $S_{9}$, addressed in Proposition 3.3, we include the output of Function 8.1 as well as an abbreviated table of the Gurobi output. A complete table of this output can be found at http://www.math.binghamton.edu/menger/coverings/. The corresponding output of Function 8.1, together with the generators for the subgroups in the minimal cover of elements with cycle structure $(6,6)$ in $M_{12}$ and the linear programs produced for $J_{1}$, can be found at the same website.

Function 8.1. GAP function to create the output files to be read by Gurobi.

```
#SubgroupCoveringNumber takes as input a group $G$, a list of
#elements $L$, a list of maximal subgroups $M$, and the name of a
#file of type .lp to which output is written.
SubgroupCoveringNumber:= function(G, ElementList,
MaximalSubgroupList, filename)
local maxs, maxconjs, x, y, temp, elts, eltconjs, output,
NumberSubgroups, NumberElements, i, j, FilteredSubgroupIndices;
#Subgroup covering number first computes all conjugate subgroups
#of those in the list MaximalSubgroupList.
maxs:= [];
for x in MaximalSubgroupList do
maxconjs:= ConjugateSubgroups (G,x);
for y in maxconjs do
Add(maxs, y);
od;
od;
NumberSubgroups:= Length(maxs);
#All cyclic subgroups generated by the conjugates of the elements
#in ElementList are stored in the irredundant list elts.
elts:= [];
for x in ElementList do
eltconjs:= AsList(ConjugacyClass (G, x));
for y in eltconjs do
if not Group(y) in elts then
Add(elts, Group(y));
fi;
od;
od;
NumberElements:= Length(elts);
#SubgroupCoveringNumber now begins writing to the output file.
#Each variable r1, r2,... represents a binary variable that takes
#on the value 0 or 1. (A 1 represents the subgroup being included
# in the covering; a O means it's not included.)
#First, we write that we want to minimize the sum of all the
#variables, i.e., we want to minimize the number of subgroups
```

```
#included in the covering.
output := OutputTextFile( filename, false );;
    SetPrintFormattingStatus(output, false);
    AppendTo(output,"Minimize\n");
for i in [1..NumberSubgroups] do
        AppendTo(output, Concatenation( " + r", String(i)));
    od;
    AppendTo(output,"\n Subject To\n");
#For each subgroup H in elts, we require that H is a subgroup
#Of at least one maximal subgroup in the covering. This
#corresponds to the sum over all the variables representing
#maximal subgroups containing H being at least 1. Note that
#Gurobi interprets > as ``less than or equal."
for i in [1..NumberElements] do
FilteredSubgroupIndices:= Filtered([1..NumberSubgroups],
    j -> (IsSubgroup(maxs[j],elts[i])));
for j in FilteredSubgroupIndices do
AppendTo(output, " + r", String(j));
od;
AppendTo(output, " > 1\n");
od;
#This last part specifies that each variable is "`Binary," i.e., that
#it can only take on the value 0 or the value 1.
AppendTo(output, "\\ Variables\n");
    AppendTo(output,"Binary\n");
    for i in [1..NumberSubgroups] do
        AppendTo(output, Concatenation( "r", String(i), "\n"));
    od;
    AppendTo(output,"End\n");
    CloseStream(output);
    return maxs;
#The function returns the list of maximal subgroups.
end;
```

As a sample of the output of Function 8.1 we will show how the calculations proceed for the elements with cycle structure $(3,6)$ in the group $S_{9}$. First, we use GAP to create a file that is readable by the optimization software Gurobi:

```
gap> G:= SymmetricGroup(9);
Sym( [ 1 .. 9 ] )
gap> max:= MaximalSubgroupClassReps(G);
[ Alt( [ 1 .. 9 ] ), Group([ (1,2,3,4,5), (1,2), (6,7,8,9), (6,7) ]),
    Group([ (1,2,3,4,5,6), (1,2), (7,8,9), (7,8) ]),
    Group([ (1,2,3,4,5,6,7), (1,2), (8,9) ]),
    Group([ (1,2,3,4,5,6,7,8), (1,2) ]),
    Group([ (1,2,3), (1,2), (4,5,6), (4,5), (7,8,9), (7,8),
        (1,4,7)(2,5,8) (3,6,9), (1,4) (2,5) (3,6) ]),
    Group([ (4,7)(5,8)(6,9), (2,7,6) (3,4,8), (1,2,3) (4,5,6)(7,8,9) ]) ]
gap> M:= [max[3], max[6], max[7]];
```

```
[ Group([ (1,2,3,4,5,6), (1,2), (7,8,9), (7,8) ]),
    Group([ (1,2,3), (1,2), (4,5,6), (4,5), (7, 8,9), (7,8),
        (1,4,7) (2,5,8) (3,6,9), (1,4) (2,5) (3,6) ]),
    Group([ (4,7)(5,8)(6,9), (2,7,6)(3,4,8), (1,2,3) (4,5,6)(7,8,9) ]) ]
gap> g:= (1,2,3) (4,5,6,7,8,9);
(1,2,3) (4,5,6,7,8,9)
gap> L:= [g];
[ (1,2,3)(4,5,6,7,8,9) ]
gap> Read("Programs/SubgroupCoveringNumber.g");
gap> l:= SubgroupCoveringNumber(G,L,M, "S9.lp");;
gap> time;
218128
```

Note that only one element with cycle structure $(3,6)$ is needed in the list $L$ since all elements with the same cycle structure are conjugate in a symmetric group. We next use Gurobi to optimize this system of linear equations. We have removed some lines of the output here for the sake of brevity, although the full output is available online at http://www.math.binghamton.edu/menger/coverings/.

```
gurobi> m = read("S9.lp")
Read LP format model from file S9.lp
Reading time = 0.09 seconds
(null): 10080 rows, 1204 columns, 80640 nonzeros
gurobi> m.optimize()
Optimize a model with 10080 rows, 1204 columns and 80640 nonzeros
Found heuristic solution: objective 423
Presolve time: 0.10s
Presolved: 10080 rows, 1204 columns, 80640 nonzeros
Variable types: 0 continuous, 1204 integer (1204 binary)
Root relaxation: objective 7.000000e+01, 2182 iterations, 0.19 seconds
```

| Nodes | Current Node | Objective Bounds | Work |  |
| :---: | :---: | :---: | :---: | :---: |
| Expl Unexpl | Obj Depth IntInf | Incumbent | BestBd | Gap | It/Node Time



```
Cutting planes:
    Zero half: 107
```

Explored 717 nodes (514185 simplex iterations) in 453.03 seconds
Thread count was 8 (of 8 available processors)
Optimal solution found (tolerance 1.00e-04)
Best objective $8.400000000000 e+01$, best bound $8.400000000000 \mathrm{e}+01$, gap $0.0 \%$

The "Best objective" is the best actual solution that was found by Gurobi, and the size of this solution is 84 . The "best bound" is the size of the best lower bound that Gurobi could determine for a solution to this system of equations, and this lower bound is also 84 . Therefore, we conclude that the covering of the elements with cycle structure $(3,6)$ in $S_{9}$ by the 84 subgroups in MS3 is minimal.

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Department of Mathematical Sciences, State University of New York at Binghamton, Binghamton, NY 13902-6000, USA.

E-mail address: menger@math.binghamton.edu
Department of Mathematial Sciences, Florida Atlantic University, Boca Raton, FL 33431, USA.

E-mail address: dpopova@fau.edu
Centre for the Mathematics of Symmetry and Computation, School of Mathematics and Statistics, The University of Western Australia, 35 Stirling Highway, Crawley, W.A. 6009, Australia.

E-mail address: eric.swartz@uwa.edu.au


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