On irreducible algebraic sets over linearly ordered semilattices

Artem N. Shevlyakov

October 11, 2018

Abstract

Equations over linearly ordered semilattices are studied. For any equation t(X) = s(X) we find irreducible components of its solution set and compute the average number of irreducible components of all equations in n variables.

1 Introduction

This paper is devoted to the following problem. One can define a notion of an equation over a linearly ordered semilattice $L_l = \{a_1, a_2, \ldots, a_l\}$ (the formal definition of an equation is given below). A set Y is algebraic if it is the solution set of some system of equations over L_l . Let us consider an equation t(X) = s(X) over L_l , and Y be the solution set of t(X) = s(X). One can find algebraic sets Y_1, Y_2, \ldots, Y_m such that $Y = \bigcup_{i=1}^m Y_i$. One can decompose each Y_i into a union of other algebraic sets, etc. This process terminates after a finite number of steps and gives a decomposition of Y into a union of irreducible algebraic sets Y_i (the sets Y_i are called the irreducible components of Y). Roughly speaking, irreducible algebraic sets are "atoms" which form any algebraic set. The size and the number of such "atoms" are important characteristics of the semilattices L_l , since there are connections between irreducible algebraic sets and universal theory of linearly ordered semilattices (see [1]). Moreover, the number of irreducible components was involved in the estimation of lower bounds of algorithm complexity (see [2] for more details).

In this paper (Section 4) we study the properties of the irreducible components of the solution set Y of an equation t(X) = s(X). Precisely, we prove that the union of irreducible algebraic sets $Y = \bigcup_{i=1}^m Y_i$ is redundant, i.e. the intersections $\bigcap_{i \in I} Y_i$ (|I| < m) consists of many points (Proposition 4.5). Moreover, for any equation t(X) = s(X) in n variables we count the number m of irreducible components (see (6)), and in Section 5 we count the average number $\overline{\operatorname{Irr}}(n,l)$ of irreducible components of the solution sets of equations in n variables.

2 Main definitions

Let $L_l = \{a_1, a_2, \ldots, a_l\}$ be the linearly ordered semilattice of l elements and $a_1 < a_2 < \ldots < a_l$. The multiplication in L_l is defined by $a_i \cdot a_j = a_{\min(i,j)}$. Obviously, the linear order on L_l can be expressed by the multiplication as follows

$$a_i \le a_j \Leftrightarrow a_i a_j = a_i$$
.

A term t(X) in variables $X = \{x_1, x_2, \dots, x_n\}$ is a commutative word in letters x_i .

Let $\operatorname{Var}(t)$ be the set of all variables occurring in a term t(X). Following [1], an equation is an equality of terms t(X) = s(X). Below we consider inequalities $t(X) \leq s(X)$ as equations, since $t(X) \leq s(X)$ is the short form of t(X)s(X) = t(X). Notice that we consider equations as ordered pairs of terms, i.e. the expressions t(X) = s(X), s(X) = t(X) are different equations. Let Eq(n) denote the set of all equations in $X = \{x_1, x_2, \ldots, x_n\}$ variables (we assume that each $t(X) = s(X) \in Eq(n)$ contains the occurrences of all variables x_1, x_2, \ldots, x_n). An equation $t(X) = s(X) \in Eq(n)$ is said to be a (k_1, k_2) -equation if $|\operatorname{Var}(t) \setminus \operatorname{Var}(s)| = k_1$ and $|\operatorname{Var}(s) \setminus \operatorname{Var}(t)| = k_2$. For example, $x_1x_2 = x_1x_3x_4$ is a (1, 2)-equation. Let $Eq(k_1, k_2, n) \subseteq Eq(n)$ be the set of all (k_1, k_2) -equations in n variables. Obviously,

$$Eq(n) = \bigcup_{(k_1, k_2) \in K_n} Eq(k_1, k_2, n), \tag{1}$$

where

$$K_n = \{(k_1, k_2) \mid k_1 + k_2 \le n\} \setminus \{(0, n), (n, 0)\}.$$

Each equation $t(X) = s(X) \in Eq(k_1, k_2, n)$ is uniquely defined by k_1 variables in the left part and by k_2 other variables in the right part (the residuary $n - k_1 - k_2$ variables should occur in both parts of the equation). Thus,

$$#Eq(k_1, k_2, n) = \binom{n}{k_1} \binom{n - k_1}{k_2}.$$

By (1), one can compute

$$#Eq(n) = 3^n - 2.$$

Remark 2.1. In this paper we consider only equations t(X) = s(X) with n > l, i.e. the number of variables occurring in t(X) = s(X) is more than the order of the semilattice L_l . The case $n \leq l$ needs the different technic and was announced in [3].

A point $P \in L_l^n$ is a solution of an equation t(X) = s(X) if t(P), s(P) define the same element in the semilattice L_l . By the properties of linearly ordered semilattices, a point $P = (p_1, p_2, \ldots, p_n)$ is a solution of t(X) = s(X) iff there exist variables $x_i \in \text{Var}(t), x_j \in \text{Var}(s)$ such that $p_i = p_j$ and $p_i \leq p_k$ for all $1 \leq k \leq n$. The set of all solutions of an equation t(X) = s(X) is denoted by V(t(X) = s(X)).

An arbitrary set of equations is called a *system*. The set of all solutions V(S) of a system $S = \{t_i(X) = s_i(X) \mid i \in I\}$ is defined as $\bigcap_{i \in I} V(t_i(X) = s_i(X))$. A set $Y \subseteq L_l^n$ is called *algebraic over* L_l if there exists a system S in n variables with V(S) = Y. An algebraic set Y is *irreducible* if Y is not a proper finite union of other algebraic sets.

Proposition 2.2. Any algebraic set Y over L_l is a finite union of irreducible sets

$$Y = Y_1 \cup Y_2 \cup \ldots \cup Y_m, \quad Y_i \not\subseteq Y_i \text{ for all } i \neq j,$$
 (2)

and this decomposition is unique up to a permutation of components.

Proof. A semilattice S is equationally Noetherian if for any infinite system \mathbf{S} in variables $X = \{x_1, x_2, \dots, x_n\}$ there exists a finite subsystem $\mathbf{S}' \subseteq \mathbf{S}$ with the same solution set. According to [1], the decomposition (2) holds for any algebraic set Y over an equationally Noetherian semilattice S. Thus, it is sufficient to prove that L_l is equationally Noetherian.

However the condition $|Eq(n)| < \infty$ gives that there is not any infinite system over L_l . Thus, L_l is equationally Noetherian.

The subsets Y_i from the union (2) are called the *irreducible components* of Y. Let Y be an algebraic set over L_l defined by a system $\mathbf{S}(X)$. One can define an equivalence relation \sim_Y over the set of all terms in variables X as follows

$$t(X) \sim_Y s(X) \Leftrightarrow t(P) = s(P)$$
 for any point $P \in Y$.

The set of \sim_Y -equivalence classes is called the coordinate semilattice of Y and denoted by $\Gamma(Y)$ (see [1] for more details). The following statement describes the coordinate semilattices of irreducible algebraic sets.

Proposition 2.3. A set Y is irreducible over L_l iff $\Gamma(Y)$ is embedded into L_l

Proof. Following [1], $\Gamma(Y)$ is discriminated by L_l iff Y is irreducible (see [1] for the definition of the discrimination). However for a finite semilattice L_l the discrimination is equivalent to the embedding.

There are different algebraic sets over L_l with isomorphic coordinate semilattices. Such sets are called *isomorphic*. For example, the following sets

$$Y_1 = V(\{x_1 \le x_2 \le x_3\}), Y_2 = V(\{x_3 \le x_2 \le x_1\})$$

has the isomorphic coordinate semilattices

$$\Gamma(Y_1) = \langle x_1, x_2, x_3 \mid x_1 \le x_2 \le x_3 \rangle \cong L_3,$$

$$\Gamma(Y_2) = \langle x_1, x_2, x_3 \mid x_3 \le x_2 \le x_1 \rangle \cong L_3.$$

Thus, Y_1, Y_2 are isomorphic.

3 Example

Let n=3, l=2. We have exactly $Eq(3)=3^3-2=25$ equations in three variables over L_2 . The following table contains the information about such equations over L_2 . The second column contains systems which define irreducible components of the solution set of an equation in the first column. A cell of the table contains \uparrow if

an information in this cell is similar to the cell above.

information in this cell is similar to the cell above.		
Equations	Irreducible components (IC)	Number of IC
$x_1 x_2 x_3 = x_1 x_2 x_3$	$x_1 \le x_2 = x_3 \cup x_1 = x_2 \le x_3 \cup x_3 \cup x_4 = x_4 $	6
	$x_2 \le x_1 = x_3 \cup x_3 \le x_1 = x_2 \cup x_3$	
	$x_1 = x_3 \le x_2 \cup x_2 = x_3 \le x_1$	
$x_1 = x_1 x_2 x_3,$	$x_1 \le x_2 = x_3 \cup x_1 = x_2 \le x_3 \cup x_3 \cup x_4 = x_3 \cup x_4 = x_3 \cup x_4 = x_3 \cup x_4 = x_4 $	3
$x_1 x_2 x_3 = x_1$	$x_1 = x_3 \le x_2$	
$x_2 = x_1 x_2 x_3,$	↑	3
$x_1 x_2 x_3 = x_2$		
$x_3 = x_1 x_2 x_3,$	↑	3
$x_1 x_2 x_3 = x_3$		
$x_1 = x_2 x_3,$	$x_1 = x_2 \le x_3 \cup x_1 = x_3 \le x_2$	2
$x_2x_3 = x_1$		
$x_2 = x_1 x_3,$	↑	2
$x_1x_3 = x_2$		
$x_3 = x_1 x_2,$	↑	2
$x_1 x_2 = x_3$		
$x_1x_2 = x_1x_3,$	$x_1 = x_2 \le x_3 \cup x_1 = x_3 \le x_2 \cup x_3 = x_3 = x_3 \cup x_1 = x_3 = x_2 \cup x_3 = x_3 $	4
$x_1x_3 = x_1x_2$	$x_1 \le x_2 = x_3 \cup x_2 = x_3 \le x_1$	
$x_1x_2 = x_2x_3,$	↑	4
$x_2x_3 = x_1x_2$		
$x_1x_3 = x_2x_3,$	↑	4
$x_2x_3 = x_1x_3$		
$x_1 x_2 = x_1 x_2 x_3,$	$x_1 = x_2 \le x_3 \cup x_1 = x_3 \le x_2 \cup x_3 = x_3 = x_3 \cup x_1 = x_3 = x_2 \cup x_3 = x_3 $	5
$x_1 x_2 x_3 = x_1 x_2$	$x_1 \le x_2 = x_3 \cup x_2 = x_3 \le x_1 \cup x_1 = x_2 = x_3 = x_1 \cup x_2 = x_3 = x_1 \cup x_2 = x_3 = x_1 \cup x_2 = x_3 = x_2 = x_3 $	
	$x_2 \le x_1 = x_3$	
$x_1 x_3 = x_1 x_2 x_3,$	<u></u>	5
$x_1 x_2 x_3 = x_1 x_3$		
$x_2 x_3 = x_1 x_2 x_3,$	<u></u>	5
$x_1 x_2 x_3 = x_2 x_3$		

One can directly compute the average number of irreducible components of algebraic sets defined by equations in three variables:

$$\overline{\operatorname{Irr}}(3,2) = \frac{6 + 2(3 + 3 + 3 + 2 + 2 + 2 + 2 + 4 + 4 + 4 + 5 + 5 + 5)}{25} = \frac{90}{25} = 3.6 \quad (3)$$

Recall that in Section 5 we obtain the general expression for $\overline{\operatorname{Irr}}(n,l)$ (7). Clearly, (7) gives (3) for n=3, l=2 (see the proof in (8) and (9)).

4 Decompositions of algebraic sets

Let Y denote the solution set of an equation t(X) = s(X) over the semilattice $L_l = \{a_1, a_2, \ldots, a_l\}$. The table above shows that any irreducible component divides the variables X into l classes and sorts the classes in some order. The following definition formalizes such properties of irreducible components.

A disjoint partition $\sigma = (X_1, X_2, \dots, X_l)$ of the set $X = \{x_1, x_2, \dots, x_n\}$ is called ordered if there is a linear order \leq_{σ} on σ : $X_1 \leq_{\sigma} X_2 \leq_{\sigma} \dots \leq_{\sigma} X_l$. Let $\chi_{\sigma}(x_i)$ denote the class X_k with $x_i \in X_k$.

We shall denote $x_i =_{\sigma} x_j$ $(x_i \leq_{\sigma} x_j)$ if $\chi(x_i) = \chi(x_j)$ (respectively, $\chi_{\sigma}(x_i) \leq_{\sigma} \chi_{\sigma}(x_j)$).

An ordered partition σ is Y-irreducible if the set X_1 (the minimal set of the order \leq_{σ}) contains a variable from t(X) and a variable from s(X).

For example, an equation $x_1x_2x_3 = x_1$ over L_2 has the following Y-irreducible partitions: $(\{x_1\}, \{x_2, x_3\}), (\{x_1, x_2\}, \{x_3\}), (\{x_1, x_3\}, \{x_2\})$. Such partitions obviously correspond to irreducible components of $V(x_1x_2x_3 = x_1)$ in the table above.

Any Y-irreducible partition σ defines an algebraic set Y_{σ} as follows

$$Y_{\sigma} = V(\mathbf{S}_{\sigma}) = V(\bigcup_{x_i = \sigma x_j} \{x_i = x_j\} \bigcup_{x_i < \sigma x_j} \{x_i \le x_j\}).$$

For example, the partition $\sigma = (\{x_2, x_3\}, \{x_1\})$ defines the system

$$\mathbf{S}_{\sigma} = \{x_2 = x_3, x_2 \le x_1, x_3 \le x_1\}.$$

for $Y = V(\{x_1x_2 = x_1x_3\}).$

Lemma 4.1. The set Y_{σ} defined by a Y-irreducible partition σ is an irreducible algebraic set, and moreover $\Gamma(Y_{\sigma}) \cong L_l$.

Proof. By the definition of a coordinate semilattice, $\Gamma(Y_{\sigma})$ is generated by the elements $\{x_1, x_2, \dots, x_n\}$ and has the following defined relations

$$\{x_i=x_j\mid \text{ if } x_i=_\sigma x_j\}\cup\{x_i\leq x_j\mid \text{ if } x_i\leq_\sigma x_j\}.$$

It is easy to see that all elements x_i are linearly ordered in $\Gamma(Y_{\sigma})$. Thus, $\Gamma(Y_{\sigma})$ is a linearly ordered semilattice, and it is isomorphic to L_l . By Proposition 2.3, the set Y_{σ} is irreducible.

The following lemma gives the decomposition of the set Y = V(t(X) = s(X)) via ordered partitions.

Lemma 4.2. The set Y = V(t(X) = s(X)) is a union

$$Y = \bigcup_{\sigma \text{ is } Y \text{-}irreducible} Y_{\sigma} \tag{4}$$

Proof. Let $P = (p_1, p_2, \dots, p_n) \in Y$. One can define an equivalence relation \sim_P as follows

$$x_i \sim_P x_j \Leftrightarrow p_i = p_j$$
.

Thus, we obtain equivalence classes $\{X_1^P, X_2^P, \dots, X_k^P\}$. Since $p_i \in L_l$, $k \leq l$. One can define a linear order $x_i \leq_P x_j$ if $p_i \leq p_j$. The order \leq_P induces a linear order over the classes $\{X_i\}$. Let us fix a pair of variables $x_t, x_s \in X_1^P$ (probably, x_t, x_s is the same variable) such that $x_t \in \text{Var}(t)$ and $x_s \in \text{Var}(s)$ (such pair (x_t, x_s) always exists, since P satisfies the equation t(X) = s(X)). Let us find a set Y_σ with $P \in Y_\sigma$ by the following procedure.

Procedure

Input: a set of k equivalence classes $\sigma_0 = (X_1^P, X_2^P, \dots, X_k^P)$ with the linear order \leq_P .

Output: $\sigma = (X_1, X_2, \dots, X_l)$ with a linear order \leq_{σ} .

Step 0: Put $\sigma = \sigma_0$. If l = k terminate the procedure, otherwise go to the step 1.

Step j $(1 \le j \le l - k)$:

- 1. Take an arbitrary equivalence class $X_i \in \sigma = (X_1, X_2, \dots, X_{k+j-1})$ such that $|X_i| \geq 2$ and X_i contains a variable $x \in X \setminus \{x_t, x_s\}$. Such class always exists, since n > l > k + j 1.
- 2. Move x from X_i to a new class X' and define a linear order \leq_{σ} by $X_i \leq_{\sigma} X' \leq X_{i+1}$. Put $\sigma = (X_1, X_2, \dots, X_i, X', X_{i+1}, \dots, X_{l+j-1})$. Go to the next step.

Roughly speaking, the procedure increases the number of classes preserving the relation $<_{\sigma}$.

After the procedure we obtain an ordered partition σ of l equivalence classes X_i . The procedure does not move the variables x_t, x_s , therefore $x_t, x_s \in X_1$ and σ is a Y-irreducible partition.

Let us prove $P \in Y_{\sigma} = V(\mathbf{S}_{\sigma})$. An equation $x_i \leq x_j \in \mathbf{S}_{\sigma}$ (one can similarly consider an equality $x_i = x_j \in \mathbf{S}_{\sigma}$) is not satisfied by P if $p_i > p_j$ or equivalently $x_j <_P x_i$. Since the procedure preserves the relation $<_{\sigma}$, we have $x_j <_{\sigma} x_i$, and by the definition of \mathbf{S}_{σ} , the equation $x_i \leq x_j$ can not occur in \mathbf{S}_{σ} . Thus, we came to the contradiction.

Let us prove now $Y_{\sigma} \subseteq Y$ for each σ . Consider a point $P = (p_1, p_2, \ldots, p_n) \in Y_{\sigma}$. Since $\sigma = (X_1, X_2, \ldots, X_l)$ is a Y-irreducible partition, the class X_1 contains variables $x_t \in \text{Var}(t)$, $x_s \in \text{Var}(s)$ and $p_t = p_s$. Since X_1 is the minimal class of the order \leq_{σ} ,

$$x_t \le x_i \in \mathbf{S}_{\sigma}, \ x_s \le x_i \in \mathbf{S}_{\sigma} \text{ for any } i \in [1, n] \setminus \{t, s\}.$$

Thus, $p_t = p_s \le p_i$ for any $1 \le i \le n$, and we have

$$t(P) = p_t = p_s = s(P) \Rightarrow P \in V(t(X) = s(X)) = Y.$$

Let $\sigma = (X_1, X_2, \dots, X_l)$ be a Y-irreducible partition of X. Let us define a point $P_{\sigma} = (p_1, p_2, \dots, p_n) \in L_l^n$ by

$$p_i = a_k \text{ if } x_i \in X_k.$$

Lemma 4.3. The point P_{σ} belongs to the set Y_{σ} , and $P_{\sigma} \notin Y_{\sigma'}$ for each Y-irreducible partition $\sigma' \neq \sigma$. Thus, in the union (4) $Y_{\sigma} \nsubseteq Y_{\sigma'}$ for distinct partitions σ, σ' .

Proof. One can directly prove that $P_{\sigma} \in V(\mathbf{S}_{\sigma}) = Y_{\sigma}$.

Let us take an irreducible partition

$$\sigma' = (X_1', X_2', \dots, X_l') \neq \sigma = (X_1, X_2, \dots, X_l).$$

There exist variables x_i, x_j such that $x_i <_{\sigma} x_j$ but $x_i \ge_{\sigma'} x_j$. For the point P_{σ} we have $p_i < p_j$, therefore P_{σ} does not satisfy the equation $x_i \ge x_j \in \mathbf{S}_{\sigma'}$, and $P_{\sigma} \notin Y_{\sigma'}$.

According to Lemmas 4.1, 4.2, 4.3, we obtain the following statement.

Theorem 4.4. The number of Y-irreducible partitions of a set Y = V(t(X) = s(X)) is equal to the number of irreducible components of Y.

The next statement describes the properties the union (4).

Proposition 4.5. Let (4) be a union of the irreducible components of a set Y = V(t(X) = s(X)) over L_l . Then

1. a point P belongs to all Y_{σ} iff $P = (a, a, \ldots, a)$ for some $a \in L_l$;

2.

$$Y_{\sigma} \setminus \bigcup_{\sigma' \neq \sigma} Y_{\sigma'} = \{P_{\sigma}\}$$

(it follows that the decomposition (4) is redundant, i.e. each point of $Y \setminus \bigcup_{\sigma} \{P_{\sigma}\}$ is covered by at least two irreducible components);

- 3. all irreducible components are isomorphic to each other;
- 4. $|Y_{\sigma}| = {2l-1 \choose l}$ for each σ .
- Proof. 1. Obviously, P = (a, a, ..., a) satisfies all systems \mathbf{S}_{σ} , so $P \in \bigcap_{\sigma} Y_{\sigma}$. Let us consider a point $Q = (q_1, q_2, ..., q_n)$ with $q_i < q_j$. It is clear that Q does not satisfy any set Y_{σ} with $x_i \geq_{\sigma} x_j$. Thus, $Q \notin \bigcap_{\sigma} Y_{\sigma}$.
 - 2. In Lemma 4.3 we proved $P_{\sigma} \in Y_{\sigma}$. By the definition, only the point P_{σ} makes all inequalities \leq of the system \mathbf{S}_{σ} strict. Thus, for any point $P = (p_1, p_2, \ldots, p_n) \in Y_{\sigma} \setminus \{P_{\sigma}\}$ there exists an equation $x_i \leq x_j \in \mathbf{S}_{\sigma}$ such that $p_i = p_j$. Below we find an irreducible partition σ' with $P \in Y_{\sigma'}$. Let $\sigma = (X_1, X_2, \ldots, X_l), x_i \in X_{i'}$ and without loss of generality one can

Let $\sigma = (X_1, X_2, \dots, X_l)$, $x_i \in X_{i'}$ and without loss of generality one can assume that $x_j \in X_{i'+1}$. If $i' \neq 1$ we put $\sigma' = (X'_1, X'_2, \dots, X'_l)$ where

$$X'_{k} = \begin{cases} X_{k} \text{ if } k \neq i', \ k \neq i' + 1, \\ (X_{i'+1} \setminus \{x_{j}\}) \cup \{x_{i}\} \text{ if } k = i' + 1, \\ (X_{i'} \setminus \{x_{i}\}) \cup \{x_{j}\} \text{ if } k = i' \end{cases}$$
 (5)

Since $X_1' = X_1$, σ' is a Y-irreducible partition. The system $\mathbf{S}_{\sigma'}$ contains $x_j \leq x_i$ instead of $x_i \leq x_j \in \mathbf{S}_{\sigma}$. Since other relations in the systems $\mathbf{S}_{\sigma'}, \mathbf{S}_{\sigma}$ are the same, $P \in V(\mathbf{S}_{\sigma'}) = Y_{\sigma'}$.

Suppose now i' = 1. Without loss of generality we assume $x_i \in \text{Var}(t)$. By the definition of a Y-irreducible partition, there exists a variable $x_k \in X_1 \cap \text{Var}(s)$. If $x_j \in \text{Var}(t)$ we can define σ' by (5). In this case X'_1 contains variables $x_j \in \text{Var}(t)$, $x_k \in \text{Var}(s)$, so σ' is an Y-irreducible partition and $P \in Y_{\sigma'}$. Otherwise $(x_j \in \text{Var}(s))$, one can take x_k instead x_i and repeat all reasonings above.

- 3. The statement immediately follows from Lemma 4.1.
- 4. For $\sigma = (X_1, X_2, \dots, X_l)$ the number $|Y_{\sigma}|$ is equal to the number of sequences $X_1 \leq X_2 \leq \dots \leq X_l$ with $X_i \in \{a_1, a_2, \dots, a_l\}$. According to combinatorics, the number of such monotone sequences is $\binom{2l-1}{l}$.

5 Average number of irreducible components

Let $\binom{n}{m}$ be the Stirling number of the second kind. By the definition, $\binom{n}{m}$ is the number of all partitions of an n-element set into m non-empty unlabelled subsets. The number $\binom{n}{m}^* = m! \binom{n}{m}$ obviously equals the number of all partitions of n-element set into m labelled non-empty subsets. Thus, there are exactly $\binom{n}{l}^*$ ordered partitions $\sigma = (X_1, X_2, \ldots, X_l)$ of the set of variables X, |X| = n into l equivalence classes. An ordered partition $\sigma = (X_1, X_2, \ldots, X_l)$ is not Y-irreducible if either

 $X_1 \subseteq \operatorname{Var}(t) \setminus \operatorname{Var}(s)$ or $X_1 \subseteq \operatorname{Var}(s) \setminus \operatorname{Var}(t)$ For a (k_1, k_2) -equation t(X) = s(X) there exists

$$\sum_{i=1}^{k_1} \binom{k_1}{i} \binom{n-i}{l-1}^*$$

partitions σ with $X_1 \subseteq \text{Var}(t) \setminus \text{Var}(s)$. Similarly, there exist

$$\sum_{i=1}^{k_2} \binom{k_2}{i} \binom{n-i}{l-1}^*$$

partitions σ with $X_1 \subseteq \text{Var}(s) \setminus \text{Var}(t)$.

By Theorem 4.4, for a (k_1, k_2) -equation t(X) = s(X) the number of irreducible components (Y-irreducible partitions) equals

$$\operatorname{Irr}(k_1, k_2, n, l) = {n \brace l}^* - \sum_{i=1}^{k_1} {k_1 \choose i} {n-i \brace l-1}^* - \sum_{i=1}^{k_2} {k_2 \choose i} {n-i \brack l-1}^*.$$
 (6)

The average number of irreducible components of algebraic sets defined by equations from Eq(n) is

$$\begin{split} \overline{\operatorname{Irr}}(n,l) &= \frac{\sum_{(k_1,k_2) \in K_n} \#Eq(k_1,k_2,n) \operatorname{Irr}(k_1,k_2,n,l)}{\#Eq(n)} = \\ &\underline{\sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \#Eq(k_1,k_2,n) \operatorname{Irr}(k_1,k_2,n,l) - \#Eq(0,n,n) \operatorname{Irr}(0,n,n,l)}}{\#Eq(n)} \end{split}$$

Below we compute Irr using the following denotations:

1. $A \stackrel{(1)}{=} B$: an expression B is obtained from A by the binomial theorem

$$(a+b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i}.$$

2. $A \stackrel{(2)}{=} B$: an expression B is obtained from A by the following identity of binomial coefficients

$$\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c}.$$

3. $A \stackrel{(3)}{=} B$: an expression B is obtained from A by the recurrence relation of Stirling numbers

$${a+1 \brace b} = b {a \brace b} + {a \brace b-1}.$$

4. $A \stackrel{(4)}{=} B$: an expression B is obtained from A by the following identity of Stirling numbers

$${a+1 \brace b+1} = \sum_{i=0}^{a} {a \choose i} {i \brace b}.$$

Remark that in the last formula one can change the sum $\sum_{i=0}^{a}$ to $\sum_{i=c}^{a}$ (c < b), since $\binom{c}{b} = 0$ for c < b.

8

We have

$$\begin{split} \#Eq(0,n,n) & \text{Irr}(0,n,n,l) = \binom{n}{0} \binom{n}{n} \left(\binom{n}{l}^* - \sum_{i=1}^n \binom{n}{i} \binom{n-i}{l-1}^* \right) = \\ \left\{ \binom{n}{l}^* - \sum_{i=1}^n \binom{n}{n-i} \binom{n-i}{l-1}^* = \binom{n}{l}^* - \sum_{j=0}^{n-1} \binom{n}{j} \binom{j}{l-1}^* = \binom{n}{l}^* - (l-1)! \sum_{j=0}^{n-1} \binom{n}{j} \binom{j}{l-1}^* = \binom{n}{l}^* - (l-1)! \binom{n}{j} \binom{j}{l-1}^* = \binom{n}{l}^* - (l-1)! \binom{n}{l}^* = 0, \end{split}$$

$$\sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \# Eq(k_1, k_2, n) \operatorname{Irr}(k_1, k_2, n) = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} \left(\begin{Bmatrix} n \\ l \end{Bmatrix}^* - \sum_{i=1}^{k_1} \binom{k_1}{i} \begin{Bmatrix} n-i \\ l-1 \end{Bmatrix}^* - \sum_{i=1}^{k_2} \binom{k_2}{i} \begin{Bmatrix} n-i \\ l-1 \end{Bmatrix}^* \right) = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} \begin{Bmatrix} n \\ l \end{Bmatrix}^* - \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} \sum_{i=1}^{k_1} \binom{k_1}{i} \begin{Bmatrix} n-i \\ l-1 \end{Bmatrix}^* - \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1} \binom{n-k_1}{k_2} \sum_{i=1}^{k_2} \binom{k_2}{i} \begin{Bmatrix} n-i \\ l-1 \end{Bmatrix}^* = S_1 - S_2 - S_3,$$

where

$$S_1 = \begin{Bmatrix} n \\ l \end{Bmatrix}^* \sum_{k_1=0}^{n-1} \binom{n}{k_1} 2^{n-k_1} \stackrel{(1)}{=} \begin{Bmatrix} n \\ l \end{Bmatrix}^* (3^n - 1),$$

$$S_{2} \stackrel{(2)}{=} \sum_{k_{1}=0}^{n-1} \sum_{i=1}^{k_{1}} \binom{n}{k_{1}} \binom{k_{1}}{i} \begin{Bmatrix} n-i \\ l-1 \end{Bmatrix}^{*} \sum_{k_{2}=0}^{n-k_{1}} \binom{n-k_{1}}{k_{2}} \stackrel{(1)}{=}$$

$$\sum_{k_{1}=0}^{n-1} \sum_{i=1}^{k_{1}} \binom{n}{i} \binom{n-i}{k_{1}-i} \begin{Bmatrix} n-i \\ l-1 \end{Bmatrix}^{*} 2^{n-k_{1}} = \sum_{i=1}^{n-1} \binom{n}{i} \begin{Bmatrix} n-i \\ l-1 \end{Bmatrix}^{*} \sum_{k_{1}=i}^{n-1} \binom{n-i}{k_{1}-i} 2^{n-k_{1}} = \sum_{i=1}^{n-1} \binom{n}{i} \begin{Bmatrix} n-i \\ l-1 \end{Bmatrix}^{*} \sum_{k_{1}=i}^{n-1} \binom{n-i}{k_{1}-i} 2^{n-k_{1}} = \sum_{i=1}^{n-1} \binom{n}{i} \begin{Bmatrix} n-i \\ l-1 \end{Bmatrix}^{*} \binom{n-i}{n-i-j} 2^{n-i-j} - 1 \stackrel{(1)}{=} \sum_{i=1}^{n-1} \binom{n}{i} \begin{Bmatrix} n-i \\ l-1 \end{Bmatrix}^{*} (3^{n-i}-1).$$

Computing

$$\sum_{i=1}^{n-1} \binom{n}{i} \binom{n-i}{l-1}^* = (l-1)! \sum_{j=1}^{n-1} \binom{n}{j} \binom{j}{l-1} \stackrel{(4)}{=} (l-1)! \left(\binom{n+1}{l} - \binom{n}{l-1} \right) \stackrel{(3)}{=} (l-1)! \binom{n}{l} = \binom{n}{l}^*,$$

we obtain

$$S_2 = \sum_{i=1}^{n-1} \binom{n}{i} \binom{n-i}{l-1}^* 3^{n-i} - \binom{n}{l}^* = S(n,l) - \binom{n}{l}^*,$$

where

$$S(n,l) = \sum_{i=1}^{n-1} {n \choose i} {n-i \choose l-1}^* 3^{n-i}.$$

Let us compute

$$S_{3} = \sum_{k_{1}=0}^{n-1} \sum_{i=1}^{n-k_{1}} \sum_{k_{2}=i}^{n-k_{1}} \binom{n}{k_{1}} \binom{n-k_{1}}{i} \binom{n-k_{1}-i}{k_{2}-i} \binom{n-i}{k_{1}-1}^{*} = \sum_{k_{1}=0}^{n-1} \sum_{i=1}^{n-k_{1}} \binom{n}{k_{1}} \binom{n-k_{1}}{i} \binom{n-k_{1}}{i} \binom{n-k_{1}}{i} \binom{n-k_{1}-i}{k_{2}-i} \binom{n-k_{1}-i}{i} \stackrel{(1)}{=} \sum_{k_{1}=0}^{n-1} \sum_{i=1}^{n-k_{1}} \binom{n}{k_{1}} \binom{n-k_{1}}{i} \binom{n-i}{k_{1}-1}^{*} 2^{n-k_{1}-i} \stackrel{(2)}{=} \sum_{k_{1}=0}^{n-1} \sum_{i=1}^{n-k_{1}} \binom{n}{i} \binom{n-i}{n-k_{1}-i} \binom{n-i}{k_{1}-1}^{*} 2^{n-k_{1}-i} = \sum_{i=1}^{n} \binom{n}{i} \binom{n-i}{l-1}^{*} 2^{n-i} \sum_{k_{1}=0}^{n-i} \binom{n-i}{k_{1}} 2^{-k_{1}} \stackrel{(1)}{=} \sum_{i=1}^{n} \binom{n}{i} \binom{n-i}{l-1}^{*} 2^{n-i} \binom{n-i}{l-1}^{*} 2^{n-i} \binom{n-n}{l-1}^{*} = S(n,l) + \binom{n}{n} \binom{n-n}{l-1}^{*} = S(n,l).$$

Finally, we obtain

$$\overline{\operatorname{Irr}}(n,l) = \frac{S_1 - S_2 - S_3 - 0}{3^n - 2} = \frac{\binom{n}{l}^* (3^n - 1) - (S(n,l) - \binom{n}{l}^*) - S(n,l)}{3^n - 2} = \frac{3^n \binom{n}{l}^* - 2S(n,l)}{3^n - 2}.$$
(7)

Let us compute $\overline{\operatorname{Irr}}(n,2)$ using the following identities of the Stirling numbers

$${n \brace 1} = 1, \ {n \brace 2} = 2^{n-1} - 1.$$

We have

$$S(n,2) = \sum_{i=1}^{n-1} \binom{n}{i} \cdot 1 \cdot 3^{n-i} = \sum_{i=1}^{n-1} \binom{n}{i} 3^{n-i} \stackrel{(1)}{=} 4^n - 3^n - 1,$$

therefore

$$\overline{\operatorname{Irr}}(n,2) = \frac{3^n \cdot 2(2^{n-1} - 1) - 2(4^n - 3^n - 1)}{3^n - 2} = \frac{6^n - 2 \cdot 4^n + 2}{3^n - 2}.$$
 (8)

In particular, n = 3 gives

$$\overline{\text{Irr}}(3,2) = \frac{6^3 - 2 \cdot 4^3 + 2}{3^3 - 2} = \frac{90}{25} = 3.6 \tag{9}$$

that coincides with (3).

The following statement gives the estimation of $\overline{\operatorname{Irr}}(n,l)$.

Proposition 5.1. The number $\overline{Irr}(n, l)$ satisfies

$$\frac{1}{3} \binom{n}{l}^* \le \overline{\operatorname{Irr}}(n, l) \le \binom{n}{l}^*$$

Proof. One can bound S(n, l) as follows

$$S(n,l) \le 3^{n-1} \sum_{i=1}^{n-1} \binom{n}{j} \binom{j}{l-1}^* \stackrel{(4)}{=} 3^{n-1} (l-1)! \left(\binom{n+1}{l} - \binom{n}{l-1} \right) \stackrel{(3)}{=} 3^{n-1} (l-1)! \binom{n}{l} = 3^{n-1} \binom{n}{l}^*,$$

and similarly

$$S(n,l) \ge 3 \sum_{i=1}^{n-1} {n \choose j} {j \brace l-1}^* = 3 {n \brace l}^*.$$

Thus,

$$\overline{\mathrm{Irr}}(n,l) \leq \frac{3^n \binom{n}{l}^* - 2 \cdot 3 \binom{n}{l}^*}{3^n - 2} = \binom{n}{l}^* \frac{3^n - 6}{3^n - 2} \leq \binom{n}{l}^*,$$

and

$$\overline{\operatorname{Irr}}(n,l) \ge \frac{3^n \binom{n}{l}^* - 2 \cdot 3^{n-1} \binom{n}{l}^*}{3^n - 2} = \binom{n}{l}^* \frac{3^n - 2 \cdot 3^{n-1}}{3^n - 2} \ge \binom{n}{l}^* \frac{3^n - 2 \cdot 3^{n-1}}{3^n} = \frac{1}{3} \binom{n}{l}^*.$$

Proposition 5.2. For a fixed l and $n \to \infty$ we have the asymptotic equivalence

$$\overline{\operatorname{Irr}}(n,l) \sim l^n$$
.

Proof. Using the following explicit formula for Stirling numbers

$${n \brace l} = \frac{1}{l!} \sum_{j=0}^{l} (-1)^{l-j} {l \choose j} j^{n},$$

we obtain $\binom{n}{l} \sim l^n$ for fixed l and $n \to \infty$. By Proposition 5.1, we have

$$\overline{\operatorname{Irr}}(n,l) \sim \begin{Bmatrix} n \\ l \end{Bmatrix}^* = l! \begin{Bmatrix} n \\ l \end{Bmatrix} \sim l! l^n \sim l^n.$$

References

- [1] E. Yu. Daniyarova, A. G. Myasnikov, V. N. Remeslennikov, Algebraic geometry over algebraic structures. II. Foundations, J. Math. Sci., 185:3 (2012), 389–416.
- [2] M. Ben-Or, Lower bounds for algebraic computation trees, Proc. 15th Annual Symposium on Theory of Computing (1983), 80–86.
- [3] M. V. Malov, On irreducible algebraic sets over infinite linearly ordered semilattices, to appear.

The information of the author:

Artem N. Shevlyakov

Sobolev Institute of Mathematics

644099 Russia, Omsk, Pevtsova st. 13

Phone: +7-3812-23-25-51. e-mail: a_shevl@mail.ru