

Hydra group doubles are not residually finite

Kristen Pueschel

April 23, 2022

Abstract

In 2013, Kharlampovich, Myasnikov, and Sapir constructed the first examples of finitely presented residually finite groups with large Dehn functions. Given any recursive function f , they produce a finitely presented residually finite group with Dehn function dominating f . There are no known elementary examples of finitely presented residually finite groups with super-exponential Dehn function. Dison and Riley's hydra groups can be used to construct a sequence of groups for which the Dehn function of the k^{th} group is equivalent to the k^{th} Ackermann function. Kharlampovich, Myasnikov, and Sapir asked whether or not these groups are residually finite. We show that these constructions do not produce residually finite groups.

Classification: 20E26, 20E06 Keywords: residual finiteness, hydra groups, Dehn function, separable subgroup

1 Introduction

The first examples of finitely presented residually finite groups with super-exponential Dehn function were constructed in [8]:

Theorem (Kharlampovich, Myasnikov, and Sapir). *For any recursive function $f : \mathbb{N} \rightarrow \mathbb{N}$, there is a finitely presented residually finite solvable group G of derived length 3 for which the Dehn function $\delta_G \succ f$.*

Their examples are sufficiently complicated that it remains interesting to find elementary examples that arise ‘in nature’. One place to look is among known elementary examples of groups with large Dehn function. In [6], Dison and Riley introduced the hydra groups

$$G_k := \langle a_1, \dots, a_k, t \mid a_1^t = a_1, a_i^t = a_i a_{i-1}, i > 1 \rangle,$$

where we use the conventions $a^b := b^{-1}ab$ and $[a, b] = a^{-1}b^{-1}ab$. They proved that the HNN extension

$$\Gamma_k = \langle G_k, p \mid [a_i t, p] = 1, \forall 1 \leq i \leq k \rangle$$

over the subgroup $H_k = \langle a_1 t, \dots, a_k t \rangle$ has Dehn function equivalent to the Ackermann function $A_k(n)$.

In [8], the authors commented that it was unknown whether or not Γ_k is residually finite for all $k > 1$, but that they expected Γ_k would not be residually finite for $k > 1$. We confirm this.

Theorem 1.1. *For all $k > 1$, the group Γ_k is not residually finite.*

The free product with amalgamation

$$\Gamma'_k = \langle G_k, \overline{G_k} \mid a_i t = \overline{a_i t}, 1 \leq i \leq k \rangle$$

also enjoys a fast growing Dehn function. An analogous theorem holds for these groups:

Theorem 1.2. *For all $k > 1$, the group Γ'_k is not residually finite.*

Definition 1. The subgroup $H \leq G$ is separable in G if for all $g \notin H$, there is a finite quotient $\phi : G \rightarrow Q$ such that $\phi(g) \notin \phi(H)$. Equivalently, $H \leq G$ is separable if and only if it is closed in the profinite topology of G , which means that

$$H = \bigcap_{\substack{H < K < G \\ [G:K] < \infty}} K.$$

In the next section we will see that separability of the subgroup H_k in G_k is necessary for the residual finiteness of Γ_k and Γ'_k . Therefore, Theorems 1.1 and 1.2 are proven via:

Lemma 1.3. *The group H_k is not separable in G_k for any $k > 1$.*

In particular, we will show that the non-separability of H_2 in G_2 implies non-separability of H_k in G_k . To see that H_2 is not a separable subgroup of G_2 , we recognize (G_2, H_2) as isomorphic to an important group-subgroup pair (G_{BKS}, H_{BKS}) studied by Burns, Karrass, and Solitar in [4]. Burns, Karrass, and Solitar proved that H_{BKS} is a non-separable subgroup of G_{BKS} . The group G_{BKS} was the first example of a 3-manifold group containing a finitely presented non-separable subgroup [4] and it has been an important tool for verifying other examples of non subgroup-separable groups. For example, Niblo and Wise showed that G_{BKS} virtually embeds in the fundamental group of the complement of the link of 4 circles, L . Therefore, L is not subgroup separable [11]. Further, they showed that the fundamental groups of compact graph manifolds have only one obstruction to subgroup separability: the existence of an embedding of L (and hence a virtual embedding of G_{BKS}). Niblo and Wise have also shown that G_{BKS} contains finitely presented subgroups which are contained in no proper finite-index subgroups. That is, there is a proper subgroup K such that finite quotients of G_{BKS} will not witness that K is a proper subgroup. [10].

Dison and Riley have constructed variations on their group-subgroup pairs that also have large distortion. These too can be used to produce candidates for elementary examples of finitely presented groups with fast-growing Dehn functions that might be residually finite. For $\mathbf{w} = (w_1, \dots, w_k)$, where w_i is a positive word on letters in $\{a_1, \dots, a_{i-1}\}$, consider the group

$$G_k(\mathbf{w}) = \langle a_1, \dots, a_k \mid a_i^t = a_i w_i, 1 \leq i \leq k \rangle.$$

For powers $\mathbf{r} = (r_1, \dots, r_k)$, where $r_i \geq 0$, consider the subgroup

$$H_k(\mathbf{r}) = \langle a_1 t^{r_1}, \dots, a_k t^{r_k} \rangle.$$

We prove that these cannot be used to produce residually finite groups with large Dehn function. In particular:

Theorem 1.4. *H_k is a separable subgroup of $G_k(\mathbf{w})$ if and only if $\mathbf{w} = (1, \dots, 1)$. Therefore the HNN extension $\Gamma_k(\mathbf{w}) = \langle G_k(\mathbf{w}), p \mid h^p = h, h \in H_k \rangle$ is residually finite only if $G_k(\mathbf{w}) = F_k \times \mathbb{Z}$.*

Theorem 1.5. *Suppose that $\mathbf{w} = (1, \dots, 1, w_c, \dots, w_k)$ where $w_c \neq 1$ and $\mathbf{r} = (r_1, \dots, r_k)$. Let $[w_c]_i$ denote the index of a_i in w_c . If*

$$\sum_{i=1}^{c-1} [w_c]_i r_i \neq 0$$

then $H_k(\mathbf{r})$ is a non-separable subgroup of $G_k(\mathbf{w})$.

Corollary 1.6. *$H_k(\mathbf{r})$ is a separable subgroup of G_k if and only if $\mathbf{r} = (0, \dots, 0)$, that is, the subgroup is separable only in the obvious case that $H_k(\mathbf{r}) = \langle a_1, \dots, a_k \rangle$.*

Remark 1. The case where $\sum_{i=1}^{c-1} [w_c]_i r_i = 0$ is not understood. In particular, we do not know whether or not $H_3(\mathbf{r}) \leq G_3$ is separable for $\mathbf{r} = (0, 1, 0)$.

We conclude that an example of a residually finite group with super-exponential Dehn function is unlikely to be found as an HNN extension over a subgroup of a hydra-like group.

The failure of residual finiteness for the groups Γ_k and Γ'_k leads us to ask if the word problem for Γ_k and Γ'_k is solvable. After all, residual finiteness of the group G always provides a solution to the word problem for G via McKinsey's Algorithm [9]. Given a word w in the generators of G , the algorithm runs two processes in parallel: one lists trivial words, looking for w , and the other lists homomorphisms to finite groups, looking to see if w ever has non-trivial image. Our result shows that there are non-trivial elements of Γ_k and Γ'_k for which finite quotients cannot be used to distinguish them from the identity element. Still, the word problems for Γ_k and Γ'_k are decidable. Indeed, Dison and Riley showed that the distortion of H_k in G_k is bounded above by a recursive function, which implies the Membership Problem for H_k is decidable, and therefore that the Word Problem for Γ_k and Γ'_k is decidable [6]. In fact, Dison, Einstein, and Riley have found a polynomial time solution to the Word Problem [5].

I wish to thank my advisor, Tim Riley, for his help, suggestions, and corrections, and Mark Sapir, for a helpful conversation.

2 Preliminaries

Definition 2. A group G is *residually finite* if every element $x \in G - \{1\}$ has a non-trivial image in some finite quotient of G . Equivalently, the intersection of all finite index subgroups of G is trivial.

Lemma 2.1. If $\Gamma = \langle G, p \mid h^p = h, h \in H \rangle$ is residually finite, then H is separable in G and G is residually finite.

Proof. It is obvious that G is residually finite, since residual finiteness is inherited by subgroups. Suppose that H is not separable. We can find $g \in G - H$ such that for every homomorphism ϕ from G to an arbitrary finite group, $\phi(g) \in \phi(H)$. Consider an arbitrary map $\Phi : \Gamma \rightarrow Q$ for Q finite. Since Φ restricts to a homomorphism on G , $\Phi(g) = \Phi(h)$ for some $h \in H$, and

$$\Phi([p, g]) = [\Phi(p), \Phi(g)] = [\Phi(p), \Phi(h)] = \Phi([p, h]) = \Phi(1) = 1.$$

Although $[p, g]$ is non-trivial in the free amalgamated product Γ , it is trivialized in every finite quotient of Γ . Therefore, if H is not separable in G , Γ is not residually finite. \square

By a theorem of Baumslag and Tretkoff, this necessary condition is actually sufficient: if G is residually finite and H is separable, then Γ is residually finite [1].

Remark 2. For any property \mathcal{P} , if H is not \mathcal{P} -separable, then Γ is not residually \mathcal{P} . Berlai has shown that for \mathcal{P} the properties of solvability and amenability, so long as G is residually \mathcal{P} , failure to be \mathcal{P} -separable is the only obstruction to Γ being residually \mathcal{P} [2].

Lemma 2.2. Suppose that $H < G$. Take $\overline{H} < \overline{G}$ to be another copy of our group-subgroup pair. If H is not separable in G , then $\Gamma' = \langle G, \overline{G} \mid a_i t = \overline{a_i t}, 1 \leq i \leq k \rangle = G *_H \overline{G}$ is not residually finite.

Proof. We will show that if $g \in G - H$ is an element that cannot be separated from H in finite quotients, that the non-trivial element $g^{-1}\overline{g} \in \Gamma'$ is trivialized in every finite quotient, so Γ' is not residually finite. An arbitrary map Φ from Γ' to a finite group Q will factor as a pair of maps $\psi, \phi : G \rightarrow Q$. By the definition of amalgamation, $\psi(h) = \phi(h)$ for all $h \in H$. To see that $g^{-1}\overline{g}$ is trivialized, we show that the functions ϕ and ψ agree on g as well. Construct $\Psi : G \rightarrow Q \times Q$, which is (ψ, ϕ) . This new target group is still finite, and the image of H is contained in the diagonal. As $\Psi(g) \in \Psi(H) \subset \Delta$, this implies that $\psi(g) = \phi(g)$. Therefore $\Phi(g^{-1}\overline{g}) = 1$. \square

Remark 3. If taking the direct product of groups preserves property \mathcal{P} (eg. solvability, amenability), then if H is not \mathcal{P} -separable in G , the same proof as above implies that $\Gamma' = \langle G, \overline{G} \mid H = \overline{H} \rangle$ is not residually \mathcal{P} . Kahrobaei has shown that for \mathcal{P} the properties of solvability and amenability, so long as G is residually \mathcal{P} , failure to be \mathcal{P} -separable is the only obstruction to Γ' being residually \mathcal{P} [7].

We will use Lemmas 2.1 and 2.2 to show that the HNN extensions Γ_k and the amalgamated products Γ'_k are not residually finite, by recognizing that the subgroup H_k is not separable in G_k .

Definition 3. A group G is Hopfian if every surjective endomorphism of G is an automorphism.

In 1971 Gilbert Baumslag proved in [3]:

Lemma 2.3. Finitely generated free-by-cyclic groups are residually finite, and therefore Hopfian.

In particular, as the groups G_k are free-by-cyclic, they are Hopfian, so to check that endomorphisms are automorphisms we need only check that they are surjective.

3 H_k is not a separable subgroup of G_k

Lemma 3.1. H_2 is not a separable subgroup of G_2 .

Proof. To show that G_2 is not H_2 -separable, we use the result of Burns, Karrass, and Solitar [4] that

$$H_{BKS} = \langle \alpha^{-1}, y\alpha^{-1}y^{-2}\alpha \rangle \leq G_{BKS} = \langle \alpha, \beta, y \mid \alpha^y = \alpha\beta, \beta^y = \beta \rangle$$

is not separable. We demonstrate an automorphism ϕ of G_2 from which the isomorphism carrying $(G_2, \phi(H_2))$ to (G_{BKS}, H_{BKS}) is clear. Recall the presentation

$$H_2 = \langle a_1 t, a_2 t \rangle \leq G_2 = \langle a_1, a_2, t \mid a_1^t = a_1, a_2^t = a_2 a_1 \rangle = \langle a_1, a_2, t \mid [a_1, t] = 1, [a_2, t] = a_1 \rangle.$$

Consider the map:

$$\begin{aligned} a_2 &\mapsto a_2^{-2}ta_2 \\ t &\mapsto a_2^{-1}t^{-1}a_2 = a_1t^{-1} \\ a_1 &\mapsto a_1. \end{aligned}$$

We verify that ϕ is an endomorphism:

$$\begin{aligned} [\phi(a_1), \phi(t)] &= [a_1, a_1t^{-1}] = a_1^{-1}ta_1^{-1}a_1a_1t^{-1} = 1 = \phi([a_1, t]), \\ [\phi(a_2), \phi(t)] &= [a_2^{-2}ta_2, a_2^{-1}t^{-1}a_2] = [a_2^{-1}t, t^{-1}]^{a_2} = a_2^{-1}(t^{-1}a_2ta_2^{-1}tt^{-1})a_2 = [a_2, t] = a_1 = \phi(a_1). \end{aligned}$$

ϕ is also surjective:

$$\begin{aligned} a_1 &= \phi(a_1) \\ a_2 &= \phi((a_2t)^{-1}) \\ t &= \phi(t^{-1}a_1) \end{aligned}$$

Because G_2 is Hopfian, and ϕ is a surjective endomorphism, it must be an automorphism. The image of a_1t is

$$a_1a_2^{-1}t^{-1}a_2 = [a_2, t]a_2^{-1}t^{-1}a_2 = a_2^{-1}t^{-1}a_2ta_2^{-1}t^{-1}a_2 = (a_2^{-1}t^{-1}a_2)t(a_2^{-1}t^{-1}a_2),$$

and noting that t and $a_2^{-1}t^{-1}a_2$ commute, we can write $\phi(a_1t) = ta_2^{-1}t^{-2}a_2$. The image of a_2t is a_2^{-1} . Therefore

$$\phi(H_2) = \langle a_2^{-1}, ta_2^{-1}t^{-2}a_2 \rangle.$$

The isomorphism between $(G_2, \phi(H_2))$ and (G_{BKS}, H_{BKS}) is apparent from their definitions. By [4], H_{BKS} is not a separable subgroup of G_{BKS} , and therefore H_2 is not a separable subgroup of G_2 . \square

Next we show that the non-separability of H_k in G_k follows from the non-separability of H_2 in G_2 . There is a natural inclusion $G_2 \hookrightarrow G_k$ which is just $a_1 \mapsto a_1, a_2 \mapsto a_2, t \mapsto t$. In the following we will abuse notation and write G_2 and H_2 for the image in G_k of G_2 and H_2 under the inclusion. Dison and Riley develop a description of elements of $G_2 \cap H_k$ in [6], which we include here for the convenience of the reader:

Assign an order to the elements $\{a_1, \dots, a_k\}$, which we will call ‘priority’: $a_{i+1} > a_i$ for all i .

Definition 4. The piece decomposition of a word $u \in F_k = \langle a_1, \dots, a_k \rangle$, is a grouping $u \cong \pi_1 \cdots \pi_l$ where π_i are maximal words without occurrences of $a_k^{\pm 1}$, except possibly with prefix a_k or suffix a_k^{-1} .

For example, $a_1a_2^{-1}a_1^2a_2a_1^{-1}a_2a_1^2a_2^{-2}$ has piece decomposition $(a_1a_2^{-1})(a_1^2)(a_2a_1^{-1})(a_2a_1^2a_2^{-1})(a_2^{-1})$, where the parentheses indicate the different pieces. This piece-decomposition can be recursively defined. In particular, words containing no a_k can be broken into pieces with respect to the next highest priority letter occurring.

Lemma 3.2 (Dison and Riley). *A word $w = t^r u$ represents an element of H_k if and only if u has piece decomposition $u = \pi_1 \cdots \pi_l$ and these pieces satisfy that for $p_0 = r$, there is a p_i such that $t^{p_i}\pi_{i+1} \in H_k t^{p_{i+1}}$ and $p_l = 0$. When p_{i+1} exists satisfying $t^{p_i}\pi_{i+1} \in H_k t^{p_{i+1}}$, it is unique.*

Lemma 3.3 (Dison and Riley). $G_2 \cap H_k = H_2$.

Proof. From the definition, it is clear that $H_2 \subset G_2 \cap H_k$. Suppose that $w \in G_2 \cap H_k$. Using the free-by-cyclic normal form for G_k , rewrite $w = t^r u$ where $u \in \langle a_1, \dots, a_k \rangle$. Observe that as $w \in G_2$, this is also in the normal form for G_2 , so $u \in \langle a_1, a_2 \rangle$. By Lemma 3.2, this word is in H_k if and only if u has a piece decomposition $u = \pi_1 \cdots \pi_l$ and a tuple $(p_0 = r, p_1, \dots, p_{l-1}, p_l = 0)$, such that $t^{p_i}\pi_{i+1} \in H_k t^{p_{i+1}}$. The maximum priority letter that can occur in a word in $G_2 \cap H_k$ is a_2 . As $t^{p_i}\pi_{i+1} \in H_k t^{p_{i+1}}$ contains no letters of priority greater than 2, we get that $t^{p_i}\pi_{i+1} \in H_2 t^{p_{i+1}}$ for all i . From Lemma 3.2 we have that $w \in H_2$. Therefore $G_2 \cap H_k = H_2$. \square

Lemma 3.4. *The intersection of a subgroup $H < G$ with a separable subgroup $S < G$ is separable in H .*

Proof. As S is separable, $S = \bigcap_{\substack{S \leq K < G \\ [G:K] < \infty}} K$. So

$$S \cap H = \left(\bigcap_{\substack{S \leq K < G \\ [G:K] < \infty}} K \right) \cap H = \bigcap_{\substack{S \leq K < G \\ [G:K] < \infty}} (K \cap H)$$

and we have expressed $S \cap H$ as the intersection of a family of finite-index subgroups in H , since $K \cap H$ is finite index in H . \square

Proof of Lemma 1.3. By Lemma 3.4, if $G_2 \cap H_k$ is not separable in G_2 , then H_k is not separable in G_k . By Lemma 3.3, $G_2 \cap H_k = H_2$, so since H_2 is not separable in G_2 , H_k is not separable in G_k . \square

Lemmas 2.1 and 2.2 showed that separability of H_k in G_k is a necessary condition for residual finiteness of Γ_k and Γ'_k , so the non-separability of H_k in G_k shown in Lemma 1.3 implies Theorems 1.1 and 1.2.

4 Generalizations of the Hydra Groups

In the last section we saw that for every k , H_k is not a separable subgroup of G_k . We were interested in these groups because Dison and Riley showed that H_k is distorted like the Ackermann function A_k in G_k , which forces the Dehn function of the doubles to be large. In this section we consider other pairs for which the machinery of Dison and Riley show that the analogous HNN extensions and free products with amalgamation will have exponential or superexponential Dehn function. We will show that these groups too are not residually finite.

The following proposition is an extension of the example of Burns, Karrass, and Solitar in [4]. It is the key to proving Theorem 1.4: the subgroup H_k is separable in the generalized hydra group $G_k(\mathbf{w})$ only when $G_k(\mathbf{w}) = F_k \times \mathbb{Z}$.

Proposition 4.1. *If $r > 0$, the subgroup $H_2(r, 1) = \langle a_1 t^r, a_2 t \rangle < G_2$ is not separable.*

Remark 4. For every $s \in \mathbb{Z}$ there is an automorphism, η_s of G_2 which carries $H_2(r, 0)$ to $H_2(r, s)$, defined by $\eta_s(a_2) = a_2 t^{s-1}$, $\eta_s(t) = t$. Observe that $\eta_s(a_1) = \eta_s([a_2, t]) = [a_2 t^s, t] = (a_2 a_1^s)^{-1} (a_2 a_1^{s+1}) = a_1$. Below we will actually prove that $H_2(r, 0)$ is not separable.

Remark 5. We restrict to the case $r > 0$ in order to be able to apply the techniques of Dison and Riley. In particular, we wish to have the analogue of Lemma 3.2 in order to prove Lemma 4.3.

Remark 6. If there was an automorphism ϕ of G_2 carrying H_2 onto $H_2(r, 0)$, the result would follow immediately. Suppose that there was such an automorphism ϕ . Let $q : G_2 \rightarrow G_2^{ab} = \langle a_2, t \mid [a_2, t] = 1 \rangle$ be the abelianization map. The automorphism $\phi : G_2 \rightarrow G_2$ descends to ϕ_{ab} , an automorphism of the abelianization G_2^{ab} . The restriction of ϕ to H_2 , called ϕ^{res} , will descend to an isomorphism from $q(H_2)$ to $q(H_2(r, 0))$, which agrees with the restriction to $q(H_2)$ of ϕ_{ab} . Note that $q(H_2) = q(G_2)$, and $q(H_2(r, 0)) = \langle t^r, a_2 \rangle$. When $r > 1$, this is a proper subgroup of G_2^{ab} . This is a contradiction. Because ϕ_{ab} and ϕ_{ab}^{res} have the same domain and ϕ_{ab}^{res} is a restriction of ϕ_{ab} , they should be the same function. However, these maps have different ranges. Therefore the proof of the proposition requires more than an application of Lemma 1.3.

Lemma 4.2. $\langle t \rangle \cap H_2(r, 0) = \{1\}$

Proof. Suppose for the contradiction that there is a non-trivial element of the intersection. It can be expressed either as an element of $H_2(r, 0)$ or as an element of $\langle t \rangle$:

$$(a_1 t^r)^{\alpha_1} a_2^{\beta_1} \cdots (a_1 t^r)^{\alpha_n} a_2^{\beta_n} = t^m$$

for some α_i, β_i and $m \neq 0$. There is a van Kampen diagram for the word $w = (a_1 t^r)^{\alpha_1} a_2^{\beta_1} \cdots (a_1 t^r)^{\alpha_n} a_2^{\beta_n} t^{-m}$ over the G_2 presentation: $\langle a_1, a_2, t \mid a_1^t = a_1, a_2^t = a_2 a_1 \rangle = \langle a_1, a_2, t \mid a_1^t = a_1, t^{a_2} = t a_1^{-1} \rangle$. From the second presentation it is clear that there are a_2 corridors in any van Kampen diagram for which the boundary word contains either an a_2 or a_2^{-1} . See Bridson and Gersten [3] for a detailed account of corridors in van Kampen diagrams. Since the word w contains the letter a_2 , there are a_2 -corridors. A corridor is innermost if the boundary word it cuts off is a word on only the generators a_1 and t . There are always at least two innermost corridors, so at least one of them will cut off a word δ , which is of the form $(a_1 t^r)^{\alpha_i}$ for some i . The word along the side of the a_2 corridor will either be a power $t^k a_1^{-k}$ or t^k , call it γ . The word $\delta \gamma^{-1} = 1$, but we note that this equality is not possible, as there is a non-zero index sum of either a_1 or t in $\delta \gamma^{-1}$. \square

Lemma 4.3. *The word $[t^{-1}, a_2^{-2} t^{-1} a_2^2] \notin H_2(r, 0)$ when $r \geq 1$.*

Proof. The analogue of Lemma 3.2 holds for $H_2(r, 0)$. That is, we can decide whether or not a word in $\langle a_1, a_2 \rangle$ is in $H_2(r, 0)$ by considering whether the rewriting can be carried out on each successive piece.

$$ta_2^{-2}a_2t^{-1}a_2^{-2}t^{-1}a_2^2 = (a_2a_1^{-1})^{-2}(a_2a_1^{-2})^2(a_2a_1^{-1})^{-2}a_2^2 = (a_1a_2^{-1})(a_1^{-1})(a_2a_1^{-1}a_2^{-1})(a_1)(a_2)$$

where the parentheses in the final line separate the different pieces. There is no p such that $a_1a_2^{-1} \in H_2(r, 0)t^{-p}$. Suppose that there is. Then

$$ht^{-p} = a_1a_2^{-1} \Rightarrow h = a_1a_2^{-1}t^p = t^pa_1^{1-p}a_2^{-1} \Rightarrow t^pa_1^{1-p} = ha_2 \in H_2(r, 0).$$

We can rewrite $t^pa_1^{1-p} \in H_2(r, 0)$, as $(a_1t^r)^{1-p}t^{r(p-1)+p} \in H_2(r, 0)$, and since $a_1t^r \in H_2(r, 0)$, it follows that $t^{r(p-1)+p} \in H_2(r, 0)$. Lemma 4.2 implies that $r(p-1)+p = 0$, so $p(r+1) = r$. Since $r > 0$, we get $p = \frac{r}{1+r}$, which is not an integer. Thus $[t^{-1}, a_2^{-2}t^{-1}a_2^2] \notin H_2(r, 0)$. \square

Proof of Proposition 4.1. For $r > 0$, the arguments of Burns, Karrass, and Solitar can be translated directly to work for this easy variation of their example [4]. For the convenience of the reader, we repeat their argument (almost) verbatim. We drop all decoration and use H to refer to $H_2(r, 0)$ throughout this proof.

Let \mathcal{T} be the infinitely generated group $\langle t_k \mid [t_k, t_{k+1}] = 1, k \in \mathbb{Z} \rangle$. To make calculations easier, Burns, Karrass, and Solitar rewrite G_2 as the HNN extension $G_2 = \langle \mathcal{T}, a_2 \mid k \in \mathbb{Z}, t_k^{a_2} = t_{k+1} \rangle$. In our original presentation, $t_k = t^{a_2^k}$. This implies that $a_1 = [a_2, t] = a_2^{-1}t^{-1}a_2t = t_1^{-1}t_0$, and the word $[t^{-1}, a_2^{-2}t^{-1}a_2^2] = [t_0^{-1}, t_2^{-1}]$.

Given an arbitrary finite-index subgroup L satisfying $H < L$, Burns, Karrass, and Solitar find a subgroup $N < L \cap \mathcal{T}$ such that

$$H \cap \mathcal{T} < N \triangleleft \mathcal{T}.$$

Analysis of the quotient \mathcal{T}/N will imply that the word $[t^{-1}, a_2^{-2}t^{-1}a_2^2]$ is contained in $N \subset L$. According to Lemma 4.3, $[t^{-1}, a_2^{-2}t^{-1}a_2^2]$ is not an element of H . As L is an arbitrary finite index subgroup containing H , this implies that H is not separable.

If L is a finite-index subgroup $L < G$, the core of L , $\text{core}(L) = \bigcap_{g \in G} L^g$, is a finite-index normal subgroup. Moreover, $\text{core}(L) \cap \mathcal{T}$ is still normal and finite index in \mathcal{T} . The group N above is given by $(H \cap \mathcal{T})(\text{core}(L) \cap \mathcal{T})$. The majority of the work of this proof is in showing that N is normal in \mathcal{T} .

Lemma 4.4. $H \cap \mathcal{T} = \langle t_i^{-1}t_{i-1}^{r+1} \mid i \in \mathbb{Z} \rangle$.

Proof. Notice that all elements of \mathcal{T} have trivial a_2 index sum, since every element in the generating set has zero a_2 index sum: $t_i = t^{a_2^i}$. The elements of H with trivial a_2 index sum are all generated by a_2 conjugates of a_1t^r , so $H \cap \mathcal{T} \leq \langle (a_1t^r)^{a_2^{i-1}} \mid i \in \mathbb{Z} \rangle = \langle (t_1^{-1}t_0^{r+1})^{a_2^{i-1}} \mid i \in \mathbb{Z} \rangle = \langle t_i^{-1}t_{i-1}^{r+1} \mid i \in \mathbb{Z} \rangle$. The other inclusion is clear. \square

Claim. $N = (H \cap \mathcal{T})(\text{core}(L) \cap \mathcal{T})$ is normal in \mathcal{T} .

Notice that $H \cap \mathcal{T}$ and $\text{core}(L) \cap \mathcal{T}$ are invariant under conjugation by a_2 . Therefore N is invariant under conjugation by a_2 . We will next establish that $(H \cap \mathcal{T})t_0^{\pm 1} \subset N$ by considering where conjugation by $t_0^{\pm 1}$ sends the generators $t_i^{-1}t_{i-1}^{r+1}$.

Lemma 4.5. If $i < 0$, then $(t_i^{-1}t_{i-1}^{r+1})t_0^{\pm 1} \in H \cap \mathcal{T}$.

Proof. For $i < 0$, both $t_0^{-1}t_i^{(r+1)^i} \in H \cap \mathcal{T}$ and $t_i^{(r+1)^i}t_0^{-1} \in H \cap \mathcal{T}$, as

$$t_0^{-1}t_i^{(r+1)^i} = t_0^{-1}t_{-1}^{(r+1)}(t_{-1}^{-(r+1)}t_{-2}^{(r+1)^2}) \cdots (t_{i+1}^{-(r+1)^{i-1}}t_i^{(r+1)^i}) = t_0^{-1}t_{-1}^{(r+1)}(t_{-1}^{-1}t_{-2}^{(r+1)})^{(r+1)} \cdots (t_{i+1}^{-1}t_i^{r+1})^{(r+1)^{i-1}},$$

where the second equality holds since $[t_k, t_{k+1}] = 1$ for all k . The same kind of rewriting shows $t_i^{(r+1)^i}t_0^{-1} \in H \cap \mathcal{T}$.

Next $(t_i^{-1}t_{i-1}^{r+1})t_0$ can be rewritten using words of the form $t_0^{-1}t_i^{(r+1)^i}$:

$$t_0^{-1}t_i^{-1}t_{i-1}^{r+1}t_0 = t_0^{-1}t_i^{(r+1)^i}(t_i^{-(r+1)^i}t_{i-1}^{r+1}t_i^{(r+1)^i})t_i^{-(r+1)^i}t_0 = (t_0^{-1}t_i^{(r+1)^i})(t_i^{-1}t_{i-1}^{r+1})(t_0^{-1}t_i^{(r+1)^i})^{-1}.$$

Since $t_i^{-1}t_{i-1}^{r+1}$ and $t_0^{-1}t_i^{(r+1)^i}$ are in $H \cap \mathcal{T}$, so is $(t_i^{-1}t_{i-1}^{r+1})^{t_0} \in H \cap \mathcal{T}$. Similarly,

$$t_0 t_i^{-1} t_{i-1}^{r+1} t_0^{-1} = (t_i^{(r+1)^i} t_0^{-1})^{-1} (t_i^{-1} t_{i-1}^{r+1}) (t_i^{(r+1)^i} t_0^{-1})^{-1} \in H \cap \mathcal{T}.$$

□

For the other half of the generators, we can only show the following weaker lemma:

Lemma 4.6. *If $l > 0$, then $(t_l^{-1}t_{l-1}^{r+1})^{t_0^{\pm 1}} \in (H \cap \mathcal{T})(\text{core}(L) \cap \mathcal{T})$.*

Proof. Because $\text{core}(L) \cap \mathcal{T}$ is finite index in \mathcal{T} , there exists $t_i t_j^{-1} \in \text{core}(L) \cap \mathcal{T}$ with $i - j < 0$. Indeed there are infinitely many generators and only finitely many cosets of $\text{core}(L) \cap \mathcal{T}$. Since $\text{core}(L) \cap \mathcal{T}$ is normal, $t_{i-j} t_0^{-1} = (t_i t_j^{-1})^{a_2^{-j}} \in \text{core}(L) \cap \mathcal{T}$. By conjugating $t_{i-j} t_0^{-1}$ by $a_2^{(i-j)k}$ we get $t_{(i-j)(k+1)} t_{(i-j)k}^{-1} \in \text{core}(L) \cap \mathcal{T}$ for all $k \in \mathbb{Z}$. Stringing these elements together, we get that $t_{(i-j)k} t_0^{-1} \in \text{core}(L) \cap \mathcal{T}$ for all $k \in \mathbb{Z}$.

Given $l > 0$, choose $n = (i - j)k$ such that $n > l$. Then

$$t_0 t_l^{-1} t_{l-1}^{r+1} t_0^{-1} = (t_0 t_n^{-1}) (t_n t_l^{-1} t_{l-1}^{r+1} t_n^{-1}) (t_n t_0^{-1}) = (t_0 t_n^{-1}) (t_0 t_{l-n}^{-1} t_{l-1-n}^{r+1} t_0^{-1})^{a_2^n} (t_n t_0^{-1})$$

Lemma 4.5 implies that the middle term is an element of $H \cap \mathcal{T}$, as $l - n < 0$, and $H \cap \mathcal{T}$ is invariant under conjugation by a_2 . The conjugating terms $t_n t_0^{-1} \in \text{core}(L) \cap \mathcal{T}$ and so $t_0 t_l^{-1} t_{l-1}^{r+1} t_0^{-1} \in (H \cap \mathcal{T})(\text{core}(L) \cap \mathcal{T})$. □

From Lemmas 4.5 and 4.6, we have that each of the generators of $H \cap \mathcal{T}$ is conjugated by t_0 and t_0^{-1} into $N = (H \cap \mathcal{T})(\text{core}(L) \cap \mathcal{T})$. From the normality of $\text{core}(L) \cap \mathcal{T}$, we get $((H \cap \mathcal{T})(\text{core}(L) \cap \mathcal{T}))^{t_0^{\pm 1}} = N$, and we can conjugate $N^{t_0^{\pm 1}} \subset N$ by a_2^k for $k \in \mathbb{Z}$ to get $N^{t_k^{\pm 1}} \subset N$. Therefore N is a normal subgroup of \mathcal{T} .

That $[t_0^{-1}, t_2^{-1}]$ is in N follows easily from N being normal in \mathcal{T} . Indeed, since $t_0^{-1} t_i^{(r+1)^i} \in H \cap \mathcal{T}$ for $i < 0$, it follows that $t_i^{(r+1)^i} N = t_0 N$. In the quotient \mathcal{T}/N , the images of t_i and t_0 commute when $i < 0$. When $i > 0$, we can rewrite $[t_0, t_i] = [t_{-i}, t_0]^{a_2^i} = 1^{a_2^i} = 1$, so they too commute in the quotient. Therefore \mathcal{T}/N is an abelian group and $[t_0^{-1}, t_2^{-1}] \in N$. Since H and $\text{core}(L)$ are subgroups of L and $N = (H \cap \mathcal{T})(\text{core}(L) \cap \mathcal{T})$, it follows that $N \leq L$. Therefore $[t_0^{-1}, t_2^{-1}]$ is an element of L but not of $H_2(r, 0)$, and so $H_2(r, 0)$ is not a separable subgroup of G_2 . □

Consider the group $G_k(\mathbf{w}) = \langle a_1, \dots, a_k, t \mid a_1^t = a_1 w_1, \dots, a_k^t = a_k w_k \rangle$, where $\mathbf{w} = (w_1, \dots, w_k)$, with each w_i a positive word on the generators $\{a_1 \dots a_{i-1}\}$. Recall the statement of Theorem 1.4: The subgroup $H_k = H_k(1, \dots, 1)$ is separable in $G_k(\mathbf{w})$ if and only if $\mathbf{w} = (1, \dots, 1)$.

Proof of Theorem 1.4. If $\mathbf{w} = (1, \dots, 1)$, then $G_k = F_k \times \langle t \rangle$. G_k is subgroup separable, so in particular, H_k is separable. If $\mathbf{w} \neq (1, \dots, 1)$, then $G_k(\mathbf{w})$ has \mathbf{w} with initial segment of the form $(1, \dots, 1, w_c, \dots, w_k)$ where w_c is the first non-trivial word. The subgroup $\langle w_c, a_c, t \rangle$ is isomorphic to G_2 and the subgroup $\langle w_c t^{|w_c|}, a_c t \rangle$ is isomorphic to the subgroup $H_2(|w_c|, 1)$, where $|w_c|$ is the length of w_c in $\langle a_1, \dots, a_{c-1} \rangle$. Proposition 4.1 implies that this subgroup is not separable in $\langle w_c, a_c, t \rangle$. Since $\langle w_c t^{|w_c|}, a_c t \rangle$ is the intersection $H_k(\mathbf{w}) \cap \langle w_c, a_c, t \rangle$, Lemma 3.4 implies that H_k is not separable in $G_k(\mathbf{w})$. □

The most general form for which the methods of Dison and Riley apply are $H_k(\mathbf{r}) \leq G_k(\mathbf{w})$. We are able to get only a partial characterization of separability in this case, which is a generalization of Theorem 1.4 and its proof.

Recall the statement of Theorem 1.5: Suppose that $\mathbf{w} = (1, \dots, 1, w_c, \dots, w_k)$ and $\mathbf{r} = (r_1, \dots, r_k)$, with conditions on \mathbf{w} , \mathbf{r} as above. Let $[w_c]_i$ denote the index sum of a_i in w_c . If $\sum_{i=1}^{c-1} [w_c]_i r_i \neq 0$, then $H_k(\mathbf{r})$ is not a separable subgroup of $G_k(\mathbf{w})$.

Proof of Theorem 1.5. We examine the subgroup $\langle a_c, w_c, t \mid w_c^t = w_c, a_c^t = a_c w \rangle$, which is isomorphic to G_2 . The subgroup given by

$$\langle w_c t^{\sum_{i=1}^{c-1} [w_c]_i r_i}, a_c t \rangle$$

is isomorphic to $H_2(\sum_{i=1}^c [w_c]_i r_i, 1)$, and by Lemma 4.1, this subgroup is not separable if $\sum_{i=1}^c [w_c]_i r_i \neq 0$. □

Remark 7. Separability of H_k in G_k for the case that $\sum_{i=1}^c [w_c]_i r_i = 0$ is not established by the argument above. The most basic examples for which our method fails are those of the form

$$G_{c+1}(\mathbf{w}) = \langle a_1, \dots, a_c, a_{c+1}, t \mid a_i^t = a_i, i < c, a_c^t = a_c w_c, a_{c+1}^t = a_{c+1} w_{c+1} \rangle$$

and

$$H_{c+1}(\mathbf{r}) = \langle a_1, \dots, a_{c-1}, a_c t^{r_c}, a_{c+1} \rangle.$$

The simplest case of this failure is the group $G_3 = \langle a_1, a_2, t \mid a_1^t = a_1, a_2^t = a_2 a_1, a_3^t = a_3 a_2 \rangle$ with subgroup $H_3(\mathbf{r}) = \langle a_1, a_2 t, a_3 \rangle$.

References

- [1] B. Baumslag and M. Tretkoff. Residually finite HNN extensions. *Communications in Algebra*, 6(2):179–194, 1978.
- [2] F. Berlai. Residual properties of free products. arXiv:1405.0244, May 2014.
- [3] M.R. Bridson and S.M. Gersten. The optimal isoperimetric inequality for torus bundles over the circle. *Quarterly Journal of Math*, 47:1–23, 1996.
- [4] R. G. Burns, A. Karrass, and D. Solitar. A note on groups with separable finitely generated subgroups. *Bulletin of the Australian Mathematical Society*, 36:153–160, 8 1987.
- [5] W. Dison, E. Einstein, and T. R. Riley. Taming the hydra: the word problem and extreme integer compression.
- [6] W. Dison and T. R. Riley. Hydra groups. *Commentarii Mathematici Helvetici. A Journal of the Swiss Mathematical Society*, 88(3):507–540, 2013.
- [7] D. Kahrobaei. Doubles of residually solvable groups. In *Aspects of infinite groups*, volume 1 of *Algebra and Discrete Mathematics*, pages 192–200. World Sci. Publ., Hackensack, NJ, 2008.
- [8] O. Kharlampovich, A. Myasnikov, and M. Sapir. Algorithmically complex residually finite groups. arXiv:1204.6506, March 2013.
- [9] J.C.C. McKinsey. The decision problem for some classes of sentences without quantifiers. *Journal of Symbolic Logic*, 8:61–76, 1943.
- [10] G. A. Niblo and D. T. Wise. The engulfing property for 3-manifolds. In *The Epstein birthday schrift*, volume 1 of *Geometry and Topology Monographs*, pages 413–418 (electronic). Geometry and Topology Publishing, Coventry, 1998.
- [11] G. A. Niblo and D. T. Wise. Subgroup separability, knot groups and graph manifolds. *Proceedings of the American Mathematical Society*, 129(3):685–693, 2001.