On group automorphisms in universal algebraic geometry

Artem N. Shevlyakov*

March 27, 2020

Abstract

In this paper we study group equations with occurrences of automorphisms. We describe equational domains in this class of equations. Moreover, we solve a number of open problem posed in universal algebraic geometry

1 Introduction

In the classic approach to algebraic geometry over groups we are dealing with equations over a group G as expressions $w(X)=1$, where $w(X)$ is an element of $G*F(X)$ ($F(X)$ is the free group generated by a set of variables X). This class of equations was studied in many papers (see [1, 2] and the survey [4] for more details).

In the current paper we consider a different class of equations over a group G: now $w(X)$ may contain the occurrences of symbols $\{\phi \mid \phi\in \mathrm{Aut}(G)\}$. Any equation of this type is called below an *equation with automorphisms*. The study of such equations is justified by many important problems in group theory. For example, the twisted conjugacy problem for a group G is equivalent to the solution of the following equation $\phi(x)u=vx$ for given $u,v\in G$, $\phi\in\mathrm{Aut}(G)$ (also, see this problem in [5] for equations with endomorphisms).

There is a connection between the "standard" group equations and equations with automorphisms. Indeed, for an equation $c_0x_1c_1x_2c_2\ldots c_{k-1}x_kc_k=1$ ($c_i\in G$) we have

$$c_0x_1c_1x_2c_2\ldots c_{k-1}x_kc_k=(c_0x_1c_0^{-1})(c_0c_1x_2c_1^{-1}c_0^{-1})\ldots$$
$$(c_0c_1c_2\ldots c_{k-1}x_kc_{k-1}^{-1}\ldots c_2^{-1}c_1^{-1}c_0^{-1})c_0c_1c_2\ldots c_k=x_1^{c_0^{-1}}x_2^{(c_0c_1)^{-1}}\ldots x_k^{(c_0c_1c_2\ldots c_{k-1})^{-1}}\prod c_i$$
$$=\phi_1(x_1)\phi_2(x_2)\ldots\phi_k(x_k)\prod c_i,$$

where each ϕ_i is an inner automorphism of a group G. Thus, the equation above is equivalent to the following equation with automorphisms

$$\phi_1(x_1)\phi_2(x_2)\ldots\phi_k(x_k)c=1\ (c\in G).$$

This correspondence allows us to study equations with automorphisms by methods developed for group equations. For instance, in [2] equational domains for group equations were described. In Section 3 we solve the similar problem for equations

*The author was supported by Russian Fund of Fundamental Research (project 18-31-00330, the results of Sections 3) and Russian Science Foundation (project 18-71-10028, the results of Section 4)

with automorphisms. The results of Section 3 provide various examples of equational domains, so it allows to solve a number of open problems posed in universal algebraic geometry (Section 4). Namely, we solve Problem 4.4.7 from [2]. Notice that our solution implies negative answers for Problems 5.3.1-4 in [2] (the reduction of Problem 4.4.7 to Problems 5.3.1-4 was shown in [2]).

2 Definitions

All definitions below are derived from [2], where all notions of algebraic geometry were formulated for algebraic structures of arbitrary languages.

Denote by $\mathcal{L}=\{\cdot,^{-1},1\}$ the standard language of group theory. Let us fix a group G and consider the extended language $\mathcal{L}(A)=\{\cdot,^{-1},1\}\cup\{\phi^{(1)}\mid\phi^{(1)}\in A\}$, where the unary functional symbols $\phi^{(1)}$ correspond to a group of automorphisms $A\subseteq\mathrm{Aut}(G)$. Any group G of the language $\mathcal{L}(A)$ is called an $\mathcal{L}(A)$-group (implicitly we fix an interpretation of the symbols ϕ to the elements of the group $A\subseteq\mathrm{Aut}(G)$).

Using the properties of automorphisms, any $\mathcal{L}(A)$-term in variables $X=\{x_1,x_2,\ldots,x_n\}$ is equivalent to a product

$$\phi_1(x_{i_1}^{\varepsilon_1})\phi_2(x_{i_2}^{\varepsilon_2})\ldots\phi_k(x_{i_k}^{\varepsilon_k}), \tag{1}$$

where $\phi_j\in A$, $x_{i_j}\in X$, $\varepsilon_j\in\{-1,1\}$.

An $\mathcal{L}(A)$-*equation* is an expression $t(X)=1$, where $t(X)$ is an $\mathcal{L}(A)$-term. *An* $\mathcal{L}(A)$-*system* is an arbitrary set of $\mathcal{L}(A)$-equations. The set of all solutions of an $\mathcal{L}(A)$-system $\mathbf{S}$ in G is denoted by $\mathrm{V}_G(\mathbf{S})$. A set $Y\subseteq G^n$ is called $\mathcal{L}(A)$-*algebraic* if there exists an $\mathcal{L}(A)$-system $\mathbf{S}$ in variables $X=\{x_1,x_2,\ldots,x_n\}$ with $Y=\mathrm{V}_G(\mathbf{S})$.

An $\mathcal{L}(A)$-group G is an $\mathcal{L}(A)$-*equational domain* if for any n and arbitrary $\mathcal{L}(A)$-algebraic sets $Y_1,Y_2\subseteq G^n$ the union $Y=Y_1\cup Y_2$ is also $\mathcal{L}(A)$-algebraic.

Theorem 2.1. ([2]) *An $\mathcal{L}(A)$-group G is an $\mathcal{L}(A)$-equational domain iff there exists an $\mathcal{L}(A)$-system $\mathbf{S}$ in variables x,y such that*

$$\mathrm{V}_G(\mathbf{S})=\{(x,y)\mid x=1 \textit{ or } y=1\}.$$

Remark 2.2. Actually, Theorem 2.1 was proved for group languages with constants, but its proof is valid for arbitrary group languages.

Let us recall the results of [2] related to group equations with no automorphisms.

Let H be a fixed subgroup of a group G. We pick elements of H as constants in the language $\mathcal{L}(H)=\mathcal{L}\cup\{h\mid h\in H\}$. Any $\mathcal{L}(H)$-term in variables X is actually an element of the free product $H*F(X)$, where $F(X)$ is the free group generated by the set X. An $\mathcal{L}(H)$-equation is an expression $t(X)=1$, where $t(X)$ is an $\mathcal{L}(H)$-term. Naturally, one can define the notions of algebraic sets and equational domains in the language $\mathcal{L}(H)$. As we mentioned in Remark 2.2, Theorem 2.1 holds for $\mathcal{L}(H)$-equational domains.

It was found in [2] the complete description of $\mathcal{L}(H)$-equational domains.

Theorem 2.3. ([2]) *An $\mathcal{L}(H)$-group G is an $\mathcal{L}(H)$-equational domain iff there are not $a,b\in G$, $a,b\neq 1$ such that*

$$[a,h^{-1}bh]=1 \textit{ for any } h\in H \tag{2}$$

(here $[x,y]=x^{-1}y^{-1}xy$).

According to Theorem 2.3, one can obtain (see [2]) few examples of $\mathcal{L}(H)$-equational domains:

1. the free group F_2 of rank 2 (for $H = F_2$),
2. the alternating group A_5 (for $H = A_5$).

Both examples will be used below in the paper.

3 Equational domains

Let us study equations with automorphisms, and the following theorem describes equational domains in the class of $\mathcal{L}(A)$-groups. Its proof is similar to Theorem 2.3 from [2].

Theorem 3.1. *An $\mathcal{L}(A)$-group G is an $\mathcal{L}(A)$-equational domain iff there are not $a, b \in G$, $a, b \neq 1$ such that*

$$[a, \phi(b)] = 1 \text{ for any } \phi \in A. \tag{3}$$

Proof. Let us prove the "if" statement. We have that any solution (x, y) of the $\mathcal{L}(A)$-system $\mathbf{S} = \{[x, \phi(y)] = 1 \mid \phi \in A\}$ satisfies $x = 1$ or $y = 1$. Thus, $\mathrm{V}_G(\mathbf{S}) = \{(x, y) \mid x = 1 \text{ or } y = 1\}$, and Theorem 2.1 concludes the proof.

Now, we prove the "only if" part of the theorem. By Theorem 2.1, there exists an $\mathcal{L}(A)$-system $\mathbf{S}$ with the solution set $\{(x, y) \mid x = 1 \text{ or } y = 1\}$. Let $w(x, y) = 1$ be an arbitrary $\mathcal{L}(A)$-equation of $\mathbf{S}$. Using the following commutator identities,

$$[s, t]^{-1} = [t, s],\ [sp, t] = [s, t]^p[p, t],\ [s^{-1}, t] = [t, s]^{s^{-1}}$$

one can equivalently rewrite $w(x, y)$ as a product

$$w(x, y) = u(x)v(y) \prod_i [\phi_i(x), \psi_i(y)]^{w_i(x,y)}$$

where $\phi_i, \psi_i \in A$, and $w_i(x, y)$, $u(x)$, $v(y)$ are $\mathcal{L}(A)$-terms.

Since $(1, y), (x, 1) \in \mathrm{V}_G(\mathbf{S})$ for any $x, y \in G$, then $u(x) = 1$ and $v(y) = 1$ for all $x, y \in G$. Hence, one can assume that any equation $w(x, y) = 1 \in \mathbf{S}$ is of the form

$$\prod_i [\phi_i(x), \psi_i(y)]^{w_i(x,y)} = 1.$$

Assume there exist $a, b \in G$, $a, b \neq 1$ with (3). We have

$$\begin{aligned}[][a, {\phi_i}^{-1}(\psi_i(b))] = 1 \Leftrightarrow a{\phi_i}^{-1}(\psi_i(b)) = {\phi_i}^{-1}(\psi_i(b))a \Leftrightarrow \\ \phi_i(a{\phi_i}^{-1}(\psi_i(b))) = \phi_i({\phi_i}^{-1}(\psi_i(b))a) \Leftrightarrow \\ \phi_i(a)\psi_i(b) = \psi_i(b)\phi_i(a) \Leftrightarrow [\phi_i(a), \psi_i(b)] = 1\end{aligned}$$

and $(a, b) \in \mathrm{V}_G(w(x, y) = 1)$. Thus, the point (a, b) satisfies any equation of $\mathbf{S}$, and we obtain a contradiction $\mathrm{V}_G(\mathbf{S}) \neq \{(x, y) \mid x = 1 \text{ or } y = 1\}$. □

Let us compare Theorem 3.1 and Theorem 2.3. Obviously, Theorem 2.3 follows from Theorem 3.1 for $A = \mathrm{Inn}_H(G)$, where $\mathrm{Inn}_H(G) = \{\phi_h(g) = h^{-1}gh \mid h \in H\}$ is a subgroup of the group $\mathrm{Inn}(G)$ of inner automorphisms.

Moreover, if a group G is an $\mathcal{L}(H)$-equational domain, then G is an $\mathcal{L}(A)$-equational domain for any $A \supseteq \mathrm{Inn}_H(G)$. Therefore, the alternating group A_5 is an $\mathcal{L}(A)$-equational domain for $A = \mathrm{Aut}(A_5)$. The free group F_2 of rank 2 is also an $\mathcal{L}(A)$-equational domain for $A = \mathrm{Aut}(F_2)$. However, the following statement provides F_2 to be an equational domain with a cyclic group of automorphisms.

Example 3.2. Let F_2 be the free group of rank 2, and a, b be free generators. Let ϕ denote the automorphism $\phi(a) = b$, $\phi(b) = a$. Then Theorem 3.1 states that F_2 is an $\mathcal{L}(A)$-equational domain for $A = \langle\phi\rangle$.

The following two statements also follow from Theorem 3.1.

Corollary 3.3. *If a group G has a nontrivial center $Z(G)$, then G is not an $\mathcal{L}(A)$-equational domain for any $A \subseteq \mathrm{Aut}(G)$.*

Proof. Let $a \in Z(G) \setminus \{1\}$ be a central element. Hence, a commute with any $\phi(a)$, and the pair (a, a) satisfies (3) for all ϕ. □

Corollary 3.4. *Let G be an $\mathcal{L}(A_0)$-equational domain for some $A_0 \subseteq \mathrm{Aut}(G)$, and $H = \Pi G$ be a direct power of G indexed by a set I. In other words, any element of H is an ordered tuple $(g_i \mid i \in I)$. Let $\mathcal{P}$ be a set of permutations of I such that $\mathcal{P}$ is transitive on I (i.e. for any pair $i, j \in I$ there exists $\pi \in \mathcal{P}$ with $\pi(i) = j$). Let us define automorphisms of H as follows:*

$$f_\phi((g_i \mid i \in I)) = (\phi(g_i) \mid i \in I), \tag{4}$$

$$\sigma_\pi((g_i \mid i \in I)) = (g_{\pi(i)} \mid i \in I), \tag{5}$$

where $\phi \in A_0$, $\pi \in \mathcal{P}$. Let $A \subseteq \mathrm{Aut}(H)$ denote the group generated by $\{f_\phi, \sigma_\pi \mid \phi \in A_0, \pi \in \mathcal{P}\}$. Then the $\mathcal{L}(A)$-group H is an $\mathcal{L}(A)$-equational domain.

Proof. Let us take $\mathbf{a} = (a_i \mid i \in I), \mathbf{b} = (b_i \mid i \in I) \in H$, $\mathbf{a}, \mathbf{b} \neq 1$. Since $\mathcal{P}$ transitively acts on I, there exists $\psi \in A$ and an index $i \in I$ such that $a_i \neq 1$, $c_i \neq 1$ where $\psi(\mathbf{b}) = \mathbf{c} = (c_i \mid i \in I)$.

Since G is an $\mathcal{L}(A_0)$-equational domain, there exists $\phi \in A_0$ with

$$[a_i, \phi(c_i)] \neq 1.$$

Therefore,

$$[\mathbf{a}, f_\phi(\psi(\mathbf{b}))] \neq 1,$$

and Theorem 3.1 completes the proof.

□

4 One problem from universal algebraic geometry

The book [2] contains an open problem (Problem 4.4.7), which can be equivalently formulated as follows: is there an algebraic structure $\mathcal{A}$ of an appropriate language L such that

1. $\mathcal{A}$ is an L-equational domain;
2. $\mathcal{A}$ is q_ω-compact;
3. $\mathcal{A}$ is not u_ω-compact.

We solve this problem in the class of $\mathcal{L}(A)$-groups. Let us give all necessary definitions.

Let $A \subseteq \mathrm{Aut}(H)$ be a subgroup of automorphisms of a group H. An $\mathcal{L}(A)$-group H is q_ω-*compact* if for any $\mathcal{L}(A)$-system $\mathbf{S}$ and an $\mathcal{L}(A)$-equation $w(X)=1$ such that

$$\mathrm{V}_H(\mathbf{S}) \subseteq \mathrm{V}_H(w(X)=1) \tag{6}$$

there exists a finite subsystem $\mathbf{S}' \subseteq \mathbf{S}$ with

$$\mathrm{V}_H(\mathbf{S}') \subseteq \mathrm{V}_H(w(X)=1). \tag{7}$$

An $\mathcal{L}(A)$-group H is u_ω-*compact* if for any $\mathcal{L}(A)$-system $\mathbf{S}$ and $\mathcal{L}(A)$-equations $w_i(X)=1$ $(1 \leq i \leq m)$ such that

$$\mathrm{V}_H(\mathbf{S}) \subseteq \bigcup_{i=1}^{m} \mathrm{V}_H(w_i(X)=1) \tag{8}$$

there exists a finite subsystem $\mathbf{S}' \subseteq \mathbf{S}$ with

$$\mathrm{V}_H(\mathbf{S}') \subseteq \bigcup_{i=1}^{m} \mathrm{V}_H(w_i(X)=1) \tag{9}$$

Let us define a group solving the problem above. Let G be a finite group such that G is an $\mathcal{L}(A_0)$-equational domain for $A_0 = \mathrm{Aut}(G)$ (for example, one may take $G = A_5$). Following Corollary 3.4, we define the $\mathcal{L}(A)$-group $H = \Pi G$ for $I = \mathbb{Z}$, $\mathcal{P} = \{\pi\}$ (where π is a permutation $\pi(n) = n+1$ over $\mathbb{Z}$), and A is generated by the automorphisms f_ϕ, σ_π (4,5).

We denote the subgroup generated by $\{f_\phi \mid \phi \in \mathrm{Aut}(G)\} \subseteq A$ by A_G. The automorphism σ_π is denoted by σ below. By the definition, σ acts on an element $(g_i \mid i \in \mathbb{Z})$ by

$$\sigma(g_i) = g_{i+1}.$$

Thus, we should prove that H is

1. an $\mathcal{L}(A)$-equational domain (it immediately follows from Corollary 3.4);
2. q_ω-compact (Lemma 4.7);
3. not u_ω-compact (Lemma 4.1).

Below we will use the following denotation

$$\sigma_k(x) = \begin{cases} \underbrace{\sigma(\sigma(\ldots\sigma}_{k \text{ times}}(x)\ldots)) \text{ for } k>0, \\ \underbrace{\sigma^{-1}(\sigma^{-1}(\ldots\sigma^{-1}}_{k \text{ times}}(x)\ldots)) \text{ for } k<0, \\ x \text{ for } k=0 \end{cases}$$

The automorphism σ commute with any f_ϕ, i.e. $\sigma(f_\phi(h)) = f_\phi(\sigma(h))$ for all $h \in H$. Hence any equation over the $\mathcal{L}(A)$-group H can be written in the following form

$$\sigma_{k_1}(f_1(x_{j_1}^{\varepsilon_1}))\sigma_{k_2}(f_2(x_{j_2}^{\varepsilon_2}))\ldots\sigma_{k_l}(f_l(x_{j_l}^{\varepsilon_l})) = 1, \tag{10}$$

where $f_i \in A_G$, $\varepsilon_i \in \{-1,1\}$, $k_j \in \mathbb{Z}$.

Lemma 4.1. *The $\mathcal{L}(A)$-group H is not* u_ω*-compact.*

Proof. Since H is an $\mathcal{L}(A)$-equational domain, there are no $\mathbf{a},\mathbf{b}\in H$ such that $\mathbf{a},\mathbf{b}\neq 1$ and $[\mathbf{a},\phi(\mathbf{b})]=1$ for any $\phi\in A$. Hence, any solution of the $\mathcal{L}(A)$-system $\mathbf{S}=\{[x,\phi(y)]=1\mid \phi\in A\}$ satisfies either $x=1$ or $y=1$. Thus, the following inclusion

$$\mathrm{V}_H(\mathbf{S})\subseteq \mathrm{V}_H(x=1)\cup \mathrm{V}_H(y=1).$$

holds.

Let $\mathbf{S}'$ be a finite subsystem of $\mathbf{S}$ and $n=\max\{|k|\mid \sigma_k \text{ occurs in } \mathbf{S}'\}$. Define $\mathbf{a}=(a_i\mid i\in\mathbb{Z})$, $\mathbf{b}=(b_i\mid i\in\mathbb{Z})$ such that

$$a_i=\begin{cases} g, & \text{if } i=0\\ 1, & \text{otherwise}\end{cases}\qquad b_i=\begin{cases} g, & \text{if } i=n+1\\ 1, & \text{otherwise}\end{cases}$$

where $g\in G\setminus\{1\}$.

Let $A'=\{\phi\mid [x,\phi(y)]=1\in\mathbf{S}'\}$ (i.e. A' is the set of all ϕ such that the equation $[x,\phi(y)]=1$ belongs to S') be a finite set of automorphisms. By the choice of $\mathbf{b}$, the element $\phi(\mathbf{b})$ has 1 at the 0-th coordinate for each $\phi\in A'$. Therefore, $\phi(\mathbf{b})$ commutes with $\mathbf{a}$ and we obtain $(\mathbf{a},\mathbf{b})\in \mathrm{V}_H(\mathbf{S}')$. Since $\mathbf{a}\neq 1$, $\mathbf{b}\neq 1$, then the inclusion

$$\mathrm{V}_H(\mathbf{S}')\subseteq \mathrm{V}_H(x=1)\cup \mathrm{V}_H(y=1)$$

fails. Thus, H is not u_ω-compact. □

There is a correspondence between $\mathcal{L}(A)$-systems over H and $\mathcal{L}(A_0)$-systems over G. Let $\mathbf{S}$ be an $\mathcal{L}(A)$-system in variables $X=\{x_1,x_2,\dots,x_n\}$. The system $\mathbf{S}$ defines an $\mathcal{L}(A_0)$-system $\gamma(\mathbf{S})$ over G in infinite number of variables $Y=\{y_{ij}\mid i\in\mathbb{Z}, 1\leq j\leq n\}$ (below $\varepsilon_i\in\{-1,1\}$):

$$\sigma_{k_1}(f_1(x_{j_1}^{\varepsilon_1}))\sigma_{k_2}(f_2(x_{j_2}^{\varepsilon_2}))\dots\sigma_{k_l}(f_l(x_{j_l}^{\varepsilon_l}))=1\Leftrightarrow$$
$$f_1(y_{k_1+k\ j_1}^{\varepsilon_1})f_2(y_{k_2+k\ j_2}^{\varepsilon_2})\dots f_l(y_{k_l+k\ j_l}^{\varepsilon_l})=1\in\gamma(\mathbf{S}) \text{ for all } k\in\mathbb{Z}. \quad (11)$$

In other words, $\gamma(\mathbf{S})$ is the coordinate-wise version of $\mathbf{S}$ over the direct power $H=\Pi G$.

Example 4.2. If $\mathbf{S}=\{\sigma(x_1)x_2=1\}$ then

$$\gamma(\mathbf{S})=\{\dots,y_{-11}y_{-22}=1,y_{01}y_{-12}=1,y_{11}y_{02}=1,y_{21}y_{12}=1,y_{31}y_{22}=1,\dots\}=$$
$$\{y_{k1}y_{(k-1)2}=1\mid k\in\mathbb{Z}\}.$$

By the definition, any $\mathcal{L}(A_0)$-equation $u(Y)=1\in\gamma(\mathbf{S})$ may come from several $\mathcal{L}(A)$-equations $W=\{w_i(X)=1\}$ of the system $\mathbf{S}$. Let us take an arbitrary equation $w_i(X)=1$ from W and denote this correspondence by $\gamma^{-1}(u(Y)=1)=\{w_i(X)=1\}$.

Remark 4.3. Below we will omit brackets in map compositions, i.e. we will write $\alpha\beta(x)$ instead of $\alpha(\beta(x))$.

Lemma 4.4. *For any $\mathcal{L}(A_0)$-equation*

$$f_1(y_{i_1\ j_1}^{\varepsilon_1})f_2(y_{i_2\ j_2}^{\varepsilon_2})\dots f_l(y_{i_l\ j_l}^{\varepsilon_l})=1 \quad (12)$$

and any number $k\in\mathbb{Z}$ the equation

$$f_1(y_{i_1+k\ j_1}^{\varepsilon_1})f_2(y_{i_2+k\ j_2}^{\varepsilon_2})\dots f_l(y_{i_l+k\ j_l}^{\varepsilon_l})=1 \quad (13)$$

also belongs to $\gamma(\mathbf{S})$.

Further, if $P=(p_{ij}\mid i\in\mathbb{Z},1\leq j\leq n)\in \mathrm{V}_G(\gamma(\mathbf{S}))$ then any shift $\sigma_k(P)=(s_{ij}\mid i\in\mathbb{Z},1\leq j\leq n)$, $s_{ij}=p_{i+k\ j}$ ($k\in\mathbb{Z}$) is also a solution of $\gamma(\mathbf{S})$.

Proof. Observe that the system $\mathbf{S}$ from Example 4.2 clearly satisfies the statements of this lemma.

The first statement directly follows from the definition of the system $\gamma(\mathbf{S})$. Let us prove the second one.

Assume there exists an $\mathcal{L}(A_0)$-equation (12) with $f_1(s_{i_1\ j_1}^{\varepsilon_1})f_2(s_{i_2\ j_2}^{\varepsilon_2})\dots f_l(s_{i_l\ j_l}^{\varepsilon_l}) \neq 1$ or, equivalently,

$$f_1(p_{i_1+k\ j_1}^{\varepsilon_1})f_2(p_{i_2+k\ j_2}^{\varepsilon_2})\dots f_l(p_{i_l+k\ j_l}^{\varepsilon_l}) \neq 1 \tag{14}$$

However, $\gamma(\mathbf{S})$ contains the equation $u(Y) = 1$ (13), and, by (14), we have $u(P) \neq 1 \Rightarrow P \notin \mathrm{V}_G(\gamma(\mathbf{S}))$. □

Let $\mathbf{S}_0, \mathbf{S}_1$ be $\mathcal{L}(A_0)$-systems in variables $Y = \{y_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n\}$. We say that $\mathbf{S}_0, \mathbf{S}_1$ are *Z-equivalent* for a given $Z \subseteq Y$ if the projections of $\mathrm{V}_G(\mathbf{S}_0)$ and $\mathrm{V}_G(\mathbf{S}_1)$ onto the coordinates Z are the same (in other words, for each $P = (p_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n) \in \mathrm{V}_G(\mathbf{S}_k)$ there exists $Q = (q_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n) \in \mathrm{V}_G(\mathbf{S}_{1-k})$ with $p_{ij} = q_{ij}$ for each $y_{ij} \in Z$, $k \in \{0, 1\}$.

Lemma 4.5. *Let $\mathbf{S}_0$ be an $\mathcal{L}(A_0)$-system in variables $Y = \{y_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n\}$ over a finite group G. Then for any finite $Z \subseteq Y$ there exists a finite Z-equivalent subsystem $\mathbf{S}_1 \subseteq \mathbf{S}_0$.*

Proof. The statement immediately follows from the finiteness of the group G. □

Let us denote a subsystem of an $\mathcal{L}(A_0)$-system $\gamma(\mathbf{S})$ by $\gamma_Z(\mathbf{S})$, if $\gamma_Z(\mathbf{S})$ is Z-equivalent to $\gamma(\mathbf{S})$.

Let Z be a set of variables occurring in an $\mathcal{L}(A_0)$-system $\gamma(\mathbf{S})$. The system $\gamma(\mathbf{S})$ may contain subsystems which are Z-equivalent to $\gamma(\mathbf{S})$ (as it proved above, for finite Z such subsystems always exist). Let us denote the class of such systems by $Z(\gamma(\mathbf{S}))$. We pick an arbitrary system from $Z(\gamma(\mathbf{S}))$ and denote it by $\gamma_Z(\mathbf{S})$.

Suppose an $\mathcal{L}(A_0)$-system $\gamma_Z(\mathbf{S})$ was constructed by an $\mathcal{L}(A)$-system $\mathbf{S}$ and a finite set Z. By the definition, $\gamma^{-1}\gamma_Z(\mathbf{S}) \subseteq \mathbf{S}$ is the set of equations from $\mathbf{S}$ which were essentially used in the construction of $\gamma_Z(\mathbf{S})$. One can apply the operator γ to $\gamma^{-1}\gamma_Z(\mathbf{S})$ and obtain a new $\mathcal{L}(A_0)$-system $\gamma\gamma^{-1}\gamma_Z(\mathbf{S})$.

Let us summarize all simple properties of the systems $\mathbf{S}, \gamma(\mathbf{S}), \gamma_Z(\mathbf{S}), \gamma^{-1}\gamma_Z(\mathbf{S}), \gamma\gamma^{-1}\gamma_Z(\mathbf{S})$:

1. $\mathbf{S}, \gamma^{-1}\gamma_Z(\mathbf{S})$ are $\mathcal{L}(A)$-systems and their solutions belong to H^n;
2. $\gamma(\mathbf{S}), \gamma_Z(\mathbf{S}), \gamma\gamma^{-1}\gamma_Z(\mathbf{S})$ are $\mathcal{L}(A_0)$-systems and coordinates of their solutions belong to G;
3. the systems $\gamma_Z(\mathbf{S}), \gamma^{-1}\gamma_Z(\mathbf{S})$ are finite for finite Z;
4. we have the inclusions $\gamma^{-1}\gamma_Z(\mathbf{S}) \subseteq \mathbf{S}$, $\gamma_Z(\mathbf{S}) \subseteq \gamma\gamma^{-1}\gamma_Z(\mathbf{S}) \subseteq \gamma(\mathbf{S})$.
5. the $\mathcal{L}(A_0)$-systems $\gamma\gamma^{-1}\gamma_Z(\mathbf{S})$, $\gamma_Z(\mathbf{S})$, $\gamma(\mathbf{S})$ are Z-equivalent.

Lemma 4.6. *Let C be a finite set of pairs (i, j), $i \in \mathbb{Z}$, $1 \leq j \leq n$ Then $\gamma\gamma^{-1}\gamma_Z(\mathbf{S})$ is Z_k-equivalent to $\gamma(\mathbf{S})$ for any set $Z_k = \{y_{i+k\ j} \mid (i, j) \in C\}$, $k \in \mathbb{Z}$.*

Proof. Let us take a point $P = (p_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n) \in \mathrm{V}_G(\gamma\gamma^{-1}\gamma_Z(\mathbf{S}))$ and consider the shift $R = \sigma_k(P) = (r_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n)$, $r_{ij} = p_{i+k\ j}$. According to Lemma 4.4, R is a solution of $\mathrm{V}_G(\gamma\gamma^{-1}\gamma_Z(\mathbf{S}))$. By the Z-equivalence, there exists a point $R' = (r'_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n) \in \mathrm{V}_G(\gamma(\mathbf{S}))$ with $r'_{ij} = r_{ij} = p_{i+k\ j}$ for any $(i, j) \in C$. By Lemma 4.4, the point $R'' = \sigma_{-k}(R')$, $R'' = (r''_{ij} \mid i \in \mathbb{Z}, 1 \leq j \leq n)$ is a solution of $\gamma(\mathbf{S})$. By the definition of R'', for each $(i, j) \in C$ we have $r''_{i+k\ j} = r'_{ij} = r_{ij} = p_{i+k\ j}$, and, therefore, $\gamma\gamma^{-1}\gamma_Z(\mathbf{S})$ is Z_k-equivalent to $\gamma(\mathbf{S})$. □

Lemma 4.7. *The $\mathcal{L}(A)$-group H is* q_ω*-compact.*

Proof. Suppose an $\mathcal{L}(A)$-system $\mathbf{S}$ and an $\mathcal{L}(A)$-equation $w(X)=1$ (10) satisfy (6).

The $\mathcal{L}(A)$-term $w(X)$ defines the set of pairs

$$C=\{(k_1,j_1),(k_2,j_2),\ldots,(k_l,j_l)\}.$$

Let us put $\mathbf{S}'=\gamma^{-1}\gamma_Z(\mathbf{S})$ for $Z=\{y_{ij}\mid(i,j)\in C\}$ and prove (7). Assume there exists a point $(\mathbf{h}_1,\mathbf{h}_2,\ldots,\mathbf{h}_n)\in \mathrm{V}_H(\mathbf{S}')\setminus \mathrm{V}_H(w(X)=1)$, $\mathbf{h}_j\in H$. In other words, there exists $P=(p_{ij}\mid i\in\mathbb{Z},1\leq j\leq n)\in \mathrm{V}_G(\gamma\gamma^{-1}\gamma_Z(\mathbf{S}))\setminus \mathrm{V}_G(\gamma(w(X))=1)$.

We have

$$\gamma(w(X)=1)=\{f_1(y^{\varepsilon_1}_{k_1+k\ j_1})f_2(y^{\varepsilon_2}_{k_2+k\ j_2})\ldots f_l(y^{\varepsilon_l}_{k_l+k\ j_l})=1\mid k\in\mathbb{Z}\}$$

and there exists $k\in\mathbb{Z}$ such that

$$f_1(p^{\varepsilon_1}_{k_1+k\ j_1})f_2(p^{\varepsilon_2}_{k_2+k\ j_2})\ldots f_l(p^{\varepsilon_l}_{k_l+k\ j_l})\neq 1$$

By Lemma 4.6, there exists a point $Q=(q_{ij}\mid i\in\mathbb{Z},1\leq j\leq n)\in \mathrm{V}_G(\gamma(\mathbf{S}))$ with $q_{i+k\ j}=p_{i+k\ j}$ for any $(i,j)\in C$. Therefore, $w(Q)\neq 1$.

Thus, $Q\in \mathrm{V}_G(\gamma(\mathbf{S}))\setminus \mathrm{V}_G(\gamma(w(X)=1))$. The point Q defines $R=(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_n)\in H^n$, $\mathbf{r}_j=(q_{ij}\mid i\in\mathbb{Z})$ such that $R\in \mathrm{V}_G(\mathbf{S})\setminus \mathrm{V}_G(w(X)=1)$, and we obtain a contradiction with (6). □

5 Conclusions

The construction of the group H from Corollary 3.4 is close to the notion of wreath product. In particular, the group H from Section 4 is structurally similar to the wreath product $G\wr\mathbb{Z}$.

This correspondence allows us to remind an important problem of universal algebraic geometry posed by B. Plotkin [3].

Problem (B. Plotkin [3]). Let $H=A\wr B$ be the wreath product of the groups A and B.

1. When H is q_ω-compact?
2. When H is q_ω-compact but not equationally Noetherian (a group is equationally Noetherian if the subsystem $\mathbf{S}'\subseteq\mathbf{S}$ in (7) does not depend on an equation $w(X)=1$)?
3. Is H necessarily q_ω-compact if both A,B q_ω-compact?

Let us explain the assertion of the problem above. Originally, B. Plotkin posed it for group equations in the "standard" language $\mathcal{L}=\{\cdot,^{-1},1\}$. However, in [6] the Problem was partially solved for languages with constants.

Theorem 5.1. [6] *If a group A is not abelian and B is infinite, then H is not* q_ω*-compact in the language with constants $\mathcal{L}(H)=\mathcal{L}\cup\{h\mid h\in H\}$.*

Thus, for the language $\mathcal{L}(H)$ the following problem remains open.

Problem. Let us consider the class of $\mathcal{L}(H)$-equations. Is H q_ω-compact (equationally Noetherian) for abelian A?

In the conclusion of the whole paper, we should discuss other ways to solve Problems 4.4.7, 5.3.1-4 from [2]. Usually (see [2]), the negative solution of a problem in universal algebraic geometry may be found in structures of pure relational languages, since such languages admit a very simple view of equations.

However, we cannot solve Problems 5.3.1-4 in relational languages. For this reason, we had to develop the algebraic geometry over equations with automorphisms. Thus, one can formulate a problem.

Problem. Is there an algebraic structure $\mathcal{A}$ of pure relational language L such that

1. $\mathcal{A}$ is an L-equational domain,
2. $\mathcal{A}$ is q_ω-compact,
3. $\mathcal{A}$ is not u_ω-compact?

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The information of the author:
Artem N. Shevlyakov
Sobolev Institute of Mathematics
644099 Russia, Omsk, Pevtsova st. 13
Phone: +7-3812-23-25-51.

Omsk State Technical University
644050 Russia, Omsk, pr. Mira, 11

e-mail: `a_shevl@mail.ru`