# Actions of the braid group, and new algebraic proofs of results of Dehornoy and Larue. 

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#### Abstract

This article surveys many standard results about the braid group with emphasis on simplifying the usual algebraic proofs.

We use van der Waerden's trick to illuminate the Artin-Magnus proof of the classic presentation of the algebraic mapping-class group of a punctured disc.

We give a simple, new proof of the Dehornoy-Larue braid-group trichotomy, and, hence, recover the Dehornoy right-ordering of the braid group.

We then turn to the Birman-Hilden theorem concerning braid-group actions on free products of cyclic groups, and the consequences derived by Perron-Vannier, and the connections with the Wada representations. We recall the very simple Crisp-Paris proof of the Birman-Hilden theorem that uses the Larue-Shpilrain technique. Studying ends of free groups permits a deeper understanding of the braid group; this gives us a generalization of the Birman-Hilden theorem. Studying Jordan curves in the punctured disc permits a still deeper understanding of the braid group; this gave Larue, in his PhD thesis, correspondingly deeper results, and, in an appendix, we recall the essence of Larue's thesis, giving simpler combinatorial proofs.

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## 1 General Notation

Let $\mathbb{N}$ denote the set of finite cardinals, $\{0,1,2, \ldots\}$.
Throughout, we fix an element $n$ of $\mathbb{N}$.

Let $i, j \in \mathbb{Z}$ and let $v$ be a symbol. We define

$$
\begin{aligned}
& {[i \uparrow j]:=\{k \in \mathbb{Z} \mid i \leq k \text { and } k \leq j\},} \\
& {[i \downarrow j]:=\{k \in \mathbb{Z} \mid i \geq k \text { and } k \geq j\},} \\
& ([i \uparrow j]):= \begin{cases}(i, i+1, \ldots, j-1, j) \in \mathbb{Z}^{j-i+1} & \text { if } i \leq j, \\
() \in \mathbb{Z}^{0} & \text { if } i>j,\end{cases} \\
& ([i \downarrow j]):= \begin{cases}(i, i-1, \ldots, j+1, j) \in \mathbb{Z}^{i-j+1} & \text { if } i \geq j, \\
() \in \mathbb{Z}^{0} & \text { if } i<j .\end{cases}
\end{aligned}
$$

Also, $v_{[i \uparrow j]}:=\left\{v_{k} \mid k \in[i \uparrow j]\right\}$, and this will usually be a subset of some ambient set, $G$. If $i \leq j, v_{([i \uparrow j])}:=\left(v_{i}, v_{i+1}, \ldots, v_{j-1}, v_{j}\right) \in G^{j-i+1}$, and, if $G$ is a group, $\Pi v_{[i \uparrow j]}:=v_{i} v_{i+1} \cdots v_{j-1} v_{j} \in G$. If $i>j, v_{([i \uparrow j])}:=()$, the 0 -tuple, and $\Pi v_{[i \uparrow j]}:=1$, the empty product. We define $v_{[i \downarrow j]}, v_{([i \downarrow j])}$ and $\Pi v_{[i \downarrow j]}$, analogously. Thus, if $i \geq j, \Pi v_{[i \downarrow j]}:=v_{i} v_{i-1} \cdots v_{j+1} v_{j}$. Finally, $[i \uparrow \infty[:=\{k \in \mathbb{Z} \mid i \leq k\}$.

For elements $a, b$ of a group $G, \bar{a}:=a^{-1}, a^{b}:=\bar{b} a b, a^{n b}:=\bar{b} a^{n} b$, and $[a]:=$ $\left\{a^{g} \mid g \in G\right\}$, the conjugacy class of $a$ in $G$. The group of all automorphisms of $G$ will be denoted by $\operatorname{Aut}(G)$.

An ordering of a set will mean a total order for the set. An ordered set is one endowed with a specific ordering. We will speak of $n$-tuples for a given set and $n$-tuples of elements of a given set.

## 2 Outline

Let $\Sigma_{0,1, n}:=\left\langle\left\{z_{1}\right\} \cup t_{[1 \uparrow n]} \mid z_{1} \Pi t_{[1 \uparrow n]}=1\right\rangle$. Then $\Sigma_{0,1, n}$ is a one-relator group which is freely generated by the set $t_{[1 \uparrow n]}$.

Let Out ${ }_{0,1, n}^{+}$denote the subgroup of $\operatorname{Aut}\left(\Sigma_{0,1, n}\right)$ consisting of all automorphisms of $\Sigma_{0,1, n}$ which map the set $\left\{z_{1}\right\} \cup\left\{\left[t_{i}\right]\right\}_{i \in[1 \uparrow n]}$ to itself. Let Out ${ }_{0,1,0}$ denote $\operatorname{Aut}(\mathbb{Z})$, and, for $n \geq 1$, let $\operatorname{Out}_{0,1, n}$ denote the group of all automorphisms of $\Sigma_{0,1, n}$ which map the set $\left\{z_{1}, \bar{z}_{1}\right\} \cup\left\{\left[t_{i}\right],\left[\bar{t}_{i}\right]\right\}_{i \in[1 \uparrow n]}$ to itself. Then Out ${ }_{0,1, n}^{+}$is a subgroup of index two in Out ${ }_{0,1, n}$. We call Out ${ }_{0,1, n}$ the algebraic mapping-class group of the surface of genus 0 with 1 boundary component and $n$ punctures; see [18] for background on algebraic mapping-class groups.

Frequently, Out ${ }_{0,1, n}$ will be denoted $\mathcal{B}_{n}$ and called the $n$-string braid group. (The similar symbol $B_{n}$ denotes a certain Coxeter diagram.)

In Section 3, we define $\sigma_{[1 \uparrow n-1]} \subseteq$ Out $_{0,1, n}^{+}$, we review Artin's 1925 proof that $\sigma_{[1 \uparrow n-1]}$ generates Out $_{0,1, n}^{+}$, and we present intermediate results that we shall apply in subsequent sections. In Section [4, we recall the definition of Artin groups, specifically $\operatorname{Artin}\left\langle A_{n}\right\rangle, \operatorname{Artin}\left\langle B_{n}\right\rangle$ and $\operatorname{Artin}\left\langle D_{n}\right\rangle$. In Section 5, we verify Artin's 1925 result that Out ${ }_{0,1, n}^{+} \simeq \operatorname{Artin}\left\langle A_{n-1}\right\rangle$, by combining Magnus' 1934 proof, Manfredini's observation that Out ${ }_{0,1,(n-1) \perp 1} \simeq \operatorname{Artin}\left\langle B_{n-1}\right\rangle$, and the van der Waerden trick, to condense the calculations involved.

In Section 6, we use results of Section 4 to recover the Dehornoy-Larue trichotomy for $\mathcal{B}_{n}$ and the Dehornoy right-ordering of $\mathcal{B}_{n}$; this represents a substantial simplification. Let us emphasize that we verify directly that Out ${ }_{0,1, n}^{+}$ satisfies the trichotomy, in contrast with the approach by Larue [22] of using the trichotomy for $\operatorname{Artin}\left\langle A_{n-1}\right\rangle$ to verify that $\operatorname{Artin}\left\langle A_{n-1}\right\rangle$ acts faithfully on $\Sigma_{0,1, n}$.

In Section 7, we review the action of $\mathcal{B}_{n}$ on the set of ends of $\Sigma_{0,1, n}$. The argument of Thurston given in [27] yields the Dehornoy right-ordering of $\mathcal{B}_{n}$, but not the trichotomy. By analysing further, we obtain new results about the $\mathcal{B}_{n}$-orbit of $t_{1}$ in $\Sigma_{0,1, n}$.

In Section 8, for each $m \geq 2$, we introduce $\mathrm{Out}_{0,1, n^{(m)}}$, the algebraic mapping-class group of the disc with $n C_{m}$-points. We recall the Crisp-Paris proof of the Birman-Hilden result that the natural map from Out ${ }_{0,1, n}$ to Out $_{0,1, n^{(m)}}$ is injective, and then modify an argument of Steve Humphries to show that there is a natural identification $\mathrm{Out}_{0,1, n^{(m)}}=\mathrm{Out}_{0,1, n}$. The new results obtained in Section 7 then provide additional information in this context.

In Section 9, we review some applications by Perron-Vannier [26] of the above Birman-Hilden result, and discuss connections with the actions given by Wada [29] and studied by Shpilrain [28] and Crisp-Paris [10], 11].

Following a kind suggestion of Patrick Dehornoy, we studied the analysis of the $\mathcal{B}_{n}$-orbit of $t_{1}$ in $\Sigma_{0,1, n}$ given by David Larue [21]. Larue's approach is combinatorial and uses polygonal curves in the punctured disc. By combining Larue's approach with Whitehead's use of graphs, we were able to simplify Larue's main arguments, and we record our combinatorial approach in an appendix. We also show how Larue's results imply the results we had obtained, more easily, by studying ends, in Section 7.

## 3 Artin's generators of $\mathcal{B}_{n}$

In this section we describe the famous generating set of $\mathcal{B}_{n}$. Let us fix more notation related to $\Sigma_{0,1, n}=\left\langle\left\{z_{1}\right\} \cup t_{[1 \uparrow n]} \mid z_{1} \Pi t_{[1 \uparrow n]}=1\right\rangle$ and $\mathcal{B}_{n} \leq \operatorname{Aut}\left(\Sigma_{0,1, n}\right)$.
3.1 Notation. Let $m \in \mathbb{N}$. Consider an $m$-tuple $a_{([1 \uparrow m])}$ for $t_{[1 \uparrow n]} \cup \bar{t}_{[1 \uparrow n]}$, and an element $w$ of $\Sigma_{0,1, n}$.

If $\Pi a_{[1 \uparrow m]}=w$ in $\Sigma_{0,1, n}$, we say that $a_{([1 \uparrow m])}$ is an expression for $w$. We say that the expression $a_{([1 \uparrow m])}$ is reduced if, for all $j \in[1 \uparrow n-1], a_{j+1} \neq \bar{a}_{j}$ in $t_{[1 \uparrow n]} \cup \bar{t}_{[1 \uparrow n]}$. For each element of $\Sigma_{0,1, n}$, there exists a unique reduced expression called the normal form.

Suppose that $a_{([1 \uparrow m])}$ is the normal form for $w$. We define the length of $w$ to be $|w|:=m$. The set of elements of $\Sigma_{0,1, n}$ whose normal forms have $a_{([1 \uparrow m])}$ as an initial segment is denoted $(w \star)$; and, the set of elements of $\Sigma_{0,1, n}$ whose normal forms have $a_{([1 \uparrow m])}$ as a terminal segment is denoted $(\star w)$. The elements of $(w \star)$ are said to begin with $w$, and the elements of $(\star w)$ are said to end with $w$.

Let $\mathrm{Sym}_{n}$ denote the group of permutations of $[1 \uparrow n]$ acting on the right (on $[1 \uparrow n]$ ).

Let $\phi \in \mathcal{B}_{n}$. There exists a unique permutation $\pi \in \operatorname{Sym}_{n}$, and a unique $(n+2)$-tuple $\left(w_{([0 \uparrow n+1])}\right)$ for $\Sigma_{0,1, n}$ such that $w_{0}=1$ and $w_{n+1}=1$, and, for each $i \in[1 \uparrow n], w_{i} \notin\left(t_{i^{\pi}}\right) \cup\left(\bar{t}_{i^{\pi}}\right)$ and $t_{i}^{\phi}=t_{i^{\pi}}^{w_{i}}$. For each $i \in[0 \uparrow n]$, let $u_{i}=w_{i} \bar{w}_{i+1}$. If $j \in[i \uparrow n]$, then $\Pi u_{[i \uparrow j]}=w_{i} \bar{w}_{j+1}$. In particular, $\Pi u_{[i \uparrow n]}=w_{i}$. We define $\pi(\phi):=\pi, w_{i}(\phi):=w_{i}, i \in[0 \uparrow n+1]$, and $u_{i}(\phi):=u_{i}, i \in[0 \uparrow n]$. We write $\|\phi\|:=\sum_{i \in[1 \uparrow n]}\left|t_{i}^{\phi}\right|=n+2 \sum_{i \in[1 \uparrow n]}\left|w_{i}(\phi)\right|$.

Let $\sigma_{[1 \uparrow n-1]} \subseteq \mathcal{B}_{n}$ be the subset determined by, for all $i \in[1 \uparrow n-1]$ and all $k \in[1 \uparrow n]$,

$$
t_{k}^{\sigma_{i}}= \begin{cases}t_{k} & \text { if } k \in[1 \uparrow i-1] \cup[i+2 \uparrow n] \\ t_{i+1} & \text { if } k=i \\ t_{i}^{t_{i+1}} & \text { if } k=i+1\end{cases}
$$

In the literature, $\sigma_{i}$ is sometimes represented in $2 \times n$-matrix notation, for example, in the format

$$
\sigma_{i}=\left(\begin{array}{cccccccc}
t_{1} & \ldots & t_{i-1} & t_{i} & t_{i+1} & t_{i+2} & \ldots & t_{n} \\
t_{1} & \ldots & t_{i-1} & t_{i+1} & t_{i}^{i+1} & t_{i+2} & \ldots & t_{n}
\end{array}\right) .
$$

We shall often find it convenient to compress the dots and say that $\sigma_{i}$ and $\bar{\sigma}_{i}$ are determined by the expressions

$$
\begin{array}{llllllll}
\frac{k \in[1 \uparrow i-1]}{\left(t_{k}\right.} & t_{i} & t_{i+1} & \frac{k \in[i+1 \uparrow n]}{\left.t_{k}\right)^{\sigma_{i}}} & \text { and } & \frac{k \in[1 \uparrow i-1]}{\left(t_{k}\right.} & t_{i} & t_{i+1} \\
=\left(\begin{array}{llll}
t_{k} & t_{i+1} & t_{i}^{t_{i+1}} & t_{k}
\end{array}\right), & & =\left(t_{k}\right. & t_{i+1}^{t_{i}} & t_{i} & \left.t_{k}\right) .
\end{array}
$$

We shall apply the following result in different situations.
3.2 Lemma (Artin [3]). Let $\phi \in \mathcal{B}_{n}$. Let $\pi=\pi(\phi)$ and, for each $i \in[0 \uparrow n]$, let $u_{i}=u_{i}(\phi)$.
(i). Suppose that there exists some $i \in[1 \uparrow n-1]$ such that $u_{i} \in\left(\star \bar{t}_{(i+1)^{\pi}}\right)$. Then $\left\|\sigma_{i} \phi\right\| \leq\|\phi\|-2$; moreover, for each $j \in[1 \uparrow i], t_{j}^{\sigma_{i} \phi}$ and $t_{j}^{\phi}$ both begin with the same element of $t_{[1 \uparrow n]} \cup \bar{t}_{[1 \uparrow n]}$.
(ii). Suppose that there exists some $i \in[1 \uparrow n-1]$ such that $u_{i} \in\left(\bar{t}_{i^{\pi} \star}\right)$. Then $\left\|\bar{\sigma}_{i} \phi\right\| \leq\|\phi\|-2$; moreover, for each $j \in[1 \uparrow i-1], t_{j}^{\bar{\sigma}_{i} \phi}$ and $t_{j}^{\phi}$ both begin with the same element of $t_{[1 \uparrow n]} \cup \bar{t}_{[1 \uparrow n]}$.
(iii). Suppose that, for each $i \in[1 \uparrow n-1], u_{i} \notin\left(\bar{t}_{i \pi \star}\right) \cup\left(\star \bar{t}_{(i+1)^{\pi}}\right)$. Then $\phi=1$.

Proof. (i). There exists some $v \in \Sigma_{0,1, n}-\left(\star t_{(i+1)^{\pi}}\right)$ such that $u_{i}=v \bar{t}_{(i+1)^{\pi}}$. Since $w_{i}(\phi)=u_{i} w_{i+1}(\phi)$, we have

$$
\begin{equation*}
w_{i}(\phi)=v \bar{t}_{(i+1)^{\pi}} w_{i+1}(\phi) . \tag{3.2.1}
\end{equation*}
$$

Since $v \notin\left(\star t_{(i+1)^{\pi}}\right)$ and $w_{i+1}(\phi) \notin\left(t_{(i+1)^{\pi} \star}\right)$, there is no cancellation in the expression $t_{i^{\pi}}^{v t_{(i+1)} \pi w_{i+1}(\phi)}$ for $t_{i}^{\phi}$; hence

$$
\begin{equation*}
t_{i}^{\phi} \in\left(\bar{w}_{i+1}(\phi) t_{\left.(i+1)^{\pi \star}\right)} \text { and }\left|t_{i}^{\phi}\right|=1+2|v|+2+2\left|w_{i+1}(\phi)\right| .\right. \tag{3.2.2}
\end{equation*}
$$

For all $j \in[1 \uparrow i-1] \cup[i+2 \uparrow n], t_{j}^{\sigma_{i} \phi}=t_{j}^{\phi}$; hence, $t_{j}^{\sigma_{i} \phi}$ has the same first letter as $t_{j}^{\phi}$, and, $\left|t_{j}^{\sigma_{i} \phi}\right|=\left|t_{j}^{\phi}\right|$.

Since $t_{i}^{\sigma_{i} \phi}=t_{i+1}^{\phi} \in\left(\bar{w}_{i+1}(\phi) t_{(i+1)^{\pi} \star}\right)$, we see, from (3.2.2), that $t_{i}^{\sigma_{i} \phi}$ has the same first letter as $t_{i}^{\phi}$. Also, $\left|t_{i}^{\sigma_{i} \phi}\right|=\left|t_{i+1}^{\phi}\right|$.

By (3.2.1), $w_{i}(\phi) \bar{w}_{i+1}(\phi) t_{(i+1)^{\pi}}=v$; hence

$$
t_{i+1}^{\sigma_{i} \phi}=\left(t_{i}^{t_{i+1}}\right)^{\phi}=\left(t_{i \pi}^{w_{i}(\phi)}\right)^{\left(t_{(i+1) \pi}^{w_{i+1}(\phi)}\right)}=t_{i \pi}^{v w_{i+1}(\phi)} .
$$

Hence, $\left|t_{i+1}^{\sigma_{i} \phi}\right| \leq 1+2|v|+2\left|w_{i+1}(\phi)\right| \stackrel{(3.2 .2)}{=}\left|t_{i}^{\phi}\right|-2$.
It now follows that $\left\|\sigma_{i} \phi\right\| \leq\|\phi\|-2$, and (i) is proved.
(ii). There exists some $v \in \Sigma_{0,1, n}-\left(t_{i \pi} \star\right)$ such that $u_{i}=\bar{t}_{i \pi} v$. Since $w_{i+1}(\phi)=$ $\bar{u}_{i} w_{i}(\phi)$, we have

$$
\begin{equation*}
w_{i+1}(\phi)=\bar{v} t_{i \pi} w_{i}(\phi) . \tag{3.2.3}
\end{equation*}
$$

Since $\bar{v} \notin\left(\star \bar{t}_{i^{\pi}}\right)$ and $w_{i}(\phi) \notin\left(\bar{t}_{i \pi \star}\right)$, there is no cancellation in the expression $t_{(i+1)^{\pi}}^{\overline{t_{i} \pi} w_{i}(\phi)}$ for $t_{i+1}^{\phi}$; hence

$$
\begin{equation*}
\left|t_{i+1}^{\phi}\right|=1+2|\bar{v}|+2+2\left|w_{i}(\phi)\right| . \tag{3.2.4}
\end{equation*}
$$

For all $j \in[1 \uparrow i-1] \cup[i+2 \uparrow n], t_{j}^{\overline{\sigma_{i}} \phi}=t_{j}^{\phi}$; hence, $t_{j}^{\overline{\sigma_{i}} \phi}$ has the same first letter as $t_{j}^{\phi}$, and, $\left|t_{j}^{\bar{\sigma}_{i} \phi}\right|=\left|t_{j}^{\phi}\right|$.

Since $t_{i+1}^{\bar{\sigma}_{i+1}}=t_{i}^{\phi}$, we see that $\left|t_{i+1}^{\bar{\sigma}_{i} \phi}\right|=\left|t_{i}^{\phi}\right|$.
By (3.2.3), $w_{i+1}(\phi) \bar{w}_{i}(\phi) \bar{t}_{i \pi}=\bar{v}$; hence

$$
t_{i}^{\bar{\sigma}_{i} \phi}=\left(t_{i+1}^{\bar{t}_{i}}\right)^{\phi}=\left(t_{(i+1)^{\pi}}^{w_{i+1}(\phi)}\right)^{\left(\bar{t}_{i}^{w_{i}(\phi)}\right)}=t_{i^{\pi}}^{\bar{v} w_{i}(\phi)} .
$$

Hence, $\left|t_{i}^{\bar{\sigma}_{i} \phi}\right| \leq 1+2|\bar{v}|+2\left|w_{i}(\phi)\right| \stackrel{(3.2 .4]}{=}\left|t_{i+1}^{\phi}\right|-2$.
It now follows that $\left\|\bar{\sigma}_{i} \phi\right\| \leq\|\phi\|-2$, and (ii) is proved.
(iii). Since $u_{0}=\bar{w}_{1}(\phi) \notin\left(\star \bar{t}_{1}^{\pi}\right)$ and $u_{n}=w_{n}(\phi) \notin\left(\bar{t}_{n}^{\pi} \star\right)$, we see that there is no cancellation anywhere in the expression $u_{0} \prod_{i \in[1 \uparrow n]}\left(t_{i} \pi u_{i}\right)$. Hence,

$$
\left|u_{0} \prod_{i \in[1 \uparrow n]}\left(t_{i \pi} u_{i}\right)\right|=\sum_{i \in[0 \uparrow n]}\left|u_{i}\right|+n \text {, that is, } \sum_{i \in[0 \uparrow n]}\left|u_{i}\right|=\left|u_{0} \prod_{i \in[1 \uparrow n]}\left(t_{i} u_{i}\right)\right|-n .
$$

Recall that $u_{0} \prod_{i \in[1 \uparrow n]}\left(t_{i} u_{i}\right)=\prod_{i \in[1 \uparrow n]}\left(t_{i}^{w_{i}(\phi)}\right)=\left(\prod_{i \in[1 \uparrow n]} t_{i}\right)^{\phi}=\prod_{i \in[1 \uparrow n]} t_{i}$. Hence

$$
\left|u_{0} \prod_{i \in[1 \uparrow n]}\left(t_{i \pi} u_{i}\right)\right|=n \text { and } \sum_{i \in[0 \uparrow n]}\left|u_{i}\right|=n-n=0 .
$$

Hence, all the elements of $u_{[0 \uparrow n]}$ are trivial.
For each $i \in[0 \uparrow n+1], w_{i}=\Pi u_{[i \uparrow n]}$; hence, all the elements of $w_{[1 \uparrow n]}$ are trivial. Also, $\prod_{i \in[1 \uparrow n]} t_{i^{\pi}}=u_{0} \prod_{i \in[1 \uparrow n]}\left(t_{i \pi} u_{i}\right)=\prod_{i \in[1 \uparrow n]} t_{i}$. Hence $\pi$ is trivial. Thus $\phi=1$.

The following is then immediate.
3.3 Proposition (Artin [3]). For each $\phi \in \mathcal{B}_{n}$, either $\phi=1$, or there exists some $\sigma_{i}^{\epsilon} \in \sigma_{[1 \uparrow n-1]} \cup \bar{\sigma}_{[1 \uparrow n-1]}$ such that $\left\|\sigma_{i}^{\epsilon} \phi\right\| \leq\|\phi\|-2$. Hence, $\left\langle\sigma_{[1 \uparrow n-1]}\right\rangle=\mathcal{B}_{n}$.
3.4 Remarks. If $w \in \Sigma_{0,1, n}$ has odd length, then $w^{\sigma_{i}}$ has odd length, and $\left|w^{\sigma_{i}}\right| \leq 2|w|+1$, with equality being achieved only if every odd letter of $w$ equals $t_{i+1}$. Similar statements hold with $\bar{\sigma}_{i}$ in place of $\sigma_{i}$.

Let $\phi \in \mathcal{B}_{n}$ and let $|\phi|$ denote the minimum length of $\phi$ as a word in $\sigma_{[1 \uparrow n-1]}$. Thus, $\left|t_{i}^{\phi}\right| \leq 2^{|\phi|+1}-1$. Hence, $\|\phi\| \leq n 2^{|\phi|+1}-n$. Proposition 3.3 gives an algorithm for writing $\phi$ as a word in $\sigma_{[1 \uparrow n-1]}$ of length at most $\frac{\|\phi\|-n}{2}$, and we have now seen that $\frac{\|\phi\|-n}{2} \leq \frac{n 2^{|\phi|+1}-2 n}{2}=n 2^{|\phi|}-n$.

## 4 Definition of Artin groups

4.1 Definition. A Coxeter diagram $X$ consists of a set $V$ together with a function $V \times V \rightarrow \mathbb{N} \cup\{\infty\}, \quad(x, y) \mapsto m_{x, y}$, such that, for all $x, y \in V, m_{x, x}=0$ and $m_{x, y}=m_{y, x}$. The elements of $V$ are called the vertices of $X$, and, for $(x, y) \in V \times V$, we say that $m_{x, y}$ is the number of edges joining $x$ and $y$; we can depict $X$ in a natural way. We then define the Artin group of $X$, denoted $\operatorname{Artin}\langle X\rangle$, to be the group presented with generating set $V$ and relations saying that, for all $(x, y) \in V \times V$,

$$
\begin{aligned}
& x y=y x \\
& x y x=y x y \\
& \text { if } m_{x, y}=0, \\
& m_{x, y}=1, \\
& x y x y=y x y x
\end{aligned} \text { if } m_{x, y}=2,
$$

Notice that if $m_{x, y}=\infty$, then no relation is imposed. Notice also that if $V$ is empty, then $\operatorname{Artin}\langle X\rangle$ is the trivial group.
4.2 Notation. (i). Let $A_{n}$ denote the Coxeter diagram

$$
a_{1}-a_{2}-\cdots-a_{n-1}-a_{n}
$$

Clearly, $A_{0}$ is empty. We define $A_{-1}$ to be empty also.
Thus, in $A_{n}$, the vertex set is $a_{[1 \uparrow n]}$, and, for each $(i, j) \in[1 \uparrow n]^{2}$, the number of edges joining $a_{i}$ to $a_{j}$ is $\begin{cases}1 & \text { if }|i-j|=1, \\ 0 & \text { if }|i-j| \neq 1 .\end{cases}$

Thus, $\operatorname{Artin}\left\langle A_{n}\right\rangle$ has a presentation with generating set $a_{[1 \uparrow n]}$ and relations saying that, for each $(i, j) \in[1 \uparrow n]^{2}$,

$$
\begin{aligned}
a_{i} a_{j}=a_{j} a_{i} & \text { if }|i-j| \neq 1, \\
a_{i} a_{j} a_{i}=a_{j} a_{i} a_{j} & \text { if }|i-j|=1 .
\end{aligned}
$$

(ii). Let $B_{n}$ denote the Coxeter diagram

$$
b_{1}-b_{2}-\cdots-b_{n-1}=b_{n} .
$$

Here, the vertex set is $b_{[1 \uparrow n]}$, and, for each $(i, j) \in[1 \uparrow n]^{2}$, the number of edges joining $b_{i}$ to $b_{j}$ is $\begin{cases}2 & \text { if }\{i, j\}=\{n-1, n\}, \\ 1 & \text { if }|i-j|=1 \text { and }\{i, j\} \neq\{n-1, n\}, \\ 0 & \text { if }|i-j| \neq 1 .\end{cases}$
(iii). For $n \geq 2$, let $D_{n}$ denote the Coxeter diagram

$$
d_{1}-d_{2}-\cdots-d_{n-3}-d_{n-2}^{d_{n}}-d_{n-1} .
$$

Here, the vertex set is $d_{[1 \uparrow n]}$, and, for each $(i, j) \in[1 \uparrow n]^{2}$, the number of edges joining $d_{i}$ to $d_{j}$ is

$$
\begin{cases}1 & \text { if }\{i, j\} \in\{\{1,2\},\{2,3\}, \ldots,\{n-2, n-1\},\{n-2, n\}\} \\ 0 & \text { otherwise. }\end{cases}
$$

## 5 Artin's presentation of $\mathcal{B}_{n}$

In this section, we verify Artin's result that there exists an isomorphism $\gamma_{n}: \operatorname{Artin}\left\langle A_{n-1}\right\rangle \rightarrow \mathcal{B}_{n}$ determined by $\frac{i \in[1 \uparrow n-1]}{\left(a_{i}\right)^{\gamma_{n}}}$. We express this by writ$=\left(\sigma_{i}\right)$
ing $\mathcal{B}_{n}=\operatorname{Artin}\left\langle\sigma_{1}-\sigma_{2}-\cdots-\sigma_{n-1}\right\rangle$.
5.1 Proposition. There exists a homomorphism $\gamma_{n}: \operatorname{Artin}\left\langle A_{n-1}\right\rangle \rightarrow \mathcal{B}_{n}$ de-
termined by $\frac{i \in[1 \uparrow n-1]}{\left(a_{i}\right)^{\gamma_{n}}}$, and $\gamma_{n}$ is surjective. $=\left(\sigma_{i}\right)$

Proof. (a). Suppose that $1 \leq i \leq i+2 \leq j \leq n-1$. We have the following.
$\left.\begin{array}{lcccccc} & \frac{k \in[1 \uparrow i-1]}{\left(t_{k}\right.} & & t_{i} & t_{i+1} & \frac{k \in[i+2 \uparrow j-1]}{} & \\ t_{k} & t_{j} & t_{j+1} & \left.\frac{0 k \in[j+2 \uparrow n]}{t_{k}}\right)^{\sigma_{i} \sigma_{j}} \\ =\left(t_{k}\right. & t_{i+1} & t_{i}^{t_{i+1}} & t_{k} & t_{j} & t_{j+1} & \left.t_{k}\right)^{\sigma_{j}} \\ =\left(t_{k}\right. & t_{i+1} & t_{i}^{t_{i+1}} & t_{k} & t_{j+1} & t_{j}^{t_{j+1}} & t_{k}\end{array}\right)$.
(b). Suppose that $1 \leq i \leq n-2$. We have the following.

$$
\begin{array}{lcccl} 
& \frac{k \in[1 \uparrow i-1]}{\left(t_{k}\right.} & t_{i} & t_{i+1} & t_{i+2} \\
& \left(t_{k}\right. & t_{i+1} & t_{i}^{t_{i+1}} & t_{i+2} \\
\left.t_{k}\right)^{\sigma_{i} \sigma_{i+1} \sigma_{i}} \\
=\left(t_{k}\right. & t_{i+2} & t_{i}^{t_{i+2}} & \left.t_{k}\right)^{t_{i+2}} & \left.t_{k}\right)^{\sigma_{i} \sigma_{i}} \\
=\left(t_{k}\right. & t_{i+2} & t_{i+1}^{t_{i+2}} & t_{i}^{t_{i+1} t_{i+2}} & \left.t_{k}\right) \\
=\left(t_{k}\right. & t_{i+1} & t_{i+2} & t_{i}^{t_{i+1} t_{i+2}} & \left.t_{k}\right)^{\sigma_{i+1}} \\
=\left(t_{k}\right. & t_{i} & t_{i+2} & t_{i+1}^{t_{i+2}} & \left.t_{k}\right)^{\sigma_{i} \sigma_{i+1}} \\
=\left(t_{k}\right. & t_{i} & t_{i+1} & t_{i+2} & \left.t_{k}\right)^{\sigma_{i+1} \sigma_{i} \sigma_{i+1}}
\end{array}
$$

Together, (a) and (b) show that there exists a homomorphism
$\gamma_{n}: \operatorname{Artin}\left\langle A_{n-1}\right\rangle \rightarrow \mathcal{B}_{n}$ determined by $\frac{\frac{i \in[1 \uparrow n-1]}{\left(a_{i}\right)^{\gamma_{n}}}}{}$. By Artin's Proposition 3.3, $=\left(\sigma_{i}\right)$
$\left\langle\sigma_{[1 \uparrow n-1]}\right\rangle=\mathcal{B}_{n}$, and, hence, $\gamma_{n}$ is surjective.
In the remainder of this section, we shall use induction on $n$ to show that the surjective homomorphism $\gamma_{n}: \operatorname{Artin}\left\langle A_{n-1}\right\rangle \rightarrow \mathcal{B}_{n}$ of Proposition 5.1 is an isomorphism. Notice that $\gamma_{n}$ endows $\operatorname{Artin}\left\langle A_{n-1}\right\rangle$ with a canonical action on $\Sigma_{0,1, n}$.

The following is precisely [24, Proposition 1] and, also, [10, Proposition 2.1(2)].
5.2 Lemma (Manfredini [24). If $n \geq 1$, then
$\operatorname{Artin}\left\langle A_{n-1}\right\rangle \ltimes \Sigma_{0,1, n}=\operatorname{Artin}\left\langle a_{1}-a_{2}-\cdots-a_{n-1}=\bar{t}_{n}\right\rangle \simeq \operatorname{Artin}\left\langle B_{n}\right\rangle$.
Proof. For $n=1$, the result is clear.
For $n=2$, we have the following.

$$
\begin{aligned}
& \operatorname{Artin}\left\langle A_{1}\right\rangle \ltimes \Sigma_{0,1,2}=\left\langle\left\{a_{1}\right\} \cup t_{[1 \uparrow 2]} \mid t_{1}^{a_{1}}=t_{2}, t_{2}^{a_{1}}=\bar{t}_{2} t_{1} t_{2}\right\rangle \\
& =\left\langle a_{1}, t_{2} \mid t_{2}^{a_{1}}=\bar{t}_{2} t_{2}^{\bar{a}_{1}} t_{2}\right\rangle=\left\langle a_{1}, t_{2} \mid\left(\bar{a}_{1} t_{2}\right)\left(a_{1}\right)=\left(\bar{t}_{2} a_{1}\right)\left(t_{2} \bar{a}_{1} t_{2}\right)\right\rangle \\
& =\left\langle a_{1}, t_{2} \mid\left(a_{1}\right)\left(\bar{t}_{2} a_{1} \bar{t}_{2}\right)=\left(\bar{t}_{2} a_{1}\right)\left(\bar{t}_{2} a_{1}\right)\right\rangle=\operatorname{Artin}\left\langle a_{1}=\bar{t}_{2}\right\rangle .
\end{aligned}
$$

From the case $n=2$, we see that there exists a homomorphism

$$
\begin{aligned}
& \frac{i \in[1 \uparrow n-1]}{\left(b_{i}\right.} \\
& =\left(\begin{array}{ll}
\left.b_{n}\right)^{\mu} \\
a_{i} & \bar{t}_{n}
\end{array}\right) .
\end{aligned}
$$

For each $k \in[1 \uparrow n]$, let $\mathfrak{t}_{k}$ denote the element $\bar{b}_{n}^{\Pi \overline{b_{[n-1 \downarrow k]}}}$ of $\operatorname{Artin}\left\langle B_{n}\right\rangle$. For each $i \in[1 \uparrow n-1]$ and $k \in[1 \uparrow n]$, let $\mathfrak{t}_{k}^{\bar{\sigma}_{i}}$ denote $\mathfrak{t}_{k}$, resp. $\mathfrak{t}_{i}$, resp. $\mathfrak{t}_{i+1}^{\bar{t}_{i}}$, if $k \in[1 \uparrow i-1] \cup[i+2 \uparrow n]$, resp. $k=i+1$, resp. $k=i$. We shall see that $\mathfrak{t}_{k}^{\bar{b}_{i}}=\mathfrak{t}_{k}^{\bar{\sigma}_{i}} ;$ this immediately implies that there exists a homomorphism
$\bar{\mu}: \operatorname{Artin}\left\langle A_{n-1}\right\rangle \ltimes \Sigma_{0,1, n} \rightarrow \operatorname{Artin}\left\langle B_{n}\right\rangle$ determined by $\frac{i \in[1 \uparrow n-1]}{\left(a_{i}\right.} \frac{k \in[1 \uparrow n]}{\left.t_{k}\right)^{\bar{\mu}}}$, which $=\left(\begin{array}{ll}b_{i} & \mathfrak{t}_{k}\end{array}\right)$ is then clearly inverse to $\mu$, and the result will be proved.

For each $m \in[n \downarrow 1]$, we shall show, by decreasing induction on $m$, that, for each $k \in[n \downarrow m]$ and each $i \in[n-1 \downarrow m], \mathrm{t}_{k}^{\bar{b}_{i}}=\mathfrak{t}_{k}^{\bar{\sigma}_{i}}$. For $m=n$, this is trivial, and, for $m=n-1$, it follows from the case $n=2$. Suppose that $m \in[n-2 \downarrow 1]$.
(a). For each $k \in[n \downarrow m+1]$ and each $i \in[n-1 \downarrow m+1], \mathfrak{t}_{k}^{\bar{b}_{i}}=\mathfrak{t}_{k}^{\bar{\sigma}_{i}}$, by hypothesis.
(b). For each $k \in[n \downarrow m+2], \mathfrak{t}_{k} \in\left\langle b_{[n \downarrow m+2]}\right\rangle$ and, hence, $\mathfrak{t}_{k}^{\bar{b}_{m}}=\mathfrak{t}_{k}=\mathfrak{t}_{k}^{\bar{\sigma}_{m}}$.
(c). $\mathfrak{t}_{m+1}^{\bar{b}_{m}}=\bar{b}_{n}^{\Pi \bar{b}_{[n-1 \downarrow m+1]} \bar{b}_{m}}=\mathfrak{t}_{m}=\mathfrak{t}_{m+1}^{\bar{\sigma}_{m}}$.
(d). For each $i \in[n-1 \downarrow m+2], \mathfrak{t}_{m}^{\bar{t}_{i}} \stackrel{(\text { c) })}{=} \mathfrak{t}_{m+1}^{\bar{b}_{m}} \bar{b}_{i}=\mathfrak{t}_{m+1}^{\bar{b}_{i} \bar{b}_{m}} \stackrel{(\text { a) })}{=} \mathfrak{t}_{m+1} \stackrel{\text { (c) }}{=} \mathfrak{t}_{m}=\mathfrak{t}_{m}^{\bar{\sigma}_{i}}$.
(e). $\mathfrak{t}_{m}^{\bar{b}_{m+1}} \stackrel{(\mathrm{c})}{=} \mathfrak{t}_{m+1}^{\bar{b}_{m+1} \bar{b}_{m+1}} \stackrel{(\mathrm{a})}{=} \mathfrak{t}_{m+2}^{\bar{b}_{m+1} \bar{b}_{m} \bar{b}_{m+1}}=\mathfrak{t}_{m+2}^{\bar{b}_{m+1} \bar{b}_{m+1} \bar{b}_{m}} \stackrel{(\mathrm{~b})}{=} \mathfrak{t}_{m+2}^{\bar{b}_{m+1}} \bar{b}_{m} \stackrel{(\mathrm{a})}{=} \mathfrak{t}_{m+1}^{\bar{b}_{m}} \stackrel{(\mathrm{c})}{=} \mathfrak{t}_{m}=\mathfrak{t}_{m}^{\bar{\sigma}_{m+1}}$.


$$
\stackrel{(\mathrm{a})}{=}\left(\mathfrak{t}_{m+1} \mathfrak{t}_{m+2} \overline{\mathfrak{t}}_{m+1}\right)^{\bar{b}_{m} \bar{b}_{m+1}} \stackrel{(\mathrm{c}),(\mathrm{b}),(\mathrm{c})}{=}\left(\mathfrak{t}_{m} \mathfrak{t}_{m+2} \overline{\mathfrak{t}}_{m}\right)^{\bar{b}_{m+1}} \stackrel{(\mathrm{e}),(\mathrm{a}),(\mathrm{e})}{=} \mathfrak{t}_{m} \mathfrak{t}_{m+1} \overline{\mathfrak{t}}_{m}=\mathfrak{t}_{m}^{\bar{\sigma}_{m}} .
$$

Now the result follows by induction.

We write $\operatorname{Stab}\left(\operatorname{Artin}\left\langle A_{n}\right\rangle ;\left[t_{n+1}\right]\right)$ to denote the $\operatorname{Artin}\left\langle A_{n}\right\rangle$-stabilizer of the conjugacy class $\left[t_{n+1}\right]$ under the $\operatorname{Artin}\left\langle A_{n}\right\rangle$-action on $\Sigma_{0,1, n+1}$. The Reidemeis-ter-Schreier rewriting technique automatically gives a useful presentation of $\operatorname{Stab}\left(\operatorname{Artin}\left\langle A_{n}\right\rangle ;\left[t_{n+1}\right]\right)$, but applying the technique can be rather tedious. Once the presentation has been found, we can verify it directly using the van der Waerden trick, as in the following proof.
5.3 Theorem (Magnus [23]). If $n \geq 1$, then there exists a homomorphism

$$
\left.\phi_{n}: \operatorname{Artin}\left\langle A_{n-1}\right\rangle \ltimes \Sigma_{0,1, n} \rightarrow \operatorname{Artin}\left\langle A_{n}\right\rangle \text { determined by } \begin{array}{ll}
\frac{i \in[1 \uparrow n-1]}{\left(a_{i}\right.} & \left.t_{n}\right)^{\phi_{n}} \\
& =\left(a_{i}\right. \\
a_{n}^{2}
\end{array}\right) .
$$

## Moreover, the following hold.

(i). $\phi_{n}$ is injective.
(ii). For each $i \in[1 \uparrow n], t_{i}^{\phi_{n}}=\bar{a}_{i}^{2 \Pi a_{[i+1 \uparrow n]}}$ in $\operatorname{Artin}\left\langle A_{n}\right\rangle$.
(iii). The image of $\phi_{n}$ is $\operatorname{Stab}\left(\operatorname{Artin}\left\langle A_{n}\right\rangle ;\left[t_{n+1}\right]\right)$.

Proof. Let us write $G=\operatorname{Artin}\left\langle A_{n}\right\rangle$ and $H=\operatorname{Artin}\left\langle A_{n-1}\right\rangle \ltimes \Sigma_{0,1, n}$.
In $G$,
$\left(a_{n-1} a_{n}^{2} a_{n-1}\right)^{a_{n}}=\left(\bar{a}_{n} a_{n-1} a_{n}\right)\left(a_{n} a_{n-1} a_{n}\right)=\left(a_{n-1} a_{n} \bar{a}_{n-1}\right)\left(a_{n-1} a_{n} a_{n-1}\right)=a_{n-1} a_{n}^{2} a_{n-1}$, and, hence, $a_{n-1} a_{n}^{2} a_{n-1} a_{n}^{2}=a_{n}^{2} a_{n-1} a_{n}^{2} a_{n-1}$. By Lemma 5.2, $H \simeq \operatorname{Artin}\left\langle B_{n}\right\rangle$, and we see that there exist a homomorphism $\phi_{n}: H \rightarrow G$ determined by $\underline{i \in[1 \uparrow n-1]}$

$$
\left.\begin{array}{rl} 
& \left(a_{i}\right. \\
= & \left.\bar{t}_{n}\right)^{\phi_{n}} . \\
= & \left(a_{i}\right.
\end{array} a_{n}^{2}\right) .
$$

Let $v_{([1 \uparrow n+2])}=([1 \uparrow n+1])$, thought of as a generic $(n+1)$-tuple, and consider the free left $H$-set $H \times v_{[1 \uparrow n+1]}$, with left $H$-transversal $v_{[1 \uparrow n+1]}$.

We construct a right $G$-action on $H \times v_{[1 \uparrow n+1]}$ such that $H \times v_{[1 \uparrow n+1]}$ becomes an $(H, G)$-bi-set. For each $i \in[1 \uparrow n]$, we define the right action of the generator $a_{i} \in G$ on the left $H$-set $H \times v_{[1 \uparrow n+1]}$, by specifying the action on the given left $H$-transversal as follows.

$$
\begin{aligned}
& \frac{k \in[1 \uparrow i-1]}{\left(v_{k}\right.} & v_{i} & v_{i+1} \\
\left(\begin{array}{ll} 
& \frac{k \in[i+2 \uparrow n+1]}{\left.v_{k}\right) a_{i}} \\
= & \left(a_{i-1} v_{k}\right.
\end{array}\right. & v_{i+1} & \bar{t}_{i} v_{i} & \left.a_{i} v_{k}\right) .
\end{aligned}
$$

We now verify that the relations of $G$ are respected.
(a). Suppose that $1 \leq i<i+2 \leq j \leq n$. We have the following.

| $k \in[1 \uparrow i-1]$ |  |  | $\underline{k \in[i+2 \uparrow j-1]}$ |  | $k \in[j+2 \uparrow n+1]$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $v_{k}$ | $v_{i}$ | $v_{i+1}$ | $v_{k}$ | $v_{j}$ | $v_{j+1}$ | $\left.v_{k}\right) a_{i} a_{j}$ |
| $a_{i-1} v_{k}$ | $v_{i+1}$ | $\bar{t}_{i} v_{i}$ | $a_{i} v_{k}$ | $a_{i} v_{j}$ | $a_{i} v_{j+1}$ | $\left.a_{i} v_{k}\right) a_{j}$ |
| $=\left(a_{i-1} a_{j-1} v_{k}\right.$ | $a_{j-1} v_{i+1}$ | $\bar{t}_{i} a_{j-1} v_{i}$ | $a_{i} a_{j-1} v_{k}$ | $a_{i} v_{j+1}$ | $a_{i} \bar{t}_{j} v_{j}$ | $\left.a_{i} a_{j} v_{k}\right)$ |
| $=\left(a_{j-1} a_{i-1} v_{k}\right.$ | $a_{j-1} v_{i+1}$ | $a_{j-1} \bar{t}_{i} v_{i}$ | $a_{j-1} a_{i} v_{k}$ | $a_{i} v_{j+1}$ | $\bar{t}_{j} a_{i} v_{j}$ | $\left.a_{j} a_{i} v_{k}\right)$ |
| $=\left(\quad a_{j-1} v_{k}\right.$ | $a_{j-1} v_{i}$ | $a_{j-1} v_{i+1}$ | $a_{j-1} v_{k}$ | $v_{j+1}$ | $\bar{t}_{j} v_{j}$ | $\left.a_{j} v_{k}\right) a_{i}$ |
| $=\left(\quad v_{k}\right.$ | $v_{i}$ | $v_{i+1}$ | $v_{k}$ | $v_{j}$ | $v_{j+1}$ | $\left.v_{k}\right) a_{j} a_{i}$ |

(b). Suppose that $1 \leq i \leq n-1$. We have the following.
$\underline{k \in[1 \uparrow i-1]}$

$$
\begin{aligned}
& \text { ( } \quad v_{k} \\
& =\left(\quad a_{i-1} v_{k}\right. \\
& =\left(\quad a_{i-1} a_{i} v_{k}\right. \\
& =\left(a_{i-1} a_{i} a_{i-1} v_{k}\right. \\
& =\left(a_{i} a_{i-1} a_{i} v_{k}\right. \\
& =\left(\begin{array}{ll}
\quad a_{i} a_{i-1} v_{k} & a_{i} v_{i+1}
\end{array}\right. \\
& \begin{array}{l}
=\left(\begin{array}{rr}
a_{i} v_{k} & a_{i} v_{i} \\
& v_{k}
\end{array} v_{i}\right.
\end{array}
\end{aligned}
$$

Now (a) and (b) prove that the relations of $G$ are respected. Hence, we have a right $G$-action on $H \times v_{[1 \uparrow n+1]}$.

Notice that $v_{n+1} \bar{t}_{n}^{\phi_{n}}=v_{n+1} a_{n}^{2}=\bar{t}_{n} v_{n} a_{n}=\bar{t}_{n} v_{n+1}$. Also, for each $i \in[1 \uparrow n-1]$, $v_{n+1} a_{i}^{\bar{\phi}_{n}}=v_{n+1} a_{i}=a_{i} v_{n+1}$. It follows that, for each $h \in H, v_{n+1} h^{\phi_{n}}=h v_{n+1}$. Hence, $\phi_{n}$ is injective. This proves (i).

Recall that $G=\operatorname{Artin}\left\langle A_{n}\right\rangle$.
Let $i \in[1 \uparrow n]$.
We shall show by decreasing induction on $i$ that

$$
\begin{equation*}
a_{n}^{\Pi \bar{a}_{[n-1, i]}}=a_{i}^{\Pi a_{[i+1 \uparrow n]}} . \tag{5.3.1}
\end{equation*}
$$

If $i=n$, then (5.3.1) holds. Now suppose that $i \geq 2$, and that (5.3.1) holds. Conjugating (5.3.1) by $\bar{a}_{i-1}$ yields

$$
a_{n}^{\Pi \bar{a}_{[n-1 \downarrow i-1]}}=a_{i}^{\Pi a_{[i+1 \uparrow n]} \bar{a}_{i-1}}=a_{i}^{\bar{a}_{i-1} \Pi a_{[i+1 \uparrow n]}}=a_{i-1}^{a_{i} \Pi a_{[i+1 \uparrow n]}}=a_{i-1}^{\Pi a_{[i \uparrow n]}} .
$$

By induction, (5.3.1) holds.
Now $\bar{t}_{i}^{\phi_{n}}=\left(\bar{t}_{n}^{\left.\Pi \bar{a}_{[n-1, i]}\right]}\right)^{\phi_{n}}=a_{n}^{\left.2 \Pi \bar{a}_{[n-1, i]}\right]} \stackrel{[5.3 .1]}{=} a_{i}^{2 \Pi a_{[i+1 \uparrow n]}}$. This proves (ii). Also, $\bar{t}_{i}^{\phi_{n}} \Pi \bar{a}_{[n \downarrow i]}=\Pi \bar{a}_{[n \downarrow i+1]} a_{i}$.

If $k \in[1 \uparrow i-1]$, then

$$
a_{i}^{\Pi a_{[k \uparrow n]}}=a_{i}^{\Pi a_{[k \uparrow i-2]} \Pi a_{[i-1 \uparrow i} \Pi a_{[i+1 \uparrow \uparrow]}}=a_{i}^{\Pi a_{[i-1 \uparrow i]} \Pi a_{[i+1 \uparrow n]}}=a_{i-1}^{\Pi a_{[i+1 \uparrow n]}}=a_{i-1} .
$$

Hence, $a_{i-1} \Pi \bar{a}_{[n \downarrow k]}=\Pi \bar{a}_{[n \downarrow k]} a_{i}$.
Let $\psi_{n}$ denote the map of sets

$$
\psi_{n}: H \times v_{[1 \uparrow n+1]} \rightarrow G, \quad h v_{k} \mapsto h^{\phi_{n}} \Pi \bar{a}_{[n \downarrow k]} \text { for all } h v_{k}=\left(h, v_{k}\right) \in H \times v_{[1 \uparrow n+1]} .
$$

Hence, for each $h \in H$, we have the following, in $G$.

| $\underline{k \in[1 \uparrow i-1]}$ |  |  | $\underline{k \in[i+2 \uparrow n+1]}$ |
| :---: | :---: | :---: | :---: |
| (h) $v_{k}$ | $v_{i}$ | $v_{i+1}$ | $\left.\left.v_{k} \quad\right)\right)^{\psi_{n}} a_{i}$ |
| $=\left(h^{\phi_{n}}\left(\quad \Pi \bar{a}_{[n \downarrow k]}\right.\right.$ | $\Pi \bar{a}_{[n \downarrow i]}$ | $\Pi \bar{a}_{[n \downarrow i+1]}$ | $\left.\left.\Pi \bar{a}_{[n \downarrow k]}\right)\right) a_{i}$ |
| $=\left(h^{\phi_{n}}\left(a_{i-1} \Pi \bar{a}_{[n \downarrow k]}\right.\right.$ | $\Pi \bar{a}_{[n \downarrow i+1]}$ | $\bar{t}_{i}^{\phi_{n}} \Pi \bar{a}_{[n \downarrow i]}$ | $\left.\left.a_{i} \Pi \bar{a}_{[n \downarrow k]}\right)\right)$ |
| $=(h) a_{i-1} v_{k}$ | $v_{i+1}$ | $\bar{t}_{i} v_{i}$ | $\left.\left.a_{i} v_{k} \quad\right)\right)^{\psi_{n}}$ |
| $=\left(\begin{array}{ll}h\end{array} \quad v_{k}\right.$ | $v_{i}$ | $v_{i+1}$ | $\left.\left.v_{k} \quad\right) a_{i}\right)^{\psi_{n}}$ |

This proves that $\psi_{n}$ is a map of right $G$-sets, and, hence, $\psi_{n}$ must be surjective. Thus, $G=\bigcup_{k \in[1 \uparrow n+1]} H^{\phi_{n}} v_{k}^{\psi_{n}}$, and, hence, the index of $H^{\phi_{n}}$ in $G$ is at most $n+1$.

Consider the action of $G$ on the set of conjugacy classes $\left\{\left[t_{k}\right]\right\}_{k \in[1 \uparrow n+1]}$ in $\Sigma_{0,1, n+1}$. For any $i \in[1 \uparrow n], a_{i}$ acts as the transposition $\left(\left[t_{i}\right],\left[t_{i+1}\right]\right)$. In particular, the index of $\operatorname{Stab}\left(G ;\left[t_{n+1}\right]\right)$ in $G$ is $n+1$. Also, the elements of $a_{[1 \uparrow n-1]} \cup\left\{a_{n}^{2}\right\}$ fix $\left[t_{n+1}\right]$, and, hence, $H^{\phi_{n}} \leq \operatorname{Stab}\left(G ;\left[t_{n+1}\right]\right)$. By comparing indices, we see that $H^{\phi_{n}}=\operatorname{Stab}\left(G ;\left[t_{n+1}\right]\right)$. This proves (iii).
5.4 Theorem (Artin). $\mathcal{B}_{n}=\operatorname{Artin}\left\langle\sigma_{1}-\sigma_{2}-\cdots-\sigma_{n-1}\right\rangle$.

Proof. This is trivial for $n \leq 1$. Hence, we may assume that $n \geq 1$ and that the homomorphism $\gamma_{n}: \operatorname{Artin}\left\langle A_{n-1}\right\rangle \rightarrow \mathcal{B}_{n}$, of Proposition 5.1, determined $\underline{i \in[1 \uparrow n-1]}$
by $\left(a_{i}\right)^{\gamma_{n}}$ is an isomorphism; and it remains to show that the surjective $=\left(\sigma_{i}\right)$
homomorphism $\gamma_{n+1}: \operatorname{Artin}\left\langle A_{n}\right\rangle \rightarrow \mathcal{B}_{n+1}$ is injective.
Consider an element $w$ of the kernel of $\gamma_{n+1}$. In particular, $w$ fixes $t_{n+1}$ in the $\operatorname{Artin}\left\langle A_{n}\right\rangle$-action on $\Sigma_{0,1, n+1}$. By Theorem [5.3(iii), $w$ lies in the image of the homomorphism $\phi_{n}: \operatorname{Artin}\left\langle A_{n-1}\right\rangle \ltimes \Sigma_{0,1, n} \rightarrow \operatorname{Artin}\left\langle A_{n}\right\rangle$ determined by $\underline{i \in[1 \uparrow n-1]}$
$\left(\begin{array}{ll}a_{i} & t_{n}\end{array}\right)^{\phi_{n}}$. Thus, we may express $w$ as a product of two words $=\left(\begin{array}{ll}a_{i} & \bar{a}_{n}^{2}\end{array}\right)$

$$
\begin{equation*}
w=w_{1}\left(a_{([1 \uparrow n-1])}\right) w_{2}\left(t_{([1 \uparrow n])}^{\phi_{n}}\right) . \tag{5.4.1}
\end{equation*}
$$

Now,
in $\operatorname{Artin}\left\langle A_{n}\right\rangle \ltimes \Sigma_{0,1, n+1}, t_{n+1}=t_{n+1}^{w}=t_{n+1}^{w_{1}\left(a_{([1 \uparrow n-1])}\right) w_{2}\left(t_{(1 \uparrow n])}^{\phi n}\right)}=t_{n+1}^{w_{2}\left(t_{(1 \uparrow n)]}^{\phi n}\right)}$.
Consider the homomorphism $\phi_{n+1}: \operatorname{Artin}\left\langle A_{n}\right\rangle \ltimes \Sigma_{0,1, n+1} \rightarrow \operatorname{Artin}\left\langle A_{n+1}\right\rangle$ de$i \in[1 \uparrow n]$
termined by $\quad\left(\begin{array}{ll}a_{i} & t_{n+1}\end{array}\right)^{\phi_{n+1}}$. Let $i \in[1 \uparrow n]$. By Theorem 5.3(ii),

$$
=\left(\begin{array}{ll}
a_{i} & \bar{a}_{n+1}^{2}
\end{array}\right)
$$

$$
\left(t_{i}^{\phi_{n}}\right)^{\phi_{n+1} a_{n+1}}=\left(\bar{a}_{i}^{2 \Pi a_{[i+1 \uparrow n]}}\right)^{\phi_{n+1} a_{n+1}}=\left(\bar{a}_{i}^{\left.2 \Pi a_{[i+1 \uparrow n]}\right)^{a_{n+1}}}\right.
$$

$$
=\left(\bar{a}_{i}^{2 \Pi a_{[i+1 \uparrow n+1]}}\right)=\left(t_{i}\right)^{\phi_{n+1}},
$$

$$
\left(t_{n+1}\right)^{\phi_{n+1} a_{n+1}}=\left(\bar{a}_{n+1}^{2}\right)^{a_{n+1}}=\bar{a}_{n+1}^{2}=\left(t_{n+1}\right)^{\phi_{n+1}} .
$$

Thus the two $(n+1)$-tuples $\left(t_{([1 \uparrow n])}^{\phi_{n}}, t_{n+1}\right)$ and $t_{([1 \uparrow n+1])}$ for $\operatorname{Artin}\left\langle A_{n}\right\rangle \ltimes \Sigma_{0,1, n+1}$ become conjugate in $\operatorname{Artin}\left\langle A_{n+1}\right\rangle$ under $\phi_{n+1}$. By Theorem 5.3)(i), $\phi_{n+1}$ is injective. Since $t_{([1 \uparrow n+1])}$ freely generates a free subgroup of $\operatorname{Artin}\left\langle A_{n}\right\rangle \ltimes \Sigma_{0,1, n+1}$, we see that $\left(t_{([1 \uparrow n])}^{\phi_{n}}, t_{n+1}\right)$ also freely generates a free subgroup of $\operatorname{Artin}\left\langle A_{n}\right\rangle \ltimes \Sigma_{0,1, n+1}$. From (5.4.1), we see that $w_{2}$ must be the trivial word.

Hence, $w=w_{1}\left(a_{([1 \uparrow n-1])}\right)$ in $\operatorname{Artin}\left\langle A_{n}\right\rangle$. By the induction hypothesis, $w_{1}\left(a_{([1 \uparrow n-1])}\right)=1$ in $\operatorname{Artin}\left\langle A_{n-1}\right\rangle$. Hence $w=1$ in $\operatorname{Artin}\left\langle A_{n}\right\rangle$.

Now the result holds by induction.
Combining Lemma5.2, Theorem 5.3 and Theorem 5.4, we have the following.
5.5 Corollary (Artin-Magnus-Manfredini). If $n \geq 1$, then

$$
\begin{aligned}
& \mathcal{B}_{n}=\operatorname{Artin}\left\langle\sigma_{1}-\sigma_{2}-\cdots-\sigma_{n-2}-\sigma_{n-1}\right\rangle \\
& \simeq \operatorname{Artin}\left\langle A_{n-1}\right\rangle, \\
& \operatorname{Stab}\left(\mathcal{B}_{n} ;\left[t_{n}\right]\right)=\operatorname{Artin}\left\langle\sigma_{1}-\sigma_{2}-\cdots-\sigma_{n-2}=\sigma_{n-1}^{2}\right\rangle \\
& \mathcal{B}_{n-1} \ltimes \Sigma_{0,1, n-1}=\operatorname{Artin}\left\langle\sigma_{1}-\sigma_{2}-\cdots-B_{n-1}\right\rangle, \\
&\left.=\bar{t}_{n-1}\right\rangle \simeq \operatorname{Artin}\left\langle B_{n-1}\right\rangle .
\end{aligned}
$$

5.6 Historical Remarks. In 1925, Artin [3] found the above presentation of $\mathcal{B}_{n}$ by an intuitive topological argument but, later, in [4], he indicated that there were difficulties that could be corrected. In 1934, Magnus [23] gave an algebraic proof that the relations suffice. In 1945, Markov [25] gave a similar algebraic proof. In 1947, Bohnenblust [7] gave a similar algebraic proof; in 1948, Chow [8] simplified the latter proof. All these algebraic proofs of the sufficiency of the relations involve the Reidemeister-Schreier rewriting process for the subgroup of index $n$.

Larue [22] gave a new algebraic proof of the sufficiency of the relations, by using the Dehornoy-Larue trichotomy [14] for braid groups. We shall proceed in the opposite direction. Proofs of the trichotomy for $\operatorname{Artin}\left\langle A_{n-1}\right\rangle$ tend to be more difficult than proofs that Out ${ }_{0,1, n}^{+}=\operatorname{Artin}\left\langle A_{n-1}\right\rangle$, and we shall now see that Artin's generation argument easily gives the trichotomy for Out ${ }_{0,1, n}^{+}$.

## 6 The Dehornoy-Larue trichotomy

### 6.1 Definitions. Let $\phi \in \mathcal{B}_{n}$.

We say that $\phi$ is $\sigma_{1}$-neutral if $\phi$ lies in the subgroup of $\mathcal{B}_{n}$ generated by $\sigma_{[2 \uparrow n-1]}$.

We say that $\phi$ is $\sigma_{1}$-positive if $n \geq 2$ and $\phi$ can be expressed as the product of a finite sequence of elements of $\sigma_{[1 \uparrow n-1]} \cup \bar{\sigma}_{[2 \uparrow n-1]}$ such that at least one term of the sequence is $\sigma_{1}$. We say that $\phi$ is $\sigma$-positive if $n \geq 2$ and, for some $i \in[1 \uparrow n-1], \phi$ can be expressed as the product of a finite sequence of elements of $\sigma_{[i \uparrow n-1]} \cup \bar{\sigma}_{[i+1 \uparrow n-1]}$ such that at least one term of the sequence is $\sigma_{i}$.

We say that $\phi$ is $\sigma_{1}$-negative if $\bar{\phi}$ is $\sigma_{1}$-positive, that is, $n \geq 2$ and $\phi$ can be expressed as the product of a finite sequence of elements of $\sigma_{[2 \uparrow n-1]} \cup \bar{\sigma}_{[1 \uparrow n-1]}$ such that at least one term of the sequence is $\bar{\sigma}_{1}$.

If $\phi$ satisfies exactly one of the properties of being $\sigma_{1}$-neutral, $\sigma_{1}$-positive $\sigma_{1}$-negative, we say that $\phi$ satisfies the $\sigma_{1}$-trichotomy.
6.2 Historical Remarks. View $\operatorname{Artin}\left\langle A_{n}\right\rangle$ as a subgroup of $\operatorname{Artin}\left\langle A_{n+1}\right\rangle$ in a natural way, and let $\operatorname{Artin}\left\langle A_{\infty}\right\rangle$ denote the union of the resulting chain; thus $\operatorname{Artin}\left\langle A_{\infty}\right\rangle=\left\langle a_{[1 \uparrow \infty \mid}\right\rangle$. Dehornoy [14, Theorem 6] gave a one-sided ordering of $\operatorname{Artin}\left\langle A_{\infty}\right\rangle$; the positive semigroup for this ordering is the set of ' $a$-positive' elements of $\operatorname{Artin}\left\langle A_{\infty}\right\rangle$.

Let $\phi \in \mathcal{B}_{n}$. By replacing $\phi$ with $\bar{\phi}$ if necessary, we can apply Dehornoy's result to deduce that there exists some $n^{\prime} \geq n$ such that $\phi$ is $\sigma$-negative in $\mathcal{B}_{n^{\prime}}$, or $\phi=1$. Larue [21] showed that this implies that $t_{1}^{\phi} \in\left(t_{1} \star\right)$, and that this in turn implies that $\phi$ can be expressed as the product of a finite sequence, of length at most $|\phi|+\frac{1}{4} n^{2} 3^{|\phi|}$, of elements of $\sigma_{[2 \uparrow n-1]} \cup \bar{\sigma}_{[1 \uparrow n-1]}$. Thus, every element of $\mathcal{B}_{n}$ satisfies the $\sigma_{1}$-trichotomy. Larue's work is surveyed in [16, Chapter 5]. Topological versions of these results can be found in [19] and [16, Chapter 6].

We shall give elementary direct proofs of the foregoing results and replace Larue's bound $|\phi|+\frac{1}{4} n^{2} 3^{|\phi|}$ with the much smaller bound $n 2^{|\phi|}-n$. Larue's proof contains interesting information that we shall rework in the Appendix.

Part (iii) of the following seems to be new.
6.3 Lemma. Let $n \geq 1$ and let $\phi$ be an element of $\mathcal{B}_{n}$ such that $t_{1}^{\phi} \in\left(t_{1} \star\right)$. Let $\pi=\pi(\phi)$ and, for each $i \in[1 \uparrow n]$, let $u_{i}=u_{i}(\phi)$.
(i). Suppose that there exists some $i \in[1 \uparrow n-1]$ such that $u_{i} \in\left(* \bar{t}_{(i+1) \pi}\right)$. Then $\left\|\sigma_{i} \phi\right\| \leq\|\phi\|-2$ and $t_{1}^{\sigma_{i} \phi} \in\left(t_{1} \star\right)$; moreover, if $t_{1}^{\phi}=t_{1}$, then $i \in[2 \uparrow n-1]$.
(ii). Suppose that there exists some $i \in[2 \uparrow n-1]$ such that $u_{i} \in\left(\bar{t}_{i^{\pi} \star}\right)$. Then $\left\|\bar{\sigma}_{i} \phi\right\| \leq\|\phi\|-2$ and $t_{1}^{\bar{\sigma}_{i} \phi} \in\left(t_{1} \star\right)$.
(iii). Suppose that, for each $i \in[1 \uparrow n-1]$, $u_{i} \notin\left(\star \bar{t}_{(i+1)^{\pi}}\right)$ and, for each $i \in$ $[2 \uparrow n-1], u_{i} \notin\left(\bar{t}_{i^{\pi}}\right)$. Then $\phi=1$.

Proof. For each $i \in[0 \uparrow n+1]$, let $w_{i}=w_{i}(\phi)$.
(i). The first part follows from Artin's Lemma 3.2(i). Notice that, if $t_{1}^{\phi}=t_{1}$, then $w_{1}=1$ and $u_{1}=\bar{w}_{2} \notin\left(\star \bar{t}_{2^{\pi}}\right)$.
(ii) follows from Lemma 3.2(ii).
(iii). Recall that $u_{0} \prod_{i \in[1 \uparrow n]}\left(t_{i \pi} u_{i}\right)=\prod_{i \in[1 \uparrow n]}\left(t_{i^{\pi}}^{w_{i}}\right)=\left(\prod_{i \in[1 \uparrow n]} t_{i}\right)^{\phi}=\prod_{i \in[1 \uparrow n]} t_{i}$. Hence, $u_{0} t_{1^{\pi}} u_{1} \prod_{i \in[2 \uparrow n]}\left(t_{i} u_{i}\right)=t_{1} \prod_{i \in[2 \uparrow n]} t_{i}$, and, hence, $u_{1} \prod_{i \in[2 \uparrow n]}\left(t_{i \pi} u_{i}\right)=\bar{t}_{1^{\pi}} \bar{u}_{0} t_{1} \prod_{i \in[2 \uparrow n]} t_{i}$. Since $u_{n}=w_{n} \notin\left(\bar{t}_{n \pi \star}\right)$, the hypotheses imply that there is no cancellation anywhere in the expression $u_{1} \prod_{i \in[2 \uparrow n]}\left(t_{i \pi} u_{i}\right)$. Hence,

$$
\begin{equation*}
\sum_{i \in[1 \uparrow n]}\left|u_{i}\right|+n-1=\left|u_{1} \prod_{i \in[2 \uparrow n]}\left(t_{i \pi} u_{i}\right)\right|=\left|\bar{t}_{1 \pi} \bar{u}_{0} t_{1} \prod_{i \in[2 \uparrow n]} t_{i}\right| \leq\left|\bar{t}_{1 \pi} \bar{u}_{0} t_{1}\right|+n-1 . \tag{6.3.1}
\end{equation*}
$$

Since $t_{1^{\pi}}^{\bar{u}_{0}}=t_{1^{\pi}}^{w_{1}}=t_{1}^{\phi} \in\left(t_{1} \star\right)$, we see that $u_{0} t_{1^{\pi}} \in\left(t_{1} \star\right)$, and

$$
\begin{equation*}
\left|\bar{t}_{1} u_{0} t_{1^{\pi}}\right|=-1+\left|u_{0} t_{1^{\pi}}\right| \leq-1+\left|u_{0}\right|+1=\left|u_{0}\right| . \tag{6.3.2}
\end{equation*}
$$

Since $\prod_{i \in[0 \uparrow n]} u_{i}=w_{0} \bar{w}_{n+1}=1$, we see that

$$
\begin{equation*}
\prod_{i \in[1 \uparrow n]} u_{i}=\bar{u}_{0}=w_{1} \notin\left(\bar{t}_{1 \pi \star}\right) . \tag{6.3.3}
\end{equation*}
$$

Now, $\sum_{i \in[1 \uparrow n]}\left|u_{i}\right| \stackrel{\sqrt{6.3 .1)}}{\leq}\left|\bar{t}_{1 \pi} \bar{u}_{0} t_{1}\right| \stackrel{\sqrt{6.3 .2]}}{\leq}\left|\bar{u}_{0}\right| \stackrel{\sqrt{6.3 .3]}}{=}\left|\prod_{i \in[1 \uparrow n]} u_{i}\right|$. Therefore, there is no cancellation in $\prod_{i \in[1 \uparrow n]} u_{i}$, and, by (6.3.3), $u_{1} \notin\left(\bar{t}_{1 \pi \star}\right)$. By Lemma3.3(iii), $\phi=1$.

As in Remarks [3.4, we deduce the following from Lemma 6.3 by induction on $\|\phi\|$.
6.4 Corollary (Larue [21]). Let $n \geq 1$ and let $\phi \in \mathcal{B}_{n}$.
(i). Suppose that $t_{1}^{\phi} \in\left(t_{1} \star\right)$. Then $\phi$ is $\sigma_{1}$-negative or $\sigma_{1}$-neutral. In more detail, $\phi$ can be expressed as the product of a sequence, of length at most $n 2^{|\phi|}-n$, of elements of $\sigma_{[2 \uparrow n-1]} \cup \bar{\sigma}_{[1 \uparrow n-1]}$.
(ii). Moreover, $\phi$ is $\sigma_{1}$-neutral if and only if $t_{1}^{\phi}=t_{1}$.
6.5 Notation. For each $i \in[1 \uparrow n-1]$, let $\sigma_{i}^{\prime}$ and $\sigma_{i}^{\prime \prime}$ be the automorphisms of $\Sigma_{0,1, n}$ determined by

$$
\left.\begin{array}{llllll}
\frac{k \in[1 \uparrow i]}{\left(t_{k}\right.} & t_{i+1} & \frac{k \in[i+2 \uparrow n]}{\left.t_{k}\right)^{\sigma_{i}^{\prime}}} & \frac{k \in[1 \uparrow i-1]}{\left(t_{k}\right.} & t_{i} & t_{i+1} \\
=\left(\begin{array}{llll}
t_{k} & t_{i+1}^{t_{i}} & t_{k}
\end{array}\right), & =\left(t_{k}\right. & t_{i+1} & t_{i} & t_{k}
\end{array}\right) .
$$

Then $\sigma_{i}=\sigma_{i}^{\prime} \sigma_{i}^{\prime \prime}$. The normal form in $t_{[1 \uparrow n]}$ factorizes into an alternating product with factors which are normal forms of non-trivial elements of $\left\langle t_{[i \uparrow i+1]}\right\rangle$ alternating with factors which are normal forms of non-trivial elements of $\left\langle t_{[1 \uparrow i-1] \cup[i+2 \uparrow n]}\right\rangle$. On $\left\langle t_{[i \uparrow i+1]}\right\rangle, \sigma_{i}^{\prime}$ acts as conjugation by $t_{i}$, while $\sigma_{i}^{\prime \prime}$ interchanges the two free generators. On $\left\langle t_{[1 \uparrow i-1] \cup[i+2 \uparrow n]}\right\rangle, \sigma_{i}^{\prime}$ and $\sigma_{i}^{\prime \prime}$ act as the identity map.

The next result gives three trichotomies, called (a), (b) and (c), which hold for elements of $\mathcal{B}_{n}$. Attribution is not sharply defined, but it is reasonable to attribute (b) to Dehornoy [14], and (a) and (c) to Larue [21].
6.6 Theorem (Dehornoy-Larue [14], [21]). Let $n \geq 1$, let $\phi \in \mathcal{B}_{n}$ and consider the following nine conditions.
(a1). $t_{1}^{\phi}=t_{1} . \quad(\mathrm{a} 2) . t_{1}^{\phi} \in\left(t_{1} \star\right)-\left\{t_{1}\right\} . \quad(\mathrm{a} 3) . t_{1}^{\phi} \notin\left(t_{1} \star\right)$.
(b1). $\phi$ is $\sigma_{1}$-neutral. (b2). $\phi$ is $\sigma_{1}$-negative. (b3). $\phi$ is $\sigma_{1}$-positive.
(c1). $\left(t_{1} \star\right)^{\phi}=\left(t_{1} \star\right) \quad(\mathrm{c} 2) \cdot\left(t_{1} \star\right)^{\phi} \subset\left(t_{1} \star\right) . \quad(\mathrm{c} 3) .\left(t_{1} \star\right)^{\phi} \supset\left(t_{1} \star\right)$.
Then: $(\mathrm{a} 1) \Leftrightarrow(\mathrm{b} 1) \Leftrightarrow(\mathrm{c} 1) ;(\mathrm{a} 2) \Leftrightarrow(\mathrm{b} 2) \Leftrightarrow(\mathrm{c} 2) ;(\mathrm{a} 3) \Leftrightarrow(\mathrm{b} 3) \Leftrightarrow(\mathrm{c} 3)$.
Exactly one of (b1), (b2), (b3), holds; that is, $\phi$ satisfies the $\sigma_{1}$-trichotomy in $\mathcal{B}_{n}$.

Proof. (a1) $\Leftrightarrow$ (b1) by Corollary 6.4(ii). We shall use (a1) and (b1) interchangeably in the remainder of the proof.
(b1) $\Rightarrow(\mathrm{c} 1)$. If $\phi$ is $\sigma_{1}$-neutral, then so is $\bar{\phi}$. It follows that $\left(t_{1} \star\right)^{\phi} \subseteq\left(t_{1} \star\right)$ and $\left(t_{1} \star\right)^{\bar{\phi}} \subseteq\left(t_{1} \star\right)$. Thus, $\left(t_{1} \star\right)^{\phi}=\left(t_{1} \star\right)$.
$(\mathrm{a} 2) \Rightarrow(\mathrm{b} 2)$. If (a2) holds, then Corollary $6.4(\mathrm{i})$ shows that (b1) or (b2) holds. Since (a1) fails, (b1) fails. Thus (b2) holds.
(b2) $\Rightarrow(\mathrm{c} 2)$. Using Notation 6.5, we see that

$$
\left(t_{1} \star\right)^{\bar{\sigma}_{1}}=\left(t_{1} \star\right)^{\bar{\sigma}_{1}^{\prime \prime} \bar{\sigma}_{1}^{\prime}}=\left(t_{2} \star\right)^{\bar{\sigma}_{1}^{\prime}} \subseteq\left(t_{1} t_{2} \star\right) \subset\left(t_{1} \star\right) .
$$

Since the composition of injective self-maps of $\left(t_{1} \star\right)$ can be bijective only if all the factors are bijective, we see that (b2) $\Rightarrow(\mathrm{c} 2)$.
$(\mathrm{a} 3) \Rightarrow(\mathrm{b} 3)$. We translate into algebra the crucial reflection argument of [16, Corollary 5.2.4].

Suppose that (a3) holds.
With Notation 3.1, let $w_{1}=w_{1}(\phi)$ and $\pi=\pi(\phi)$. Then $\bar{w}_{1} t_{1^{\pi}} w_{1}=t_{1}^{\phi} \notin\left(t_{1} \star\right)$. It follows that $\bar{w}_{1} t_{1^{\pi}} \notin\left(t_{1} \star\right)$. Hence, $\bar{w}_{1} \bar{t}_{1^{\pi}} \notin\left(t_{1} \star\right)$. Hence, $\bar{t}_{1}^{\phi}=\bar{w}_{1} \bar{t}_{1 \pi} w_{1} \notin\left(t_{1} \star\right) \cup\{1\}$. On conjugating by $t_{1}$, we see that $\bar{t}_{1}^{\phi t_{1}} \in\left(\bar{t}_{1} \star\right)$.

Let $\zeta$ be the automorphism of $\Sigma_{0,1, n}$ determined by $\quad\left(t_{k}\right)^{\zeta}$. For $=\left(\bar{t}_{k}^{\Pi \bar{t}_{[k-1,1]}}\right)$
each $k \in[1 \uparrow n],\left(\Pi t_{[1 \uparrow k]}\right)^{\zeta}=\Pi \bar{t}_{[k \downarrow 1]}$. It follows that $\zeta^{2}=1$. Notice that $\zeta$ belongs to Out ${ }_{0,1, n}^{-}:=$Out $_{0,1, n}-$ Out $_{0,1, n}^{+}$. Also, $\quad\left(\begin{array}{ll}t_{1} & \frac{k \in[2 \uparrow n]}{t_{k}}\end{array}\right)^{\bar{t}_{1} \zeta}$. Hence, $=\left(\begin{array}{ll}\bar{t}_{1} & \bar{t}_{k}^{\Pi \Pi \bar{t}_{[k-1 \downarrow 2]}}\end{array}\right)$

$$
t_{1}^{\phi \zeta}=t_{1}^{\zeta \phi \zeta}=\bar{t}_{1}^{\phi_{1} \bar{t}_{1} \zeta} \in\left(\bar{t}_{1} \star\right)^{\bar{t}_{1} \zeta} \subseteq\left(t_{1} \star\right) .
$$

By Corollary 6.4(i), $\phi^{\zeta}$ can be expressed as the product of a finite sequence of elements of $\sigma_{[2 \uparrow n-1]} \cup \bar{\sigma}_{[1 \uparrow n-1]}$. It is not difficult to check that, for each $i \in[1 \uparrow n-1]$, $\sigma_{i}^{\zeta}=\bar{\sigma}_{i}$ in Out $_{0,1, n}$. Hence $\phi^{\zeta^{2}}(=\phi)$ can be expressed as the product of a finite sequence of elements of $\sigma_{[2 \uparrow n-1]}^{\zeta} \cup \bar{\sigma}_{[1 \uparrow n-1]}^{\zeta}\left(=\bar{\sigma}_{[2 \uparrow n-1]} \cup \sigma_{[1 \uparrow n-1]}\right)$. Hence, (b3) or (b1) holds. Since (a3) holds, (a1) fails, and (b1) fails. Thus (b3) holds.
$(\mathrm{b} 3) \Rightarrow(\mathrm{c} 3)$. If $\phi$ is $\sigma_{1}$-positive, then $\bar{\phi}$ is $\sigma_{1}$-negative, and, by $(\mathrm{b} 2) \Rightarrow(\mathrm{c} 2)$, $\left(t_{1} \star\right)^{\bar{\phi}} \subset\left(t_{1} \star\right)$ and, hence, $\left(t_{1} \star\right) \subset\left(t_{1} \star\right)^{\phi}$.
(c1) $\Rightarrow$ (a1). Suppose that (a1) fails. Then (a2) or (a3) holds. Hence (c2) or (c3) holds. Hence (c1) fails.
$(\mathrm{c} 2) \Rightarrow(\mathrm{a} 2)$ and $(\mathrm{c} 3) \Rightarrow(\mathrm{a} 3)$ are proved similarly.
Thus the desired equivalences hold.
Since exactly one of (a1), (a2), (a3) holds, exactly one of (b1), (b2), (b3) holds.

The following gives the Dehornoy right-ordering of $\mathcal{B}_{n}$; recall the definition of $\sigma$-positive from Definitions 6.1,
6.7 Theorem. For each $\phi \in \mathcal{B}_{n}$ exactly one of the following holds: $\phi=1 ; \phi$ is $\sigma$-positive; $\bar{\phi}$ is $\sigma$-positive. The set of $\sigma$-positive elements of $\mathcal{B}_{n}$ is the positive cone of a right-ordering of $\mathcal{B}_{n}$.

Proof. Suppose that $\phi \neq 1$.
Let $i$ be the largest element of $[1 \uparrow n-1]$ such that $\phi \in\left\langle\sigma_{[i, n-1]}\right\rangle$. The natural subscript-shifting isomorphism from $\left\langle t_{[i \uparrow n]}\right\rangle$ to $\Sigma_{0,1, n-i+1}$ induces an isomorphism from $\left\langle\sigma_{[i \uparrow n-1]}\right\rangle$ to $B_{n-i+1}$. Notice that $\phi$ is mapped to an element of $B_{n-i+1}$ which
is not $\sigma_{1}$-neutral; by Theorem 6.6, this image is $\sigma_{1}$-positive or $\sigma_{1}$-negative but not both. Hence exactly one of $\phi, \bar{\phi}$ is $\sigma$-positive.

It is easy to see that the product of two $\sigma$-positive elements of $\mathcal{B}_{n}$ is $\sigma$-positive.

Hence the set of $\sigma$-positive elements of $\mathcal{B}_{n}$ is the positive cone for a right-ordering of $\mathcal{B}_{n}$, the Dehornoy right-ordering.

## 7 Ends, right-orderings and squarefreeness

7.1 Review. A (reduced) end of $\Sigma_{0,1, n}$ is a function

$$
\left[1 \uparrow \infty \left[\rightarrow \quad t_{[1 \uparrow n]} \cup \bar{t}_{[1 \uparrow n]}, \quad i \mapsto a_{i},\right.\right.
$$

such that, for each $i \in\left[1 \uparrow \infty\left[, a_{i+1} \neq \bar{a}_{i}\right.\right.$. The function is then represented as a right-infinite reduced product, $a_{1} a_{2} \cdots$ or $\Pi a_{[1 \uparrow \infty[ }$.

We denote the set of ends of $\Sigma_{0,1, n}$ by $\mathfrak{E}\left(\Sigma_{0,1, n}\right)$, or simply by $\mathfrak{E}$ if there is no risk of confusion.

An element of $\Sigma_{0,1, n} \cup \mathfrak{E}\left(\Sigma_{0,1, n}\right)$ is said to be squarefree if, in its reduced expression, no two consecutive terms are equal; for example: $\left(t_{1} t_{2}\right)^{\infty}$ is a squarefree end; $t_{1} t_{2} t_{2} t_{3}$ is a non-squarefree word.

For each $w \in \Sigma_{0,1, n}$, we define the shadow of $w$ in $\mathfrak{E}$ to be

$$
(w \mathbb{4}):=\left\{\Pi a_{[1 \uparrow \infty[ } \in \mathfrak{E} \mid \Pi a_{[1 \uparrow|w|]}=w\right\} .
$$

Thus, for example, $(1 \boldsymbol{4})=\mathfrak{E}$.
We shall now give $\mathfrak{E}$ an ordering, $<$. The first step is, for each $w \in \Sigma_{0,1, n}$, to assign an ordering, $<$, to a partition of $(w \mathbb{4})$ into $2 n$ or $2 n-1$ subsets, depending as $w=1$ or $w \neq 1$, as follows. We set

$$
\left(t_{1} \mathbf{\triangleleft}\right)<\left(\bar{t}_{1} \mathbb{4}\right)<\left(t_{2} \mathbb{\triangleleft}\right)<\left(\bar{t}_{2} \mathbb{4}\right)<\cdots<\left(t_{n} \mathbb{4}\right)<\left(\bar{t}_{n} \mathbb{4}\right) .
$$

If $i \in[1 \uparrow n]$ and $w \in\left(\star \bar{t}_{i}\right)$, then we set

$$
\begin{aligned}
& \left(w \bar{t}_{i} \mathbf{4}\right)<\left(w t_{i+1} \mathbf{4}\right)<\left(w \bar{t}_{i+1} \mathbf{4}\right)<\left(w t_{i+2} \mathbf{4}\right)<\left(w \bar{t}_{i+2} \mathbf{4}\right)<\cdots \\
& \cdots<\left(w t_{n} \mathbf{4}\right)<\left(w \bar{t}_{n} \mathbf{4}\right)<\left(w t_{1} \mathbf{4}\right)<\left(w \bar{t}_{1} \mathbf{4}\right)<\left(w t_{2} \mathbb{4}\right)<\cdots \\
& \cdots<\left(w t_{i-1} \mathbf{4}\right)<\left(w \bar{t}_{i-1} \mathbf{4}\right) \text {. }
\end{aligned}
$$

If $i \in[1 \uparrow n]$ and $w \in\left(\star t_{i}\right)$, then we set

$$
\begin{aligned}
& \left(w t_{i+1} \text { 【 }\right)<\left(w \bar{t}_{i+1} \text { ৫ }\right)<\left(w t_{i+2} \text { ৫ }\right)<\left(w \bar{t}_{i+2} \text { ৫ }\right)<\cdots \\
& \cdots<\left(w t_{n} \mathbf{4}\right)<\left(w \bar{t}_{n} \mathbf{4}\right)<\left(w t_{1} \mathbf{4}\right)<\left(w \bar{t}_{1} \mathbf{4}\right)<\left(w t_{2} \mathbf{4}\right)<\cdots \\
& \cdots<\left(w t_{i-1} \mathbf{4}\right)<\left(w \bar{t}_{i-1} \mathbb{4}\right)<\left(w t_{i} \mathbb{4}\right) \text {. }
\end{aligned}
$$

Hence, for each $w \in \Sigma_{0,1, n}$, we have an ordering $<$ of a partition of ( $w \mathbb{4}$ ) into $2 n$ or $2 n-1$ subsets.

If $\Pi a_{[1 \uparrow \infty[ }$ and $\Pi b_{[1 \uparrow \infty]}$ are two different (reduced) ends, then there exists $i \in \mathbb{N}$ such that $\Pi a_{[1 \uparrow i]}=\Pi b_{[1 \uparrow i]}$ in $\Sigma_{0,1, n}$, and $a_{i+1} \neq b_{i+1}$ in $t_{[1 \uparrow n]} \cup \bar{t}_{[1 \uparrow n]}$. Let $w=\Pi a_{[1 \uparrow i]}=\Pi b_{[1 \uparrow i]}$. Then $\Pi a_{[1 \uparrow \infty[ }$ and $\Pi b_{[1 \uparrow \infty[ }$ lie in $(w \mathbf{4})$, but lie in different elements of the partition of $(w \mathbb{4})$ into $2 n$ or $2 n-1$ subsets. We then order $\Pi a_{[1 \uparrow \infty[ }$ and $\Pi b_{[1 \uparrow \infty[ }$ according to the order of the elements of the partition of $(w \mathbb{4})$ that they belong to. This completes the definition of the ordering $<$ of $\mathfrak{E}$.

We remark that the smallest element of $\mathfrak{E}$ is $\bar{z}_{1}^{\infty}=\left(\Pi t_{[1 \uparrow n]}\right)^{\infty}$ and the largest element of $\mathfrak{E}$ is $z_{1}^{\infty}=\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}$.
7.2 Review. Following Nielsen-Thurston [9], [27], we now define the action of $\mathcal{B}_{n}$ on $\mathfrak{E}\left(\Sigma_{0,1, n}\right)$ and show that it respects the ordering; our treatment will be quite elementary compared to the usual approaches.

We assume that $n \geq 2$, and we first define the action of $\sigma_{1}$ on $\mathfrak{E}$.
Consider any reduced end $\mathfrak{e} \in \mathfrak{E}$. There is then a unique factorization $\mathfrak{e}=$ $\Pi w_{[1 \uparrow i]}$ or $\mathfrak{e}=\Pi w_{[1 \uparrow \infty[ }$, where, in the former case, $w_{([1 \uparrow i-1])}$ is a finite sequence of non-trivial words, and $w_{i}$ is a reduced end, and, in the latter case, $w_{([1 \uparrow \infty[)}$ is an infinite sequence of non-trivial words, and in both cases, the $w_{j}$ alternate between elements of $\left\langle t_{[1 \uparrow \uparrow]}\right\rangle \cup \mathfrak{E}\left(\left\langle t_{[1 \uparrow 2]}\right\rangle\right)$, and elements of $\left\langle t_{[3 \uparrow n]}\right\rangle \cup \mathfrak{E}\left(\left\langle t_{[3 \uparrow n]}\right\rangle\right)$. We shall express this factorization as $\mathfrak{e}=\left[w_{1}\right]\left[w_{2}\right] \cdots$.

Recall, from Notation [6.5, that we have the factorization $\sigma_{1}=\sigma_{1}^{\prime} \sigma_{1}^{\prime \prime}$. On $\left\langle t_{[1 \uparrow 2]}\right\rangle \cup \mathfrak{E}\left(\left\langle t_{[1 \uparrow 2]}\right\rangle\right), \sigma_{1}^{\prime}$ acts as conjugation by $t_{1}$, while $\sigma_{1}^{\prime \prime}$ interchanges the two free generators. On $\left\langle t_{[3 \uparrow n]}\right\rangle, \sigma_{1}^{\prime}$ and $\sigma_{1}^{\prime \prime}$ act as the identity map. This completes the description of the action of $\sigma_{1}^{\prime}, \sigma_{1}^{\prime \prime}$ and $\sigma_{1}$ on $\mathfrak{E}$.

It is not difficult to show that, for any reduced ends $\Pi a_{[1 \uparrow \infty[ }$ and $\Pi b_{[1 \uparrow \infty}$, if $\left(\Pi a_{[1 \uparrow \infty( }\right)^{\sigma_{1}}=\Pi b_{[1 \uparrow \infty[ }$, then for all $i, j \in \mathbb{N}$, if $j \geq 2 i$, then $\left(\Pi a_{[1 \uparrow j]}\right)^{\sigma_{1}} \in\left(\Pi b_{[1 \uparrow i]} \star\right)$. Thus, $\left(\Pi a_{[1 \uparrow \infty \mid}\right)_{1}^{\sigma_{1}}$ is the limit of $\left(\Pi a_{[1 \uparrow j]}\right)^{\sigma_{1}}$ as $j$ tends to $\infty$.

It is clear that $\sigma_{1}^{\prime}, \sigma_{1}^{\prime \prime}$ and, hence, $\sigma_{1}$ act bijectively on $\mathfrak{E}$. Hence we have the action of $\bar{\sigma}_{1}$ on $\mathfrak{E}$. It is then not difficult to verify that we have an action of $\mathcal{B}_{n}$ on $\mathfrak{E}$.

We next show that $\sigma_{1}$ respects the ordering of $\mathfrak{E}$. We do this by considering all the ways that two reduced ends can be compared, and the resulting effect of $\sigma_{1}^{\prime}$ and $\sigma_{1}$. We represent the information in tables. In all of the following, we understand that $t_{1} a, \bar{t}_{1} b, t_{2} c$, and $\bar{t}_{2} d$ are reduced expressions for elements of $\left\langle t_{[1 \uparrow 2]}\right\rangle \cup \mathfrak{E}\left(\left\langle t_{[1 \uparrow 2]}\right\rangle\right)$, and $b \neq 1$. Since $a$ does not begin with $\bar{t}_{1}, a^{\sigma_{1}^{\prime \prime}} t_{2}$ begins with $t_{1}$ or $\bar{t}_{1}$ or $t_{2}$. We make the convention that $\Sigma_{0,1, n}$ acts trivially on the right on $\mathfrak{E}$.

| $(\cdots]\left[w t_{1}\right.$ ¢ $)$ | $(\cdots]\left[w t_{1} \mathbb{4}\right)^{\sigma_{1}^{\prime}}$ | $(\cdots]\left[w t_{1} \mathbb{4}\right)^{\sigma_{1}}$ |
| :---: | :---: | :---: |
| $\cdots]\left[w t_{1} t_{2} c\right][$ | $\cdots]\left[\left(\bar{t}_{1} w\right) t_{1} \quad t_{2}\left(c t_{1}\right)\right]\left[{ }^{\text {a }}\right.$ | $\cdots]\left[\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) t_{2} t_{1}\left(c^{\sigma^{\prime \prime}} t_{2}\right)\right]$ |
| $\cdots]\left[w t_{1} \bar{t}_{2} d\right][$ [ | $\cdots]\left[\left(\bar{t}_{1} w\right) t_{1} \bar{t}_{2}\left(d t_{1}\right)\right][\cdots$ | $\cdots]\left[\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) t_{2} \bar{t}_{1}\left(d^{\sigma_{1}^{\prime \prime}} t_{2}\right)\right][$. |
| $\cdots]\left[w t_{1}\right]\left[t_{3} \uparrow \bar{t}_{n}\right.$ | $\left.\cdots]\left[\bar{t}_{1} w\right) t_{1} t_{1}\right]\left[t_{3} \uparrow \bar{t}_{n} \cdots\right.$ | $\cdots]\left[\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) t_{2} t_{2}\right]\left[t_{3} \uparrow \bar{t}_{n} \cdots\right.$ |
| $\cdots]\left[\begin{array}{ll}w \\ 1 & \left.t_{1} a\right][ \end{array}\right.$ | $\cdots]\left[\left(\bar{t}_{1} w\right) t_{1} \quad t_{1}\left(a t_{1}\right)\right][\cdots$ | $\cdots]\left[\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) t_{2} t_{2}\left(a^{\sigma_{1}^{\prime \prime}} t_{2}\right)\right][\cdots$ |

Here, the case $w=1$ does not present any problems.

| $(\cdots]\left[w \bar{t}_{1} \mathbf{4}\right)$ | $(\cdots]\left[w \bar{t}_{1} \mathbb{4}\right)^{\sigma_{1}^{\prime}}$ | $(\cdots]\left[w \bar{t}_{1} \text { ¢ }\right)^{\sigma_{1}}$ |
| :---: | :---: | :---: |
| $\cdots]\left[\begin{array}{ll} \\ \bar{t}_{1} & \left.\bar{t}_{1} b\right][ \end{array}\right.$ | $\cdots]\left[\left(\bar{t}_{1} w\right) \bar{t}_{1} \bar{t}_{1}\left(b t_{1}\right)\right][\cdots$ | $\cdots]\left[\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) \bar{t}_{2} \bar{t}_{2}\left(b^{\sigma_{1}^{\prime \prime}} t_{2}\right)\right]$ |
| $\cdots]\left[\begin{array}{ll}w \bar{t}_{1} & \bar{t}_{1}\end{array}\right]\left[t_{3} \uparrow \bar{t}_{n} \cdots\right.$ | $\cdots]\left[\left(\bar{t}_{1} w\right) \bar{t}_{1}\right]\left[t_{3} \uparrow \bar{t}_{n}\right.$. | $\cdots]\left[\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) \bar{t}_{2}\right]\left[t_{3} \uparrow \bar{t}^{\prime}\right.$ |
| $\cdots]\left[\begin{array}{ll}w \bar{t}_{1} & t_{2} c\end{array}\right][\cdots$ | $\cdots]\left[\left(\bar{t}_{1} w\right) \bar{t}_{1} t_{2}\left(c t_{1}\right)\right][\cdots$ | $\cdots]\left[\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) \bar{t}_{2} t_{1}\left(c^{\sigma_{1}^{\prime \prime}} t_{2}\right)\right][$ |
| $\cdots]\left[w \bar{t}_{1} \bar{t}_{2} d\right][\cdots$ | $\cdots]\left[\left(\bar{t}_{1} w\right) \bar{t}_{1} \bar{t}_{2}\left(d t_{1}\right)\right][\cdots$ | $\cdots]\left[\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) \bar{t}_{2} \bar{t}_{1}\left(d^{\sigma_{1}^{\prime \prime}} t_{2}\right)\right][$. |
| $\cdots]\left[w \bar{t}_{1}\right]\left[t_{3} \uparrow \bar{t}_{n}\right.$. | $\cdots]\left[\left(\bar{t}_{1} w\right)\right]\left[t_{3} \uparrow \bar{t}_{n}\right.$. | $\cdots]\left[\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right)\right]\left[t_{3} \uparrow \bar{t}_{n}\right.$. |

Here, $w$ does not end with $t_{1}$, and, hence, $\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right)$ ends with $t_{1}, \bar{t}_{1}$ or $\bar{t}_{2}$.

| $(\cdots]\left[w t_{2}\right.$ ¢ $)$ | $(\cdots]\left[w t_{2} \text { ¢ }\right)^{\sigma_{1}^{\prime}}$ | $(\cdots]\left[w t_{2} \mathbf{4}\right)^{\sigma_{1}}$ |
| :---: | :---: | :---: |
|  | $]\left[\left(\bar{t}_{1} w\right) t_{2} t_{1}\right]\left[t^{\prime}\right.$ | $\cdots]\left[\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) t_{1} t_{2}\right]\left[t_{3}\right.$ |
| $\cdots]\left[w t_{2} t_{1} a\right][$. | $\cdots]\left[\left(\bar{t}_{1} w\right) t_{2} t_{1}\left(a t_{1}\right)\right]$ | $\cdots]\left[\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) t_{1} t_{2}\left(a^{\sigma_{1}^{\prime \prime}} t_{2}\right)\right]$ |
| $\cdots]\left[w t_{2} \overline{1}_{1} b\right][$ | $\cdots]\left[\left(\bar{t}_{1} w\right) t_{2} \bar{t}_{1}\left(b t_{1}\right)\right][$ | $\left.\cdots]\left[\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) t_{1} \bar{t}_{2}\left(b^{\sigma_{1}^{\prime \prime}} t_{2}\right)\right][$ |
| $\cdots]\left[w t_{2} \bar{t}_{1}\right]\left[t_{3} \uparrow \bar{t}_{n}\right.$ | $\cdots\left[\left(\bar{t}_{1} w\right) t_{2}\right]\left[t_{3} \uparrow \bar{t}_{n}\right.$ | $]\left[\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) t_{1}\right]\left[t_{3} \uparrow \bar{t}_{n}\right.$ |
| $\cdots\left[\begin{array}{ll}w t_{2} & t_{2} c\end{array}\right][\cdots$ | $]\left[\left(\bar{t}_{1} w\right) t_{2} \quad t_{2}\left(c t_{1}\right.\right.$ | $]\left[\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) t_{1} t_{1}\left(c^{\sigma_{1}^{\prime \prime}}\right.\right.$ |
| $(\cdots]\left[w \bar{t}_{2}\right.$ ¢ $)$ | $(\cdots]\left[w \bar{t}_{2} \mathbb{4}\right)^{\sigma_{1}^{\prime}}$ | $(\cdots]\left[w \bar{t}_{2} \mathbf{4}\right)^{\sigma_{1}}$ |
| $]\left[w \bar{t}_{2} \bar{t}_{2} d\right]$ | $\cdots]\left[\left(\bar{t}_{1} w\right) \bar{t}_{2} \bar{t}_{2}(d t\right.$ | $\cdots]\left[\left(\bar{t}_{2} w^{\sigma_{1 \prime \prime}^{\prime \prime}}\right) \bar{t}_{1} \bar{t}_{1}\left(d^{\sigma_{1}^{\prime \prime}}\right.\right.$ |
| $\cdots]\left[w \bar{t}_{2}\right]\left[t_{3} \uparrow \bar{t}_{n}\right.$ | $\cdots]\left[\left(\bar{t}_{1} w\right) \bar{t}_{2} t_{1}\right]\left[\bar{t}_{3} \uparrow \bar{t}_{n}\right.$ | $\cdots]\left[\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) \bar{t}_{1} t_{2}\right]\left[t_{3} \uparrow \bar{t}^{\prime}\right.$ |
| $\cdots]\left[\begin{array}{ll}\text { t } & \left.t_{1} a\right][ \end{array}\right.$ | $\cdots]\left[\left(\bar{t}_{1} w\right) \bar{t}_{2} t_{1}\left(a t_{1}\right)\right][$ | $\left.\cdots]\left[\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) \bar{t}_{1} t_{2}\left(a^{\sigma_{1}^{\prime \prime}} t_{2}\right)\right]$ |
| $\cdots]\left[\begin{array}{ll} \\ t_{2} & \left.\overline{1}_{1} b\right][ \end{array}\right.$ | $\cdots]\left[\left(\bar{t}_{1} w\right) \bar{t}_{2} \bar{t}_{1}\left(b t_{1}\right)\right]$ | $\cdots]\left[\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) \bar{t}_{1} \bar{t}_{2}\left(b^{\sigma_{1}^{\prime \prime}} t_{2}\right)\right][$. |
| $\cdots]\left[\begin{array}{ll}w \bar{t}_{2} & \bar{t}_{1}\end{array}\right]\left[t_{3} \uparrow \bar{t}_{n}\right.$ | $]\left[\left(\bar{t}_{1} w\right) \bar{t}_{2}\right]\left[t_{3} \uparrow \bar{t}_{n}\right.$. | $]\left[\left(\bar{t}_{2} w^{\sigma_{1}^{\prime \prime}}\right) \bar{t}_{1}\right]\left[t_{3} \uparrow \bar{t}_{n}\right.$ |


| $\left(\cdots t_{3} \boldsymbol{4}\right)$ | $\left(\cdots t_{3} \mathbb{4}\right)^{\sigma_{1}^{\prime}}$ | $\left(\cdots t_{3} \mathbb{4}\right)^{\sigma_{1}}$ |
| :--- | :--- | :--- |
| $\cdots t_{3} t_{4} \uparrow \bar{t}_{n} \cdots$ | $\cdots t_{3} t_{4} \uparrow \bar{t}_{n} \cdots$ | $\cdots t_{3} t_{4} \uparrow \bar{t}_{n} \cdots$ |
| $\left.\cdots t_{3}\right]\left[t_{1} a\right][\cdots$ | $\left.\cdots t_{3}\right]\left[\left(a t_{1}\right)\right][\cdots$ | $\left.\cdots t_{3}\right]\left[\left(a^{\sigma_{1}^{\prime \prime}} t_{2}\right)\right][\cdots$ |
| $\left.\cdots t_{3}\right]\left[\bar{t}_{1} b\right][\cdots$ | $\left.\cdots t_{3}\right]\left[\bar{t}_{1} \bar{t}_{1}\left(b t_{1}\right)\right][\cdots$ | $\left.\cdots t_{3}\right]\left[\bar{t}_{2} \bar{t}_{2}\left(b^{\sigma_{1}^{\prime \prime}} t_{2}\right)\right][\cdots$ |
| $\left.\cdots t_{3}\right]\left[\bar{t}_{1}\right]\left[t_{3} \uparrow \bar{t}_{n} \cdots\right.$ | $\left.\left.\cdots t_{3}\right] \bar{t}_{1}\right]\left[t_{3} \uparrow \bar{t}_{n} \cdots\right.$ | $\left.\cdots t_{3}\right]\left[t_{2}\right]\left[t_{3} \uparrow \bar{t}_{n} \cdots\right.$ |
| $\left.\cdots t_{3}\right]\left[t_{2} c\right][\cdots$ | $\left.\left.\cdots t_{3}\right] \bar{t}_{1} t_{2}\left(c t_{1}\right)\right][\cdots$ | $\left.\cdots t_{3}\right]\left[t_{2} t_{1}\left(c^{\sigma_{1}^{\prime}} t_{2}\right)\right][\cdots$ |
| $\left.\cdots t_{3}\right]\left[\bar{t}_{2} d\right][\cdots$ | $\left.\cdots t_{3}\right]\left[\bar{t}_{1} \bar{t}_{2}\left(d t_{1}\right)\right][\cdots$ | $\left.\cdots t_{3}\right]\left[\bar{t}_{2} \bar{t}_{1}\left(d^{\sigma_{1}^{\prime \prime}} t_{2}\right)\right][\cdots$ |
| $\cdots t_{3} t_{3} \cdots$ | $\cdots t_{3} t_{3} \cdots$ | $\cdots t_{3} t_{3} \cdots$ |

The remaining tables are clearly of the same form as the last one. Thus we have proved that the action of $\sigma_{1}$ respects the ordering of $\mathfrak{E}$. It follows that the action of $\bar{\sigma}_{1}$ respects the ordering of $\mathfrak{E}$. Similarly, the actions of $\sigma_{[2 \uparrow n-1]} \cup \bar{\sigma}_{[2 \uparrow n-1]}$ respect the ordering of $\mathfrak{E}$. Hence $\mathcal{B}_{n}$ acts on $(\mathfrak{E}, \leq)$.
7.3 Remarks (Thurston [27]). The (right) action of $\mathcal{B}_{n}$ on $(\mathfrak{E}, \leq)$ gives rise to many right orderings of $\mathcal{B}_{n}$.

Let us use the left-to-right lexicographic ordering on ( $\mathfrak{E}^{n}, \leq$ ), and consider the $\mathcal{B}_{n}$-orbit of $t_{([1 \uparrow n])}^{\infty}:=\left(t_{i}^{\infty}\right)_{i \in[1 \uparrow n]}$. It is not difficult to show that the $\mathcal{B}_{n}$-stabilizer
of $t_{([1 \uparrow n])}^{\infty}$ is trivial．Thus we have an injective map

$$
\mathcal{B}_{n} \rightarrow \mathfrak{E}^{n}, \quad \phi \mapsto t_{([1 \uparrow n])}^{\infty \phi}:=\left(\left(t_{i}^{\infty}\right)^{\phi}\right)_{i \in[1 \uparrow n]} .
$$

Let $\leq$ denote the ordering of $\mathcal{B}_{n}$ induced by pullback from $\mathfrak{E}^{n}$ ．Clearly $\leq$ is a right－ordering of $\mathcal{B}_{n}$ ．

If $n \geq 2$ and $\phi \in \mathcal{B}_{n}$ is $\sigma_{1}$－negative，then，as in the proof of Theo－ rem 6．6（b2）$\Rightarrow(\mathrm{c} 2)$ ，we have $\left(t_{1} \mathbb{4}\right)^{\phi} \subset\left(t_{1}\right.$ 《）．Since $\max \left(t_{1} \mathbb{4}\right)=t_{1}^{\infty}$ ，we see that $\left(t_{1}^{\infty}\right)^{\phi}<t_{1}^{\infty}$ ．Hence $\phi<1$ and $1<\bar{\phi}$ ．Similar arguments with $\left(t_{i} \mathbb{\triangleleft}\right)$ ， $i \in[2 \uparrow n]$ ，show that，if $\phi \in \mathcal{B}_{n}$ is $\sigma$－positive（resp．$\sigma$－negative），then $1<\phi$ （resp． $1>\phi$ ）．Hence the right－ordering we have obtained from（ $\mathfrak{E}^{n}, \leq$ ）coin－ cides with the Dehornoy right－ordering．However，the study of ends does not seem to readily yield the $\sigma_{1}$－trichotomy．

The following will be useful in the study of squarefreeness．
7．4 Lemma．Let $n \geq 1$ ，let $i \in[1 \uparrow n]$ ，and let $w \in \Sigma_{0,1, n}-\left(\star t_{i}\right)-\left(\star \bar{t}_{i}\right)$ ．Then， in $\mathfrak{E}\left(\Sigma_{0,1, n}\right)$ ，the following hold：
（i）．$w t_{i} \bar{w}\left(\left(\Pi t_{[1 \uparrow n]}\right)^{\infty}\right) \leq w t_{i}\left(\left(\Pi t_{[i \uparrow n]} \Pi t_{[1 \uparrow i-1]}\right)^{\infty}\right)=\min \left(w t_{i} t_{i} \boldsymbol{\triangleleft}\right)$ ；
（ii）． $\min \left(w t_{i} t_{i} \boldsymbol{\triangleleft}\right)<\max \left(w \bar{t}_{i} \bar{t}_{i} \boldsymbol{\triangleleft}\right)$ ；
（iii）． $\max \left(w \bar{t}_{i} \bar{t}_{i} \boldsymbol{\triangleleft}\right)=w \bar{t}_{i}\left(\left(\Pi \bar{t}_{[i \downarrow 1]} \Pi \bar{t}_{[n \downarrow i+1]}\right)^{\infty}\right) \leq w \bar{t}_{i} \bar{w}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right)$ ；
（iv）．$\left(w t_{i} t_{i} \boldsymbol{\triangleleft}\right) \cup\left(w \bar{t}_{i} \bar{t}_{i} \boldsymbol{\triangleleft}\right) \subseteq\left[w t_{i} \bar{w}\left(\left(\Pi t_{[1 \uparrow n]}\right)^{\infty}\right), w \bar{t}_{i} \bar{w}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right)\right]$ ．
（v）．If $n \geq 3$ ，then one of the following holds：
（a）．$t_{1}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right)<w t_{i} \bar{w}\left(\left(\Pi t_{[1 \uparrow n]}\right)^{\infty}\right) ;$
（b）．$t_{1}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right)>w \bar{t}_{i} \bar{w}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right)$ ；
and，hence，$t_{1}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right) \notin\left[w t_{i} \bar{w}\left(\left(\Pi t_{[1 \uparrow n]}\right)^{\infty}\right)\right.$ ，$\left.w \bar{t}_{i} \bar{w}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right)\right]$ ，that is， $t_{1}\left(z_{1}^{\infty}\right) \notin\left[w t_{i} \bar{w}\left(\bar{z}_{1}^{\infty}\right), w \bar{t} \bar{w}\left(z_{1}^{\infty}\right)\right]$

Proof．Recall that：

$$
\begin{aligned}
& \left(t_{1} \mathbf{4}\right)<\left(\bar{t}_{1} \mathbf{4}\right)<\left(t_{2} \mathbf{4}\right)<\cdots<\left(t_{n} \mathbb{4}\right)<\left(\bar{t}_{n} \mathbb{4}\right), \\
& \left(t_{i} t_{i+1} \mathbb{4}\right)<\left(t_{i} \bar{t}_{i+1} \mathbf{4}\right)<\cdots<\left(t_{i} \bar{t}_{n} \mathbb{4}\right)<\left(t_{i} t_{1} \mathbb{4}\right)<\cdots<\left(t_{i} \bar{t}_{i-1} \mathbb{4}\right)<\left(t_{i} t_{i} \mathbb{4}\right),
\end{aligned}
$$

（ii）．It is straightforward to see that $w t_{i}\left(\left(\Pi t_{[i \uparrow n]} \Pi t_{[1 \uparrow i-1]}\right)^{\infty}\right)=\min \left(w t_{i} t_{i} \mathbb{4}\right)$ ．
Let $x$ denote the element of $t_{[1 \uparrow n]} \cup \bar{t}_{[1 \uparrow n]}$ such that $\bar{w}\left(\left(\Pi t_{[1 \uparrow n]}\right)^{\infty}\right) \in(x \mathbb{4})$ ； notice that $x \neq \bar{t}_{i}$ ．
If $x \neq t_{i}$ ，then $\left(w t_{i} x \mathbb{4}\right)<\left(w t_{i} t_{i} \mathbb{⿶}\right)$ ，and we have

$$
w t_{i} \bar{w}\left(\left(\Pi t_{[1 \uparrow n]}\right)^{\infty}\right) \in\left(w t_{i} x \boldsymbol{\triangleleft}\right)<\left(w t_{i} t_{i} \boldsymbol{\triangleleft}\right) \ni \min \left(w t_{i} t_{i} \boldsymbol{⿶}\right)
$$

If $x=t_{i}$ ，then $\bar{w}$ is completely cancelled in $\bar{w}\left(\left(\Pi t_{[1 \uparrow n]}\right)^{\infty}\right)$ ，and，moreover，

$$
w t_{i} \bar{w}\left(\left(\Pi t_{[1 \uparrow n]}\right)^{\infty}\right)=w t_{i}\left(\left(\Pi t_{[i \uparrow n]} \Pi t_{[1 \uparrow i-1]}\right)^{\infty}\right)=\min \left(w t_{i} t_{i} \boldsymbol{⿶}\right) .
$$

Thus，（i）holds．
（iii）is clear．
（iii）．It is straightforward to see that $w \bar{t}_{i}\left(\left(\Pi \bar{t}_{[i \downarrow 1]} \Pi \bar{t}_{[n \downarrow i+1]}\right)^{\infty}\right)=\max \left(w \bar{t}_{i} \bar{t}_{i} \boldsymbol{4}\right)$ ．
Let $x$ denote the element of $t_{[1 \uparrow n]} \cup \bar{t}_{[1 \uparrow n]}$ such that $\bar{w}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right) \in(x \mathbb{4})$ ； notice that $x \neq t_{i}$ ．
If $x \neq \bar{t}_{i}$ ，then $\left(w \bar{t}_{i} \bar{t}_{i} \boldsymbol{⿶}\right)<\left(w \bar{t}_{i} x \boldsymbol{⿶}\right)$ ，and we have

$$
\max \left(w \bar{t}_{i} \bar{t}_{i} \boldsymbol{⿶}\right) \in\left(w \bar{t}_{i} \bar{t}_{i} \boldsymbol{⿶}\right)<\left(w \bar{t}_{i} x \boldsymbol{\iota}\right) \ni w \bar{t}_{i} \bar{w}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right) .
$$

If $x=\bar{t}_{i}$ ，then $\bar{w}$ is completely cancelled in $\bar{w}\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}$ ，and，moreover，

$$
w \bar{t}_{i} \bar{w}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right)=w \bar{t}_{i}\left(\left(\Pi \bar{t}_{[i \downarrow 1]} \Pi \bar{t}_{[n \downarrow i+1]}\right)^{\infty}\right)=\max \left(w \bar{t}_{i} \bar{t}_{i} \boldsymbol{⿶}\right)
$$

Thus，（iiii）holds．
（iv）follows from（ii）－（iii）．
（v）．It is not difficult to see that

$$
w t_{i} \bar{w}\left(\left(\Pi t_{[1 \uparrow n]}\right)^{\infty}\right) \in\left(w t_{i} \boldsymbol{\triangleleft}\right) \quad \text { and } \quad w \bar{t} \bar{i} \bar{w}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right) \in\left(w \bar{t}_{i} \boldsymbol{⿶}\right) .
$$

Case 1．$w \notin\left(t_{1} \star\right)$ ．
If $w=1$ ，then
$t_{1}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right) \in\left(t_{1} \bar{t}_{n} \mathbb{4}\right)<\left(t_{i} t_{1} \boldsymbol{\triangleleft}\right) \ni t_{i}\left(\left(\Pi t_{[1 \uparrow n]}\right)^{\infty}\right)=w t_{i} \bar{w}\left(\left(\Pi t_{[1 \uparrow n]}\right)^{\infty}\right)$.
If $w \neq 1$ ，then $t_{1}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right) \in\left(t_{1} \mathbb{4}\right)<(w \mathbb{4}) \ni w t_{i} \bar{w}\left(\left(\Pi t_{[1 \uparrow n]}\right)^{\infty}\right)$ ．
In both subcases，（a）holds．
Case 2．$w \in\left(t_{1} \star\right)$ ．
Here，$w \bar{t}_{i} \bar{w}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right) \in(w \mathbb{4}) \subseteq\left(t_{1} \mathbb{4}\right)$ ．Hence，

$$
w \bar{t}_{i} \bar{w}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right) \leq \max \left(t_{1} \mathbb{4}\right)=t_{1}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right)
$$

To prove that（b）holds，it remains to show that

$$
w \bar{t}_{i} \bar{w}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right) \neq t_{1}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right),
$$

that is， $\bar{t}_{1} w \bar{t}_{i} \bar{w}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right) \neq\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}$ ，that is， $\bar{t}_{1} w \bar{t}_{i} \bar{w} \notin\left\langle\Pi \bar{t}_{[n \downarrow 1]}\right\rangle$ ．We can write $w=t_{1} u$ where $u \notin\left(\bar{t}_{1} \star\right)$ ．Then $\bar{t}_{1} w \bar{t}_{i} \bar{w}=u \bar{t}_{i} \bar{u}_{1}$ ，in normal form．Thus it suffices to show that $u \bar{t}_{i} \bar{u} \bar{t}_{1} \notin\left\langle\Pi \bar{t}_{[n \downarrow 1]}\right\rangle$ ．

If $u=1$ ，then $u \bar{t}_{i} \bar{u} \bar{t}_{1}=\bar{t}_{i} \bar{t}_{1} \notin\left\langle\Pi \bar{t}_{[n \downarrow 1]}\right\rangle$ ，since $n \geq 3$ ．
If $u \neq 1$ ，then $u \bar{t}_{i} \bar{u} \bar{t}_{1} \notin\left\langle\Pi \bar{t}_{[n \downarrow 1]}\right\rangle$ ，since $u \bar{t}_{i} \bar{u}_{t_{1}}$ does not lie in the submonoid of $\Sigma_{0,1, n}$ generated by $t_{[1 \uparrow n]}$ ，nor in the submonoid generated by $\bar{t}_{[1 \uparrow n]}$ ．

In both subcases，（b）holds．
In both cases，（ $\mathbf{v}$ ）holds．
The following appeared as［5，Lema 2．2．17］．
7．5 Theorem．If $n \geq 1$ then，for each $\phi \in \mathcal{B}_{n}, t_{1}^{\phi}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right)$ is a squarefree end．

Proof. This is clear if $n=1$.
For $n=2, \mathcal{B}_{2}=\left\langle\sigma_{1}\right\rangle$, and

$$
t_{1}^{\mathcal{B}_{2}}=\left\{t_{1}^{\sigma_{1}^{2 m}}, t_{1}^{\sigma_{1}^{1+2 m}} \mid m \in \mathbb{Z}\right\}=\left\{t_{1}^{\left(t_{1} t_{2}\right)^{m}}, t_{2}^{\left(t_{1} t_{2}\right)^{m}} \mid m \in \mathbb{Z}\right\} .
$$

Thus, every word in $t_{1}^{\mathcal{B}_{2}}$ is squarefree and does not end in $\bar{t}_{2}$. Hence, every end in $t_{1}^{\mathcal{B}_{2}}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right)$ is squarefree.

Thus, we may assume that $n \geq 3$.
Recall that $z_{1}=\Pi \bar{t}_{[n \uparrow 1]}$, and, hence, $\bar{z}_{1}=\Pi t_{[1 \uparrow n]}$. Let $\cup[t]_{[1 \uparrow n]}$ denote $\bigcup_{i \in[1 \uparrow n]}\left[t_{i}\right]$. By Lemma 7.4(园), $t_{1}\left(z_{1}^{\infty}\right)$ does not lie in

$$
\bigcup_{x \in \cup[t]]_{[1 \uparrow n]}}\left[x\left(\bar{z}_{1}^{\infty}\right), \bar{x}\left(z_{1}^{\infty}\right)\right] \quad\left(=\bigcup_{i=1}^{n} \bigcup_{w \in \Sigma_{0,1, n}-\left(\nless t_{i}\right)-\left(\star \bar{t}_{i}\right)}\left[w t_{i} \bar{w}\left(\bar{z}_{1}^{\infty}\right), w \bar{t} \bar{w}\left(z_{1}^{\infty}\right)\right]\right) .
$$

Notice that $\phi$ permutes the elements of each of the following sets: $U[t]_{[1 \uparrow n]}$; $\left\{\bar{z}_{1}^{\infty}\right\} ;\left\{z_{1}^{\infty}\right\} ;$ and, $\bigcup_{x \in \cup[t][1 \uparrow n]}\left[x\left(\bar{z}_{1}^{\infty}\right), \bar{x}\left(z_{1}^{\infty}\right)\right]$. Hence $\left(t_{1}\left(z_{1}^{\infty}\right)\right)^{\phi}$ does not lie in $\bigcup_{x \in \mathrm{U}[t][1 \uparrow n]}\left[x\left(\bar{z}_{1}^{\infty}\right), \bar{x}\left(z_{1}^{\infty}\right)\right]$. By Lemma 7.4(iiv),

$$
\bigcup_{x \in \cup[t][1 \uparrow n]}\left[x\left(\bar{z}_{1}^{\infty}\right), \bar{x}\left(z_{1}^{\infty}\right)\right] \quad \supseteq \bigcup_{i=1}^{n} \bigcup_{w \in \Sigma_{0,1, n}-\left(\star t_{i}\right)-\left(\star \bar{t}_{i}\right)}\left(\left(w t_{i} t_{i} \boldsymbol{\triangleleft}\right) \cup\left(w \bar{t}_{i} \bar{t}_{i} \boldsymbol{\triangleleft}\right)\right) .
$$

Hence, $\left(t_{1}\left(z_{1}^{\infty}\right)\right)^{\phi}$ does not lie in the latter set either, and, hence, $\left(t_{1}\left(z_{1}^{\infty}\right)\right)^{\phi}$ is a squarefree end. Since $\left(t_{1}\left(z_{1}^{\infty}\right)\right)^{\phi}=t_{1}^{\phi}\left(z_{1}^{\infty}\right)$, the desired result holds.

We now obtain new information about the $\mathcal{B}_{n}$-orbit of $t_{1}$ in $\Sigma_{0,1, n}$.
7.6 Corollary. Let $n \geq 1$, let $\phi \in \mathcal{B}_{n}$, and let $k \in[1 \uparrow n]$.
(i). $t_{1}^{\phi}$ is a squarefree word in $\Sigma_{0,1, n}$.
(ii). $t_{1}^{\phi} \notin\left(\Pi \bar{t}_{[n \downarrow k+1]} t_{k} \star\right)-\left\{t_{k}^{\Pi t_{[k+1 \uparrow n]}}\right\}$.
(iii). $t_{1}^{\phi} \notin\left(\Pi t_{[1 \uparrow k-1]} \bar{t}_{k} \star\right)$.

Proof. Recall from Notation 3.1 that we write $t_{1}^{\phi}=t_{1 \pi(\phi)}^{w_{1}(\phi)}$. Let $\pi=\pi(\phi)$ and $w_{1}=w_{1}(\phi)$.

It is not difficult to see that

$$
t_{1}^{\phi}\left(z_{1}^{\infty}\right)=\bar{w}_{1} t_{1^{\pi}} w_{1}\left(\left(\Pi \bar{t}_{[n \downarrow 1]}\right)^{\infty}\right) \in\left(\bar{w}_{1} \mathbb{\triangleleft}\right) .
$$

By Theorem 7.5, $t_{1}^{\phi}\left(z_{1}^{\infty}\right)$ is a squarefree end. Hence, $\bar{w}_{1}$ is a squarefree word, and $w_{1} \notin\left(\star \bar{t}_{k} \Pi t_{[k+1 \downarrow \eta]}\right)$.

Since $\bar{w}_{1}$ is a squarefree word, $t_{1}^{\phi}$ is also a squarefree word. Hence (i) holds. Also, $w_{1} \notin\left(\star \bar{t}_{k} \Pi t_{[k+1 \uparrow n]}\right)$ implies that $\bar{w}_{1} \notin\left(\Pi \bar{t}_{[n \downarrow k+1]} t_{k} \star\right)$ and, hence, $t_{1}^{\phi} \notin$ $\left(\Pi \bar{t}_{[n \downarrow k+1]} t_{k} \star\right)-\left\{t_{k}^{\Pi t_{[k+1 \uparrow n]}}\right\}$ and, also, $\bar{t}_{1}^{\phi} \notin\left(\Pi \bar{t}_{[n \downarrow k+1]} t_{k} \star\right)$. In particular, (ii) holds.

Let $\xi$ be the automorphism of $\Sigma_{0,1, n}$ determined by $\frac{j \in[1 \uparrow n]}{\left(t_{j}\right)^{\xi}}$. Then $=\left(\bar{t}_{n+1-j}\right)$ $\xi^{2}=1$ and $\xi \in \mathrm{Out}_{0,1, n}^{-}:=\mathrm{Out}_{0,1, n}-\mathrm{Out}_{0,1, n}^{+}$. Also,

$$
t_{n}^{\phi \xi}=t_{n}^{\xi \phi \xi}=\bar{t}_{1}^{\phi \xi} \notin\left(\Pi \bar{t}_{[n \downarrow k+1]} t_{k^{\star}}\right)^{\xi}=\left(\Pi t_{[1 \uparrow n-k]} \bar{t}_{n+1-k} \star\right) .
$$

It follows that $t_{n}^{\mathcal{B}_{n}^{\xi}} \cap\left(\Pi t_{[1 \uparrow n-k]} \bar{t}_{n+1-k} \star\right)=\emptyset$. Since $\mathcal{B}_{n}^{\xi}=\mathcal{B}_{n}$ and $t_{n}^{\mathcal{B}_{n}}=t_{1}^{\mathcal{B}_{n}}$, we see that $t_{1}^{\phi} \notin\left(\Pi t_{[1 \uparrow n-k]} \bar{t}_{n+1-k} \star\right)$. Now replacing $k$ with $n+1-k$ gives (iii).

In Remark IV.3, we shall give a second proof of Corollary 7.6 using Larue-Whitehead diagrams.

## 8 Actions on free products of cyclic groups

8.1 Notation. Throughout this section, we assume that $n \geq 1$ and we fix a positive integer $N$.

Let $p_{([1 \uparrow N])}$ be a partition of $n$. Thus, $p_{([1 \uparrow N])}$ is an $N$-tuple for $[1 \uparrow \infty[$ such that $p_{1}+\cdots+p_{N}=n$.

Let $m_{([1 \uparrow N])}$ be an $N$-tuple for $\mathbb{N}-\{1\}$.
We let $\Sigma_{0,1, p_{1}^{\left(m_{1}\right)} \perp p_{2}^{\left(m_{2}\right)} \perp \cdots \perp p_{N}^{\left(m_{n}\right)}}$ denote the group with presentation

$$
\left\langle z, \tau_{[1 \uparrow n]} \mid z \Pi \tau_{[1 \uparrow n]},\left\{\tau_{j+\sum p_{[1 \uparrow i-1]}}^{m_{i}}\right\}_{i \in[1 \uparrow N], j \in\left[1 \uparrow p_{i}\right]}\right\rangle .
$$

Thus, $\Sigma_{0,1, p_{1}^{\left(m_{1}\right)} \perp p_{2}^{\left(m_{2}\right)} \perp \ldots \perp p_{N}^{\left(m_{N}\right)}}$ is isomorphic to a free product of cyclic groups, $C_{m_{1}}^{* p_{1}} * C_{m_{2}}^{* p_{2}} * \cdots C_{m_{N}}^{* p_{N}}$, where $C_{0}$ is interpreted as $C_{\infty}$, and $p_{i}^{(0)}$ is also written $p_{i}$ with no exponent.

We let Out $_{0,1, p_{1}^{\left(m_{1}\right)} \perp p_{2}^{\left(m_{2}\right)} \perp \ldots \perp p_{N}^{\left(m_{N}\right)}}$ denote the group of all automorphisms of $\Sigma_{0,1, p_{1}^{\left(m_{1}\right)} \perp p_{2}^{\left(m_{2}\right)} \perp \cdots \perp p_{N}^{\left(m_{n}\right)}}$ which map $\{z, \bar{z}\}$ and

$$
\left\{\left\{\left\{\left[\tau_{i}\right],\left[\bar{\tau}_{i}\right]\right\} \mid i \in\left[p_{1}+\ldots+p_{j-1}+1 \uparrow p_{1}+\ldots+p_{j}\right]\right\} \mid j \in[1 \uparrow N]\right\}
$$

to themselves.
We let Out ${ }_{0,1, p_{1}^{\left(m_{1}\right)} \perp p_{2}^{\left(m_{2}\right)} \perp \ldots \perp p_{N}^{\left(m_{n}\right)}}$ denote the group of all automorphisms of $\Sigma_{0,1, p_{1}^{\left(m_{1}\right)} \perp p_{2}^{\left(m_{2}\right)} \perp \cdots \perp p_{N}^{\left(m_{n}\right)}}$ which map $\{z\}$ and

$$
\left\{\left\{\left[\tau_{i}\right] \mid i \in\left[p_{1}+\ldots+p_{j-1}+1 \uparrow p_{1}+\ldots+p_{j}\right]\right\} \mid j \in[1 \uparrow N]\right\}
$$

to themselves.
In the case where all the $m_{i}$ are 0 , we get groups denoted Out $0_{0,1, p_{1} \perp p_{2} \perp \cdots \perp p_{N}}$ and Out $0_{0,1, p_{1} \perp p_{2} \perp \cdots \perp p_{N}}^{+}$. Notice that Out ${ }_{0,1, p_{1} \perp p_{2} \perp \cdots \perp p_{N}}$ is the subgroup of Out ${ }_{0,1, n}$ consisting of those elements such that the permutation in $\mathrm{Sym}_{n}$, arising from
the permutation of $\left\{\left\{\left[t_{1}\right],\left[t_{1}\right]\right\}, \ldots,\left\{\left[t_{n}\right],\left[t_{n}\right]\right\}\right\}$, lies in the natural image of $\operatorname{Sym}_{p_{1}} \times \operatorname{Sym}_{p_{2}} \times \cdots \times \operatorname{Sym}_{p_{N}}$ in $\operatorname{Sym}_{n}$.

There are natural maps

$$
\begin{align*}
& \text { Out }_{0,1, p_{1} \perp p_{2} \perp \cdots \perp p_{N}} \rightarrow \text { Out }_{0,1, p_{1}^{\left(m_{1}\right)} \perp p_{2}^{\left(m_{2}\right)} \perp \cdots \perp p_{N}^{\left(m_{n}\right)}},  \tag{8.1.1}\\
& \text { Out }_{0,1, p_{1} \perp p_{2} \perp \cdots \perp p_{N}}^{+} \rightarrow \text { Out }_{0,1, p_{1}^{\left(m_{1}\right)} \perp p_{0}^{\left(m_{2}\right)} \perp \cdots \perp p_{N}^{\left(m_{n}\right)}}^{+} . \tag{8.1.2}
\end{align*}
$$

Since (8.1.2) is of index two in (8.1.1), we see that (8.1.1) is injective, resp. surjective, resp. bijective, if and only if (8.1.2) is.

For topological reasons, we suspect that (8.1.1) and (8.1.2) are isomorphisms. In this section, we shall prove that this holds in the case where all the $m_{i}$ are equal or $N=1$. We begin by proving that (8.1.1) and (8.1.2) are injective, which seems to be new.
8.2 Theorem. With Notation 8.1, the maps

$$
\begin{align*}
& \text { Out }_{0,1, p_{1} \perp p_{2} \perp \cdots \perp p_{N}} \rightarrow \text { Out }_{0,1, p_{1}^{\left(m_{1}\right)} \perp p_{2}^{\left(m_{2}\right)} \perp \cdots \perp p_{N}^{\left(m_{n}\right)},}^{\text {Out }_{0,1, p_{1} \perp p_{2} \perp \cdots \perp p_{N}}^{+} \rightarrow \text { Out }_{0,1, p_{1}^{\left(m_{1}\right)} \perp p_{2}^{\left(m_{2}\right)} \perp \cdots \perp p_{N}^{\left(m_{n}\right)}}^{+}} \tag{8.1.1}
\end{align*}
$$

are injective.
Proof. Suppose that $\phi$ is an element of the kernel of (8.1.1) or (8.1.2). Clearly, $\phi \in$ Out $_{0,1, n}^{+}$, and $t_{([1 \uparrow n])}^{\phi}, t_{([1 \uparrow n])}$ have the same image in $\Sigma_{0,1, p_{1}^{\left(m_{1}\right)} \perp p_{2}^{\left(m_{2}\right)} \perp \cdots \perp p_{N}^{\left(m_{N}\right)}}$. By Theorem [7.5, $\left(t_{([1 \uparrow n])}\right)^{\phi}$ is an $n$-tuple of squarefree words in $\Sigma_{0,1, n}$, and, hence, has the same normal form in $\Sigma_{0,1, n}$ and in $\Sigma_{0,1, p_{1}^{\left(m_{1}\right)} \perp p_{2}^{\left(m_{2}\right)} \perp \cdots \perp p_{N}^{\left(m_{N}\right)}}$. Hence $t_{([1 \uparrow n])}^{\phi}=t_{([1 \uparrow n])}$ as $n$-tuples for $\Sigma_{0,1, n}$. Thus $\phi=1$, and the result is proved.
8.3 Historical Remarks. Let us now restrict to the classic case where $N=1$. Here, for an integer $m \geq 2$, we are considering the action of $\mathrm{Out}_{0,1, n}$ on $C_{m}^{* n}$, and it induces maps

$$
\begin{align*}
& \text { Out }_{0,1, n} \rightarrow \text { Out }_{0,1, n^{(m)}}  \tag{8.3.1}\\
& \text { Out }_{0,1, n}^{+} \rightarrow \text { Out }_{0,1, n^{(m)}}^{+} . \tag{8.3.2}
\end{align*}
$$

Theorem 8.2]shows that these maps are injective. Birman-Hilden [6, Theorem 7] gave a topological proof that (8.3.2) is injective, thus answering a question of Magnus. Crisp-Paris [11] gave an elegant algebraic proof of the injectivity of (8.3.2) using the trichotomy argument of Larue [22] and Shpilrain [28]. The Crisp-Paris argument can be summarized as follows.

For each $i \in[1 \uparrow n]$, let $\left(\left\langle\tau_{i}\right\rangle \star\right)$ denote the set of elements of $\Sigma_{0,1, n^{(m)}}$ whose free-product normal form begins with an element of $\left\langle\tau_{i}\right\rangle-\{1\}$.

Suppose that $\phi$ is a non-trivial element of $\mathcal{B}_{n}=\mathrm{Out}_{0,1, n}^{+}$. We will show that $\phi$ acts non-trivially on $\Sigma_{0,1, n^{(m)}}$.

We may assume that $n \geq 3$. By Theorem 6.7, by replacing $\phi$ with $\bar{\phi}$ if necessary, we may assume that $\phi$ is $\sigma$-negative. Thus there exists some $i \in[1 \uparrow n-1]$ such that $\phi$ is the product of a finite sequence of elements of $\sigma_{[i+1 \uparrow n-1]} \cup \bar{\sigma}_{[i \uparrow n-1]}$, and $\bar{\sigma}_{i}$ appears at least once in the sequence.

With Notation 6.5.

$$
\left(\left\langle\tau_{i}\right\rangle \star\right)^{\bar{\sigma}_{i}}=\left(\left\langle\tau_{i}\right\rangle \star\right)^{\bar{\sigma}_{i}^{\prime \prime} \bar{\sigma}_{i}^{\prime}}=\left(\left\langle\tau_{i+1}\right\rangle \star\right)^{\bar{\sigma}_{i}^{\prime}} \subseteq\left(\tau_{i}\left(\left\langle\tau_{i+1}\right\rangle \star\right)\right) \subset\left(\left\langle\tau_{i}\right\rangle \star\right),
$$

since $n \geq 3$. Because the elements of $\sigma_{[i+1 \uparrow n-1]} \cup \bar{\sigma}_{[i \uparrow n-1]}$ act as injective self-maps on $\left(\left\langle\tau_{i}\right\rangle \star\right)$, it follows that $\left(\left\langle\tau_{i}\right\rangle \star\right)^{\phi} \subset\left(\left\langle\tau_{i}\right\rangle \star\right)$, and, hence, $\phi$ acts non-trivially on $\Sigma_{0,1, n^{(m)}}$, as desired.

Let us now verify the surjectivity of the maps (8.3.1) and (8.3.2). The case where $m=2$ was verified by Stephen Humphries [2, Lemma 2.1.7].
8.4 Notation. Let $m, n \in \mathbb{N}$ with $n \geq 1$ and $m \geq 2$. Let $\left\lfloor\frac{m}{2}\right\rfloor$ denote the greatest integer not exceeding $\frac{m}{2}$. Then $\left[0 \uparrow\left\lfloor\frac{m}{2}\right\rfloor\right] \cup\left[-1 \downarrow\left(-\left\lfloor\frac{m-1}{2}\right\rfloor\right)\right]$ is a set of representatives for the integers modulo $m$. For $\tau^{k} \in\left\langle\tau \mid \tau^{m}=1\right\rangle$, we define $\left|\tau^{k}\right|$ $\frac{k \in\left[0 \uparrow\left\lfloor\frac{m}{2}\right\rfloor\right]}{\left(\left|\tau^{k}\right|\right.} \quad \frac{k \in\left[-1 \downarrow-\left\lfloor\frac{m-1}{2}\right\rfloor\right]}{\left.\left|\tau^{k}\right|\right)}$; we extend $|-|$ to all of $\Sigma_{0,1, n(m)}$ by using normal $=(2 k \quad-2 k-1)$
forms for the free product $C_{m}^{*}$.
Let $\phi \in$ Out $_{0,1, n^{(m)}}^{+}$. There exists a unique permutation $\pi \in \operatorname{Sym}_{n}$, and a unique $(n+2)$-tuple $\left(w_{([0 \uparrow n+1])}\right)$ for $\Sigma_{0,1, n^{(m)}}$ such that $w_{0}=1$ and $w_{n+1}=1$, and, for each $i \in[1 \uparrow n], w_{i} \notin\left(t_{i \pi} \star\right) \cup\left(\bar{t}_{i \pi} \star\right)$ and $t_{i}^{\phi}=t_{i \pi}^{w_{i}}$. For each $i \in[0 \uparrow n]$, let $u_{i}=w_{i} \bar{w}_{i+1}$. We define $\pi(\phi):=\pi, w_{i}(\phi):=w_{i}, i \in[0 \uparrow n+1]$, and $u_{i}(\phi):=u_{i}$, $i \in[0 \uparrow n]$. We write $\|\phi\|:=n+2 \sum_{i \in[1 \uparrow n]}\left|w_{i}(\phi)\right|$.

The following is similar to Artin's Lemma 3.2,
8.5 Lemma. Let $n \geq 1, m \geq 2$ and let $\phi \in \mathrm{Out}_{0,1, n^{(m)}}$. Let $\pi=\pi(\phi)$. For each $i \in[0 \uparrow n]$, let $u_{i}=u_{i}(\phi)$. For each $i \in[1 \uparrow n]$, let $a_{i}, b_{i}$ denote the elements of $[0, m-1]$ determined by the following: there exists some $u_{i}^{\prime} \in \Sigma_{0,1, n^{(m)}}-\left(\star\left\langle\tau_{i \pi}\right\rangle\right)$ such that $u_{i-1}=u_{i}^{\prime} \tau_{i^{\pi}}^{a_{i}}$; there exists some $u_{i}^{\prime \prime} \in \Sigma_{0,1, n^{(m)}}-\left(\left\langle\tau_{i^{\pi}}\right\rangle \star\right)$ such that $u_{i}=\tau_{i \pi}^{b_{i}} u_{i}^{\prime \prime}$. In particular, $a_{1}=b_{n}=0$.
(i). Suppose that there exists some $i \in[2 \uparrow n]$ such that $a_{i} \in\left[\left\lfloor\frac{m}{2}\right\rfloor \uparrow m-1\right]$. Then $\left\|\sigma_{i-1} \phi\right\|<\|\phi\|$.
(ii). Suppose that there exists some $i \in[1 \uparrow n-1]$ such that $b_{i} \in\left[\left\lfloor\frac{m+1}{2}\right\rfloor \uparrow m-1\right]$. Then $\left\|\bar{\sigma}_{i} \phi\right\|<\|\phi\|$.
(iii). If $\phi \neq 1$, there exists some $\sigma_{i}^{\epsilon} \in \sigma_{[1 \uparrow n-1]} \cup \bar{\sigma}_{[1 \uparrow n-1]}$ such that $\left\|\sigma_{i}^{\epsilon} \phi\right\|<\|\phi\|$.

Proof. (i). Let $a=a_{i}$. There exists some $v \in \Sigma_{0,1, n^{(m)}}-\left(\star\left\langle\tau_{i^{\pi}}\right\rangle\right)$ such that $u_{i-1}=v \tau_{i^{\pi}}^{a}$. Since $w_{i-1}(\phi)=u_{i-1} w_{i}(\phi)$, we have

$$
\begin{equation*}
w_{i-1}(\phi)=v \tau_{i \pi}^{a} w_{i}(\phi) ; \tag{8.5.1}
\end{equation*}
$$

since $w_{i}(\phi) \notin\left(\left\langle\tau_{i \pi}\right\rangle \star\right)$ and $v \notin\left(\star\left\langle\tau_{i^{\pi}}\right\rangle\right), v \tau_{i \pi}^{a} w_{i}(\phi)$ is a free-product normal form for $w_{i-1}(\phi)$.
Claim. $\left|\tau_{i^{\pi}}^{a+1}\right|<\left|\tau_{i^{\pi}}^{a}\right|$.
Proof. If $a \in\left[\left\lfloor\frac{m}{2}\right\rfloor+1 \uparrow m-1\right]$, then $a-m \in\left[-\left\lfloor\frac{m-1}{2}\right\rfloor \uparrow-1\right]$, and, hence,

$$
\left|\tau_{i \pi}^{a}\right|=\left|\tau_{i^{\pi}}^{a-m}\right|=-2(a-m)-1=2 m-2 a-1 .
$$

Therefore, if $a \in\left[\left\lfloor\frac{m}{2}\right\rfloor \uparrow m-2\right],\left|\tau_{i^{\pi}}^{a+1}\right|=2 m-2(a+1)-1=2 m-2 a-3$.
Thus, $\left|\tau_{i^{\pi}}^{a+1}\right|<\left|\tau_{i^{\pi}}^{a}\right|$ if $a \in\left[\left\lfloor\frac{m}{2}\right\rfloor+1 \uparrow m-2\right]$.
For $a=\left\lfloor\frac{m}{2}\right\rfloor, a \geq \frac{m-1}{2}$, and $\left|\tau_{i \pi}^{a}\right|=2 a>2 m-2 a-3=\left|\tau_{i^{\pi}}^{a+1}\right|$.
For $a=m-1,\left|\tau_{i^{\pi}}^{a}\right|=1$ and $\left|\tau_{i^{\pi}}^{a+1}\right|=0$.
Thus,

$$
\left|w_{i-1}(\phi)\right|=|v|+\left|\tau_{i^{\pi}}^{a}\right|+\left|w_{i}(\phi)\right|>|v|+\left|\tau_{i \pi}^{a+1}\right|+\left|w_{i}(\phi)\right| .
$$

By (8.5.1), $w_{i-1}(\phi) \bar{w}_{i}(\phi) \tau_{i^{\pi}}=v \tau_{i^{\pi}}^{a+1}$; hence

$$
\tau_{i}^{\sigma_{i-1} \phi}=\left(\tau_{i-1}^{\tau_{i}}\right)^{\phi}=\left(\tau_{(i-1)^{\pi}}^{w_{i-1}(\phi)}\right)^{\left(\tau_{i \pi}^{w_{i}(\phi)}\right)}=\tau_{(i-1)^{\pi}}^{v v_{i}^{a+1} w_{i}(\phi)} .
$$

Hence, $\left|w_{i}\left(\sigma_{i-1} \phi\right)\right|=\left|v \tau_{i \pi}^{a+1} w_{i}(\phi)\right| \leq|v|+\left|\tau_{i \pi}^{a+1}\right|+\left|w_{i}(\phi)\right|<\left|w_{i-1}(\phi)\right|$.
For each $j \in[1 \uparrow i-2] \cup[i+1 \uparrow n], \tau_{j}^{\sigma_{i-1} \phi}=\tau_{j}^{\phi}$, and, hence, $\left|w_{j}\left(\sigma_{i-1} \phi\right)\right|=$ $\left|w_{j}(\phi)\right|$.

Also, $\tau_{i-1}^{\sigma_{i-1} \phi}=\tau_{i}^{\phi}$; in particular, $\left|w_{i-1}\left(\sigma_{i-1} \phi\right)\right|=\left|w_{i}(\phi)\right|$.
It now follows that $\left\|\sigma_{i-1} \phi\right\|<\|\phi\|$.
(ii). Let $b=b_{i}$. There exists some $v \in \Sigma_{0,1, n^{(m)}}-\left(\left\langle\tau_{i^{\pi}}\right\rangle \star\right)$ such that $u_{i}=\tau_{i^{\pi}}^{b} v$. Since $w_{i+1}(\phi)=\bar{u}_{i} w_{i}(\phi)$, we have

$$
\begin{equation*}
w_{i+1}(\phi)=\bar{v} \bar{\tau}_{i^{\pi}}^{b} w_{i}(\phi) \tag{8.5.2}
\end{equation*}
$$

Since $w_{i}(\phi) \notin\left(\left\langle\tau_{i^{\pi}}\right\rangle \star\right)$ and $\bar{v} \notin\left(\star\left\langle\tau_{i^{\pi}}\right\rangle\right), \bar{v} \bar{\tau}_{i^{\pi}}^{b} w_{i}(\phi)$ is a free-product normal form for $w_{i+1}(\phi)$. Hence, $\left|w_{i+1}(\phi)\right|=|\bar{v}|+\left|\bar{\tau}_{i^{\pi}}^{b}\right|+\left|w_{i}(\phi)\right|$.
Claim. $\left|\bar{\tau}_{i^{\pi}}^{b+1}\right|<\left|\bar{\tau}_{i^{\pi}}^{b}\right|$.
Proof. For any $b \in\left[\left\lfloor\frac{m+1}{2}\right\rfloor \uparrow m\right]$, then $m-b \in\left[\left\lfloor\frac{m}{2}\right\rfloor \downarrow 0\right]$, and, hence,

$$
\left|\bar{\tau}_{i^{\pi}}^{b}\right|=\left|\tau_{i^{\pi}}^{m-b}\right|=2(m-b)=2 m-2 b .
$$

Therefore, since $b \in\left[\left\lfloor\frac{m+1}{2}\right\rfloor \uparrow m-1\right]$,

$$
\left|\bar{\tau}_{i^{\pi}}^{b+1}\right|=2 m-2(b+1)=2 m-2 b-2<\left|\bar{\tau}_{i^{\pi}}^{b}\right|,
$$

as claimed.

Hence $\left|w_{i+1}(\phi)\right|>|\bar{v}|+\left|\bar{\tau}_{i^{\pi}}^{b+1}\right|+\left|w_{i}(\phi)\right|$.
For all $j \in[1 \uparrow i-1] \cup[i+2 \uparrow n], \tau_{j}^{\bar{\sigma}_{i} \phi}=\tau_{j}^{\phi}$; hence, $\left|w_{j}\left(\bar{\sigma}_{i} \phi\right)\right|=\left|w_{j}(\phi)\right|$.
Since $\tau_{i+1}^{\bar{\sigma}_{i} \phi}=\tau_{i}^{\phi}$, we see that $\left|w_{i+1}\left(\bar{\sigma}_{i} \phi\right)\right|=\left|w_{i}(\phi)\right|$.
By (8.5.2), $w_{i+1}(\phi) \bar{w}_{i}(\phi) \bar{\tau}_{i^{\pi}}=\bar{v} \bar{\tau}_{i^{\pi}}^{b+1}$; hence

$$
\tau_{i}^{\bar{\sigma}_{i} \phi}=\left(\tau_{i+1}^{\bar{\tau}_{i}}\right)^{\phi}=\left(\tau_{(i+1)^{\pi}}^{w_{i+1}(\phi)}\right)^{\left(\bar{\tau}_{i \pi}^{w_{i}(\phi)}\right)}=\tau_{i \pi}^{\bar{v} \bar{\tau}_{i}^{b+1} w_{i}(\phi)} .
$$

Hence, $\left|w_{i}\left(\bar{\sigma}_{i} \phi\right)\right|=\left|\bar{v} \bar{\tau}_{i^{\pi}}^{b+1} w_{i}(\phi)\right| \leq|\bar{v}|+\left|\bar{\tau}_{i^{\pi}}^{b+1}\right|+\left|w_{i}(\phi)\right|<\left|w_{i+1}(\phi)\right|$.
It now follows that $\left\|\bar{\sigma}_{i} \phi\right\|<\|\phi\|$, and (ii) is proved.
(iii). If $\phi \neq 1$, we choose a distinguished element of $[1 \uparrow n]$ as follows.

If, for some $i \in[1 \uparrow n], \tau_{i^{\pi}}^{a_{i}+1+b_{i}}=1$, we take any such $i$ to be our distinguished element of $[1 \uparrow n]$.

Consider then the case where, for all $i \in[1 \uparrow n], \tau_{i^{\pi}}^{a_{i}+1+b_{i}} \neq 1$. Thus, there is no further cancellation in $\Pi \tau_{[1 \uparrow n]}^{\phi}$. Since $\phi$ fixes $\Pi \tau_{[1 \uparrow n]}$, it is not difficult to see that, for all $i \in[1 \uparrow n], \tau_{i \pi}^{a_{i}+1+b_{i}}=\tau_{i}$. Since $\phi \neq 1$, it is then not difficult to show that there exists some $i \in[1 \uparrow n]$ such that $\left(a_{i}, b_{i}\right) \neq(0,0)$. We take any such $i$ to be our distinguished element of $[1 \uparrow n]$.

Let $i$ denote our distinguished element of $[1 \uparrow n]$.
Notice that $\left(a_{i}, b_{i}\right) \neq(0,0)$ and that $\tau_{i \pi}^{a_{i}+1+b_{i}} \in\left\{1, \tau_{i^{\pi}}\right\}$. Hence, $a_{i}+1+b_{i} \in$ $\{m, m+1\}$, and, hence, $b_{i} \in\left\{m-a_{i}-1, m-a_{i}\right\}$.
Case 1. $a_{i} \in\left[\left\lfloor\frac{m}{2}\right\rfloor \uparrow m-1\right]$.
Here, $i \in[2 \uparrow n]$ and, by (i), $\left\|\sigma_{i-1} \phi\right\|<\|\phi\|$.
Case 2. $a_{i} \in\left[0 \uparrow\left\lfloor\frac{m-2}{2}\right\rfloor\right]$
Here, $m-a_{i}-1 \in\left[m-1 \downarrow\left\lfloor\frac{m+1}{2}\right\rfloor\right]$, and, hence, $b_{i} \in\left[\left\lfloor\frac{m+1}{2}\right\rfloor \uparrow m-1\right]$. Here, $i \in[1 \uparrow n-1]$ and, by (ii), $\left\|\bar{\sigma}_{i} \phi\right\|<\|\phi\|$.
8.6 Theorem. Let $n \geq 1, m \geq 2$. The natural map Out $_{0,1, n}^{+} \rightarrow$ Out $_{0,1, n^{(m)}}^{+}$ is an isomorphism, and, hence, the natural map $\mathrm{Out}_{0,1, n} \rightarrow \mathrm{Out}_{0,1, n^{(m)}}$ is an isomorphism.

With Notation 8.1, the maps Out $_{0,1, p_{1} \perp p_{2} \perp \cdots \perp p_{N}} \rightarrow$ Out $_{0,1, p_{1}^{(m)} \perp p_{2}^{(m)} \perp \cdots \perp p_{N}^{(m)}}$, and $\mathrm{Out}_{0,1, p_{1} \perp p_{2} \perp \cdots \perp p_{N}}^{+} \rightarrow \mathrm{Out}_{0,1, p_{1}^{(m)} \perp p_{2}^{(m)} \perp \ldots \perp p_{N}^{(m)}}^{+}$are isomorphisms.

The following is essentially an algebraic translation of a part of a topological argument in [26, Section 3].
8.7 Proposition. With Notation 8.1, let $H$ be a subgroup of

$$
\Sigma_{0,1, p_{1}^{\left(m_{1}\right)} \perp p_{2}^{\left(m_{2}\right)} \perp \cdots \perp p_{N}^{\left(m_{n}\right)}}
$$

of finite index, and let $A$ be the subgroup of

$$
\text { Out }_{0,1, p_{1}^{\left(m_{1}\right)} \perp p_{2}^{\left(m_{2}\right)} \perp \cdots \perp p_{N}^{\left(m_{n}\right)}}
$$

consisting of elements which map $H$ to itself. Then, either the induced map $A \rightarrow \operatorname{Aut}(H)$ is injective or $\left(n, N, m_{1}\right)=(2,1,2)$.

Proof. Suppose that $\phi \in \mathrm{Out}_{0,1, p_{1}^{\left(m_{1}\right)} \perp p_{2}^{\left(m_{2}\right)} \perp \cdots \perp p_{N}^{\left(m_{n}\right)}}$, and that $\phi$ acts as the identity on $H$. We shall show that $\phi=1$ or $\left(n, N, m_{1}\right)=(2,1,2)$.

Let $G=\Sigma_{0,1, p_{1}^{\left(m_{1}\right)}} \perp_{2}^{\left(m_{2}\right)} \perp \ldots \perp p_{N}^{\left(m_{n}\right)}$.
For any $g \in G$, right multiplication by $g$ permutes the elements of the finite set $H \backslash G$, so there exists some positive integer $k$ such that $g^{k}$ acts trivially on $H \backslash G$. In particular, $H g^{k}=H$ and, hence, $g^{k} \in H$.

Hence, there exists some positive integer $k$ such that $\left(\Pi \tau_{[1 \uparrow n]}\right)^{k} \in H$. Now $\left(\Pi \tau_{[1 \uparrow n]}\right)^{\phi}=\left(\Pi \tau_{[1 \uparrow n]}\right)^{\epsilon}$ for some $\epsilon \in\{1,-1\}$, and, hence,

$$
\left(\Pi \tau_{[1 \uparrow n]}\right)^{k}=\left(\Pi \tau_{[1 \uparrow n]}\right)^{k \phi}=\left(\Pi \tau_{[1 \uparrow n]}\right)^{\phi k}=\left(\Pi \tau_{[1 \uparrow n]}\right)^{\epsilon k}=\left(\Pi \tau_{[1 \uparrow n]}\right)^{k \epsilon} .
$$

Since $\Pi \tau_{[1 \uparrow n]}$ has infinite order in $G$, we see that $\epsilon=1$. Thus $\phi$ fixes $\Pi \tau_{[1 \uparrow n]}$.
Consider any $i \in[1 \uparrow n]$. Since $\left(\Pi \tau_{[1 \uparrow n]}\right)^{\tau_{i}} \in G$, there exists some positive integer $k$ such that $\left(\Pi \tau_{[1 \uparrow n]}\right)^{\tau_{i} k} \in H$. Hence,

$$
\left(\Pi \tau_{[1 \uparrow n]}\right)^{k \tau_{i}}=\left(\Pi \tau_{[1 \uparrow n]}\right)^{\tau_{i} k}=\left(\Pi \tau_{[1 \uparrow n]}\right)^{\tau_{i} k \phi}=\left(\Pi \tau_{[1 \uparrow n]}\right)^{k \phi \tau_{i}^{\phi}}=\left(\Pi \tau_{[1 \uparrow n]}\right)^{k \tau_{i}^{\phi}} .
$$

Hence $\tau_{i}^{\phi} \bar{\tau}_{i}$ commutes with $\left(\Pi \tau_{[1 \uparrow n]}\right)^{k}$. A straightforward normal-form argument shows that $\tau_{i}^{\phi} \bar{\tau}_{i} \in\left\langle\Pi \tau_{[1 \uparrow n]}\right\rangle$.

Hence there exists an integer $j$ such that $\tau_{i}^{\phi}=\left(\Pi \tau_{[1 \uparrow n]}\right)^{j} \tau_{i}$. Since $\tau_{i}^{\phi}$ is a conjugate of $\tau_{i^{\pi(\phi)}}$, the cyclically-reduced form of $\left(\tau_{[1, n]}\right)^{j} \tau_{i}$ is $\tau_{i^{\pi(\phi)}}$. Either $j=0$, or there must be cyclic cancellation, and a straightforward analysis then shows that $\left(n, N, m_{1}\right)=(2,1,2)$. Since $i$ was arbitrary, this completes the proof.

## 9 The $\mathcal{B}_{n+1}$-group $\Phi_{n}$

9.1 Notation. Recall that $\Sigma_{0,1,(n+1)^{(2)}}=C_{2}^{*(n+1)}=\left\langle\tau_{[1 \uparrow n+1]} \mid \tau_{[1 \uparrow n+1]}^{2}=1\right\rangle$. We define $\Phi_{n}$ to be the $\mathcal{B}_{n+1}$-group consisting of the set of elements of $\Sigma_{0,1,(n+1)^{(2)}}$ which have even exponent sum in the $\tau_{i}$. It is not difficult to see that $\Phi_{n}$ is a free group of rank $n$, and that there is induced a map from Out ${ }_{0,1, n+1}=$ Out $_{0,1,(n+1)^{(2)}}$ to Aut $\Phi_{n}$. Since $\mathcal{B}_{n+1}=\operatorname{Out}_{0,1, n+1}^{+}=\operatorname{Out}_{0,1,(n+1)^{(2)}}^{+}, \Phi_{n}$ has a $\mathcal{B}_{n+1}$-action; we say that $\Phi_{n}$ is a $\mathcal{B}_{n+1}$-group, and that $\Phi_{n}$ is a $\mathcal{B}_{n+1}$-subgroup of $\Sigma_{0,1,(n+1)^{(2)}}$.

Proposition 8.7 shows that, if $n \neq 1$, then the map from Out ${ }_{0,1, n+1}=$ Out $_{0,1,(n+1)^{(2)}}$ to Aut $\Phi_{n}$ is injective, and we say that the $\mathcal{B}_{n+1}$-action is faithful, and that $\Phi_{n}$ is a faithful $\mathcal{B}_{n+1}$-group.

Over the course of this section, we shall choose various free generating sets of $\Phi_{n}$ to obtain interesting actions. In the next two examples, we identify $\Sigma_{g, 1,0}$ with $\Phi_{2 g}$ and $\Sigma_{g, 2,0}$ with $\Phi_{2 g+1}$.
9.2 Example. Now that algebraic proofs of the requisite theorems are known to us, let us review [18, Example 15.6] which was an algebraic approximation of results in [26, Section 3].

Let $g \in \mathbb{N}$. Let

$$
\Sigma_{g, 1,0}:=\left\langle x_{1}, y_{1}, \ldots, x_{g}, y_{g}, z_{1} \mid\left[x_{1}, y_{1}\right] \cdots\left[x_{g}, y_{g}\right] z_{1}=1\right\rangle
$$

where the commutator $[x, y]$ of group elements $x, y$ is $\bar{x} \bar{y} x y$. Let Out ${ }_{g, 1,0}^{+}$denote the group of all automorphisms of $\Sigma_{g, 1,0}$ which fix $z_{1}$. Then $\Sigma_{g, 1,0}$ is free of rank $2 g$ with ordered free generating set $\left(x_{1}, y_{1}, \ldots, x_{g}, y_{g}\right)$, and Out ${ }_{g, 1,0}^{+}$is the group of all automorphisms of $\Sigma_{g, 1,0}$ which fix $\left[x_{1}, y_{1}\right] \cdots\left[x_{g}, y_{g}\right]$.

We now recall some Dehn-twist elements of Out ${ }_{g, 1,0}^{+}$from Definitions 3.10 and Remarks 5.1 of [18].

For each $i \in[1 \uparrow g]$, we define $\alpha_{i}, \beta_{i} \in$ Out $_{g, 1,0}^{+}$by

$$
\begin{aligned}
& \frac{k \in[1 \uparrow i-1]}{\left(\begin{array}{llllllll}
x_{k} & y_{k} & x_{i} & y_{i} & \frac{k \in[i+1 \uparrow g]}{x_{k}} & y_{k}
\end{array}\right)^{\alpha_{i}}} \quad \text { and } \quad \frac{k \in[1 \uparrow i-1]}{\left(\begin{array}{llllll}
x_{k} & y_{k} & x_{i} & y_{i} & x_{k} & y_{k}
\end{array}\right)^{\beta_{i}}} \\
& =\left(\begin{array}{llllll}
x_{k} & y_{k} & \bar{y}_{i} x_{i} & y_{i} & x_{k} & y_{k}
\end{array}\right), \quad=\left(\begin{array}{llllll}
x_{k} & y_{k} & x_{i} & x_{i} y_{i} & x_{k} & y_{k}
\end{array}\right) .
\end{aligned}
$$

For each $i \in[1 \uparrow g-1]$, we define $\gamma_{i} \in$ Out $_{g, 1,0}^{+}$by

\[

\]

Let us identify $\Sigma_{g, 1,0}$ with $\Phi_{2 g}$ via

$$
\begin{aligned}
& \frac{k \in[1 \uparrow g]}{} \\
&= z_{k} \\
&\left(x_{k}\right.\left(\begin{array}{ll}
\Sigma_{g, 1,0} & \\
& \\
\overbrace{[2 k+1 \downarrow 2 k]} & \tau_{2 k+1} \Pi \tau_{[1 \uparrow 2 k+1]}
\end{array}\right. \\
&\left.z_{1}^{2}\right) .
\end{aligned}
$$

Notice that $\left[x_{k}, y_{k}\right]=\bar{x}_{k} \bar{y}_{k} x_{k} y_{k}$ is then identified with

$$
\Pi \tau_{[2 k \uparrow 2 k+1]} \Pi \tau_{[2 k+1 \downarrow 1]} \tau_{2 k+1} \Pi \tau_{[2 k+1 \downarrow 2 k]} \tau_{2 k+1} \Pi \tau_{[1 \uparrow 2 k+1]}
$$

which equals $\Pi \tau_{[2 k-1 \downarrow 1]} \Pi \tau_{[2 k \uparrow 2 k+1]} \Pi \tau_{[1 \uparrow 2 k+1]}$. Hence $\prod_{k \in[1 \uparrow 9]}\left[x_{k}, y_{k}\right]$ is identified with $\left(\Pi \tau_{[1 \uparrow 2 g+1]}\right)^{2}$.

This corresponds to the surface of genus $g$ with one boundary component arising as a two-sheeted branched cover of a sphere with one boundary component and $2 g+1$ double points. Then $\mathcal{B}_{2 g+1}=$ Out $_{0,1,2 g+1}^{+}=$Out $_{0,1,(2 g+1)^{(2)}}^{+}$ becomes embedded in Out $_{g, 1,0}^{+}$via the homomorphism represented as

$$
\left(\begin{array}{ccccccccc}
\sigma_{1} & \sigma_{2} & \sigma_{3} & \sigma_{4} & \sigma_{5} & \cdots & \sigma_{2 g-2} & \sigma_{2 g-1} & \sigma_{2 g} \\
\alpha_{1} & \beta_{1} & \gamma_{1} & \beta_{2} & \gamma_{2} & \cdots & \beta_{g-1} & \gamma_{g-1} & \beta_{g}
\end{array}\right) .
$$

Clearly, in the preceding example, the subgroup $\mathcal{B}_{2 g}$ of $\mathcal{B}_{2 g+1}$ is also embedded in Out ${ }_{g, 1,0}$, but it is more natural to remove from the surface a handle containing the boundary component (a sphere with three boundary components or a 'pair of pants'), and embed $\mathcal{B}_{2 g}$ in Out ${ }_{g-1,2,0}$, as follows.
9.3 Example. Now that algebraic proofs of the requisite theorems are known to us, let us review [18, Example 15.7] which was an algebraic approximation of results in [26, Section 3].

Let $g \in \mathbb{N}$. Let

$$
\Sigma_{g, 2,0}:=\left\langle x_{[1 \uparrow g]}, y_{[1 \uparrow g]}, z_{[1 \uparrow 2]} \mid\left(\prod_{i \in[1 \uparrow g]}\left[x_{i}, y_{i}\right]\right) \Pi z_{[1 \uparrow 2]}=1\right\rangle
$$

Recall that $[x, y]:=\bar{x} \bar{y} x y$. Then $\Sigma_{g, 2,0}$ is free of rank $2 g+1$ with free generating $\operatorname{set}\left(x_{[1 \uparrow g]}, y_{[1 \uparrow g]}, z_{1}\right)$ and distinguished element $z_{2}$ such that $\bar{z}_{2}=\left(\prod_{i \in[1 \uparrow g]}\left[x_{i}, y_{i}\right]\right) z_{1}$. Let Out ${ }_{g, 1 \perp 1,0}^{+}$denote the group of all automorphisms of $\Sigma_{g, 2,0} *\left\langle e_{1} \mid\right\rangle$ which map $\Sigma_{g, 2,0}$ to itself, and fix $z_{1}^{e_{1}}$ and $z_{2}$. It can be shown that Out ${ }_{g, 1 \perp 1,0}^{+}$acts faithfully on the subset $\Sigma_{g, 2,0} \cup \Sigma_{g, 2,0} e_{1}$ of $\Sigma_{g, 2,0} *\left\langle e_{1} \mid\right\rangle$.

Here, $e_{1}$ represents an arc from the base-point of one boundary component, to the base-point of the other boundary component. Karen Vogtmann calls such an arc a 'tether joining the basepoint to the second boundary component'. For any surface-with-boundaries, A'Campo [1, Section 4, Remarque 6], [26, p.232] identifies basepoints of all the boundary components, which makes tethers into loops, to obtain a topological quotient space whose fundamental group is acted on, faithfully, by the mapping-class group of the surface-with-boundaries.

We now recall some Dehn-twist elements of Out ${ }_{g, 1 \perp 1,0}^{+}$from Definitions 3.10 and Remarks 5.1 of [18].

For each $i \in[1 \uparrow g]$, we define $\alpha_{i}, \beta_{i} \in$ Out $_{g, 1 \perp 1,0}^{+}$by

$$
\left.\begin{array}{rl} 
& \frac{k \in[1 \uparrow i-1]}{\left(x_{k}\right.} y_{k} \\
x_{i} & y_{i} \\
& \frac{k}{c} \frac{k \in[i+1 \uparrow g]}{x_{k}} \\
y_{k} & z_{1}
\end{array} e_{1}\right)^{\alpha_{i}},
$$

For each $i \in[1 \uparrow g-1]$, we define $\gamma_{i} \in$ Out $_{g, 1 \perp 1,0}^{+}$by

$$
\begin{array}{rlcccccc} 
& \left.\frac{k \in[1 \uparrow i-1]}{\left(x_{k}\right.}\right)^{\gamma_{i}} \\
= & \left(\begin{array}{llllllll}
x_{k} & y_{k} & y_{i+1}^{x_{i+1}} \bar{y}_{i} x_{i} & y_{i}^{\bar{y}_{i+1}} & x_{i+1} y_{i} \bar{y}_{i+1}^{x_{i+1}} & y_{i+1} & x_{k} & y_{k} \\
z_{1} & e_{1}
\end{array}\right),
\end{array}
$$

and we define $\gamma_{g} \in$ Out $_{g, 1 \perp 1,0}^{+}$

$$
\left.\begin{array}{rl} 
& \frac{k \in[1 \uparrow i-1]}{\left(x_{k}\right.} \quad y_{k} \\
& x_{g} \\
x_{k} & y_{k}
\end{array} \bar{z}_{1} \bar{y}_{g} x_{g} \quad y_{g}^{z_{1}} \quad z_{1}^{z_{g} z_{1}} \quad \bar{z}_{1} \bar{y}_{g} e_{1}\right) .
$$

Let us identify $\Sigma_{g, 2,0}$ with $\Phi_{2 g+1}$ and $\Sigma_{g, 2,0} \cup \Sigma_{g, 2,0} e_{1}$ with $\Sigma_{0,1,(2 g+2)^{(2)}}$ via the map $\Sigma_{g, 2,0} *\left\langle e_{1}\right\rangle \rightarrow \Sigma_{0,1,(2 g+2)^{(2)}}$ determined by

$$
\left.\begin{array}{lcccl} 
& \begin{array}{cc}
k \in[1 \uparrow g] \\
x_{k} & y_{k} \\
\left(\Pi \tau_{[2 k+1 \downarrow 2 k]}\right. & \tau_{2 k+1} \Pi \tau_{[1 \uparrow 2 k+1]}
\end{array} & z_{1}^{z_{1}} & e_{1} & \left.z_{2}\right)^{\Sigma_{g, 2,0} *\left\langle\left(e_{1}\right\rangle \rightarrow \Sigma_{0,1,(2 g+2)}(2)\right.} \\
\tau_{2 g+2} & z_{1}
\end{array}\right) .
$$

This corresponds to the surface of genus $g$ with two boundary components arising as a two-sheeted branched cover of a sphere with one boundary component and $2 g+2$ double points. Now $\mathcal{B}_{2 g+2}=$ Out $_{0,1,2 g+2}^{+}=$Out $_{0,1,(2 g+2)^{(2)}}^{+}$is embedded in Out $_{g, 1 \perp 1,0}^{+}$via a homomorphism represented as

$$
\left(\begin{array}{cccccccccc}
\sigma_{1} & \sigma_{2} & \sigma_{3} & \sigma_{4} & \sigma_{5} & \cdots & \sigma_{2 g-2} & \sigma_{2 g-1} & \sigma_{2 g} & \sigma_{2 g+1} \\
\alpha_{1} & \beta_{1} & \gamma_{1} & \beta_{2} & \gamma_{2} & \cdots & \beta_{g-1} & \gamma_{g-1} & \beta_{g} & \gamma_{g}
\end{array}\right) .
$$

For $g \geq 1$, Proposition 8.7 shows that this is an embedding. In the case where $g=0$, the interpretation of the notation is as follows: $\sigma_{1}$ is mapped to $\gamma_{0} ; \gamma_{0}$ fixes $z_{1}$ and sends $e_{1}$ to $\bar{z}_{1} e_{1}$.

Clearly, in the preceding example, the subgroup $\mathcal{B}_{2 g+1}$ of $\mathcal{B}_{2 g+2}$ is also embedded in Out $_{g, 1 \perp 1,0}^{+}$, but it is more natural to remove from the surface a disc containg the two boundary components (a sphere with three boundary components or a 'pair of pants'), and embed $\mathcal{B}_{2 g+1}$ in Out ${ }_{g, 1,0}^{+}$, as in Example 9.2.

We next discuss the Perron-Vannier isomorphism $\mathcal{B}_{n+1} \ltimes \Phi_{n} \simeq \operatorname{Artin}\left\langle D_{n+1}\right\rangle$ for $n \geq 1$. The following was shown to us by Mladen Bestvina.
9.4 Lemma. Let $n \geq 2$. Then, $\operatorname{Artin}\left\langle D_{n}\right\rangle$ has a unique automorphism $v$ of order two which fixes $d_{1}, \ldots, d_{n-2}$ and interchanges $d_{n-1}$ and $d_{n}$. The semidirect product $\operatorname{Artin}\left\langle D_{n}\right\rangle \rtimes\langle v\rangle$ has presentation

$$
\operatorname{Artin}\left\langle d_{1}-d_{2}-\cdots-d_{n-3}-d_{n-2}-d_{n-1}=v \mid v^{2}=1\right\rangle .
$$

Proof. Notice that $\left\langle d_{n-1}, d_{n}, v \mid v^{2}=1, d_{n-1}^{v}=d_{n}, d_{n-1} d_{n}=d_{n} d_{n-1}\right\rangle$ is isomorphic to $\left\langle d_{n-1}, v \mid v^{2}=1, d_{n-1} d_{n-1}^{v}=d_{n-1}^{v} d_{n-1}\right\rangle$, and the latter is $\operatorname{Artin}\left\langle d_{n-1}=v \mid v^{2}=1\right\rangle$. The result now follows easily.

Part of the following appears in [26] and [10].
9.5 Theorem (Perron-Vannier [26]). Let $n \geq 2$. The semidirect product $\mathcal{B}_{n} \ltimes$ $\Phi_{n-1}$ has presentation

$$
\operatorname{Artin}\left\langle\sigma_{1}-\sigma_{2}-\cdots-\sigma_{n-3}-\left.\right|_{n-2} ^{\sigma_{n-1} \tau_{n} \tau_{n-1}}-\sigma_{n-1}\right\rangle \simeq \operatorname{Artin}\left\langle D_{n}\right\rangle
$$

Hence, $\mathcal{B}_{n} \ltimes \Phi_{n-1}$ has a unique automorphism $v$ of order two which fixes $\sigma_{1}, \ldots, \sigma_{n-2}$ and interchanges $\sigma_{n-1}$ and $\sigma_{n-1} \tau_{n} \tau_{n-1}$. The double semidirect product $\left(\mathcal{B}_{n} \ltimes \Phi_{n-1}\right) \rtimes\langle v\rangle$ has presentation

$$
\operatorname{Artin}\left\langle\sigma_{1}-\sigma_{2}-\cdots-\sigma_{n-3}-\sigma_{n-2}-\sigma_{n-1}=v \mid v^{2}=1\right\rangle
$$

Proof. By Corollary 5.5, we have a presentation

$$
\mathcal{B}_{n} \ltimes \Sigma_{0,1, n}=\operatorname{Artin}\left\langle\sigma_{1}-\cdots-\sigma_{n-1}=\bar{t}_{n}\right\rangle .
$$

If we impose the relation $t_{n}^{2}=1$, we transform $\mathcal{B}_{n} \ltimes \Sigma_{0,1, n}$ into $\mathcal{B}_{n} \ltimes \Sigma_{0,1, n^{(2)}}$, and we have

$$
\mathcal{B}_{n} \ltimes \Sigma_{0,1, n^{(2)}}=\operatorname{Artin}\left\langle\sigma_{1}-\cdots-\sigma_{n-1}=\tau_{n} \mid \tau_{n}^{2}=1\right\rangle .
$$

Here, there exists a retraction to $\left\langle\tau_{n}\right\rangle$ with kernel the normal subgroup generated by $\sigma_{[1 \uparrow n-1]}$. This normal subgroup contains $\sigma_{i}^{\tau_{i+1}}=\sigma_{i} \tau_{i+1} \tau_{i}$ for all $i \in[1 \uparrow n-1]$. By Lemma 9.4, the normal subgroup has presentation

$$
\begin{aligned}
& \sigma_{n-1}^{\tau_{n}} \\
& \left.\sigma_{n-2}-\sigma_{n-1}\right\rangle
\end{aligned}
$$

and this agrees with the desired presentation.
9.6 Remarks. Corollary 5.5 says that, for $n \geq 1$, we can go down by index $n+1$ from $\operatorname{Artin}\left\langle A_{n}\right\rangle$ by squaring the last generator, and arrive at $\operatorname{Artin}\left\langle B_{n}\right\rangle \simeq$ $\operatorname{Artin}\left\langle A_{n-1}\right\rangle \ltimes \Sigma_{0,1, n}$.

Theorem 9.5 says that, for $n \geq 2$, we can kill the square of the new last generator, go down by index 2 , and arrive at $\operatorname{Artin}\left\langle D_{n}\right\rangle \simeq \operatorname{Artin}\left\langle A_{n-1}\right\rangle \ltimes \Phi_{n-1}$.

We now record some other free generating sets of $\Phi_{n}$ which appear in the literature.
9.7 Examples. Recall Notation 9.1. In particular, the $\mathcal{B}_{n+1}$-action on $\Phi_{n}$ is faithful if $n \neq 1$.
(1). For each $k \in[1 \uparrow n]$, set $x_{k}=\tau_{k} \tau_{k+1}$ in $\Phi_{n}$. Then $x_{[1 \uparrow n]}$ is a free generating set for $\Phi_{n}$, and, for each $i \in[1 \uparrow n]$, the action of $\sigma_{i}$ on $\Phi_{n}$ is determined by

$$
\begin{array}{llllll}
\frac{k \in[1 \uparrow i-2]}{\left(x_{k}\right.} & x_{i-1} & x_{i} & x_{i+1} & \left.\frac{k \in[i+2 \uparrow n]}{x_{k}}\right)^{\sigma_{i}} \\
=\left(\begin{array}{llll}
x_{k} & x_{i-1} x_{i} & x_{i} & x_{i} x_{i+1} \\
x_{k}
\end{array}\right)
\end{array}
$$

interpretated appropriately for $i=1$ and $i=n$.
(2). For each $k \in[1 \uparrow n]$, set $x_{k}=\tau_{n+1} \tau_{k}$ in $\Phi_{n}$, Then $x_{[1 \uparrow n]}$ is a free generating set for $\Phi_{n}$, and, for each $i \in[1 \uparrow n-1], \sigma_{i}$ acts on $x_{[1 \uparrow n]}$ as follows.

$$
\begin{array}{cccccl}
\frac{k \in[1 \uparrow i-1]}{\left(x_{k}\right.} & x_{i} & x_{i+1} & \frac{k \in[i+2 \uparrow n]}{\left.x_{k}\right)^{\sigma_{i}}} & \frac{k \in[1 \uparrow n-1]}{\left(x_{k}\right.} & \left.x_{n}\right)^{\sigma_{n}} \\
=\left(\begin{array}{lccl}
x_{k} & x_{i+1} & x_{i+1} \bar{x}_{i} x_{i+1} & x_{k}
\end{array}\right) . & =\left(\begin{array}{lll}
x_{n-1} x_{k} & x_{n}
\end{array}\right) .
\end{array}
$$

(3). We next consider the free generating set used in the proof of [11, Proposition A.1(2)].

For each $k \in[1 \uparrow n]$, set $x_{k}=\tau_{n+1}^{\Pi \tau_{[1 \uparrow k]}} \tau_{k+1}$ in $\Phi_{n}$. Then $x_{[1 \uparrow n]}$ is a free generating set for $\Phi_{n}$, and, for each $i \in[1 \uparrow n-1], \sigma_{i}$ acts on $x_{[1 \uparrow n]}$ as follows,

$$
\begin{array}{cccc}
\frac{k \in[1 \uparrow i-1]}{\left(x_{k}\right.} & x_{i} & x_{i+1} & \frac{k \in[i+2 \uparrow n]}{\left.x_{k}\right)^{\sigma_{i}}} \\
=\left(\begin{array}{llll}
x_{k} & x_{i} \Pi x_{[i \uparrow i+1]} & \Pi \bar{x}_{[i+1 \downarrow i]} x_{i+1} & x_{k}
\end{array}\right) .
\end{array}
$$

Let $w=\left(\Pi x_{[1 \uparrow n-1]}^{2} x_{n}\right)^{-1}$; then $\sigma_{n}$ acts as follows.

$$
\left.\begin{array}{rl}
\frac{k \in[1 \uparrow n-1]}{\left(x_{k}\right.} & x_{n}
\end{array}\right)^{\sigma_{n}}
$$

(4). By reflecting the previous example, we can invert the elements of $\sigma_{[1 \uparrow n]}$.

For each $k \in[1 \uparrow n]$, set $x_{k}=\left(\tau_{n+1}^{\Pi \tau_{[n \downarrow 1]}} \tau_{k}\right)^{\Pi \tau_{[k \uparrow n+1]}}$ in $\Phi_{n}$. Then $x_{[1 \uparrow n]}$ is a free generating set for $\Phi_{n}$, and, for each $i \in[1 \uparrow n-1], \sigma_{i}$ acts on $x_{[1 \uparrow n]}$ as follows.

$$
\begin{array}{cccc}
\frac{k \in[1 \uparrow i-1]}{\left(x_{k}\right.} & x_{i} & x_{i+1} & \frac{k \in[i+2 \uparrow n]}{\left.x_{k}\right)^{\sigma_{i}}} \\
=\left(\begin{array}{llll}
x_{k} & x_{i} \Pi \bar{x}_{[i+1 \downarrow i]} & \Pi x_{[i \uparrow i+1]} x_{i+1} & \left.x_{k}\right) .
\end{array}\right.
\end{array}
$$

Let $w=\Pi x_{[1 \uparrow n-1]}^{2} x_{n}$; then $\sigma_{n}$ acts as follows.

$$
\left.\begin{array}{rl}
\frac{k \in[1 \uparrow n-1]}{\left(x_{k}\right.} & x_{n}
\end{array}\right)^{\sigma_{n}}
$$

9.8 Historical Remarks. Let us view $\mathcal{B}_{n}$ as a subgroup of $\mathcal{B}_{n+1}$ by suppressing $\sigma_{n}$. Then the $\mathcal{B}_{n+1}$-group $\Phi_{n}$ becomes a faithful $\mathcal{B}_{n}$-group, even if $n=1$.

Wada [29] defined various left actions of $\mathcal{B}_{n}$ on a free group of rank $n$. All but four of them are obviously non-faithful, and two of the remaining four actions are obviously equivalent up to changing the free generating set, leaving three actions to be studied for faithfulness. Shpilrain [28] ingeniously used the $\sigma_{1}$-trichotomy to prove that these three are all faithful. Crisp-Paris [11, Proposition A.1(2)] showed that the second and third of these three Wada actions are equivalent up
to changing the free generating set. They correspond to Examples 9.7(2), (4), above, with $\sigma_{n}$ suppressed. Notice that our actions on the right are the inverses of their actions on the left. In summary, the second and third Wada actions are obtained by choosing suitable free generating sets of the Perron-Vannier $\mathcal{B}_{n+1}$-group $\Phi_{n}$.

The first Wada action is constructed by choosing a non-zero integer $m$, and, for each $1 \in[1 \uparrow n-1]$, letting $\sigma_{i}$ act on $\left\langle x_{[1 \uparrow n]} \mid\right\rangle$ via

$$
\begin{array}{lccc}
\frac{k \in[1 \uparrow i-1]}{\left(x_{k}\right.} & x_{i} & x_{i+1} & \frac{k \in[i+2 \uparrow n]}{\left.x_{k}\right)^{\sigma_{i}}} \\
=\left(\begin{array}{llll}
x_{k} & x_{i+1} & x_{i}^{x_{i+1}^{m}} & x_{k}
\end{array}\right) .
\end{array}
$$

Edward Formanek has pointed out that $x_{[1, n]}^{m}$ is then a free generating set of a faithful $\mathcal{B}_{n}$-subgroup of $\left\langle x_{[1, n]} \mid\right\rangle$, where faithfulness can be seen from the fact that the $\mathcal{B}_{n}$-action is the standard Artin action with respect to this free generating set. This gives a transparent proof that the first Wada action is faithful.

## Appendix. Larue-Whitehead diagrams

In this appendix, we rework ideas from Larue's thesis [21, Chapter 2 and Appendix A], using combinatorial arguments to obtain a description of the $\mathcal{B}_{n}$-orbit of $t_{1}$ when $n \geq 1$. A topological treatment of similar ideas was given in 19, and it was arrived at independently of [21]. See [16, Chapters 5, 6].

## I Self-homeomorphisms

This section is purely motivational. We shall briefly indicate the mapping-class viewpoint of the braid group, and the Jordan-curve nature of the Whitehead graphs of the elements in the $\mathcal{B}_{n}$-orbit of $t_{1}$ if $n \geq 1$.

Let $\mathbb{C}$ denote the complex plane, and $\widehat{\mathbb{C}}$ the Riemann sphere, or projective complex line, $\mathbb{C} \cup\{\infty\}$. For each $z \in \mathbb{C}$ and each non-negative real number $r$, let $\mathbf{D}(z, r)$, resp. $\mathbf{D}^{\circ}(z, r)$, denote the closed, resp. open, disc in $\mathbb{C}$ with centre $z$ and radius $r$.

Let $S_{0,1, n}$ denote the surface formed by deleting from a sphere an open disc and $n$ points. We shall think of the discs and points as being distinguished rather than deleted; for example, it is then meaningful to speak of the self-homeomorphisms of $S_{0,1, n}$ as permuting the points. We take as our model of $S_{0,1, n}$ the sphere $\widehat{\mathbb{C}}$ having $[1 \uparrow n]$ as its set of $n$ distinguished points, and $\mathbf{D}^{\circ}\left(0, \frac{1}{2}\right)$ as its distinguished open disc. We are particularly interested in the set $[0 \uparrow n]$, and, in our diagrams, we shall mark these points out by drawing discs of small radii around them.


For each distinguished point $k \in[0 \uparrow n]$, we have a distinguished oriented tether, or arc, $\{k-r \mathbf{i} \mid-\infty \leq r \leq 0\}$, joining $\infty$ to $k$. We label the right flank of this oriented arc $t_{k}$, and label the left flank $\bar{t}_{k}$; we then cut $\widehat{\mathbb{C}}$ open along these arcs and obtain a $(2 n+2)$-gon, with clockwise boundary label $\prod_{k \in[0 \uparrow n]}\left(t_{k} \bar{t}_{k}\right)$; see Fig. I.1.4. We shall use $t_{0}$ and $z_{1}$ interchangeably in this section. Performing the boundary identifications then gives back $\widehat{\mathbb{C}}$.

The self-homeomorphism $\lambda$ of $\mathbf{D}(0,1)$ given by $\lambda\left(r e^{\mathrm{i} \theta}\right):=r e^{\mathbf{i}(\theta-2 \pi r)}$ fixes the boundary of $\mathbf{D}(0,1)$ and interchanges $\frac{1}{2}$ and $-\frac{1}{2}$; see Fig. I.1.1. For each


Figure I.1.1: The map $\lambda: \mathbf{D}(0,1) \rightarrow \mathbf{D}(0,1), r e^{\mathrm{i} \theta} \mapsto r e^{\mathrm{i}(\theta-2 \pi r)}$.
$i \in[1 \uparrow n-1]$, let $\phi_{i}$ denote the self-homeomorphism of $\widehat{\mathbb{C}}$ which, on $\widehat{\mathbb{C}}-\mathbf{D}\left(i+\frac{1}{2}, 1\right)$, acts as the identity map, and, on $\mathbf{D}\left(i+\frac{1}{2}, 1\right)$, acts by $z \mapsto \lambda\left(z-i-\frac{1}{2}\right)+i+\frac{1}{2}$. Then $\phi_{[1 \uparrow n-1]}$ generates a group $\left\langle\phi_{[1 \uparrow n-1]}\right\rangle$ of self-homeomorphisms of $\widehat{\mathbb{C}}$, which will shed light on the $\mathcal{B}_{n}$-orbit of $t_{1}$. To describe the induced action of $\left\langle\phi_{[1 \uparrow n-1]}\right\rangle$ on the fundamental group of $S_{0,1, n}$, we first give $\widehat{\mathbb{C}}$ a CW-structure by specifying a graph $S_{0,1, n}^{(1)}$ embedded in $\mathbb{C} \subset \widehat{\mathbb{C}}$.

For each $k \in[-1 \uparrow n]$, we have vertices $w_{k}:=k+\frac{1}{2}-\mathbf{i}$ and $v_{k}:=k+\frac{1}{2}+\mathbf{i}$, and an oriented straight edge $f_{k}$ joining $w_{k}$ to $v_{k}$. For each $k \in[0 \uparrow n]$, we have an oriented straight edge $e_{k}$ joining $w_{k-1}$ to $w_{k}$, and an oriented straight edge $d_{k}$ joining $v_{k-1}$ to $v_{k}$. This completes the description of the graph $S_{0,1, n}^{(1)}$. For $n=3, S_{0,1,3}^{(1)}$ can be seen in Fig. [I.1.2, Each distinguished point $k \in[0 \uparrow n]$ is the midpoint of the rectangle in $\mathbb{C}$ cut out by the path $f_{k-1} d_{k} \bar{~}_{k} \bar{e}_{k}$.

Let $\left\langle S_{0,1, n}^{(1)} \mid\right\rangle$ denote the (free) fundamental groupoid of $S_{0,1, n}^{(1)}$, and let $\left\langle S_{0,1, n}^{(1)} \mid \quad\right\rangle\left(w_{-1}, w_{-1}\right)$ denote the (free) fundamental group of $S_{0,1, n}^{(1)}$ at $w_{-1}$. The
subgraph of $S_{0,1, n}^{(1)}$ spanned by $e_{[0 \uparrow n]} \cup f_{[-1 \uparrow n]}$ is a maximal subtree of $S_{0,1, n}^{(1)}$, and $d_{[0 \uparrow n]}$ then determines a free generating set $t_{[0 \uparrow n]}$ of $\left\langle S_{0,1, n}^{(1)} \mid\right\rangle\left(w_{-1}, w_{-1}\right)$; explicitly, for each $k \in[0 \uparrow n], t_{k}=\Pi e_{[0 \uparrow k-1]} f_{k-1} d_{k} \bar{f}_{k} \Pi \bar{e}_{[k \downarrow 0]}$.

The path $f_{-1} \Pi d_{[0 \uparrow n]} \bar{f}_{n} \Pi \bar{e}_{[n \downarrow 0]}$ cuts out a rectangle in $\mathbb{C}$; the complementary region in $\widehat{\mathbb{C}}$ together with the graph $S_{0,1, n}^{(1)}$ is then a retract of $\widehat{\mathbb{C}}-[0 \uparrow n]$. Let $\sim$ denote homotopy for closed paths at $w_{-1}$ in $\widehat{\mathbb{C}}-[0 \uparrow n]$. We can identify the fundamental groupoid of $S_{0,1, n}$ with $\left\langle S_{0,1, n}^{(1)} \mid f_{-1} \Pi d_{[0 \uparrow n]} \bar{f}_{n} \Pi \bar{e}_{[n \downarrow 0]} \sim w_{-1}\right\rangle$. We then identify $\Sigma_{0,1, n}$ with the fundamental group of $S_{0,1, n}$ at $w_{-1}$,

$$
\begin{aligned}
\Sigma_{0,1, n} & =\left\langle S_{0,1, n}^{(1)} \mid f_{-1} \Pi d_{[0 \uparrow n]} \bar{f}_{n} \Pi \bar{\Pi}_{[n \downarrow 0]} \sim w_{-1}\right\rangle\left(w_{-1}, w_{-1}\right) \\
& =\left\langle t_{[0 \uparrow n]} \mid \Pi t_{[0 \uparrow n]}=1\right\rangle .
\end{aligned}
$$

Consider the action of $\phi_{1}$ on the graph $S_{0,1, n}^{(1)}$. For $n=3$, the result can be


Figure I.1.3: $S_{0,1,3}^{(1)}$ and its image under $\phi_{1}$.
seen in Fig. I.1.3. The crucial point is that $f_{1}^{\phi_{1}} \sim e_{2} f_{2} \bar{d}_{2} \bar{f}_{1} \bar{e}_{1} f_{0} d_{1}$, and all the other elements of $S_{0,1,3}^{(1)}$ are fixed by $\phi_{1}$; this makes the action quite simple algebraically. Then, $\bar{f}_{1}^{\phi_{1}} \sim \bar{d}_{1} \bar{f}_{0} e_{1} f_{1} d_{2} \bar{f}_{2} \bar{e}_{2}$, and, for the free generator $t_{1}=$ $e_{0} f_{0} d_{1} \bar{f}_{1} \bar{e}_{[1,0]}$, we have

$$
t_{1}^{\phi_{1}} \sim e_{0} f_{0} d_{1}\left(\bar{d}_{1} \bar{f}_{0} e_{1} f_{1} d_{2} \bar{f}_{2} \bar{e}_{2}\right) \bar{e}_{[1,0]} \sim e_{[0,1]} f_{1} d_{2} \bar{f}_{2} \bar{e}_{[2,0]}=t_{2}
$$

Similarly, for this element, $t_{2}$, we have

$$
t_{2}^{\phi_{1}} \sim e_{[0,1]}\left(e_{2} f_{2} \bar{d}_{2} \bar{f}_{1} \bar{e}_{1} f_{0} d_{1}\right) d_{2} \bar{f}_{2} \bar{e}_{[2,0]} \sim e_{[0,2]} f_{2} \bar{d}_{2} \bar{f}_{1} \bar{e}_{1} f_{0} d_{[1,2]} \bar{f}_{2} \bar{e}_{[2,0]} \sim \bar{t}_{2} t_{1} t_{2}
$$

where the latter homotopy can be seen directly by collapsing the elements of $e_{[0,2]} \cup f_{[0,2]}$, which lie in the maximal subtree. Thus, we see that $\phi_{1}$ acts on $\Sigma_{0,1, n}$ as the automorphism $\sigma_{1}$.

It follows that the action of any given element of $\mathcal{B}_{n}$ on $\Sigma_{0,1, n}$ is induced by some self-homeomorphism $\phi \in\left\langle\phi_{[1 \uparrow n-1]}\right\rangle$. The interesting feature now is that $\phi$ carries the oriented Jordan curve $f_{-1} d_{[0 \uparrow 1]} \bar{f}_{1} \bar{e}_{[1 \downarrow 0]}\left(\sim t_{0} t_{1}\right)$ to an oriented Jordan curve $f_{-1} d_{[0 \uparrow 1]} \bar{f}_{1}^{\phi} \bar{e}_{[1 \downarrow 0]}\left(\sim\left(t_{0} t_{1}\right)^{\phi} \sim t_{0} t_{1}^{\phi}\right)$. Recall that $\widehat{\mathbb{C}}$ is obtained by edge


Figure I.1.4: Jordan curves for $z_{1} t_{1}^{\bar{\phi}_{1}}$ and a Whitehead graph for $t_{1}^{\bar{\sigma}_{1}}=t_{1} t_{2} \bar{t}_{1}$.
identification from the $(2 n+2)$-gon with clockwise, boundary label $\prod_{i \in[0 \uparrow n]}\left(t_{i} \bar{t}_{i}\right)$.
The Jordan curve $f_{-1} d_{[0 \uparrow 1]} \bar{f}_{1}^{\phi} \bar{e}_{[1 \downarrow 0]}$ has as its preimage, in the $(2 n+2)$-gon, the union of a family of (disjoint) oriented arcs. These arcs can be used to reconstruct $t_{1}^{\phi}$, since the Jordan curve cyclically reads off $t_{0} t_{1}^{\phi}$ from its meetings with the labelled oriented tethers; notice that the set of tethers is now dual to the set of generators $t_{[0 \uparrow n]}$. The purpose of this appendix is to define and study a combinatorial representation of the family of arcs, and recover Larue's characterization of the elements of $t_{1}^{\mathcal{B}_{n}}$.

Although it will not be used in our arguments, let us mention the fact that, on collapsing the interior of each labelled edge of the $(2 n+2)$-gon to a labelled vertex, each oriented arc in the family becomes an oriented edge, and we recover the (directed, multi-edge, non-cyclic) Whitehead graph of $t_{1}^{\phi}$; see Fig. I.1.4.

## II Nested sets

We now introduce some formal definitions that will allow us to associate a combinatorial Jordan curve to each element of $t_{1}^{\mathcal{B}_{n}}$.
II. 1 Definitions. Let $(A, \leq)$ be a finite ordered set, and let $m \in \mathbb{N}$.

Let $N$ denote the number of elements of $A$. Then $A$ is order-isomorphic to $[1 \uparrow N]$ in a unique way, and we assign to $A$ the induced metric, denoted $d_{A}$. Thus
$d_{A}\left(a_{1}, a_{2}\right)=1$ if and only if $a_{1} \neq a_{2}$ and no element of $A$ lies strictly between $a_{1}$ and $a_{2}$.

Recall that the elements of $A^{m}$ are called m-tuples for $A$.
Let $a_{1}, a_{2}, b_{1}, b_{2}$ be elements of $A$. We say that $\left\{a_{1}, b_{1}\right\}$ is nested with $\left\{a_{2}, b_{2}\right\}$ (for $(A, \leq)$ ) if $a_{1}, a_{2}, b_{1}, b_{2}$ are distinct elements of $A$, and either both of, or neither of, $a_{2}$ and $b_{2}$ lie between $a_{1}$ and $b_{1}$ in $(A, \leq)$. It is not difficult to see that, in this event, $\left\{a_{2}, b_{2}\right\}$ is nested with $\left\{a_{1}, b_{1}\right\}$.

Let $a_{([1 \uparrow m])}$ and $b_{([1 \uparrow m])}$ be $m$-tuples of $A$.
We say that $a_{([1 \uparrow m])}$ is an $m$-tuple without repetitions if $a_{i} \neq a_{j}$ for all $i \neq j$ in $[1 \uparrow m]$.

We say that $\left(a_{[1 \uparrow m]}\right)$ is an ascending $m$-tuple (for $(A, \leq)$ ) if $a_{1} \leq a_{2} \leq \cdots \leq$ $a_{m}$ in $(A, \leq)$.

We say that $\left\{\left\{a_{i}, b_{i}\right\}\right\}_{i \in[1 \uparrow m]}$ is nested (for $\left.(A, \leq)\right)$ if, for all $i \neq j$ in $[1 \uparrow m]$, $\left\{a_{i}, b_{i}\right\}$ is nested with $\left\{a_{j}, b_{j}\right\}$ for $(A, \leq)$.

We let $\operatorname{Sym}_{m}$ act on $A^{m}$, on the left, by ${ }^{\pi}\left(a_{([1 \uparrow m])}\right):=a_{([1 \uparrow m])}$. For example, ${ }^{(1,2,3)}\left(a_{1}, a_{2}, a_{3}\right)=\left(a_{3}, a_{1}, a_{2}\right)$, and, hence, ${ }^{(1,2,3)}(a, b, c)=(c, a, b)$. The ascending rearrangement of $a_{([1 \uparrow m])}$ is the unique ascending $m$-tuple for $(A, \leq)$ that lies in the $\mathrm{Sym}_{m}$-orbit of $a_{([1 \uparrow m])}$.

Let $a_{([1 \uparrow 2 m])}$ be a $2 m$-tuple for $A$.
A permutation $\pi \in \operatorname{Sym}_{2 m}$ is said to embed $a_{([1 \uparrow 2 m])}$ in a plane if ${ }^{\pi} a_{([1 \uparrow 2 m)])}$ is ascending for $(A, \leq)$, and both $\left\{[2 i-1 \uparrow 2 i]^{\pi}\right\}_{i \in[1 \uparrow m]}$ and $\left\{[2 i \uparrow 2 i+1]^{\pi}\right\}_{i \in[1 \uparrow m-1]}$ are nested in $(\mathbb{N}, \leq)$.

We say that $a_{([1 \uparrow 2 m])}$ is a planar $2 m$-tuple (for $(A, \leq)$ ) if there exists some $\pi \in \operatorname{Sym}_{2 m}$ which embeds $a_{([1 \uparrow 2 m])}$ in a plane. (If no two consecutive terms of $a_{([1 \uparrow 2 m])}$ are equal, $\pi$ is then unique, but we shall not need this fact.) There is then an associated diagram formed as follows. We assign, to each point $i \in[1 \uparrow 2 m] \subset \mathbb{R} \subset \mathbb{C}$, the label $a_{i \pi} \pi$; notice that this means that the label of $i^{\pi}$ is $a_{i}$. For each $i \in[1 \uparrow m]$, we join $(2 i-1)^{\pi}$ (labelled $\left.a_{2 i-1}\right)$ to $(2 i)^{\pi}$ (labelled $a_{2 i}$ ) by an oriented semi-circle in the upper half-plane, and for each $i \in[1 \uparrow m-1]$, we join $(2 i)^{\pi}$ (labelled $a_{2 i}$ ) to $(2 i+1)^{\pi}$ (labelled $\left.a_{2 i+1}\right)$ by an oriented semi-circle in the lower half-plane. These oriented semi-circles form an oriented arc with no crossings which traces out the $2 m$-tuple $a_{([1 \uparrow 2 m])}$.
II. 2 Example. Suppose that $a_{([1 \uparrow 8])}=\left(\bar{z}_{1}, t_{1}, \bar{t}_{1}, t_{2}, \bar{t}_{2}, \bar{t}_{1}, t_{1}, z_{1}\right)$ is an 8 -tuple for some ordered set $(A, \leq)$, and that the ascending rearrangement of $a_{([1 \uparrow 8])}$ is $\left(\bar{z}_{1}, t_{1}, t_{1}, \bar{t}_{1}, \bar{t}_{1}, t_{2}, \bar{t}_{2}, z_{1}\right)$.

The permutation $\left(\begin{array}{cccccccc}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 5 & 6 & 7 & 4 & 3 & 8\end{array}\right)=(3,5,7)(4,6)$ embeds $a_{([1 \uparrow 8])}$ in a plane since both $\{\{1,2\},\{5,6\},\{7,4\},\{3,8\}\}$ and $\{\{2,5\},\{6,7\},\{4,3\}\}$ are nested in ( $\mathbb{N}, \leq$ ), and ${ }^{(3,5,7)(4,6)}\left(\bar{z}_{1}, t_{1}, \bar{t}_{1}, t_{2}, \bar{t}_{2}, \bar{t}_{1}, t_{1}, z_{1}\right)=\left(\bar{z}_{1}, t_{1}, t_{1}, \bar{t}_{1}, \bar{t}_{1}, t_{2}, \bar{t}_{2}, z_{1}\right)$. The associated diagram can be seen in Fig. II.2.1.


Figure II.2.1: $\left(\bar{z}_{1}, t_{1}, \bar{t}_{1}, t_{2}, \bar{t}_{2}, \bar{t}_{1}, t_{1}, z_{1}\right)$.
Let us record two results which will be useful later.
II. 3 Lemma. Let $(A, \leq)$ be an ordered set, and let $m$ be a positive integer. Let $c_{[1 \uparrow m]}$ and $\bar{c}_{[1 \uparrow m]}$ be m-tuples without repetitions for $(A, \leq)$ such that $\left\{\left\{c_{i}, \bar{c}_{i}\right\}\right\}_{i \in[1 \uparrow m]}$ is nested, and $\max \left(c_{[1 \uparrow m]}\right)<\min \left(\bar{c}_{[1 \uparrow m]}\right)$. If $c_{([1 \uparrow m])}$ is ascending, then $\bar{c}_{([m \downarrow 1])}$ is also ascending.

Proof. We argue by induction on $m$. If $m=1$, the conclusion is trivial. Now, assume that $m \geq 2$ and that the implication holds with $m-1$ in place of $m$. We see that $c_{1}<c_{2} \leq \max \left(c_{[1 \uparrow m]}\right)<\min \left(\bar{c}_{[1 \uparrow m]}\right) \leq \bar{c}_{1}$. Since $\left\{c_{1}, \bar{c}_{1}\right\}$ is nested with $\left\{c_{2}, \bar{c}_{2}\right\}$, we also see that $c_{1}<\bar{c}_{2}<\bar{c}_{1}$. By the induction hypothesis, $\bar{c}_{[[m \downarrow 2])}$ is ascending, and hence $\bar{c}_{([m \downarrow 1])}$ is ascending. Hence, the result is proved.
II. 4 Lemma. Let $(A, \leq)$ be an ordered set, let $m \in \mathbb{N}$, and let $a_{([1 \uparrow 2 m])}$ be a $2 m$-tuple for $A$.

Then $a_{([1 \uparrow 2 m])}$ is planar for $(A, \leq)$ if and only if there exists an ordered set $(B, \leq)$, and a $2 m$-tuple $b_{([1 \uparrow 2 m])}$ for $B$, without repetitions, and an ordered-set map $B \rightarrow A, b \mapsto \operatorname{label}(b)$, such that $b_{[1 \uparrow 2 m]}=B$, label $\left(b_{([1 \uparrow 2 m])}\right)=a_{([1 \uparrow 2 m])}$, and $\left\{b_{[2 i \uparrow 2 i+1]}\right\}_{i \in[1 \uparrow m-1]}$ and $\left\{b_{[2 i-1 \uparrow 2 i]}\right\}_{i \in[1 \uparrow m]}$ are nested for $(B, \leq)$.

Proof. Suppose first that $a_{([1 \uparrow 2 m])}$ is planar for $(A, \leq)$, and let $\pi$ be an element of $\mathrm{Sym}_{2 m}$ that embeds $a_{([1 \uparrow 2 m])}$ in a plane. We take $B$ to be $[1 \uparrow 2 m]$ with the usual ordering. For each $i \in[1 \uparrow 2 m]$, let label $(i)=a_{i} \pi$ and let $b_{i}=i^{\pi}$; thus, $\operatorname{label}\left(b_{i}\right)=\operatorname{label}\left(i^{\pi}\right)=a_{i}$. All the conditions are satisfied.

Conversely, if $B$ exists, we can identify $B$ with $[1 \uparrow 2 m]$ with the usual ordering, in a unique way. Then the map $i \mapsto b_{i}$ is an element $\pi$ of $\mathrm{Sym}_{2 m}$ that embeds $a_{([1 \uparrow 2 m])}$ in a plane.

## III Planar words in $\Sigma_{0,1, n}$

III. 1 Definitions. Let $A$ be the monoid generating set $\left\{z_{1}, \bar{z}_{1}\right\} \cup t_{[1 \uparrow n]} \cup \bar{t}_{[1 \uparrow n]}$ of $\Sigma_{0,1, n}$. We form the ordered set $(A, \leq)$ with

$$
\bar{z}_{1}<t_{1}<\bar{t}_{1}<\cdots<t_{n}<\bar{t}_{n}<z_{1} .
$$

We remark that, for $n \neq 1$, the ordering on $A$ is reminiscent of the ordering of the ends of $\Sigma_{0,1, n}$ in Section 7. We emphasize that, even if $n=1, z_{1} \neq \bar{t}_{1}$ in $A$.

Let $m \in \mathbb{N}$. Consider an $m$-tuple $a_{([1 \uparrow m])}$ for $t_{[1 \uparrow n]} \cup \bar{t}_{[1 \uparrow n]}$, and let $w=$ $\Pi a_{[1 \uparrow m]} \in \Sigma_{0,1, n}$; thus $a_{([1 \uparrow m])}$ is an expression for $w$. We define the Whitehead expansion of $a_{([1 \uparrow m])}$ to be the $(2 m+2)$-tuple

$$
\left(\bar{z}_{1}, a_{1}, \bar{a}_{1}, a_{2}, \bar{a}_{2}, \ldots, a_{m}, \bar{a}_{m}, z_{1}\right)
$$

for $A$, and we shall express it in the format $\left(\bar{z}_{1},\left(\left(a_{i}, \bar{a}_{i}\right)\right)_{i \in[1 \uparrow m]}, z_{1}\right)$. We say that $a_{([1 \uparrow m])}$ is a planar expression for $w$ if the Whitehead expansion of $a_{([1 \uparrow m])}$ is planar for $(A, \leq)$. If the unique reduced expression for $w$ is a planar expression for $w$, then we say that $w$ is a planar word in $\Sigma_{0,1, n}$.
III. 2 Examples. (i). The word $t_{1} \bar{t}_{2} \bar{t}_{1}$ is planar, since the Whitehead expansion of the reduced expression is ( $\bar{z}_{1}, t_{1}, \bar{t}_{1}, t_{2}, \bar{t}_{2}, \bar{t}_{1}, t_{1}, z_{1}$ ), and, by Example II.2. ( $\left.\bar{z}_{1}, t_{1}, \bar{t}_{1}, t_{2}, \bar{t}_{2}, \bar{t}_{1}, t_{1}, z_{1}\right)$ is planar for $(A, \leq)$; in a sense, Fig. II.2.1 reflects Fig. I.1.4. We call Fig. II.2.1 the Larue-Whitehead diagram of $t_{1} \bar{t}_{2} \bar{t}_{1}$.
(ii). The word $t_{1} \bar{t}_{2}$ is not planar; there is only one permutation to consider.
(iii). The word $t_{1}^{2}$ is not planar; there are four permutations to consider.
(iv). The word $t_{3}^{t_{1} \bar{t}_{2} \bar{t}_{1}}$ is planar, while the word $t_{3}^{t_{1} t_{2} \bar{t}_{1}}$ is not planar, and these two words have the same Whitehead graph.
III. 3 Proposition. Let $w \in \Sigma_{0,1, n}$. If there exists some planar expression for $w$, then (the reduced expression for) $w$ is planar.

Proof. Suppose that $a_{([1 \uparrow m])}$ is a planar expression for $w$, as in Definitions III.1.
By Lemma [II.4, there exists an ordered set $(B, \leq)$, and a planar $(2 m+$ 2)-tuple $b_{([1 \uparrow 2 m+2])}$ for ( $B, \leq$ ), without repetitions, and a labelling $B \rightarrow A, b \mapsto$ label $(b)$, such that the labelling respects the orderings and $\operatorname{label}\left(b_{([1 \uparrow 2 m+2])}\right)$ is the Whitehead expansion of $a_{([1 \uparrow m])}$. Moreover, $B=b_{[1 \uparrow 2 m+2]}$.

Suppose that the given planar expression $a_{([1 \uparrow m])}$ is not reduced. We shall find a shorter planar expression for $w$.

There exists some $j \in[1 \uparrow m-1]$ such that $a_{j+1}=\bar{a}_{j}$ in $t_{[1 \uparrow n]} \cup \bar{t}_{[1 \uparrow n]}$, and we may suppose that we have chosen this $j$ in such a way that $d_{B}\left(b_{2 j+1}, b_{2 j+2}\right)$ has the minimum possible value. Notice that label $\left(b_{([2 \uparrow \uparrow 2 j+3])}\right)=\left(a_{j}, \bar{a}_{j}, \bar{a}_{j}, a_{j}\right)$.

Clearly, $w=\Pi a_{[1 \uparrow j-1]} \Pi a_{[j+1 \uparrow m]}$, and label $\left(b_{([1 \uparrow 2 j-1])}, b_{([2 j+4 \uparrow 2 m+2])}\right)$ is

$$
\begin{gathered}
\left(\bar{z}_{1},\left(\left(a_{i}, \bar{a}_{i}\right)\right)_{i \in[1 \uparrow j-1]},\left(\left(a_{i}, \bar{a}_{i}\right)\right)_{i \in[j+2 \uparrow m]}, z_{1}\right) \\
\left(\bar{z}_{1}, a_{1}, \bar{a}_{1}, \ldots, a_{j-1}, \bar{a}_{j-1}, a_{j+2}, \bar{a}_{j+2}, \ldots, a_{m}, \bar{a}_{m}, z_{1}\right) .
\end{gathered}
$$

It suffices to show that $\left(b_{([1 \uparrow 2 j-1])}, b_{([2 j+4 \uparrow 2 m+2])}\right)$ is planar for $(B, \leq)$.
Claim. $d_{B}\left(b_{2 j}, b_{2 j+3}\right)=1$.

Proof. Consider any $k \in[1 \uparrow 2 m-1]$ such that $b_{k}$ lies between $b_{2 j}$ and $b_{2 j+3}$.
Let $\eta$ denote $(-1)^{k}$.
Since label $\left(b_{2 j}\right)=\operatorname{label}\left(b_{2 j+3}\right)=a_{j}$, we see that label $\left(b_{k}\right)=a_{j}$. Hence $\operatorname{label}\left(b_{k+\eta}\right)=\bar{a}_{j}=\operatorname{label}\left(b_{2 j+1}\right)=\operatorname{label}\left(b_{2 j+2}\right)$.

Either $a_{j}<\bar{a}_{j}$ or $a_{j}>\bar{a}_{j}$ in $(A, \leq)$. Hence,

$$
\begin{aligned}
& \text { either } \max \left\{b_{2 j}, b_{k}, b_{2 j+3}\right\}<\min \left\{b_{2 j+1}, b_{k+\eta}, b_{2 j+2}\right\} \text { in }(B, \leq) \text {, } \\
& \text { or } \min \left\{b_{2 j}, b_{k}, b_{2 j+3}\right\}>\max \left\{b_{2 j+1}, b_{k+\eta}, b_{2 j+2}\right\} \text { in }(B, \leq) \text {, }
\end{aligned}
$$

respectively.
Since $\left\{\left\{b_{2 j}, b_{2 j+1}\right\},\left\{b_{2 j+2}, b_{2 j+3}\right\},\left\{b_{k}, b_{k+\eta}\right\}\right\}$ is nested, and $b_{k}$ lies between $b_{2 j}$ and $b_{2 j+3}$, we see, from Lemma II.3, that $b_{k+\eta}$ lies between $b_{2 j+1}$ and $b_{2 j+2}$.

Since $\left\{b_{2 j+1}, b_{2 j+2}\right\}$ is nested with $\left\{b_{k+\eta}, b_{k+2 \eta}\right\}$ and $b_{k+\eta}$ lies between $b_{2 j+1}$ and $b_{2 j+2}$, we see that $b_{k+2 \eta}$ lies between $b_{2 j+1}$ and $b_{2 j+2}$. Hence,

$$
d_{B}\left(b_{k+2 \eta}, b_{k+\eta}\right) \leq d_{B}\left(b_{2 j+1}, b_{2 j+2}\right),
$$

with equality holding only if $\left\{b_{k+2 \eta}, b_{k+\eta}\right\}=\left\{b_{2 j+1}, b_{2 j+2}\right\}$. Also, label $\left(b_{k+2 \eta}\right)=$ $\bar{a}_{j}$, and, hence, $\operatorname{label}\left(b_{k+3 \eta}\right)=a_{j}$. Thus

$$
\operatorname{label}\left(b_{k}, b_{k+\eta}, b_{k+2 \eta}, b_{k+3 \eta}\right)=\left(a_{j}, \bar{a}_{j}, \bar{a}_{j}, a_{j}\right) .
$$

By the minimality of $d_{B}\left(b_{2 j+1}, b_{2 j+2}\right)$, we see that $k=2 j$ or $k=2 j+3$. This proves the claim.

Now consider the passage from $b_{([1 \uparrow 2 m+2])}$ to $b_{([1 \uparrow 2 j-1])}, b_{([2 j+4 \uparrow 2 m+2])}$.
On the odd-to-even steps, we pass from $\left\{b_{[2 i-1 \uparrow 2 i]}\right\}_{i \in[1 \uparrow m+1]}$ to

$$
\left\{b_{[2 i-1 \uparrow 2 i]}\right\}_{i \in[1 \uparrow j-1] \cup[j+3 \uparrow m+1]} \cup\left\{\left\{b_{2 j-1}, b_{2 j+4}\right\}\right\} .
$$

Thus, we remove $\left\{b_{2 j-1}, b_{2 j}\right\},\left\{b_{2 j+1}, b_{2 j+2}\right\},\left\{b_{2 j+3}, b_{2 j+4}\right\}$, and we add only $\left\{b_{2 j-1}, b_{2 j+4}\right\}$. To see that, for all $k \in[1 \uparrow j-1] \cup[j+3 \uparrow m+1],\left\{b_{2 k-1}, b_{2 k}\right\}$ is nested with $\left\{b_{2 j-1}, b_{2 j+4}\right\}$, we note the following:

$$
\begin{aligned}
& \left(b_{2 j-1} \text { lies between } b_{2 k-1} \text { and } b_{2 k}\right. \text { ) } \\
& \Leftrightarrow\left(b_{2 j} \text { lies between } b_{2 k-1} \text { and } b_{2 k}\right) \\
& \text { since }\left\{b_{2 j-1}, b_{2 j}\right\} \text { is nested with }\left\{b_{2 k-1}, b_{2 k}\right\} \\
& \Leftrightarrow\left(b_{2 j+3} \text { lies between } b_{2 k-1} \text { and } b_{2 k}\right) \\
& \text { since } d_{B}\left(b_{2 j}, b_{2 j+3}\right)=1 \\
& \Leftrightarrow\left(b_{2 j+4} \text { lies between } b_{2 k-1} \text { and } b_{2 k}\right) \\
& \text { since }\left\{b_{2 j+3}, b_{2 j+4}\right\} \text { is nested with }\left\{b_{2 k-1}, b_{2 k}\right\} \text {. }
\end{aligned}
$$

On the even-to-odd steps, we pass from $\left\{b_{[2 i \uparrow 2 i+1]}\right\}_{i \in[1 \uparrow m]}$ to

$$
\left\{b_{[2 i \uparrow 2 i+1]}\right\}_{i \in[1 \uparrow j-1] \cup[j+2 \uparrow m]} .
$$

Thus, we remove $\left\{b_{2 j}, b_{2 j+1}\right\}$ and $\left\{b_{2 j+2}, b_{2 j+3}\right\}$, and we add nothing. Hence this remains nested. This completes the proof.
III. 4 Proposition. Let $w$ be a planar word in $\Sigma_{0,1, n}$, and let $k \in[1 \uparrow n]$.
(i). $w$ is a squarefree word in $\Sigma_{0,1, n}$.
(ii). $w \notin\left(\Pi \bar{t}_{[n \downarrow k+1]} t_{k} \star\right)-\left\{t_{k}^{\Pi t_{[k+1 \uparrow n]}}\right\}$.
(iii). $w \notin\left(\Pi t_{[1 \uparrow k-1]} \bar{t}_{k} \star\right)$.

Proof. For some $m \in \mathbb{N}$, there exists a reduced expression $a_{([1 \uparrow m])}$ for $w$.
(i). Suppose that $w$ is not squarefree, say $t_{i}, t_{i}$ occurs in $a_{([1 \uparrow m])}$, then $t_{i}, \bar{t}_{i}, t_{i}, \bar{t}_{i}$ occurs in

$$
\left(\bar{z}_{1},\left(\left(a_{i}, \bar{a}_{i}\right)\right)_{i \in[1 \uparrow m]}, z_{1}\right)
$$

Let $m_{i}$ be the number of occurrences of $t_{i}^{ \pm 1}$ in $a_{([1 \uparrow m])}$.
Suppose $c_{\left(\left[1 \uparrow m_{i}\right]\right)}$ are labelled $t_{i}$ and $\bar{c}_{\left[\left(m_{i} \downarrow\right]\right]}$ are such that the even-to-odd pairing contains $\left\{\left\{c_{k}, \bar{c}_{k}\right\}\right\}_{k \in\left[1 \uparrow m_{i}\right]}$. The odd-to-even pairing contains $\left\{c_{k}, \bar{c}_{j}\right\}$ for some $k, j \in\left[1 \uparrow m_{i}\right]$. Let us choose $(k, j)$ so that $k+j$ is as large as possible. Then $c_{k}<c_{k+1}<\bar{c}_{j}$. Whatever $c_{k+1}$ is paired with in the odd-to-even pairing must lie in the interval $\left[c_{k}, \bar{c}_{j}\right]$ and cannot have label $t_{i}$ since the signs alternate, so $c_{j+1}$ is paired with $\bar{c}_{k}$ for some $k>j$. This contradicts the maximality of $k+j$. Hence $k=m_{i}$. Similarly, $j=m_{i}$. Thus $\left\{c_{m_{i}}, \bar{c}_{m_{i}}\right\}$ lies in both the even-to-odd pairings and the odd-to-even pairings. This gives a sinlge component, which is a contradiction.
(ii). Suppose that $w \in\left(\Pi \bar{t}_{[n \downarrow k+1]} t_{k} \star\right)$.

Thus $\left(\bar{z}_{1},\left(\left(a_{i}, \bar{a}_{i}\right)\right)_{i \in[1 \uparrow n-k+2]}\right)$ is

$$
\left(\bar{z}_{1}, \bar{t}_{n}, t_{n}, \bar{t}_{n-1}, t_{n-1}, \ldots, \bar{t}_{k+1}, t_{k+1}, t_{k}, \bar{t}_{k}, a_{n-k+2}, \bar{a}_{n-k+2}\right)
$$

Notice that $\left\{\bar{t}_{k}, a_{n-k+2}\right\}$ must be nested with $\left\{t_{k+1}, t_{k}\right\}$, and, hence $a_{n-k+2}$ must lie in $\left\{t_{k}, \bar{t}_{k}, t_{k+1}\right\}$. By (i), $a_{n-k+2} \neq t_{k}$. Since $a_{([1 \uparrow m])}$ is a reduced expression, $a_{n-k+2} \neq \bar{t}_{k}$. Hence $a_{n-k+2}=t_{k+1}$. Let us denote this term $t_{k+1}^{\prime}$ to distinguish it from the preceding occurrence of $t_{k+1}$. $\left\{\bar{t}_{k}, t_{k+1}^{\prime}\right\}$ is nested with $\left\{t_{k+1}, t_{k}\right\}$. Hence, Then $t_{k+1}^{\prime}<t_{k+1}$. By Lemma 【I.3, $\bar{t}_{k+1}^{\prime}>\bar{t}_{k+1}$.

Thus $\left(\bar{z}_{1},\left(\left(a_{i}, \bar{a}_{i}\right)\right)_{i \in[1 \uparrow n-k+3]}\right)$ is

$$
\left(\bar{z}_{1}, \bar{t}_{n}, t_{n}, \bar{t}_{n-1}, t_{n-1}, \ldots, \bar{t}_{k+1}, t_{k+1}, t_{k}, \bar{t}_{k}, t_{k+1}^{\prime}, \bar{t}_{k+1}^{\prime}, a_{n-k+3}, \bar{a}_{n-k+3}\right)
$$

Notice that $\left\{\bar{t}_{k+1}^{\prime}, a_{n-k+3}\right\}$ must be nested with $\left\{t_{k+2}, \bar{t}_{k+1}\right\}$, and, hence, $a_{n-k+3}$ must lie in $\left\{\bar{t}_{k+1}, t_{k+2}\right\}$. Since $a_{([1 \uparrow m])}$ is a reduced rexpression, $a_{n-k+3} \neq \bar{t}_{k+1}$. Hence $a_{n-k+3}=t_{k+2}$, and we denote this by $t_{k+2}^{\prime}$. Then $t_{k+2}^{\prime}<t_{k+2}$, and, by Lemma II.3,, $\bar{t}_{k+2}^{\prime}>\bar{t}_{k+2}$.

By repeating the argument in the last paragraph, we eventually find that $w=t_{k}^{\Pi t_{k+1 \uparrow n]}}$.
(iii). Suppose that $w \in\left(\Pi t_{[1 \uparrow k-1]} \bar{t}_{k} \star\right)$.

Then $\left(\bar{z}_{1},\left(\left(a_{i}, \bar{a}_{i}\right)\right)_{i \in[1 \uparrow 2 k]}\right)=\left(\bar{z}_{1}, t_{1}, \bar{t}_{1}, t_{2}, \bar{t}_{2}, \ldots, t_{k-1}, \bar{t}_{k-1}, \bar{t}_{k}, t_{k}\right)$, and by an argument similar to that in (ii), we find that this is impossible.

## IV $\mathcal{B}_{n}$ permutes the planar words in $\Sigma_{0,1, n}$

IV. 1 Proposition. Let $w \in \Sigma_{0,1, n}$ and let $i \in[1 \uparrow n-1]$. If $w$ is a planar word in $\Sigma_{0,1, n}$, then $w^{\sigma_{i}}$ is a planar word in $\Sigma_{0,1, n}$.

Proof. Suppose that $r_{([1 \uparrow m])}$ is any planar expression for $w$, as in Definitions III.1. In applying $\sigma_{i}$ to $\left(\bar{z}_{1},\left(\left(r_{i}, \bar{r}_{i}\right)\right)_{i \in[1 \uparrow m]}, z_{1}\right)$, we
replace each $t_{i}, \bar{t}_{i}$ with $t_{i+1}, \bar{t}_{i+1}$,
replace each $\bar{t}_{i}, t_{i}$ with $\bar{t}_{i+1}, t_{i+1}$,
replace each $t_{i+1}, \bar{t}_{i+1}$ with $\bar{t}_{i+1}, t_{i+1}, t_{i}, \bar{t}_{i}, t_{i+1}, \bar{t}_{i+1}$,
replace each $\bar{t}_{i+1}, t_{i+1}$ with $\bar{t}_{i+1}, t_{i+1}, \bar{t}_{i}, t_{i}, t_{i+1}, \bar{t}_{i+1}$.
We will not perform any cancellations in the resulting sequence.
Let $\pi \in \operatorname{Sym}_{2 m+2}$ be a permutation which embeds $\left(\bar{z}_{1},\left(\left(r_{i}, \bar{r}_{i}\right)\right)_{i \in[1 \uparrow m]}, z_{1}\right)$ in a plane. By Lemma II.4, there exists an ordered set $(B, \leq)$, and a $(2 m+2)$-tuple $p_{([1 \uparrow 2 m+2])}$ without repetitions, for $(B, \leq)$, such that $\pi$ embeds $p_{([1 \uparrow 2 m+2])}$ in a plane. Moreover, there exists a labelling $B \rightarrow A, b \mapsto \operatorname{label}(b)$, such that the labelling respects the orderings and

$$
\operatorname{label}\left(p_{([1 \uparrow 2 m+2])}\right)=\left(\bar{z}_{1},\left(\left(r_{i}, \bar{r}_{i}\right)\right)_{i \in[1 \uparrow m]}, z_{1}\right) .
$$

Moreover, $B=p_{[1 \uparrow 2 m+2]}$.
Let $m_{i}$ denote the number of elements of $B$ with label $t_{i}$, and let $m_{i+1}$ denote the number of elements of $B$ with label $t_{i+1}$. To begin, we have to add $4 m_{i+1}$ elements to $B$, and we have to specify the ordering on the expanded set.

Let $c_{\left[1 \uparrow m_{i}\right]}$ denote the set, in ascending order, of those elements of $B$ which have the label $t_{i}$. Let $\bar{c}_{\left.\left[m_{i} \downarrow\right]\right]}$ denote the set, in ascending order, of those elements of $B$ which have the label $\bar{t}_{i}$. Let $d_{\left[1 \uparrow m_{i+1}\right]}$ denote the set, in ascending order, of those elements of $B$ which have the label $t_{i+1}$. Let $\bar{d}_{\left[m_{i+1} \downarrow 1\right]}$ denote the set, in ascending order, of those elements of $B$ which have the label $\bar{t}_{i+1}$. Thus we have

$$
c_{1}<\ldots<c_{m_{i}}<\bar{c}_{m_{i}}<\ldots<\bar{c}_{1}<d_{1}<\ldots<d_{m_{i+1}}<\bar{d}_{m_{i+1}}<\ldots<\bar{d}_{1}
$$

and no other element of $B$ lies in the interval $\left[c_{1} \uparrow \bar{d}_{1}\right]$. We write

$$
\left[c_{1} \uparrow \bar{d}_{1}\right]=\left(c_{\left(\left[1 \uparrow m_{i}\right]\right)}, \bar{c}_{\left(\left[m_{i} \downarrow 1\right]\right)}, d_{\left(\left[1 \uparrow m_{i+1}\right]\right)}, \bar{d}_{\left(\left[m_{i+1} \downarrow 1\right]\right)}\right)
$$

to express this.


Figure IV.1.1: $\left(c_{([1 \uparrow 1])}, \bar{c}_{([1 \downarrow 1])}, d_{([1 \uparrow 2])}, \bar{d}_{([2 \downarrow 1])}\right)$.
With the preceding notation, we create an interval of $4 m_{i+1}$ new elements denoted

$$
\left[a_{1} \uparrow \bar{b}_{1}\right]=\left(a_{\left(\left[1 \uparrow m_{i+1}\right]\right)}, \bar{a}_{\left(\left[m_{i+1} \downarrow 1\right]\right)}, b_{\left(\left[1 \uparrow m_{i+1}\right]\right)}, \bar{b}_{\left(\left[m_{i+1} \downarrow 1\right]\right)}\right) .
$$

We expand $B$ by inserting this interval just before $c_{1}$, that is, just before the interval $\left[c_{1} \uparrow \bar{d}_{1}\right]$. We now have a new ordered set $B^{\prime}$ with $2 m+2+4 m_{i+1}$ elements.

We have to specify the new labelling $B^{\prime} \rightarrow A$. On $c_{\left[1 \uparrow m_{i}\right]}$, we change the labels from $t_{i}$ to $t_{i+1}$. On $\bar{c}_{\left[m_{i \downarrow 1}\right]}$, we change the labels from $\bar{t}_{i}$ to $\bar{t}_{i+1}$. On $d_{\left[1 \uparrow m_{i+1}\right]}$, we change the labels from $t_{i+1}$ to $\bar{t}_{i+1}$. On $\bar{d}_{\left[m_{i+1} \downarrow 1\right]}$, we keep the same labels, $\bar{t}_{i+1}$. On the rest of $B-\left[c_{1} \uparrow \bar{d}_{1}\right]$, we keep the same labels. We give all the elements of $a_{\left[1 \uparrow m_{i+1}\right]}$ the label $t_{i}$; we give all the elements of $\bar{a}_{\left[m_{i+1} \downarrow\right]}$ the label $\bar{t}_{i}$; we give all the elements of $b_{\left[1 \uparrow m_{i+1}\right]}$ and $\bar{b}_{\left[m_{i+1} \downarrow 1\right]}$ the label $t_{i+1}$. The labelling clearly respects the orderings of $B^{\prime}$ and $A$.

For the even-to-odd steps, it follows from Lemma II. 3 that

$$
\left\{p_{[2 k \uparrow 2 k+1]}\right\}_{k \in[1 \uparrow m]} \supseteq\left\{\left\{c_{i}, \bar{c}_{i}\right\}\right\}_{i \in[1 \uparrow r]} \cup\left\{\left\{d_{j}, \bar{d}_{j}\right\}\right\}_{j \in[1 \uparrow s]} .
$$

Let $q_{\left(\left[1 \uparrow 2 m+4 m_{i+1}\right]\right)}$ be the $2 m+4 s$-tuple obtained from $p_{([1 \uparrow 2 m+2])}$ as follows. For each $j \in\left[1 \uparrow m_{i+1}\right]$, there exists a unique $i \in[1 \uparrow m]$ such that $p_{[2 i-1 \uparrow 2 i]}=$ $\left\{d_{j}, \bar{d}_{j}\right\}$. If $p_{([2 i-1 \uparrow 2 i])}=\left(d_{j}, \bar{d}_{j}\right)$, then it is to be expanded to $\left(d_{j}, \bar{b}_{j}, a_{j}, \bar{a}_{j}, b_{j}, \bar{d}_{j}\right)$. If $p_{([2 i-1 \uparrow 2 i])}=\left(\bar{d}_{j}, d_{j}\right)$, then it is to be expanded to $\left(\bar{d}_{j}, b_{j}, \bar{a}_{j}, a_{j}, \bar{b}_{j}, d_{j}\right)$. This completes the definition of $q_{\left(\left[1 \uparrow 2 m+4 m_{i+1}\right]\right)}$.


Figure IV.1.2: $\left(a_{([1 \uparrow \downarrow])}, \bar{a}_{([2 \downarrow 1])}, b_{([1 \uparrow 2])}, \bar{b}_{([2 \downarrow 1])}, c_{([1 \uparrow 1])}, \bar{c}_{([1 \downarrow 1])}, d_{([1 \uparrow \uparrow])}, \bar{d}_{([2 \downarrow 1])}\right)$.
In passing from $\left\{p_{[2 k \uparrow 2 k+1]}\right\}_{k \in[1 \uparrow m-1]}$ to $\left\{q_{[2 k \uparrow 2 k+1]}\right\}_{k \in\left[1 \uparrow m+2 m_{i+1}-1\right]}$, we add $\left\{\left\{\bar{b}_{j}, a_{j}\right\}\right\}_{j \in[1 \uparrow s]} \cup\left\{\left\{\bar{a}_{j}, b_{j}\right\}\right\}_{j \in[1 \uparrow s]}$. In $B^{\prime}$, for each $j \in[1 \uparrow s]$,

$$
\begin{aligned}
{\left[\bar{a}_{j} \uparrow b_{j}\right] } & =\left(\bar{a}_{([j \downarrow 1])}, b_{([1 \uparrow j])}\right) \\
& \text { and the underlying set is } \cup\left\{\bar{a}_{k}, b_{k}\right\}_{k \in[1 \uparrow j]},
\end{aligned}
$$

$$
\left[a_{j} \uparrow \bar{b}_{j}\right]=\left(a_{[j \uparrow s]}, \bar{a}_{[s \downarrow \downarrow]}, b_{[1 \uparrow s]}, \bar{b}_{[s \downarrow j]}\right)
$$

and the underlying set is $\cup\left\{\bar{a}_{k}, b_{k}\right\}_{k \in[1 \uparrow s]} \cup \cup\left\{\bar{b}_{k}, a_{k}\right\}_{k \in[j \uparrow s]}$.
Both of these intervals are closed under the pairing-off of

$$
\left\{q_{[2 k \uparrow 2 k+1]}\right\}_{k \in\left[1 \uparrow m+2 m_{i+1}-1\right]} .
$$

Thus, $\left\{q_{[2 k \uparrow 2 k+1]}\right\}_{k \in\left[1 \uparrow m+2 m_{i+1}-1\right]}$ is also nested.
In passing from $\left\{p_{[2 k-1 \uparrow 2 k]}\right\}_{k \in[1 \uparrow m]}$ to $\left\{q_{[2 k-1 \uparrow 2 k]}\right\}_{k \in\left[1 \uparrow m+m_{i+1}\right]}$, we delete $\left\{\left\{d_{j}, \bar{d}_{j}\right\}\right\}_{j \in[1 \uparrow s]}$, and add $\left\{\left\{d_{j}, \bar{b}_{j}\right\}\right\}_{j \in[1 \uparrow s]} \cup\left\{\left\{a_{j}, \bar{a}_{j}\right\}\right\}_{j \in[1 \uparrow s]} \cup\left\{\left\{b_{j}, \bar{d}_{j}\right\}\right\}_{j \in[1 \uparrow s]}$. In $B^{\prime}$, for each $j \in[1 \uparrow s]$,

$$
\begin{aligned}
{\left[a_{j}, \bar{a}_{j}\right] } & =\left(\bar{a}_{([j \downarrow \downarrow])}, a_{([1 \uparrow j])}\right) \\
& \text { and the underlying set is } \underset{k \in[1 \uparrow j]}{\cup}\left\{a_{k}, \bar{a}_{k}\right\}, \\
{\left[\bar{b}_{j}, d_{j}\right]=} & \left(\bar{b}_{([j \downarrow 1])}, c_{([1 \uparrow r])}, \bar{c}_{([r \downarrow 1])}, d_{([1 \uparrow j]])}\right) \\
& \text { and the underlying set is } \underset{k \in[1 \uparrow j]}{\cup}\left\{d_{k}, \bar{b}_{k}\right\} \cup \underset{i \in[1 \uparrow r]}{\cup}\left\{c_{i}, \bar{c}_{i}\right\}, \\
{\left[b_{j}, \bar{d}_{j}\right]=} & \left(b_{([j, s])}, \bar{b}_{([\mid \downarrow \downarrow 1])}, c_{([1 \uparrow r])}, \bar{c}_{([r \downarrow 1])}, d_{([1 \uparrow s])}, \bar{d}_{([s \downarrow j])}\right) \\
& \text { and the underlying set is } \underset{k \in[1 \uparrow s]}{\cup}\left\{d_{k}, \bar{b}_{k}\right\} \cup \underset{k \in[j \uparrow s]}{\cup}\left\{b_{k}, \bar{d}_{k}\right\} \cup \underset{i \in[1 \uparrow r]}{\cup}\left\{c_{i}, \bar{c}_{i}\right\} .
\end{aligned}
$$

Each of these intervals is closed under the pairing-off of $\left\{q_{[2 k-1 \uparrow 2 k]}\right\}_{k \in\left[1 \uparrow m+2 m_{i+1}\right]}$. Thus, $\left\{q_{[2 k-1 \uparrow 2 k]}\right\}_{k \in\left[1 \uparrow m+2 m_{i+1}\right]}$ is nested.

A similar argument shows that $\bar{\sigma}_{i}$ carries planar words to planar words.
IV. 2 Theorem. The group $\mathcal{B}_{n}$ acts on the set of planar words in $\Sigma_{0,1, n}$, and, hence, if $n \geq 1$, then every element of $t_{1}^{\mathcal{B}_{n}}$ is a planar word.
IV. 3 Remark. By combining Theorem IV. 2 and Proposition III.4, we get another proof of Corollary 7.6.

## V The $\mathcal{B}_{n}$-orbits of the planar words in $\Sigma_{0,1, n}$

In this section we rework [21, Lemma 2.3.12] and in this case our argument seems to be longer than Larue's. The object is to show that the number of $\mathcal{B}_{n}$-orbits in the set of all planar words in $\Sigma_{0,1, n}$ is $n+1$, and that $\left\{\Pi_{[1 \uparrow k]}\right\}_{k \in[0 \uparrow n]}$ is a complete set of representatives.
V. 1 Lemma. Let $i, j$ be elements of $[1 \uparrow n]$ such that $j \leq i-1$, let $\phi=\Pi \sigma_{[j \uparrow i-1]}$, and let $w$ be a planar word in $\Sigma_{0,1, n}$.
(i) If $w \in\left(\Pi t_{[1 \uparrow i]} t_{j} \star\right)$, then $\left|w^{\phi}\right|<|w|$.
(ii) If $w \in\left(\Pi t_{[1 \uparrow i]} \bar{t}_{j} \star\right)$, then $\left|w^{\phi}\right|<|w|$.

Proof. It is straightforward to show that $\phi$ acts as

$$
\begin{array}{rlcc}
\frac{k \in[1, j-1]}{\left(t_{k}\right.} & t_{j} & \frac{k \in[j+1, i]}{t_{k}} & \frac{k \in[i+1, n]}{\left.t_{k}\right)^{\phi}} \\
=\left(t_{k}\right. & t_{i} & t_{k-1}^{t_{i}} & \left.t_{k}\right) .
\end{array}
$$

(i). Suppose that $w \in\left(\Pi t_{[1 \uparrow i]} t_{j} \star\right)$.


Figure V.1.1: $w \in\left(\Pi t_{[1 \uparrow i]} t_{j} \star\right), j \leq i-1$.
Since $t_{i} t_{j}$ is a subword of $w$, every letter occurring in $w$ that belongs to $t_{[j \uparrow i]} \cup \bar{t}_{[j \uparrow i]}$ belongs to a (reduced) subword of $w$ of the form $a v \bar{b}$, where $a, b \in$
$\left\{\bar{t}_{i}, t_{j}\right\}$ and $v \in\left\langle t_{[j \uparrow i j}\right\rangle$. Since, moreover, $w$ begins with $\Pi t_{[1 \uparrow i]}$, it can be shown that it is not possible to have $a=\bar{t}_{i}$ or $b=\bar{t}_{i}$. Thus $a=b=t_{j}$. Here, $\left|(a v \bar{b})^{\phi}\right|=|a v b|-2$.

We factor $w$ into syllables consisting of all such subwords together with the individual remaining letters, all of which lie in $t_{[1 \uparrow j-1]} \cup t_{[i+1 \uparrow n]}$, and all of which are mapped to single letters by $\phi$.

Since $t_{j}$ occurs in $w$, we see that $\left|w^{\phi}\right|<|w|$.
(ii). Suppose that $w \in\left(\Pi t_{[1 \uparrow i]} \bar{t}_{j} \star\right)$.


Figure V.1.2: $w \in\left(\Pi t_{[1 \uparrow i]} \bar{t}_{j} \star\right), j \leq i-1$.
Since $t_{i} \bar{t}_{j}$ is a subword of $w$, every letter occurring in $w$ that belongs to $t_{[j+1 \uparrow i]} \cup \bar{t}_{[j+1 \uparrow i]}$ belongs to a (reduced) subword of $w$ of the form $a v \bar{b}$, where $a, b \in\left\{t_{j}, \bar{t}_{i}\right\}$ and $v \in\left\langle t_{\left[j+1 \uparrow_{i}\right\rangle}\right\rangle$. Since, moreover, $w$ begins with $\Pi t_{\left[1 \uparrow_{i}\right]}$, it can be shown that it is not possible to have $a=\bar{t}_{i}$ or $b=\bar{t}_{i}$. Thus $a=b=t_{j}$. Here, $\left|(a v \bar{b})^{\phi}\right|=|a v b|-2$.

We factor $w$ into syllables consisting of all such subwords together with the individual remaining letters, all of which lie in $t_{[1 \uparrow j]} \cup t_{[i+1 \uparrow n]}$, and all of which are mapped to single letters by $\phi$.

Since $t_{i}$ occurs in $w$, it is then clear that $\left|w^{\phi}\right| \leq|w|-2$.
V. 2 Lemma. Let $i, j$ be elements of $[1 \uparrow n]$ such that $j \geq i+2$, let $\phi=\Pi \bar{\sigma}_{[j-1 \downarrow i+1]}$, and let $w$ be a planar word in $\Sigma_{0,1, n}$.
(i) If $w \in\left(\Pi t_{[1 \uparrow i]} t_{j} \star\right)$, then $\left|w^{\phi}\right| \leq|w|$, and, moreover, if $\left|w^{\phi}\right|=|w|$ then $w^{\phi} \in\left(\Pi t_{[1 \uparrow i+1]} \star\right)$.
(ii) If $w \in\left(\Pi t_{[1 \uparrow i]} \bar{t}_{j} \star\right)$, then $\left|w^{\phi}\right|<|w|$.

Proof. It is straightforward to show that $\phi$ acts as

$$
\begin{array}{cccc}
\frac{k \in[1, i]}{\left(t_{k}\right.} & \frac{k \in[i+1, j-1]}{t_{k}} & t_{j} & \frac{k \in[j+1, n]}{\left.t_{k}\right)^{\phi}} \\
=\left(t_{k}\right. & t_{k+1}^{\bar{t}_{i+1}} & t_{i+1} & \left.t_{k}\right) .
\end{array}
$$

(i). Suppose that $w \in\left(\Pi t_{[1 \uparrow i]} t_{j} \star\right)$.


Figure V.2.1: $w \in\left(\Pi t_{[1 \uparrow i]} t_{j} \star\right), j \geq i+2$.
Since $t_{i} t_{j}$ is a subword of $w$, every letter occurring in $w$ that belongs to $t_{[i+1 \uparrow j-1]} \cup \bar{t}_{[i+1 \uparrow j-1]}$ belongs to a (reduced) subword of $w$ of the form $a v \bar{b}$, where $a, b \in\left\{t_{i}, \bar{t}_{j}\right\}$ and $v \in\left\langle t_{[i+1 \uparrow j-1]}\right\rangle$. Since, moreover, $w$ begins with $\Pi t_{[1 \uparrow i]}$, it can be shown that it is not possible to have $a=t_{i}$ or $b=t_{i}$. Thus $a=b=\bar{t}_{j}$. Here, $\left|(a v \bar{b})^{\phi}\right|=|a v b|-2$.

We factor $w$ into syllables consisting of all such subwords together with the individual remaining letters, all of which lie in $t_{[1 \uparrow i]} \cup t_{[j \uparrow n]}$, and all of which are mapped to single letters by $\phi$.

It is then clear that $\left|w^{\phi}\right| \leq|w|$.
Moreover, if $\left|w^{\phi}\right|=|w|$, then $w \in\left\langle t_{[1 \uparrow i]} \cup t_{[j \uparrow n]}\right\rangle$, and $w^{\phi} \in\left(\Pi t_{[1 \uparrow i+1] \star}\right)$.
(ii). Suppose that $w \in\left(\Pi t_{[1 \uparrow i]} \bar{t}_{j} \star\right)$.


Figure V.2.2: $w \in\left(\Pi t_{[1 \uparrow i]} \bar{t}_{j} \star\right), j \geq i+2$.
Since $t_{i} \bar{t}_{j}$ is a subword of $w$, every letter occurring in $w$ that belongs to $t_{[i+1 \uparrow j]} \cup \bar{t}_{[i+1 \uparrow j]}$ belongs to a (reduced) subword of $w$ of the form $a v \bar{b}$, where $a, b \in\left\{t_{i}, \bar{t}_{j}\right\}$ and $v \in\left\langle t_{[i+1 \uparrow j]}\right\rangle$. Since, moreover, $w$ begins with $\Pi t_{[1 \uparrow i]}$, it can be shown that it is not possible to have $a=t_{i}$ or $b=t_{i}$. Thus $a=b=\bar{t}_{j}$. Here, $\left|(a v \bar{b})^{\phi}\right|=|a v b|-2$.

We factor $w$ into syllables consisting of all such subwords together with the individual remaining letters, all of which lie in $t_{[1 \uparrow i]} \cup t_{[j+1 \uparrow n]}$, and all of which are mapped to single letters by $\phi$.

Since $\bar{t}_{j}$ occurs in $w$, it is then clear that $\left|w^{\phi}\right| \leq|w|-2$.
V. 3 Theorem (Larue). The set $\left\{\Pi t_{[1 \uparrow k]}\right\}_{k \in[0 \uparrow n]}$ is a complete set of representatives of the $\mathcal{B}_{n}$-orbits in the set of all planar words in $\Sigma_{0,1, n}$.

Proof. Let $w$ be a planar word in $\Sigma_{0,1, n}$. We wish to show that there exists some $k \in[0 \uparrow n]$ such that $t_{[1 \uparrow k]} \in w^{\mathcal{B}_{n}}$.

Let $i$ be the largest integer such that $w \in\left(\Pi t_{[1 \uparrow i]} \star\right)$.

We may assume that, for all $v \in w^{\mathcal{B}_{n}},|v| \geq|w|$, and if $|v|=|w|$, then $v \notin\left(\Pi t_{[1 \uparrow i+1] \star}\right)$.

By Lemma V.11, for all $j \in[1 \uparrow i-1], w \notin\left(\Pi t_{[1 \uparrow i]} t_{j} \star\right) \cup\left(\Pi t_{[1 \uparrow i]} \bar{t}_{j} \star\right)$.
By Proposition 【II.4(i), w $\notin\left(\Pi t_{[1 \uparrow i]} t_{i} \star\right)$.
By the maximality of $i, w \notin\left(\Pi t_{[1 \uparrow i]} t_{i+1} \star\right)$.
By Proposition 【II.4(iii), $w \notin\left(\Pi t_{[1 \uparrow i]} \bar{t}_{i+1} \star\right)$.
By Lemma $\overline{V .2}$, for all $j \in[i+2 \uparrow n], w \notin\left(\Pi t_{[1 \uparrow i]} t_{j} \star\right) \cup\left(\Pi t_{[1 \uparrow i]} \bar{t}_{j} \star\right)$.
Hence, $w=\Pi t_{[1 \uparrow i]}$, as desired.
V. 4 Remarks. (i). Let $w$ be a planar word in $\Sigma_{0,1, n}$.

Lemmas V. 1 and $V .2$ give an effective procedure for finding $\phi \in \mathcal{B}_{n}$ first to minimize $\left|w^{\phi}\right|$, and then to obtain the form $w^{\phi}=\Pi t_{[1 \uparrow k]}$ for some $k \in[0 \uparrow n]$.
(ii). Let $n \geq 1$ and let $w$ be a word in $\Sigma_{0,1, n}$.

Theorem V.3 shows that $w$ lies in the $\mathcal{B}_{n}$-orbit of $t_{1}$ if and only if the cyclically-reduced form of $w$ lies in $t_{[1 \uparrow n]}$ and $w$ is planar. Moreover, in this event, Lemmas $V .1$ and $V .2$ effectively produce a $\phi \in \mathcal{B}_{n}$ such that $w^{\phi}=t_{1}$.
(iii). There is then an algorithm which, for any $k \in[1 \uparrow n]$, and any $k$-tuple $w_{([1 \uparrow k])}$ for $\Sigma_{0,1, n}$, decides if there exists some $\phi \in \mathcal{B}_{n}$ such that $w_{([1 \uparrow k])}^{\phi}=t_{([1 \uparrow k])}$, and effectively finds such a $\phi$. We proceed as follows. We first convert $w_{1}$ to $t_{1}$ if possible, and then we restrict to $\left\langle\sigma_{[2 \uparrow n-1]}\right\rangle$.

It is interesting to compare this algorithm for $\mathcal{B}_{n}$ with the Whitehead algorithm for the much larger group $\operatorname{Aut}\left(\Sigma_{0,1, n}\right)$. The information provided by planarity is more detailed then the information carried by the Whitehead graph used in the Whitehead algorithm.

We record the following.
V. 5 Theorem (Larue). Let $n \geq 1$ and let $w \in \Sigma_{0,1, n}$. Then $w$ lies in the $\mathcal{B}_{n}$-orbit of $t_{1}$ if and only if the cyclically-reduced form of $w$ lies in $t_{[1 \uparrow n]}$ and $w$ is planar.

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