# Actions of the braid group, and new algebraic proofs of results of Dehornoy and Larue.

Lluís Bacardit and Warren Dicks

November 5, 2018

#### Abstract

This article surveys many standard results about the braid group with emphasis on simplifying the usual algebraic proofs.

We use van der Waerden's trick to illuminate the Artin-Magnus proof of the classic presentation of the algebraic mapping-class group of a punctured disc.

We give a simple, new proof of the Dehornoy-Larue braid-group trichotomy, and, hence, recover the Dehornoy right-ordering of the braid group.

We then turn to the Birman-Hilden theorem concerning braid-group actions on free products of cyclic groups, and the consequences derived by Perron-Vannier, and the connections with the Wada representations. We recall the very simple Crisp-Paris proof of the Birman-Hilden theorem that uses the Larue-Shpilrain technique. Studying ends of free groups permits a deeper understanding of the braid group; this gives us a generalization of the Birman-Hilden theorem. Studying Jordan curves in the punctured disc permits a still deeper understanding of the braid group; this gave Larue, in his PhD thesis, correspondingly deeper results, and, in an appendix, we recall the essence of Larue's thesis, giving simpler combinatorial proofs.

2000 Mathematics Subject Classification. Primary: 20F36; Secondary: 20F34, 20E05, 20F60.

*Key words.* Braid group. Automorphisms of free groups. Presentation. Ordering. Ends of groups.

## **1** General Notation

Let  $\mathbb{N}$  denote the set of finite cardinals,  $\{0, 1, 2, \ldots\}$ .

Throughout, we fix an element n of  $\mathbb{N}$ .

Let  $i, j \in \mathbb{Z}$  and let v be a symbol. We define

$$\begin{split} &[i\uparrow j] := \{k \in \mathbb{Z} \mid i \le k \text{ and } k \le j\}, \\ &[i\downarrow j] := \{k \in \mathbb{Z} \mid i \ge k \text{ and } k \ge j\}, \\ &([i\uparrow j]) := \begin{cases} (i,i+1,\ldots,j-1,j) \in \mathbb{Z}^{j-i+1} & \text{if } i \le j, \\ () \in \mathbb{Z}^0 & \text{if } i > j, \end{cases} \end{split}$$

$$([i \downarrow j]) := \begin{cases} (i, i-1, \dots, j+1, j) \in \mathbb{Z}^{i-j+1} & \text{if } i \ge j, \\ () \in \mathbb{Z}^0 & \text{if } i < j. \end{cases}$$

Also,  $v_{[i\uparrow j]} := \{v_k \mid k \in [i\uparrow j]\}$ , and this will usually be a subset of some ambient set, G. If  $i \leq j$ ,  $v_{([i\uparrow j])} := (v_i, v_{i+1}, \dots, v_{j-1}, v_j) \in G^{j-i+1}$ , and, if G is a group,  $\Pi v_{[i\uparrow j]} := v_i v_{i+1} \cdots v_{j-1} v_j \in G$ . If i > j,  $v_{([i\uparrow j])} := ()$ , the 0-tuple, and  $\Pi v_{[i\uparrow j]} := 1$ , the empty product. We define  $v_{[i\downarrow j]}$ ,  $v_{([i\downarrow j])}$  and  $\Pi v_{[i\downarrow j]}$ , analogously. Thus, if  $i \geq j$ ,  $\Pi v_{[i\downarrow j]} := v_i v_{i-1} \cdots v_{j+1} v_j$ . Finally,  $[i\uparrow \infty[$   $:= \{k \in \mathbb{Z} \mid i \leq k\}$ .

For elements a, b of a group  $G, \overline{a} := a^{-1}, a^b := \overline{b}ab, a^{nb} := \overline{b}a^n b$ , and  $[a] := \{a^g \mid g \in G\}$ , the conjugacy class of a in G. The group of all automorphisms of G will be denoted by  $\operatorname{Aut}(G)$ .

An ordering of a set will mean a total order for the set. An ordered set is one endowed with a specific ordering. We will speak of n-tuples for a given set and n-tuples of elements of a given set.

## 2 Outline

Let  $\Sigma_{0,1,n} := \langle \{z_1\} \cup t_{[1\uparrow n]} \mid z_1 \Pi t_{[1\uparrow n]} = 1 \rangle$ . Then  $\Sigma_{0,1,n}$  is a one-relator group which is freely generated by the set  $t_{[1\uparrow n]}$ .

Let  $\operatorname{Out}_{0,1,n}^+$  denote the subgroup of  $\operatorname{Aut}(\Sigma_{0,1,n})$  consisting of all automorphisms of  $\Sigma_{0,1,n}$  which map the set  $\{z_1\} \cup \{[t_i]\}_{i \in [1\uparrow n]}$  to itself. Let  $\operatorname{Out}_{0,1,0}$  denote  $\operatorname{Aut}(\mathbb{Z})$ , and, for  $n \geq 1$ , let  $\operatorname{Out}_{0,1,n}$  denote the group of all automorphisms of  $\Sigma_{0,1,n}$  which map the set  $\{z_1, \overline{z}_1\} \cup \{[t_i], [\overline{t}_i]\}_{i \in [1\uparrow n]}$  to itself. Then  $\operatorname{Out}_{0,1,n}^+$  is a subgroup of index two in  $\operatorname{Out}_{0,1,n}$ . We call  $\operatorname{Out}_{0,1,n}$  the algebraic mapping-class group of the surface of genus 0 with 1 boundary component and n punctures; see [18] for background on algebraic mapping-class groups.

Frequently,  $\operatorname{Out}_{0,1,n}^+$  will be denoted  $\mathcal{B}_n$  and called the *n*-string braid group. (The similar symbol  $B_n$  denotes a certain Coxeter diagram.)

In Section 3, we define  $\sigma_{[1\uparrow n-1]} \subseteq \operatorname{Out}_{0,1,n}^+$ , we review Artin's 1925 proof that  $\sigma_{[1\uparrow n-1]}$  generates  $\operatorname{Out}_{0,1,n}^+$ , and we present intermediate results that we shall apply in subsequent sections. In Section 4, we recall the definition of Artin groups, specifically  $\operatorname{Artin}\langle A_n\rangle$ ,  $\operatorname{Artin}\langle B_n\rangle$  and  $\operatorname{Artin}\langle D_n\rangle$ . In Section 5, we verify Artin's 1925 result that  $\operatorname{Out}_{0,1,n}^+ \simeq \operatorname{Artin}\langle A_{n-1}\rangle$ , by combining Magnus' 1934 proof, Manfredini's observation that  $\operatorname{Out}_{0,1,(n-1)\perp 1}^+ \simeq \operatorname{Artin}\langle B_{n-1}\rangle$ , and the van der Waerden trick, to condense the calculations involved. In Section 6, we use results of Section 4 to recover the Dehornoy-Larue trichotomy for  $\mathcal{B}_n$  and the Dehornoy right-ordering of  $\mathcal{B}_n$ ; this represents a substantial simplification. Let us emphasize that we verify directly that  $\operatorname{Out}_{0,1,n}^+$  satisfies the trichotomy, in contrast with the approach by Larue [22] of using the trichotomy for  $\operatorname{Artin}\langle A_{n-1}\rangle$  to verify that  $\operatorname{Artin}\langle A_{n-1}\rangle$  acts faithfully on  $\Sigma_{0,1,n}$ .

In Section 7, we review the action of  $\mathcal{B}_n$  on the set of ends of  $\Sigma_{0,1,n}$ . The argument of Thurston given in [27] yields the Dehornoy right-ordering of  $\mathcal{B}_n$ , but not the trichotomy. By analysing further, we obtain new results about the  $\mathcal{B}_n$ -orbit of  $t_1$  in  $\Sigma_{0,1,n}$ .

In Section 8, for each  $m \geq 2$ , we introduce  $\operatorname{Out}_{0,1,n^{(m)}}$ , the algebraic mapping-class group of the disc with  $n \ C_m$ -points. We recall the Crisp-Paris proof of the Birman-Hilden result that the natural map from  $\operatorname{Out}_{0,1,n}$  to  $\operatorname{Out}_{0,1,n^{(m)}}$  is injective, and then modify an argument of Steve Humphries to show that there is a natural identification  $\operatorname{Out}_{0,1,n^{(m)}} = \operatorname{Out}_{0,1,n}$ . The new results obtained in Section 7 then provide additional information in this context.

In Section 9, we review some applications by Perron-Vannier [26] of the above Birman-Hilden result, and discuss connections with the actions given by Wada [29] and studied by Shpilrain [28] and Crisp-Paris [10], [11].

Following a kind suggestion of Patrick Dehornoy, we studied the analysis of the  $\mathcal{B}_n$ -orbit of  $t_1$  in  $\Sigma_{0,1,n}$  given by David Larue [21]. Larue's approach is combinatorial and uses polygonal curves in the punctured disc. By combining Larue's approach with Whitehead's use of graphs, we were able to simplify Larue's main arguments, and we record our combinatorial approach in an appendix. We also show how Larue's results imply the results we had obtained, more easily, by studying ends, in Section 7.

## **3** Artin's generators of $\mathcal{B}_n$

In this section we describe the famous generating set of  $\mathcal{B}_n$ . Let us fix more notation related to  $\Sigma_{0,1,n} = \langle \{z_1\} \cup t_{[1\uparrow n]} \mid z_1 \Pi t_{[1\uparrow n]} = 1 \rangle$  and  $\mathcal{B}_n \leq \operatorname{Aut}(\Sigma_{0,1,n})$ .

**3.1 Notation.** Let  $m \in \mathbb{N}$ . Consider an *m*-tuple  $a_{([1\uparrow m])}$  for  $t_{[1\uparrow n]} \cup \overline{t}_{[1\uparrow n]}$ , and an element w of  $\Sigma_{0,1,n}$ .

If  $\Pi a_{[1\uparrow m]} = w$  in  $\Sigma_{0,1,n}$ , we say that  $a_{([1\uparrow m])}$  is an *expression* for w. We say that the expression  $a_{([1\uparrow m])}$  is *reduced* if, for all  $j \in [1\uparrow n-1]$ ,  $a_{j+1} \neq \overline{a}_j$  in  $t_{[1\uparrow n]} \cup \overline{t}_{[1\uparrow n]}$ . For each element of  $\Sigma_{0,1,n}$ , there exists a unique reduced expression called the *normal form*.

Suppose that  $a_{([1\uparrow m])}$  is the normal form for w. We define the *length* of w to be |w| := m. The set of elements of  $\Sigma_{0,1,n}$  whose normal forms have  $a_{([1\uparrow m])}$  as an initial segment is denoted  $(w\star)$ ; and, the set of elements of  $\Sigma_{0,1,n}$  whose normal forms have  $a_{([1\uparrow m])}$  as a terminal segment is denoted  $(\star w)$ . The elements of  $(w\star)$  are said to *begin with* w, and the elements of  $(\star w)$  are said to *end with* w.

Let  $\operatorname{Sym}_n$  denote the group of permutations of  $[1\uparrow n]$  acting on the right (on  $[1\uparrow n]$ ).

Let  $\phi \in \mathcal{B}_n$ . There exists a unique permutation  $\pi \in \operatorname{Sym}_n$ , and a unique (n+2)-tuple  $(w_{([0\uparrow n+1])})$  for  $\Sigma_{0,1,n}$  such that  $w_0 = 1$  and  $w_{n+1} = 1$ , and, for each  $i \in [1\uparrow n], w_i \notin (t_{i\pi}\star) \cup (\overline{t}_{i\pi}\star)$  and  $t_i^{\phi} = t_{i\pi}^{w_i}$ . For each  $i \in [0\uparrow n]$ , let  $u_i = w_i \overline{w}_{i+1}$ . If  $j \in [i\uparrow n]$ , then  $\Pi u_{[i\uparrow j]} = w_i \overline{w}_{j+1}$ . In particular,  $\Pi u_{[i\uparrow n]} = w_i$ . We define  $\pi(\phi) := \pi, w_i(\phi) := w_i, i \in [0\uparrow n+1]$ , and  $u_i(\phi) := u_i, i \in [0\uparrow n]$ . We write  $\|\phi\| := \sum_{i \in [1\uparrow n]} |t_i^{\phi}| = n + 2 \sum_{i \in [1\uparrow n]} |w_i(\phi)|$ .

Let  $\sigma_{[1\uparrow n-1]} \subseteq \mathcal{B}_n$  be the subset determined by, for all  $i \in [1\uparrow n-1]$  and all  $k \in [1\uparrow n]$ ,

$$t_k^{\sigma_i} = \begin{cases} t_k & \text{if } k \in [1 \uparrow i - 1] \cup [i + 2 \uparrow n], \\ t_{i+1} & \text{if } k = i, \\ t_i^{t_{i+1}} & \text{if } k = i + 1. \end{cases}$$

In the literature,  $\sigma_i$  is sometimes represented in  $2 \times n$ -matrix notation, for example, in the format

$$\sigma_i = \begin{pmatrix} t_1 & \dots & t_{i-1} & t_i & t_{i+1} & t_{i+2} & \dots & t_n \\ t_1 & \dots & t_{i-1} & t_{i+1} & t_i^{t_{i+1}} & t_{i+2} & \dots & t_n \end{pmatrix}.$$

We shall often find it convenient to compress the dots and say that  $\sigma_i$  and  $\overline{\sigma}_i$  are *determined by* the expressions

We shall apply the following result in different situations.

**3.2 Lemma** (Artin [3]). Let  $\phi \in \mathcal{B}_n$ . Let  $\pi = \pi(\phi)$  and, for each  $i \in [0\uparrow n]$ , let  $u_i = u_i(\phi)$ .

- (i). Suppose that there exists some  $i \in [1\uparrow n-1]$  such that  $u_i \in (\star \overline{t}_{(i+1)\pi})$ . Then  $\|\sigma_i \phi\| \leq \|\phi\| 2$ ; moreover, for each  $j \in [1\uparrow i]$ ,  $t_j^{\sigma_i \phi}$  and  $t_j^{\phi}$  both begin with the same element of  $t_{[1\uparrow n]} \cup \overline{t}_{[1\uparrow n]}$ .
- (ii). Suppose that there exists some  $i \in [1\uparrow n-1]$  such that  $u_i \in (\overline{t}_{i^{\pi}}\star)$ . Then  $\|\overline{\sigma}_i\phi\| \leq \|\phi\| 2$ ; moreover, for each  $j \in [1\uparrow i-1]$ ,  $t_j^{\overline{\sigma}_i\phi}$  and  $t_j^{\phi}$  both begin with the same element of  $t_{[1\uparrow n]} \cup \overline{t}_{[1\uparrow n]}$ .
- (iii). Suppose that, for each  $i \in [1\uparrow n-1]$ ,  $u_i \notin (\overline{t}_{i^{\pi}}\star) \cup (\star \overline{t}_{(i+1)^{\pi}})$ . Then  $\phi = 1$ .

*Proof.* (i). There exists some  $v \in \Sigma_{0,1,n} - (\star t_{(i+1)^{\pi}})$  such that  $u_i = v \overline{t}_{(i+1)^{\pi}}$ . Since  $w_i(\phi) = u_i w_{i+1}(\phi)$ , we have

(3.2.1) 
$$w_i(\phi) = v \overline{t}_{(i+1)^{\pi}} w_{i+1}(\phi).$$

Since  $v \notin (\star t_{(i+1)^{\pi}})$  and  $w_{i+1}(\phi) \notin (t_{(i+1)^{\pi}}\star)$ , there is no cancellation in the expression  $t_{i^{\pi}}^{v\bar{t}_{(i+1)}\pi w_{i+1}(\phi)}$  for  $t_i^{\phi}$ ; hence

(3.2.2) 
$$t_i^{\phi} \in (\overline{w}_{i+1}(\phi)t_{(i+1)^{\pi}}\star) \text{ and } |t_i^{\phi}| = 1 + 2|v| + 2 + 2|w_{i+1}(\phi)|.$$

For all  $j \in [1\uparrow i-1] \cup [i+2\uparrow n], t_j^{\sigma_i\phi} = t_j^{\phi}$ ; hence,  $t_j^{\sigma_i\phi}$  has the same first letter as  $t_j^{\phi}$ , and,  $|t_j^{\sigma_i\phi}| = |t_j^{\phi}|$ .

Since  $t_i^{\sigma_i \phi} = t_{i+1}^{\phi} \in (\overline{w}_{i+1}(\phi)t_{(i+1)\pi}\star)$ , we see, from (3.2.2), that  $t_i^{\sigma_i \phi}$  has the same first letter as  $t_i^{\phi}$ . Also,  $|t_i^{\sigma_i \phi}| = |t_{i+1}^{\phi}|$ .

By (3.2.1),  $w_i(\phi)\overline{w}_{i+1}(\phi)t_{(i+1)^{\pi}} = v$ ; hence

$$t_{i+1}^{\sigma_i\phi} = (t_i^{t_{i+1}})^{\phi} = (t_{i^{\pi}}^{w_i(\phi)})^{(t_{(i+1)\pi}^{w_{i+1}(\phi)})} = t_{i^{\pi}}^{vw_{i+1}(\phi)}$$

Hence,  $|t_{i+1}^{\sigma_i\phi}| \le 1 + 2|v| + 2|w_{i+1}(\phi)| \stackrel{(3.2.2)}{=} |t_i^{\phi}| - 2.$ It now follows that  $||\sigma_i\phi|| \le ||\phi|| - 2$ , and (i) is proved.

(ii). There exists some  $v \in \Sigma_{0,1,n} - (t_{i^{\pi}} \star)$  such that  $u_i = \overline{t}_{i^{\pi}} v$ . Since  $w_{i+1}(\phi) =$  $\overline{u}_i w_i(\phi)$ , we have

$$(3.2.3) w_{i+1}(\phi) = \overline{v}t_{i^{\pi}}w_i(\phi).$$

Since  $\overline{v} \notin (\star \overline{t}_{i^{\pi}})$  and  $w_i(\phi) \notin (\overline{t}_{i^{\pi}} \star)$ , there is no cancellation in the expression  $t_{(i+1)^{\pi}}^{\overline{v}t_i\pi w_i(\phi)}$  for  $t_{i+1}^{\phi}$ ; hence

(3.2.4) 
$$|t_{i+1}^{\phi}| = 1 + 2|\overline{v}| + 2 + 2|w_i(\phi)|.$$

For all  $j \in [1\uparrow i-1] \cup [i+2\uparrow n], t_j^{\overline{\sigma}_i\phi} = t_j^{\phi}$ ; hence,  $t_j^{\overline{\sigma}_i\phi}$  has the same first letter as  $t_j^{\phi}$ , and,  $|t_j^{\overline{\sigma}_i\phi}| = |t_j^{\phi}|$ . Since  $t_{i+1}^{\overline{\sigma}_i\phi} = t_i^{\phi}$ , we see that  $|t_{i+1}^{\overline{\sigma}_i\phi}| = |t_i^{\phi}|$ . By (3.2.3),  $w_{i+1}(\phi)\overline{w}_i(\phi)\overline{t}_{i^{\pi}} = \overline{v}$ ; hence

$$t_{i}^{\overline{\sigma}_{i}\phi} = (t_{i+1}^{\overline{t}_{i}})^{\phi} = (t_{(i+1)^{\pi}}^{w_{i+1}(\phi)})^{(\overline{t}_{i^{\pi}}^{w_{i}(\phi)})} = t_{i^{\pi}}^{\overline{v}w_{i}(\phi)}.$$

Hence,  $|t_i^{\overline{\sigma}_i \phi}| \leq 1 + 2|\overline{v}| + 2|w_i(\phi)| \stackrel{(3.2.4)}{=} |t_{i+1}^{\phi}| - 2.$ It now follows that  $\|\overline{\sigma}_i \phi\| \leq \|\phi\| - 2$ , and (ii) is proved.

(iii). Since  $u_0 = \overline{w}_1(\phi) \not\in (\star \overline{t}_1^{\pi})$  and  $u_n = w_n(\phi) \not\in (\overline{t}_n^{\pi} \star)$ , we see that there is no cancellation anywhere in the expression  $u_0 \prod_{i \in [1\uparrow n]} (t_{i\pi}u_i)$ . Hence,

$$|u_0 \prod_{i \in [1\uparrow n]} (t_{i^{\pi}} u_i)| = \sum_{i \in [0\uparrow n]} |u_i| + n, \text{ that is, } \sum_{i \in [0\uparrow n]} |u_i| = |u_0 \prod_{i \in [1\uparrow n]} (t_{i^{\pi}} u_i)| - n.$$

Recall that  $u_0 \prod_{i \in [1\uparrow n]} (t_{i\pi} u_i) = \prod_{i \in [1\uparrow n]} (t_{i\pi}^{w_i(\phi)}) = (\prod_{i \in [1\uparrow n]} t_i)^{\phi} = \prod_{i \in [1\uparrow n]} t_i$ . Hence  $|u_0 \prod_{i \in [1\uparrow n]} (t_{i^{\pi}} u_i)| = n \text{ and } \sum_{i \in [0\uparrow n]} |u_i| = n - n = 0.$ 

Hence, all the elements of  $u_{[0\uparrow n]}$  are trivial.

For each  $i \in [0\uparrow n+1]$ ,  $w_i = \Pi u_{[i\uparrow n]}$ ; hence, all the elements of  $w_{[1\uparrow n]}$  are trivial. Also,  $\prod_{i \in [1\uparrow n]} t_{i^{\pi}} = u_0 \prod_{i \in [1\uparrow n]} (t_{i^{\pi}} u_i) = \prod_{i \in [1\uparrow n]} t_i$ . Hence  $\pi$  is trivial. Thus  $\phi = 1.$ 

The following is then immediate.

**3.3 Proposition** (Artin [3]). For each  $\phi \in \mathcal{B}_n$ , either  $\phi = 1$ , or there exists some  $\sigma_i^{\epsilon} \in \sigma_{[1\uparrow n-1]} \cup \overline{\sigma}_{[1\uparrow n-1]}$  such that  $\|\sigma_i^{\epsilon}\phi\| \leq \|\phi\| - 2$ . Hence,  $\langle \sigma_{[1\uparrow n-1]} \rangle = \mathcal{B}_n$ .

**3.4 Remarks.** If  $w \in \Sigma_{0,1,n}$  has odd length, then  $w^{\sigma_i}$  has odd length, and  $|w^{\sigma_i}| \leq 2|w| + 1$ , with equality being achieved only if every odd letter of w equals  $t_{i+1}$ . Similar statements hold with  $\overline{\sigma}_i$  in place of  $\sigma_i$ .

Let  $\phi \in \mathcal{B}_n$  and let  $|\phi|$  denote the minimum length of  $\phi$  as a word in  $\sigma_{[1\uparrow n-1]}$ . Thus,  $|t_i^{\phi}| \leq 2^{|\phi|+1} - 1$ . Hence,  $\|\phi\| \leq n 2^{|\phi|+1} - n$ . Proposition 3.3 gives an algorithm for writing  $\phi$  as a word in  $\sigma_{[1\uparrow n-1]}$  of length at most  $\frac{\|\phi\|-n}{2}$ , and we have now seen that  $\frac{\|\phi\|-n}{2} \le \frac{n2^{|\phi|+1}-2n}{2} = n2^{|\phi|} - n.$ 

#### 4 **Definition of Artin groups**

**4.1 Definition.** A Coxeter diagram X consists of a set V together with a function  $V \times V \to \mathbb{N} \cup \{\infty\}$ ,  $(x, y) \mapsto m_{x,y}$ , such that, for all  $x, y \in V$ ,  $m_{x,x} = 0$ and  $m_{x,y} = m_{y,x}$ . The elements of V are called the vertices of X, and, for  $(x,y) \in V \times V$ , we say that  $m_{x,y}$  is the number of edges joining x and y; we can depict X in a natural way. We then define the Artin group of X, denoted  $\operatorname{Artin}(X)$ , to be the group presented with generating set V and relations saying that, for all  $(x, y) \in V \times V$ ,

$$\begin{array}{rcl} xy &=& yx & \text{if} & m_{x,y} = 0, \\ xyx &=& yxy & \text{if} & m_{x,y} = 1, \\ xyxy &=& yxyx & \text{if} & m_{x,y} = 2, \\ \text{etc.} \end{array}$$

Notice that if  $m_{x,y} = \infty$ , then no relation is imposed. Notice also that if V is empty, then  $\operatorname{Artin}\langle X \rangle$  is the trivial group. 

**4.2 Notation.** (i). Let  $A_n$  denote the Coxeter diagram

 $a_1 - a_2 - \cdots - a_{n-1} - a_n$ 

Clearly,  $A_0$  is empty. We define  $A_{-1}$  to be empty also.

Thus, in  $A_n$ , the vertex set is  $a_{[1\uparrow n]}$ , and, for each  $(i, j) \in [1\uparrow n]^2$ , the number of edges joining  $a_i$  to  $a_j$  is  $\begin{cases} 1 & \text{if } |i-j| = 1, \\ 0 & \text{if } |i-j| \neq 1. \end{cases}$ 

Thus,  $\operatorname{Artin}\langle A_n \rangle$  has a presentation with generating set  $a_{[1\uparrow n]}$  and relations saying that, for each  $(i, j) \in [1\uparrow n]^2$ ,

 $a_i a_j = a_j a_i \quad \text{if } |i - j| \neq 1,$  $a_i a_j a_i = a_j a_i a_j \quad \text{if } |i - j| = 1.$ 

(ii). Let  $B_n$  denote the Coxeter diagram

$$b_1 - b_2 - \cdots - b_{n-1} = b_n.$$

Here, the vertex set is  $b_{[1\uparrow n]}$ , and, for each  $(i, j) \in [1\uparrow n]^2$ , the number of edges  $\begin{cases} 2 & \text{if } \{i, j\} = \{n-1, n\}, \end{cases}$ 

joining  $b_i$  to  $b_j$  is  $\begin{cases}
2 & \text{if } \{i, j\} = \{n - 1, n\}, \\
1 & \text{if } |i - j| = 1 \text{ and } \{i, j\} \neq \{n - 1, n\}, \\
0 & \text{if } |i - j| \neq 1.
\end{cases}$ 

(iii). For  $n \ge 2$ , let  $D_n$  denote the Coxeter diagram

Here, the vertex set is  $d_{[1\uparrow n]}$ , and, for each  $(i, j) \in [1\uparrow n]^2$ , the number of edges joining  $d_i$  to  $d_j$  is

$$\begin{cases} 1 & \text{if } \{i, j\} \in \{\{1, 2\}, \{2, 3\}, \dots, \{n - 2, n - 1\}, \{n - 2, n\}\}, \\ 0 & \text{otherwise.} \end{cases}$$

## 5 Artin's presentation of $\mathcal{B}_n$

In this section, we verify Artin's result that there exists an isomorphism  $\gamma_n: \operatorname{Artin}\langle A_{n-1} \rangle \to \mathcal{B}_n$  determined by  $\frac{i \in [1\uparrow n-1]}{(a_i)^{\gamma_n}}$ . We express this by writ-  $= (\sigma_i)$ ing  $\mathcal{B}_n = \operatorname{Artin}\langle \sigma_1 - \sigma_2 - \cdots - \sigma_{n-1} \rangle$ .

**5.1 Proposition.** There exists a homomorphism  $\gamma_n$ :  $\operatorname{Artin}\langle A_{n-1}\rangle \to \mathcal{B}_n$  determined by  $\frac{i\in[1\uparrow n-1]}{(a_i)^{\gamma_n}}$ , and  $\gamma_n$  is surjective. =  $(\sigma_i)$ 

*Proof.* (a). Suppose that  $1 \le i \le i + 2 \le j \le n - 1$ . We have the following.

$k{\in}[1{\uparrow}i{-}1]$			$k{\in}[i{+}2{\uparrow}j{-}1]$		(	$k \in [j+2\uparrow n]$
$(t_k$	$t_i$	$t_{i+1}$	$t_k$	$t_{j}$	$t_{j+1}$	$(t_k)^{\sigma_i \sigma_j}$
$=(t_k$	$t_{i+1}$	$t_i^{t_{i+1}}$	$t_k$	$t_{j}$	$t_{j+1}$	$(t_k)^{\sigma_j}$
$=(t_k$	$t_{i+1}$	$t_i^{t_{i+1}}$	$t_k$	$t_{j+1}$	$t_j^{t_{j+1}}$	$t_k)$
$=(t_k$	$t_i$	$t_{i+1}$	$t_k$	$t_{j+1}$	$t_j^{t_{j+1}}$	$(t_k)^{\sigma_i}$
$=(t_k$	$t_i$	$t_{i+1}$	$t_k$	$t_j$	$t_{j+1}$	$(t_k)^{\sigma_j \sigma_i}.$

(b). Suppose that  $1 \le i \le n-2$ . We have the following.

$k{\in}[1{\uparrow}i{-}1]$	<u> </u>			$\underline{k {\in} [i {+} 3 {\uparrow} n]}$
$(t_k$	$t_i$	$t_{i+1}$	$t_{i+2}$	$(t_k)^{\sigma_i \sigma_{i+1} \sigma_i}$
$=(t_k$	$t_{i+1}$	$t_i^{t_{i+1}}$	$t_{i+2}$	$(t_k)^{\sigma_{i+1}\sigma_i}$
$=(t_k$	$t_{i+2}$	$t_i^{t_{i+2}}$	$t_{i+1}^{t_{i+2}}$	$(t_k)^{\sigma_i}$
$=(t_k$	$t_{i+2}$	$t_{i+1}^{t_{i+2}}$	$t_i^{t_{i+1}t_{i+2}}$	$t_k)$
$=(t_k$	$t_{i+1}$	$t_{i+2}$	$t_i^{t_{i+1}t_{i+2}}$	$(t_k)^{\sigma_{i+1}}$
$=(t_k$	$t_i$	$t_{i+2}$	$t_{i+1}^{t_{i+2}}$	$(t_k)^{\sigma_i\sigma_{i+1}}$
$=(t_k$	$t_i$	$t_{i+1}$	$t_{i+2}$	$(t_k)^{\sigma_{i+1}\sigma_i\sigma_{i+1}}$

Together, (a) and (b) show that there exists a homomorphism  $\underline{i \in [1 \uparrow n-1]}$ 

 $\gamma_n$ : Artin $\langle A_{n-1} \rangle \to \mathcal{B}_n$  determined by  $\frac{i \in [1 \uparrow n-1]}{(a_i)^{\gamma_n}}$ . By Artin's Proposition 3.3,  $= (\sigma_i)$  $\langle \sigma_{[1 \uparrow n-1]} \rangle = \mathcal{B}_n$ , and, hence,  $\gamma_n$  is surjective.

In the remainder of this section, we shall use induction on n to show that the surjective homomorphism  $\gamma_n$ : Artin $\langle A_{n-1} \rangle \to \mathcal{B}_n$  of Proposition 5.1 is an isomorphism. Notice that  $\gamma_n$  endows Artin $\langle A_{n-1} \rangle$  with a canonical action on  $\Sigma_{0,1,n}$ . The following is precisely [24, Proposition 1] and, also, [10, Proposition 2.1(2)].

**5.2 Lemma** (Manfredini [24]). If  $n \ge 1$ , then

 $\operatorname{Artin}\langle A_{n-1}\rangle \ltimes \Sigma_{0,1,n} = \operatorname{Artin}\langle a_1 - a_2 - \cdots - a_{n-1} = \overline{t}_n \rangle \simeq \operatorname{Artin}\langle B_n \rangle.$ 

*Proof.* For n = 1, the result is clear.

For n = 2, we have the following.

$$\begin{aligned} \operatorname{Artin} \langle A_1 \rangle &\ltimes \Sigma_{0,1,2} = \langle \{a_1\} \cup t_{[1\uparrow 2]} \mid t_1^{a_1} = t_2, t_2^{a_1} = \overline{t}_2 t_1 t_2 \rangle \\ &= \langle a_1, t_2 \mid t_2^{a_1} = \overline{t}_2 t_2^{\overline{a}_1} t_2 \rangle = \langle a_1, t_2 \mid (\overline{a}_1 t_2) (a_1) = (\overline{t}_2 a_1) (t_2 \overline{a}_1 t_2) \rangle \\ &= \langle a_1, t_2 \mid (a_1) (\overline{t}_2 a_1 \overline{t}_2) = (\overline{t}_2 a_1) (\overline{t}_2 a_1) \rangle = \operatorname{Artin} \langle a_1 = \overline{t}_2 \rangle. \end{aligned}$$

From the case n = 2, we see that there exists a homomorphism  $\mu: \operatorname{Artin}\langle B_n \rangle \to \operatorname{Artin}\langle A_{n-1} \rangle \ltimes \Sigma_{0,1,n} \text{ determined by } \begin{array}{c} \underbrace{i \in [1 \uparrow n-1]}_{(b_i \ b_n)^{\mu}} \\ = (a_i \ \overline{t}_n) \end{array}$  For each  $k \in [1\uparrow n]$ , let  $\mathfrak{t}_k$  denote the element  $\overline{b}_n^{\Pi \overline{b}_{[n-1\downarrow k]}}$  of  $\operatorname{Artin}\langle B_n \rangle$ . For each  $i \in [1\uparrow n-1]$  and  $k \in [1\uparrow n]$ , let  $\mathfrak{t}_k^{\overline{\sigma}_i}$  denote  $\mathfrak{t}_k$ , resp.  $\mathfrak{t}_i$ , resp.  $\mathfrak{t}_{i+1}^{\overline{\mathfrak{t}}_i}$ , if  $k \in [1\uparrow i-1] \cup [i+2\uparrow n]$ , resp. k=i+1, resp. k=i. We shall see that  $\mathfrak{t}_{k}^{\overline{b}_{i}} = \mathfrak{t}_{k}^{\overline{\sigma}_{i}}$ ; this immediately implies that there exists a homomorphism

 $\frac{i\in[1\uparrow n-1]}{(a_i} \quad \frac{k\in[1\uparrow n]}{t_k)^{\overline{\mu}}}, \text{ which}$  $\overline{\mu}$ : Artin $\langle A_{n-1} \rangle \ltimes \Sigma_{0,1,n} \to \operatorname{Artin} \langle B_n \rangle$  determined by

is then clearly inverse to  $\mu$ , and the result will be proved.

For each  $m \in [n \downarrow 1]$ , we shall show, by decreasing induction on m, that, for each  $k \in [n \downarrow m]$  and each  $i \in [n - 1 \downarrow m]$ ,  $\mathfrak{t}_{k}^{\overline{b}_{i}} = \mathfrak{t}_{k}^{\overline{\sigma}_{i}}$ . For m = n, this is trivial, and, for m = n - 1, it follows from the case n = 2. Suppose that  $m \in [n - 2 \downarrow 1]$ .

(a). For each  $k \in [n \downarrow m + 1]$  and each  $i \in [n - 1 \downarrow m + 1]$ ,  $\mathbf{t}_{k}^{\overline{b}_{i}} = \mathbf{t}_{k}^{\overline{\sigma}_{i}}$ , by hypothesis. (b). For each  $k \in [n \downarrow m + 2]$ ,  $\mathbf{t}_{k} \in \langle b_{[n \downarrow m + 2]} \rangle$  and, hence,  $\mathbf{t}_{k}^{\overline{b}_{m}} = \mathbf{t}_{k} = \mathbf{t}_{k}^{\overline{\sigma}_{m}}$ .

(c). 
$$\mathfrak{t}_{m+1}^{b_m} = \overline{b}_n^{11b_{[n-1\downarrow m+1]}b_m} = \mathfrak{t}_m = \mathfrak{t}_{m+1}^{\overline{\sigma}_m}$$

(d). For each  $i \in [n-1\downarrow m+2]$ ,  $\mathbf{t}_m^{\overline{b}_i} \stackrel{\text{(c)}}{=} \mathbf{t}_{m+1}^{\overline{b}_m \overline{b}_i} = \mathbf{t}_{m+1}^{\overline{b}_i \overline{b}_m} \stackrel{\text{(a)}}{=} \mathbf{t}_{m+1}^{\overline{b}_m} \stackrel{\text{(c)}}{=} \mathbf{t}_m = \mathbf{t}_m^{\overline{\sigma}_i}$ .

$$(e). \ \mathbf{t}_{m}^{\overline{b}_{m+1}} \stackrel{(c)}{=} \mathbf{t}_{m+1}^{\overline{b}_{m}\overline{b}_{m+1}} \stackrel{(a)}{=} \mathbf{t}_{m+2}^{\overline{b}_{m+1}\overline{b}_{m}\overline{b}_{m+1}} = \mathbf{t}_{m+2}^{\overline{b}_{m}\overline{b}_{m+1}\overline{b}_{m}} \stackrel{(b)}{=} \mathbf{t}_{m+2}^{\overline{b}_{m+1}\overline{b}_{m}} \stackrel{(a)}{=} \mathbf{t}_{m+1}^{\overline{b}_{m}} \stackrel{(c)}{=} \mathbf{t}_{m} = \mathbf{t}_{m}^{\overline{\sigma}_{m+1}}.$$

$$(f). \ \mathbf{t}_{m}^{\overline{b}_{m}} = \mathbf{t}_{m}^{b_{m+1}b_{m}\overline{b}_{m+1}\overline{b}_{m}\overline{b}_{m+1}} \stackrel{(e)}{=} \mathbf{t}_{m}^{b_{m}\overline{b}_{m+1}\overline{b}_{m}\overline{b}_{m+1}} \stackrel{(c)}{=} \mathbf{t}_{m+1}^{\overline{b}_{m+1}\overline{b}_{m}\overline{b}_{m+1}}$$

$$\stackrel{(a)}{=} (\mathbf{t}_{m+1}\mathbf{t}_{m+2}\overline{\mathbf{t}}_{m+1})^{\overline{b}_{m}\overline{b}_{m+1}} \stackrel{(c),(\underline{b}),(c)}{=} (\mathbf{t}_{m}\mathbf{t}_{m+2}\overline{\mathbf{t}}_{m})^{\overline{b}_{m+1}} \stackrel{(e),(\underline{a}),(e)}{=} \mathbf{t}_{m}\mathbf{t}_{m+1}\overline{\mathbf{t}}_{m} = \mathbf{t}_{m}^{\overline{\sigma}_{m}}.$$

Now the result follows by induction.

We write  $\operatorname{Stab}(\operatorname{Artin}\langle A_n\rangle; [t_{n+1}])$  to denote the  $\operatorname{Artin}\langle A_n\rangle$ -stabilizer of the conjugacy class  $[t_{n+1}]$  under the Artin $\langle A_n \rangle$ -action on  $\Sigma_{0,1,n+1}$ . The Reidemeister-Schreier rewriting technique automatically gives a useful presentation of  $\operatorname{Stab}(\operatorname{Artin}(A_n); [t_{n+1}])$ , but applying the technique can be rather tedious. Once the presentation has been found, we can verify it directly using the van der Waerden trick, as in the following proof.

**5.3 Theorem** (Magnus [23]). If  $n \ge 1$ , then there exists a homomorphism

 $\phi_n \colon \operatorname{Artin}\langle A_{n-1} \rangle \ltimes \Sigma_{0,1,n} \to \operatorname{Artin}\langle A_n \rangle \text{ determined by } \begin{bmatrix} a_i & t_n \end{pmatrix}^{\phi_n} = (a_i & \overline{a}_n^2).$ 

Moreover, the following hold.

- (i).  $\phi_n$  is injective.
- (ii). For each  $i \in [1\uparrow n]$ ,  $t_i^{\phi_n} = \overline{a_i}^{2\Pi a_{[i+1\uparrow n]}}$  in  $\operatorname{Artin}\langle A_n \rangle$ .
- (iii). The image of  $\phi_n$  is Stab(Artin $\langle A_n \rangle$ ;  $[t_{n+1}]$ ).

*Proof.* Let us write  $G = \operatorname{Artin}\langle A_n \rangle$  and  $H = \operatorname{Artin}\langle A_{n-1} \rangle \ltimes \Sigma_{0,1,n}$ . In G,

$$(a_{n-1}a_n^2a_{n-1})^{a_n} = (\overline{a}_n a_{n-1}a_n)(a_n a_{n-1}a_n) = (a_{n-1}a_n \overline{a}_{n-1})(a_{n-1}a_n a_{n-1}) = a_{n-1}a_n^2a_{n-1},$$

and, hence,  $a_{n-1}a_n^2 a_{n-1}a_n^2 = a_n^2 a_{n-1}a_n^2 a_{n-1}$ . By Lemma 5.2,  $H \simeq \operatorname{Artin}\langle B_n \rangle$ , and we see that there exist a homomorphism  $\phi_n \colon H \to G$  determined by  $\underline{i \in [1\uparrow n-1]}$ 

 $\begin{array}{ccc} (a_i & \overline{t}_n)^{\phi_n} & \cdot \\ = (a_i & a_n^2) \end{array}$ 

Let  $v_{([1\uparrow n+2])} = ([1\uparrow n+1])$ , thought of as a generic (n+1)-tuple, and consider the free left *H*-set  $H \times v_{[1\uparrow n+1]}$ , with left *H*-transversal  $v_{[1\uparrow n+1]}$ .

We construct a right G-action on  $H \times v_{[1\uparrow n+1]}$  such that  $H \times v_{[1\uparrow n+1]}$  becomes an (H, G)-bi-set. For each  $i \in [1\uparrow n]$ , we define the right action of the generator  $a_i \in G$  on the left H-set  $H \times v_{[1\uparrow n+1]}$ , by specifying the action on the given left H-transversal as follows.

$k{\in}[1{\uparrow}i{-}1]$			$k{\in}[i{+}2{\uparrow}n{+}1]$
$(v_k)$	$v_i$	$v_{i+1}$	$v_k)a_i$
$=(a_{i-1}v_k)$	$v_{i+1}$	$\overline{t}_i v_i$	$a_i v_k$ ).

We now verify that the relations of G are respected.

(a). Suppose that  $1 \le i < i + 2 \le j \le n$ . We have the following.

	$k{\in}[1{\uparrow}i{-}1]$	$k{\in}[i{+}2{\uparrow}j{-}1]$				$k \in$	$\in [j+2\uparrow n+1]$
(	$v_k$	$v_i$	$v_{i+1}$	$v_k$	$v_{j}$	$v_{j+1}$	$v_k)a_ia_j$
=(	$a_{i-1}v_k$	$v_{i+1}$	$\overline{t}_i v_i$	$a_i v_k$	$a_i v_j$	$a_i v_{j+1}$	$a_i v_k) a_j$
$=(a_i)$	$-1a_{j-1}v_k$	$a_{j-1}v_{i+1}$	$\overline{t}_i a_{j-1} v_i$	$a_i a_{j-1} v_k$	$a_i v_{j+1}$	$a_i \overline{t}_j v_j$	$a_i a_j v_k)$
$=(a_j)$	$-1a_{i-1}v_k$	$a_{j-1}v_{i+1}$	$a_{j-1}\overline{t}_i v_i$	$a_{j-1}a_iv_k$	$a_i v_{j+1}$	$\overline{t}_j a_i v_j$	$a_j a_i v_k)$
= (	$a_{j-1}v_k$	$a_{j-1}v_i$	$a_{j-1}v_{i+1}$	$a_{j-1}v_k$	$v_{j+1}$	$\overline{t}_j v_j$	$a_j v_k) a_i$
= (	$v_k$	$v_i$	$v_{i+1}$	$v_k$	$v_j$	$v_{j+1}$	$v_k)a_ja_i.$

(b). Suppose that  $1 \le i \le n-1$ . We have the following.

	$k{\in}[1{\uparrow}i{-}1]$				$k \in [i+3\uparrow n+1]$
(	$v_k$	$v_i$	$v_{i+1}$	$v_{i+2}$	$v_k)a_ia_{i+1}a_i$
=(	$a_{i-1}v_k$	$v_{i+1}$	$\overline{t}_i v_i$	$a_i v_{i+2}$	$a_i v_k) a_{i+1} a_i$
= (	$a_{i-1}a_iv_k$	$v_{i+2}$	$\overline{t}_i a_i v_i$	$a_i \overline{t}_{i+1} v_{i+1}$	$a_i a_{i+1} v_k) a_i$
$=(a_i)$	$a_{i-1}a_ia_{i-1}v_k$	$a_i v_{i+2}$	$\overline{t}_i a_i v_{i+1}$	$a_i \overline{t}_{i+1} \overline{t}_i v_i$	$a_i a_{i+1} a_i v_k)$
= (	$a_i a_{i-1} a_i v_k$	$a_i v_{i+2}$	$a_i \overline{t}_{i+1} v_{i+1}$	$\overline{t}_{i+1}\overline{t}_i a_i v_i$	$a_{i+1}a_ia_{i+1}v_k)$
=(	$a_i a_{i-1} v_k$	$a_i v_{i+1}$	$a_i v_{i+2}$	$\overline{t}_{i+1}\overline{t}_iv_i$	$a_{i+1}a_iv_k)a_{i+1}$
= (	$a_i v_k$	$a_i v_i$	$v_{i+2}$	$\overline{t}_{i+1}v_{i+1}$	$a_{i+1}v_k)a_ia_{i+1}$
= (	$v_k$	$v_i$	$v_{i+1}$	$v_{i+2}$	$v_k)a_{i+1}a_ia_{i+1}.$

Now (a) and (b) prove that the relations of G are respected. Hence, we have a right G-action on  $H \times v_{[1\uparrow n+1]}$ .

Notice that  $v_{n+1}\overline{t}_n^{\phi_n} = v_{n+1}a_n^2 = \overline{t}_n v_n a_n = \overline{t}_n v_{n+1}$ . Also, for each  $i \in [1\uparrow n-1]$ ,  $v_{n+1}a_i^{\overline{\phi}_n} = v_{n+1}a_i = a_i v_{n+1}$ . It follows that, for each  $h \in H$ ,  $v_{n+1}h^{\phi_n} = hv_{n+1}$ . Hence,  $\phi_n$  is injective. This proves (i).

Recall that  $G = \operatorname{Artin}\langle A_n \rangle$ . Let  $i \in [1 \uparrow n]$ .

We shall show by decreasing induction on i that

(5.3.1) 
$$a_n^{\Pi \overline{a}_{[n-1\downarrow i]}} = a_i^{\Pi a_{[i+1\uparrow n]}}$$

If i = n, then (5.3.1) holds. Now suppose that  $i \ge 2$ , and that (5.3.1) holds. Conjugating (5.3.1) by  $\overline{a}_{i-1}$  yields

$$a_n^{\Pi \overline{a}_{[n-1\downarrow i-1]}} = a_i^{\Pi a_{[i+1\uparrow n]}\overline{a}_{i-1}} = a_i^{\overline{a}_{i-1}\Pi a_{[i+1\uparrow n]}} = a_{i-1}^{a_i\Pi a_{[i+1\uparrow n]}} = a_{i-1}^{\Pi a_{[i\uparrow n]}}.$$

By induction, (5.3.1) holds.

Now  $\overline{t}_i^{\phi_n} = (\overline{t}_n^{\Pi \overline{a}_{[n-1\downarrow i]}})^{\phi_n} = a_n^{2\Pi \overline{a}_{[n-1\downarrow i]}} \stackrel{(5.3.1)}{=} a_i^{2\Pi a_{[i+1\uparrow n]}}$ . This proves (ii). Also,  $\overline{t}_i^{\phi_n} \Pi \overline{a}_{[n\downarrow i]} = \Pi \overline{a}_{[n\downarrow i+1]} a_i$ .

If  $k \in [1\uparrow i-1]$ , then

$$a_i^{\Pi a_{[k\uparrow n]}} = a_i^{\Pi a_{[k\uparrow i-2]}\Pi a_{[i-1\uparrow i]}\Pi a_{[i+1\uparrow n]}} = a_i^{\Pi a_{[i-1\uparrow i]}\Pi a_{[i+1\uparrow n]}} = a_{i-1}^{\Pi a_{[i+1\uparrow n]}} = a_{i-1}.$$

Hence,  $a_{i-1}\Pi \overline{a}_{[n \downarrow k]} = \Pi \overline{a}_{[n \downarrow k]} a_i$ . Let  $\psi_n$  denote the map of sets

 $\psi_n \colon H \times v_{[1\uparrow n+1]} \to G, \quad hv_k \mapsto h^{\phi_n} \prod \overline{a}_{[n\downarrow k]} \text{ for all } hv_k = (h, v_k) \in H \times v_{[1\uparrow n+1]}.$ 

Hence, for each  $h \in H$ , we have the following, in G.

$\underline{k{\in}[1{\uparrow}i{-}1]}$		$k{\in}[i{+}2{\uparrow}n{+}1]$			
$(h  (v_k)$	$v_i$	$v_{i+1}$	$v_k$	$))^{\psi_n}a_i$	
$= (h^{\phi_n} ( \Pi \overline{a}_{[n \downarrow k]})$	$\Pi \overline{a}_{[n \downarrow i]}$	$\Pi \overline{a}_{[n \downarrow i+1]}$	$\Pi \overline{a}_{[n]}$	$_{\downarrow k]}))a_i$	
$= (h^{\phi_n}(a_{i-1}\Pi\overline{a}_{[n\downarrow k]})$	$\Pi \overline{a}_{[n \downarrow i+1]}$	$\overline{t}_i^{\phi_n}\Pi\overline{a}_{[n\downarrow i]}$	$a_i \Pi \overline{a}_{[n]}$	$_{\downarrow k]}))$	
$= (h  (a_{i-1}v_k))$	$v_{i+1}$	$\overline{t}_i v_i$	$a_i v_k$	$))^{\psi_{n}}$	
$=(h ( v_k)$	$v_i$	$v_{i+1}$	$v_k$	$(a_i)^{\psi_n}.$	

This proves that  $\psi_n$  is a map of right *G*-sets, and, hence,  $\psi_n$  must be surjective. Thus,  $G = \bigcup_{k \in [1\uparrow n+1]} H^{\phi_n} v_k^{\psi_n}$ , and, hence, the index of  $H^{\phi_n}$  in *G* is at most n+1.

Consider the action of G on the set of conjugacy classes  $\{[t_k]\}_{k \in [1\uparrow n+1]}$  in  $\Sigma_{0,1,n+1}$ . For any  $i \in [1\uparrow n]$ ,  $a_i$  acts as the transposition  $([t_i], [t_{i+1}])$ . In particular, the index of  $\operatorname{Stab}(G; [t_{n+1}])$  in G is n+1. Also, the elements of  $a_{[1\uparrow n-1]} \cup \{a_n^2\}$  fix  $[t_{n+1}]$ , and, hence,  $H^{\phi_n} \leq \operatorname{Stab}(G; [t_{n+1}])$ . By comparing indices, we see that  $H^{\phi_n} = \operatorname{Stab}(G; [t_{n+1}])$ . This proves (iii).

**5.4 Theorem** (Artin).  $\mathcal{B}_n = \operatorname{Artin} \langle \sigma_1 - \sigma_2 - \cdots - \sigma_{n-1} \rangle$ .

*Proof.* This is trivial for  $n \leq 1$ . Hence, we may assume that  $n \geq 1$  and that the homomorphism  $\gamma_n$ :  $\operatorname{Artin}(A_{n-1}) \to \mathcal{B}_n$ , of Proposition 5.1, determined  $i \in [1 \uparrow n-1]$ 

by  $(a_i)^{\gamma_n}$  is an isomorphism; and it remains to show that the surjective  $=(\sigma_i)$ 

homomorphism  $\gamma_{n+1}$ : Artin $\langle A_n \rangle \to \mathcal{B}_{n+1}$  is injective.

Consider an element w of the kernel of  $\gamma_{n+1}$ . In particular, w fixes  $t_{n+1}$  in the  $\operatorname{Artin}\langle A_n \rangle$ -action on  $\Sigma_{0,1,n+1}$ . By Theorem 5.3(iii), w lies in the image of the homomorphism  $\phi_n$ :  $\operatorname{Artin}\langle A_{n-1} \rangle \ltimes \Sigma_{0,1,n} \to \operatorname{Artin}\langle A_n \rangle$  determined by  $\underline{i\in[1\uparrow n-1]}$ 

 $(a_i \ t_n \,)^{\phi_n}.$  Thus, we may express w as a product of two words  $=(a_i \ \overline{a}_n^{\ 2})$ 

$$w = w_1(a_{([1\uparrow n-1])})w_2(t_{([1\uparrow n])}^{\phi_n}).$$

Now,

(5.4.1)

in Artin $\langle A_n \rangle \ltimes \Sigma_{0,1,n+1}, t_{n+1} = t_{n+1}^w = t_{n+1}^{w_1(a_{([1\uparrow n-1])})w_2(t_{([1\uparrow n])}^{\phi_n})} = t_{n+1}^{w_2(t_{([1\uparrow n])}^{\phi_n})}$ 

Consider the homomorphism  $\phi_{n+1}$ : Artin $\langle A_n \rangle \ltimes \Sigma_{0,1,n+1} \to \operatorname{Artin}\langle A_{n+1} \rangle$  de- $\underline{i \in [1 \uparrow n]}$ 

termined by  $(a_i \quad t_{n+1})^{\phi_{n+1}}$ . Let  $i \in [1\uparrow n]$ . By Theorem 5.3(ii),  $= (a_i \quad \overline{a}_{n+1}^2)^{(i+1)}$   $(t_i^{\phi_n})^{\phi_{n+1}a_{n+1}} = (\overline{a}_i^{2\Pi a_{[i+1\uparrow n]}})^{\phi_{n+1}a_{n+1}} = (\overline{a}_i^{2\Pi a_{[i+1\uparrow n]}})^{a_{n+1}}$   $= (\overline{a}_i^{2\Pi a_{[i+1\uparrow n+1]}}) = (t_i)^{\phi_{n+1}},$  $(t_{n+1})^{\phi_{n+1}a_{n+1}} = (\overline{a}_{n+1}^2)^{a_{n+1}} = \overline{a}_{n+1}^2 = (t_{n+1})^{\phi_{n+1}}.$ 

Thus the two (n+1)-tuples  $(t_{([1\uparrow n])}^{\phi_n}, t_{n+1})$  and  $t_{([1\uparrow n+1])}$  for  $\operatorname{Artin}\langle A_n \rangle \ltimes \Sigma_{0,1,n+1}$  become conjugate in  $\operatorname{Artin}\langle A_{n+1} \rangle$  under  $\phi_{n+1}$ . By Theorem 5.3(i),  $\phi_{n+1}$  is injective. Since  $t_{([1\uparrow n+1])}$  freely generates a free subgroup of  $\operatorname{Artin}\langle A_n \rangle \ltimes \Sigma_{0,1,n+1}$ , we see that  $(t_{([1\uparrow n])}^{\phi_n}, t_{n+1})$  also freely generates a free subgroup of  $\operatorname{Artin}\langle A_n \rangle \ltimes \Sigma_{0,1,n+1}$ . From (5.4.1), we see that  $w_2$  must be the trivial word.

Hence,  $w = w_1(a_{([1\uparrow n-1])})$  in  $\operatorname{Artin}\langle A_n \rangle$ . By the induction hypothesis,  $w_1(a_{([1\uparrow n-1])}) = 1$  in  $\operatorname{Artin}\langle A_{n-1} \rangle$ . Hence w = 1 in  $\operatorname{Artin}\langle A_n \rangle$ .

Now the result holds by induction.

Combining Lemma 5.2, Theorem 5.3 and Theorem 5.4, we have the following. 5.5 Corollary (Artin-Magnus-Manfredini). If  $n \ge 1$ , then

$$\mathcal{B}_{n} = \operatorname{Artin}\langle \sigma_{1} - \sigma_{2} - \cdots - \sigma_{n-2} - \sigma_{n-1} \rangle \simeq \operatorname{Artin}\langle A_{n-1} \rangle,$$
  

$$\operatorname{Stab}(\mathcal{B}_{n}; [t_{n}]) = \operatorname{Artin}\langle \sigma_{1} - \sigma_{2} - \cdots - \sigma_{n-2} = \sigma_{n-1}^{2} \rangle \simeq \operatorname{Artin}\langle B_{n-1} \rangle,$$
  

$$\mathcal{B}_{n-1} \ltimes \Sigma_{0,1,n-1} = \operatorname{Artin}\langle \sigma_{1} - \sigma_{2} - \cdots - \sigma_{n-2} = \overline{t}_{n-1} \rangle \simeq \operatorname{Artin}\langle B_{n-1} \rangle.$$

5.6 Historical Remarks. In 1925, Artin [3] found the above presentation of  $\mathcal{B}_n$  by an intuitive topological argument but, later, in [4], he indicated that there were difficulties that could be corrected. In 1934, Magnus [23] gave an algebraic proof that the relations suffice. In 1945, Markov [25] gave a similar algebraic proof. In 1947, Bohnenblust [7] gave a similar algebraic proof; in 1948, Chow [8] simplified the latter proof. All these algebraic proofs of the sufficiency of the relations involve the Reidemeister-Schreier rewriting process for the subgroup of index n.

Larue [22] gave a new algebraic proof of the sufficiency of the relations, by using the Dehornoy-Larue trichotomy [14] for braid groups. We shall proceed in the opposite direction. Proofs of the trichotomy for  $\operatorname{Artin}\langle A_{n-1}\rangle$  tend to be more difficult than proofs that  $\operatorname{Out}_{0,1,n}^+ = \operatorname{Artin}\langle A_{n-1}\rangle$ , and we shall now see that Artin's generation argument easily gives the trichotomy for  $\operatorname{Out}_{0,1,n}^+$ .

## 6 The Dehornoy-Larue trichotomy

### **6.1 Definitions.** Let $\phi \in \mathcal{B}_n$ .

We say that  $\phi$  is  $\sigma_1$ -neutral if  $\phi$  lies in the subgroup of  $\mathcal{B}_n$  generated by  $\sigma_{[2\uparrow n-1]}$ .

We say that  $\phi$  is  $\sigma_1$ -positive if  $n \geq 2$  and  $\phi$  can be expressed as the product of a finite sequence of elements of  $\sigma_{[1\uparrow n-1]} \cup \overline{\sigma}_{[2\uparrow n-1]}$  such that at least one term of the sequence is  $\sigma_1$ . We say that  $\phi$  is  $\sigma$ -positive if  $n \geq 2$  and, for some  $i \in [1\uparrow n-1], \phi$  can be expressed as the product of a finite sequence of elements of  $\sigma_{[i\uparrow n-1]} \cup \overline{\sigma}_{[i+1\uparrow n-1]}$  such that at least one term of the sequence is  $\sigma_i$ .

We say that  $\phi$  is  $\sigma_1$ -negative if  $\phi$  is  $\sigma_1$ -positive, that is,  $n \geq 2$  and  $\phi$  can be expressed as the product of a finite sequence of elements of  $\sigma_{[2\uparrow n-1]} \cup \overline{\sigma}_{[1\uparrow n-1]}$ such that at least one term of the sequence is  $\overline{\sigma}_1$ .

If  $\phi$  satisfies exactly one of the properties of being  $\sigma_1$ -neutral,  $\sigma_1$ -positive  $\sigma_1$ -negative, we say that  $\phi$  satisfies the  $\sigma_1$ -trichotomy.

**6.2 Historical Remarks.** View  $\operatorname{Artin}\langle A_n \rangle$  as a subgroup of  $\operatorname{Artin}\langle A_{n+1} \rangle$  in a natural way, and let  $\operatorname{Artin}\langle A_{\infty} \rangle$  denote the union of the resulting chain; thus  $\operatorname{Artin}\langle A_{\infty} \rangle = \langle a_{[1\uparrow\infty[} \rangle$ . Dehornoy [14, Theorem 6] gave a one-sided ordering of  $\operatorname{Artin}\langle A_{\infty} \rangle$ ; the positive semigroup for this ordering is the set of 'a-positive' elements of  $\operatorname{Artin}\langle A_{\infty} \rangle$ .

Let  $\phi \in \mathcal{B}_n$ . By replacing  $\phi$  with  $\overline{\phi}$  if necessary, we can apply Dehornoy's result to deduce that there exists some  $n' \geq n$  such that  $\phi$  is  $\sigma$ -negative in  $\mathcal{B}_{n'}$ , or  $\phi = 1$ . Larue [21] showed that this implies that  $t_1^{\phi} \in (t_1 \star)$ , and that this in turn implies that  $\phi$  can be expressed as the product of a finite sequence, of length at most  $|\phi| + \frac{1}{4}n^{2}3^{|\phi|}$ , of elements of  $\sigma_{[2\uparrow n-1]} \cup \overline{\sigma}_{[1\uparrow n-1]}$ . Thus, every element of  $\mathcal{B}_n$  satisfies the  $\sigma_1$ -trichotomy. Larue's work is surveyed in [16, Chapter 5]. Topological versions of these results can be found in [19] and [16, Chapter 6]. We shall give elementary direct proofs of the foregoing results and replace Larue's bound  $|\phi| + \frac{1}{4}n^2 3^{|\phi|}$  with the much smaller bound  $n2^{|\phi|} - n$ . Larue's proof contains interesting information that we shall rework in the Appendix.

Part (iii) of the following seems to be new.

**6.3 Lemma.** Let  $n \ge 1$  and let  $\phi$  be an element of  $\mathcal{B}_n$  such that  $t_1^{\phi} \in (t_1 \star)$ . Let  $\pi = \pi(\phi)$  and, for each  $i \in [1 \uparrow n]$ , let  $u_i = u_i(\phi)$ .

- (i). Suppose that there exists some  $i \in [1\uparrow n-1]$  such that  $u_i \in (\star \overline{t}_{(i+1)\pi})$ . Then  $\|\sigma_i \phi\| \leq \|\phi\| 2$  and  $t_1^{\sigma_i \phi} \in (t_1 \star)$ ; moreover, if  $t_1^{\phi} = t_1$ , then  $i \in [2\uparrow n-1]$ .
- (ii). Suppose that there exists some  $i \in [2\uparrow n-1]$  such that  $u_i \in (\overline{t}_{i^{\pi}}\star)$ . Then  $\|\overline{\sigma}_i \phi\| \leq \|\phi\| 2$  and  $t_1^{\overline{\sigma}_i \phi} \in (t_1\star)$ .
- (iii). Suppose that, for each  $i \in [1\uparrow n-1]$ ,  $u_i \notin (\star \overline{t}_{(i+1)\pi})$  and, for each  $i \in [2\uparrow n-1]$ ,  $u_i \notin (\overline{t}_{i\pi}\star)$ . Then  $\phi = 1$ .

*Proof.* For each  $i \in [0\uparrow n+1]$ , let  $w_i = w_i(\phi)$ .

(i). The first part follows from Artin's Lemma 3.2(i). Notice that, if  $t_1^{\phi} = t_1$ , then  $w_1 = 1$  and  $u_1 = \overline{w}_2 \notin (\star \overline{t}_{2^{\pi}})$ .

(ii) follows from Lemma 3.2(ii).

(iii). Recall that  $u_0 \prod_{i \in [1\uparrow n]} (t_{i^{\pi}} u_i) = \prod_{i \in [1\uparrow n]} (t_{i^{\pi}}^{w_i}) = (\prod_{i \in [1\uparrow n]} t_i)^{\phi} = \prod_{i \in [1\uparrow n]} t_i$ . Hence,  $u_0 t_{1^{\pi}} u_1 \prod_{i \in [2\uparrow n]} (t_{i^{\pi}} u_i) = t_1 \prod_{i \in [2\uparrow n]} t_i$ , and, hence,  $u_1 \prod_{i \in [2\uparrow n]} (t_{i^{\pi}} u_i) = \overline{t}_{1^{\pi}} \overline{u}_0 t_1 \prod_{i \in [2\uparrow n]} t_i$ . Since  $u_n = w_n \notin (\overline{t}_{n^{\pi}} \star)$ , the hypotheses imply that there is no cancellation anywhere in the expression  $u_1 \prod_{i \in [2\uparrow n]} (t_{i^{\pi}} u_i)$ . Hence,

$$(6.3.1) \quad \sum_{i \in [1\uparrow n]} |u_i| + n - 1 = |u_1 \prod_{i \in [2\uparrow n]} (t_{i^{\pi}} u_i)| = |\overline{t}_{1^{\pi}} \overline{u}_0 t_1 \prod_{i \in [2\uparrow n]} t_i| \le |\overline{t}_{1^{\pi}} \overline{u}_0 t_1| + n - 1.$$

Since  $t_{1^{\pi}}^{\overline{u}_0} = t_{1^{\pi}}^{w_1} = t_1^{\phi} \in (t_1 \star)$ , we see that  $u_0 t_{1^{\pi}} \in (t_1 \star)$ , and

(6.3.2) 
$$|\overline{t}_1 u_0 t_{1^{\pi}}| = -1 + |u_0 t_{1^{\pi}}| \le -1 + |u_0| + 1 = |u_0|$$

Since  $\prod_{i \in [0 \uparrow n]} u_i = w_0 \overline{w}_{n+1} = 1$ , we see that

(6.3.3) 
$$\prod_{i \in [1\uparrow n]} u_i = \overline{u}_0 = w_1 \notin (\overline{t}_{1\pi} \star).$$

Now,  $\sum_{i \in [1\uparrow n]} |u_i| \stackrel{(6.3.1)}{\leq} |\overline{t}_{1\pi} \overline{u}_0 t_1| \stackrel{(6.3.2)}{\leq} |\overline{u}_0| \stackrel{(6.3.3)}{=} |\prod_{i \in [1\uparrow n]} u_i|$ . Therefore, there is no cancellation in  $\prod_{i \in [1\uparrow n]} u_i$ , and, by (6.3.3),  $u_1 \notin (\overline{t}_{1\pi} \star)$ . By Lemma 3.2(iii),  $\phi = 1$ .

As in Remarks 3.4, we deduce the following from Lemma 6.3 by induction on  $\|\phi\|$ .

**6.4 Corollary** (Larue [21]). Let  $n \ge 1$  and let  $\phi \in \mathcal{B}_n$ .

- (i). Suppose that  $t_1^{\phi} \in (t_1^{\star})$ . Then  $\phi$  is  $\sigma_1$ -negative or  $\sigma_1$ -neutral. In more detail,  $\phi$  can be expressed as the product of a sequence, of length at most  $n2^{|\phi|} n$ , of elements of  $\sigma_{[2\uparrow n-1]} \cup \overline{\sigma}_{[1\uparrow n-1]}$ .
- (ii). Moreover,  $\phi$  is  $\sigma_1$ -neutral if and only if  $t_1^{\phi} = t_1$ .

**6.5 Notation.** For each  $i \in [1\uparrow n-1]$ , let  $\sigma'_i$  and  $\sigma''_i$  be the automorphisms of  $\Sigma_{0,1,n}$  determined by

$k{\in}[1{\uparrow}i]$	$k \in$	$[i+2\uparrow n]$	$k{\in}[1{\uparrow}i{-}1]$			$\underline{k{\in}[i{+}1{\uparrow}n]}$
$(t_k$	$t_{i+1}$	$(t_k)^{\sigma'_i}$	$(t_k$	$t_i$	$t_{i+1}$	$(t_k)^{\sigma_i''}$
$=(t_k$	$t_{i+1}^{t_i}$	$t_k),$	$=(t_k$	$t_{i+1}$	$t_i$	$t_k).$

Then  $\sigma_i = \sigma'_i \sigma''_i$ . The normal form in  $t_{[1\uparrow n]}$  factorizes into an alternating product with factors which are normal forms of non-trivial elements of  $\langle t_{[i\uparrow i+1]} \rangle$  alternating with factors which are normal forms of non-trivial elements of  $\langle t_{[1\uparrow i-1]\cup[i+2\uparrow n]} \rangle$ . On  $\langle t_{[i\uparrow i+1]} \rangle$ ,  $\sigma'_i$  acts as conjugation by  $t_i$ , while  $\sigma''_i$  interchanges the two free generators. On  $\langle t_{[1\uparrow i-1]\cup[i+2\uparrow n]} \rangle$ ,  $\sigma'_i$  and  $\sigma''_i$  act as the identity map.

The next result gives three trichotomies, called (a), (b) and (c), which hold for elements of  $\mathcal{B}_n$ . Attribution is not sharply defined, but it is reasonable to attribute (b) to Dehornoy [14], and (a) and (c) to Larue [21].

**6.6 Theorem** (Dehornoy-Larue [14], [21]). Let  $n \ge 1$ , let  $\phi \in \mathcal{B}_n$  and consider the following nine conditions.

(a1). 
$$t_1^{\phi} = t_1$$
. (a2).  $t_1^{\phi} \in (t_1 \star) - \{t_1\}$ . (a3).  $t_1^{\phi} \notin (t_1 \star)$ .

(b1).  $\phi$  is  $\sigma_1$ -neutral. (b2).  $\phi$  is  $\sigma_1$ -negative. (b3).  $\phi$  is  $\sigma_1$ -positive.

(c1). 
$$(t_1 \star)^{\phi} = (t_1 \star)$$
 (c2).  $(t_1 \star)^{\phi} \subset (t_1 \star)$ . (c3).  $(t_1 \star)^{\phi} \supset (t_1 \star)$ .

Then: (a1)  $\Leftrightarrow$  (b1)  $\Leftrightarrow$  (c1); (a2)  $\Leftrightarrow$  (b2)  $\Leftrightarrow$  (c2); (a3)  $\Leftrightarrow$  (b3)  $\Leftrightarrow$  (c3).

Exactly one of (b1), (b2), (b3), holds; that is,  $\phi$  satisfies the  $\sigma_1$ -trichotomy in  $\mathcal{B}_n$ .

*Proof.* (a1)  $\Leftrightarrow$  (b1) by Corollary 6.4(ii). We shall use (a1) and (b1) interchangeably in the remainder of the proof.

(b1)  $\Rightarrow$  (c1). If  $\phi$  is  $\sigma_1$ -neutral, then so is  $\overline{\phi}$ . It follows that  $(t_1\star)^{\phi} \subseteq (t_1\star)$ and  $(t_1\star)^{\overline{\phi}} \subseteq (t_1\star)$ . Thus,  $(t_1\star)^{\phi} = (t_1\star)$ .

 $(a2) \Rightarrow (b2)$ . If (a2) holds, then Corollary 6.4(i) shows that (b1) or (b2) holds. Since (a1) fails, (b1) fails. Thus (b2) holds.

 $(b2) \Rightarrow (c2)$ . Using Notation 6.5, we see that

$$(t_1\star)^{\overline{\sigma}_1} = (t_1\star)^{\overline{\sigma}_1'\overline{\sigma}_1'} = (t_2\star)^{\overline{\sigma}_1'} \subseteq (t_1t_2\star) \subset (t_1\star).$$

Since the composition of injective self-maps of  $(t_1\star)$  can be bijective only if all the factors are bijective, we see that  $(b2) \Rightarrow (c2)$ .

 $(a3) \Rightarrow (b3)$ . We translate into algebra the crucial reflection argument of [16, Corollary 5.2.4].

Suppose that (a3) holds.

With Notation 3.1, let  $w_1 = w_1(\phi)$  and  $\pi = \pi(\phi)$ . Then  $\overline{w}_1 t_{1^{\pi}} w_1 = t_1^{\phi} \notin (t_1 \star)$ . It follows that  $\overline{w}_1 t_{1^{\pi}} \notin (t_1 \star)$ . Hence,  $\overline{w}_1 \overline{t}_{1^{\pi}} \notin (t_1 \star)$ . Hence,

 $\overline{t}_1^{\phi} = \overline{w}_1 \overline{t}_{1^{\pi}} w_1 \notin (t_1 \star) \cup \{1\}.$  On conjugating by  $t_1$ , we see that  $\overline{t}_1^{\phi t_1} \in (\overline{t}_1 \star).$ 

 $(\frac{k \in [1\uparrow n]}{(t_k)^{\zeta}})^{\zeta} . \text{ For} = (\overline{t}_k^{\Pi \overline{t}_{[k-1\downarrow 1]}})$ Let  $\zeta$  be the automorphism of  $\Sigma_{0,1,n}$  determined by

each  $k \in [1\uparrow n]$ ,  $(\Pi t_{[1\uparrow k]})^{\zeta} = \Pi \overline{t}_{[k\downarrow 1]}$ . It follows that  $\zeta^2 = 1$ . Notice that  $\zeta$  belongs

to  $\operatorname{Out}_{0,1,n}^- := \operatorname{Out}_{0,1,n}^- - \operatorname{Out}_{0,1,n}^+$ . Also,  $(t_1 \quad \frac{k \in [2\uparrow n]}{t_k})^{\overline{t}_1 \zeta}$ . Hence, =  $(\overline{t}_1 \quad \overline{t}_k^{\Pi \overline{t}_{[k-1\downarrow 2]}})$ 

$$t_1^{\phi^{\zeta}} = t_1^{\zeta\phi\zeta} = \overline{t}_1^{\phi t_1\overline{t}_1\zeta} \in (\overline{t}_1\star)^{\overline{t}_1\zeta} \subseteq (t_1\star).$$

By Corollary 6.4(i),  $\phi^{\zeta}$  can be expressed as the product of a finite sequence of elements of  $\sigma_{[2\uparrow n-1]} \cup \overline{\sigma}_{[1\uparrow n-1]}$ . It is not difficult to check that, for each  $i \in [1\uparrow n-1]$ ,  $\sigma_i^{\zeta} = \overline{\sigma}_i$  in  $\operatorname{Out}_{0,1,n}$ . Hence  $\phi^{\zeta^2}(=\phi)$  can be expressed as the product of a finite sequence of elements of  $\sigma_{[2\uparrow n-1]}^{\zeta} \cup \overline{\sigma}_{[1\uparrow n-1]}^{\zeta} (= \overline{\sigma}_{[2\uparrow n-1]} \cup \sigma_{[1\uparrow n-1]})$ . Hence, (b3) or (b1) holds. Since (a3) holds, (a1) fails, and (b1) fails. Thus (b3) holds.

(b3)  $\Rightarrow$  (c3). If  $\phi$  is  $\sigma_1$ -positive, then  $\overline{\phi}$  is  $\sigma_1$ -negative, and, by (b2)  $\Rightarrow$  (c2),  $(t_1\star)^{\overline{\phi}} \subset (t_1\star)$  and, hence,  $(t_1\star) \subset (t_1\star)^{\phi}$ .

 $(c1) \Rightarrow (a1)$ . Suppose that (a1) fails. Then (a2) or (a3) holds. Hence (c2)or (c3) holds. Hence (c1) fails.

 $(c2) \Rightarrow (a2)$  and  $(c3) \Rightarrow (a3)$  are proved similarly.

Thus the desired equivalences hold.

Since exactly one of (a1), (a2), (a3) holds, exactly one of (b1), (b2), (b3) holds. 

The following gives the Dehornoy right-ordering of  $\mathcal{B}_n$ ; recall the definition of  $\sigma$ -positive from Definitions 6.1.

**6.7 Theorem.** For each  $\phi \in \mathfrak{B}_n$  exactly one of the following holds:  $\phi = 1$ ;  $\phi$  is  $\sigma$ -positive;  $\overline{\phi}$  is  $\sigma$ -positive. The set of  $\sigma$ -positive elements of  $\mathbb{B}_n$  is the positive cone of a right-ordering of  $\mathcal{B}_n$ .

## *Proof.* Suppose that $\phi \neq 1$ .

Let i be the largest element of  $[1\uparrow n-1]$  such that  $\phi \in \langle \sigma_{[i,n-1]} \rangle$ . The natural subscript-shifting isomorphism from  $\langle t_{[i\uparrow n]}\rangle$  to  $\Sigma_{0,1,n-i+1}$  induces an isomorphism from  $\langle \sigma_{[i\uparrow n-1]} \rangle$  to  $B_{n-i+1}$ . Notice that  $\phi$  is mapped to an element of  $B_{n-i+1}$  which

is not  $\sigma_1$ -neutral; by Theorem 6.6, this image is  $\sigma_1$ -positive or  $\sigma_1$ -negative but not both. Hence exactly one of  $\phi$ ,  $\overline{\phi}$  is  $\sigma$ -positive.

It is easy to see that the product of two  $\sigma$ -positive elements of  $\mathcal{B}_n$  is  $\sigma$ -positive.

Hence the set of  $\sigma$ -positive elements of  $\mathcal{B}_n$  is the positive cone for a right-ordering of  $\mathcal{B}_n$ , the Dehornoy right-ordering.

## 7 Ends, right-orderings and squarefreeness

**7.1 Review.** A (reduced) end of  $\Sigma_{0,1,n}$  is a function

$$[1\uparrow\infty[ \rightarrow t_{[1\uparrow n]} \cup \overline{t}_{[1\uparrow n]}, \quad i \mapsto a_i,$$

such that, for each  $i \in [1\uparrow\infty[, a_{i+1} \neq \overline{a}_i]$ . The function is then represented as a right-infinite reduced product,  $a_1a_2\cdots$  or  $\prod a_{[1\uparrow\infty[}$ .

We denote the set of ends of  $\Sigma_{0,1,n}$  by  $\mathfrak{E}(\Sigma_{0,1,n})$ , or simply by  $\mathfrak{E}$  if there is no risk of confusion.

An element of  $\Sigma_{0,1,n} \cup \mathfrak{E}(\Sigma_{0,1,n})$  is said to be *squarefree* if, in its reduced expression, no two consecutive terms are equal; for example:  $(t_1t_2)^{\infty}$  is a squarefree end;  $t_1t_2t_2t_3$  is a non-squarefree word.

For each  $w \in \Sigma_{0,1,n}$ , we define the *shadow* of w in  $\mathfrak{E}$  to be

$$(w\blacktriangleleft) := \{ \Pi a_{[1\uparrow\infty[} \in \mathfrak{E} \mid \Pi a_{[1\uparrow|w|]} = w \}.$$

Thus, for example,  $(1 \blacktriangleleft) = \mathfrak{E}$ .

We shall now give  $\mathfrak{E}$  an ordering, <. The first step is, for each  $w \in \Sigma_{0,1,n}$ , to assign an ordering, <, to a partition of  $(w \blacktriangleleft)$  into 2n or 2n-1 subsets, depending as w = 1 or  $w \neq 1$ , as follows. We set

$$(t_1 \blacktriangleleft) < (\overline{t}_1 \bigstar) < (t_2 \bigstar) < (\overline{t}_2 \bigstar) < \dots < (t_n \bigstar) < (\overline{t}_n \bigstar).$$

If  $i \in [1 \uparrow n]$  and  $w \in (\star \overline{t}_i)$ , then we set

$$(w\overline{t}_{i}\blacktriangleleft) < (wt_{i+1}\blacktriangleleft) < (w\overline{t}_{i+1}\blacktriangleleft) < (wt_{i+2}\blacktriangleleft) < (w\overline{t}_{i+2}\blacktriangleleft) < \cdots$$
$$\cdots < (wt_{n}\blacktriangleleft) < (w\overline{t}_{n}\blacktriangleleft) < (wt_{1}\blacktriangleleft) < (w\overline{t}_{1}\blacktriangleleft) < (wt_{2}\blacktriangleleft) < \cdots$$
$$\cdots < (wt_{i-1}\blacktriangleleft) < (w\overline{t}_{i-1}\blacktriangleleft).$$

If  $i \in [1 \uparrow n]$  and  $w \in (\star t_i)$ , then we set

$$(wt_{i+1} \blacktriangleleft) < (w\overline{t}_{i+1} \bigstar) < (wt_{i+2} \bigstar) < (w\overline{t}_{i+2} \bigstar) < \cdots$$
$$\cdots < (wt_n \bigstar) < (w\overline{t}_n \bigstar) < (wt_1 \bigstar) < (w\overline{t}_1 \bigstar) < (wt_2 \bigstar) < \cdots$$
$$\cdots < (wt_{i-1} \bigstar) < (w\overline{t}_{i-1} \bigstar) < (wt_i \bigstar).$$

Hence, for each  $w \in \Sigma_{0,1,n}$ , we have an ordering < of a partition of  $(w\blacktriangleleft)$  into 2n or 2n-1 subsets.

If  $\Pi a_{[1\uparrow\infty[}$  and  $\Pi b_{[1\uparrow\infty[}$  are two different (reduced) ends, then there exists  $i \in \mathbb{N}$  such that  $\Pi a_{[1\uparrow i]} = \Pi b_{[1\uparrow i]}$  in  $\Sigma_{0,1,n}$ , and  $a_{i+1} \neq b_{i+1}$  in  $t_{[1\uparrow n]} \cup \overline{t}_{[1\uparrow n]}$ . Let  $w = \Pi a_{[1\uparrow i]} = \Pi b_{[1\uparrow i]}$ . Then  $\Pi a_{[1\uparrow\infty[}$  and  $\Pi b_{[1\uparrow\infty[}$  lie in  $(w\blacktriangleleft)$ , but lie in different elements of the partition of  $(w\blacktriangleleft)$  into 2n or 2n-1 subsets. We then order  $\Pi a_{[1\uparrow\infty[}$  and  $\Pi b_{[1\uparrow\infty[}$  according to the order of the elements of the partition of  $(w\blacktriangleleft)$  that they belong to. This completes the definition of the ordering < of  $\mathfrak{E}$ .

We remark that the smallest element of  $\mathfrak{E}$  is  $\overline{z}_1^{\infty} = (\Pi t_{[1\uparrow n]})^{\infty}$  and the largest element of  $\mathfrak{E}$  is  $z_1^{\infty} = (\Pi \overline{t}_{[n\downarrow 1]})^{\infty}$ .

**7.2 Review.** Following Nielsen-Thurston [9], [27], we now define the action of  $\mathcal{B}_n$  on  $\mathfrak{E}(\Sigma_{0,1,n})$  and show that it respects the ordering; our treatment will be quite elementary compared to the usual approaches.

We assume that  $n \geq 2$ , and we first define the action of  $\sigma_1$  on  $\mathfrak{E}$ .

Consider any reduced end  $\mathbf{c} \in \mathfrak{E}$ . There is then a unique factorization  $\mathbf{c} = \Pi w_{[1\uparrow i]}$  or  $\mathbf{c} = \Pi w_{[1\uparrow\infty[}$ , where, in the former case,  $w_{([1\uparrow i-1])}$  is a finite sequence of non-trivial words, and  $w_i$  is a reduced end, and, in the latter case,  $w_{([1\uparrow\infty[)})$  is an infinite sequence of non-trivial words, and in both cases, the  $w_j$  alternate between elements of  $\langle t_{[1\uparrow 2]} \rangle \cup \mathfrak{E}(\langle t_{[1\uparrow 2]} \rangle)$ , and elements of  $\langle t_{[3\uparrow n]} \rangle \cup \mathfrak{E}(\langle t_{[3\uparrow n]} \rangle)$ . We shall express this factorization as  $\mathbf{c} = [w_1][w_2] \cdots$ .

Recall, from Notation 6.5, that we have the factorization  $\sigma_1 = \sigma'_1 \sigma''_1$ . On  $\langle t_{[1\uparrow 2]} \rangle \cup \mathfrak{E}(\langle t_{[1\uparrow 2]} \rangle)$ ,  $\sigma'_1$  acts as conjugation by  $t_1$ , while  $\sigma''_1$  interchanges the two free generators. On  $\langle t_{[3\uparrow n]} \rangle$ ,  $\sigma'_1$  and  $\sigma''_1$  act as the identity map. This completes the description of the action of  $\sigma'_1$ ,  $\sigma''_1$  and  $\sigma_1$  on  $\mathfrak{E}$ .

It is not difficult to show that, for any reduced ends  $\Pi a_{[1\uparrow\infty[}$  and  $\Pi b_{[1\uparrow\infty[}$ , if  $(\Pi a_{[1\uparrow\infty[})^{\sigma_1} = \Pi b_{[1\uparrow\infty[})$ , then for all  $i, j \in \mathbb{N}$ , if  $j \geq 2i$ , then  $(\Pi a_{[1\uparrowj]})^{\sigma_1} \in (\Pi b_{[1\uparrow i]}\star)$ . Thus,  $(\Pi a_{[1\uparrow\infty[})_1^{\sigma})$  is the limit of  $(\Pi a_{[1\uparrow j]})^{\sigma_1}$  as j tends to  $\infty$ .

It is clear that  $\sigma'_1, \sigma''_1$  and, hence,  $\sigma_1$  act bijectively on  $\mathfrak{E}$ . Hence we have the action of  $\overline{\sigma}_1$  on  $\mathfrak{E}$ . It is then not difficult to verify that we have an action of  $\mathcal{B}_n$  on  $\mathfrak{E}$ .

We next show that  $\sigma_1$  respects the ordering of  $\mathfrak{E}$ . We do this by considering all the ways that two reduced ends can be compared, and the resulting effect of  $\sigma'_1$  and  $\sigma_1$ . We represent the information in tables. In all of the following, we understand that  $t_1a$ ,  $\overline{t}_1b$ ,  $t_2c$ , and  $\overline{t}_2d$  are reduced expressions for elements of  $\langle t_{[1\uparrow 2]} \rangle \cup \mathfrak{E}(\langle t_{[1\uparrow 2]} \rangle)$ , and  $b \neq 1$ . Since *a* does not begin with  $\overline{t}_1$ ,  $a^{\sigma''_1}t_2$  begins with  $t_1$  or  $\overline{t}_1$  or  $t_2$ . We make the convention that  $\Sigma_{0,1,n}$  acts trivially on the right on  $\mathfrak{E}$ .

$(\cdots][wt_1 \blacktriangleleft)$	$(\cdots][wt_1 \blacktriangleleft)^{\sigma'_1}$	$(\cdots][wt_1 \blacktriangleleft)^{\sigma_1}$
$ \begin{array}{c} \cdots ][wt_1 \ t_2c][\cdots \\ \cdots ][wt_1 \ \overline{t}_2d][\cdots \\ \cdots ][wt_1][t_3 \uparrow \overline{t}_n \cdots \\ \cdots ][wt_1][t_3 \uparrow \overline{t}_n \cdots \\ \cdots ][wt_1 \ t_1a][\cdots \end{array} $	$ \cdots ][(\overline{t}_1w)t_1 \ t_2(ct_1)][\cdots \\ \cdots ][(\overline{t}_1w)t_1 \ \overline{t}_2(dt_1)][\cdots \\ \cdots ][(\overline{t}_1w)t_1 \ t_1][t_3\uparrow \overline{t}_n\cdots \\ \cdots ][(\overline{t}_1w)t_1 \ t_1(at_1)][\cdots $	$ \begin{array}{c} \cdots ][(\overline{t}_2 w^{\sigma_1''})t_2 \ t_1(c^{\sigma_1''}t_2)][\cdots \\ \cdots ][(\overline{t}_2 w^{\sigma_1''})t_2 \ \overline{t}_1(d^{\sigma_1''}t_2)][\cdots \\ \cdots ][(\overline{t}_2 w^{\sigma_1''})t_2 \ t_2][t_3 \uparrow \overline{t}_n \cdots \\ \cdots ][(\overline{t}_2 w^{\sigma_1''})t_2 \ t_2(a^{\sigma_1''}t_2)][\cdots \\ \end{array} $

$(\cdots][w\overline{t}_1\blacktriangleleft)$	$(\cdots][w\overline{t}_1 \blacktriangleleft)^{\sigma'_1}$	$(\cdots][w\overline{t}_1\blacktriangleleft)^{\sigma_1}$
$ \cdots ][w\overline{t}_1 \ \overline{t}_1b][\cdots \\ \cdots ][w\overline{t}_1 \ \overline{t}_1][t_3 \uparrow \overline{t}_n \cdots \\ \cdots ][w\overline{t}_1 \ t_2c][\cdots \\ \cdots ][w\overline{t}_1 \ \overline{t}_2d][\cdots \\ \cdots ][w\overline{t}_1][t_3 \uparrow \overline{t}_n \cdots $	$ \cdots ][(\overline{t}_1w) \ \overline{t}_1][t_3\uparrow\overline{t}_n\cdots \\ \cdots ][(\overline{t}_1w) \ \overline{t}_1t_2(ct_1)][\cdots \\ \cdots ][(\overline{t}_1w) \ \overline{t}_1\overline{t}_2(dt_1)][\cdots $	$ \cdots ][(\overline{t}_2 w^{\sigma_1''}) \ \overline{t}_2 \overline{t}_2 (b^{\sigma_1''} t_2)][\cdots \\ \cdots ][(\overline{t}_2 w^{\sigma_1''}) \ \overline{t}_2][t_3 \uparrow \overline{t}_n \cdots \\ \cdots ][(\overline{t}_2 w^{\sigma_1''}) \ \overline{t}_2 t_1 (c^{\sigma_1''} t_2)][\cdots \\ \cdots ][(\overline{t}_2 w^{\sigma_1''}) \ \overline{t}_2 \overline{t}_1 (d^{\sigma_1''} t_2)][\cdots \\ \cdots ][(\overline{t}_2 w^{\sigma_1''})][t_3 \uparrow \overline{t}_n \cdots $
Here, $w$ does not end w	with $t_1$ , and, hence, $(\overline{t}_2 w^{\sigma_1''})$	) ends with $t_1$ , $\overline{t}_1$ or $\overline{t}_2$ .
$(\cdots][wt_2\blacktriangleleft)$	$(\cdots][wt_2\blacktriangleleft)^{\sigma'_1}$	$(\cdots][wt_2\blacktriangleleft)^{\sigma_1}$
$ \begin{array}{c} \cdots ][wt_2][t_3 \uparrow \overline{t}_n \cdots \\ \cdots ][wt_2 \ t_1 a][\cdots \\ \cdots ][wt_2 \ \overline{t}_1 b][\cdots \\ \cdots ][wt_2 \ \overline{t}_1][t_3 \uparrow \overline{t}_n \cdots \\ \cdots ][wt_2 \ t_2 c][\cdots \end{array} $	$ \cdots ][(\overline{t}_1w)t_2 \ t_1][t_3\uparrow \overline{t}_n\cdots \\ \cdots ][(\overline{t}_1w)t_2 \ t_1(at_1)][\cdots \\ \cdots ][(\overline{t}_1w)t_2 \ \overline{t}_1(bt_1)][\cdots \\ \cdots ][(\overline{t}_1w)t_2][t_3\uparrow \overline{t}_n\cdots \\ \cdots ][(\overline{t}_1w)t_2 \ t_2(ct_1)][\cdots $	$ \cdots ][(\overline{t}_2 w^{\sigma_1''})t_1 \ t_2][t_3 \uparrow \overline{t}_n \cdots \\ \cdots ][(\overline{t}_2 w^{\sigma_1''})t_1 \ t_2(a^{\sigma_1''}t_2)][\cdots \\ \cdots ][(\overline{t}_2 w^{\sigma_1''})t_1 \ \overline{t}_2(b^{\sigma_1''}t_2)][\cdots \\ \cdots ][(\overline{t}_2 w^{\sigma_1''})t_1][t_3 \uparrow \overline{t}_n \cdots \\ \cdots ][(\overline{t}_2 w^{\sigma_1''})t_1 \ t_1(c^{\sigma_1''}t_2)][\cdots $
$(\cdots][w\overline{t}_2\blacktriangleleft)$	$(\cdots][w\overline{t}_2\blacktriangleleft)^{\sigma_1'}$	$(\cdots][w\overline{t}_2\blacktriangleleft)^{\sigma_1}$
$ \begin{array}{c} \cdots ][w\overline{t}_2 \ \overline{t}_2 d][\cdots \\ \cdots ][w\overline{t}_2][t_3 \uparrow \overline{t}_n \cdots \\ \cdots ][w\overline{t}_2 \ t_1 a][\cdots \\ \cdots ][w\overline{t}_2 \ \overline{t}_1 b][\cdots \\ \cdots ][w\overline{t}_2 \ \overline{t}_1][t_3 \uparrow \overline{t}_n \cdots \end{array} $	$ \cdots ][(\overline{t}_1w)\overline{t}_2 \ \overline{t}_2(dt_1)][\cdots \\ \cdots ][(\overline{t}_1w)\overline{t}_2 \ t_1][t_3\uparrow\overline{t}_n\cdots \\ \cdots ][(\overline{t}_1w)\overline{t}_2 \ t_1(at_1)][\cdots \\ \cdots ][(\overline{t}_1w)\overline{t}_2 \ \overline{t}_1(bt_1)][\cdots \\ \cdots ][(\overline{t}_1w)\overline{t}_2][t_3\uparrow\overline{t}_n\cdots $	$ \cdots ] [(\overline{t}_2 w^{\sigma_1''}) \overline{t}_1 \ \overline{t}_1 (d^{\sigma_1''} t_2)] [\cdots \\ \cdots ] [(\overline{t}_2 w^{\sigma_1''}) \overline{t}_1 \ t_2] [t_3 \uparrow \overline{t}_n \cdots \\ \cdots ] [(\overline{t}_2 w^{\sigma_1''}) \overline{t}_1 \ t_2 (a^{\sigma_1''} t_2)] [\cdots \\ \cdots ] [(\overline{t}_2 w^{\sigma_1''}) \overline{t}_1 \ \overline{t}_2 (b^{\sigma_1''} t_2)] [\cdots \\ \cdots ] [(\overline{t}_2 w^{\sigma_1''}) \overline{t}_1] [t_3 \uparrow \overline{t}_n \cdots $
$(\cdots t_3 \blacktriangleleft)$	$(\cdots t_3 \blacktriangleleft)^{\sigma'_1}$	$(\cdots t_3 \blacktriangleleft)^{\sigma_1}$
$ \cdots t_3 t_4 \uparrow \overline{t}_n \cdots \\ \cdots t_3 ] [t_1 a] [\cdots \\ \cdots t_3] [\overline{t}_1 b] [\cdots \\ \cdots t_3] [\overline{t}_1] [t_3 \uparrow \overline{t}_n \\ \cdots t_3] [t_2 c] [\cdots \\ \cdots t_3] [\overline{t}_2 d] [\cdots \\ \cdots t_3 t_3 \cdots $	$\cdots t_3][(at_1)][\cdots$ $\cdots t_3][\overline{t_1}\overline{t_1}(bt_1)][\cdots$ $\cdots t_3][\overline{t_1}][t_3\uparrow \overline{t_n}\cdots$ $\cdots t_3][\overline{t_1}][t_1c_1(ct_1)][\cdots$	$ \cdots t_3  t_4 \uparrow \overline{t}_n \cdots \\ \cdots t_3][(a^{\sigma_1''} t_2)][\cdots \\ \cdots t_3][\overline{t}_2 \overline{t}_2 (b^{\sigma_1''} t_2)][\cdots \\ \cdots t_3][\overline{t}_2][t_3 \uparrow \overline{t}_n \cdots \\ \cdots t_3][\overline{t}_2 t_1 (c^{\sigma_1''} t_2)][\cdots \\ \cdots t_3][\overline{t}_2 \overline{t}_1 (d^{\sigma_1''} t_2)][\cdots \\ \cdots t_3  t_3 \cdots $

Here, the case w = 1 does not present any problems.

The remaining tables are clearly of the same form as the last one. Thus we have proved that the action of  $\sigma_1$  respects the ordering of  $\mathfrak{E}$ . It follows that the action of  $\overline{\sigma}_1$  respects the ordering of  $\mathfrak{E}$ . Similarly, the actions of  $\sigma_{[2\uparrow n-1]} \cup \overline{\sigma}_{[2\uparrow n-1]}$  respect the ordering of  $\mathfrak{E}$ . Hence  $\mathcal{B}_n$  acts on  $(\mathfrak{E}, \leq)$ .

**7.3 Remarks** (Thurston [27]). The (right) action of  $\mathcal{B}_n$  on  $(\mathfrak{E}, \leq)$  gives rise to many right orderings of  $\mathcal{B}_n$ .

Let us use the left-to-right lexicographic ordering on  $(\mathfrak{E}^n, \leq)$ , and consider the  $\mathcal{B}_n$ -orbit of  $t^{\infty}_{([1\uparrow n])} := (t^{\infty}_i)_{i\in[1\uparrow n]}$ . It is not difficult to show that the  $\mathcal{B}_n$ -stabilizer

of  $t^{\infty}_{([1\uparrow n])}$  is trivial. Thus we have an injective map

$$\mathcal{B}_n \to \mathfrak{E}^n, \qquad \phi \mapsto t^{\infty \phi}_{([1\uparrow n])} := ((t^{\infty}_i)^{\phi})_{i \in [1\uparrow n]}.$$

Let  $\leq$  denote the ordering of  $\mathcal{B}_n$  induced by pullback from  $\mathfrak{E}^n$ . Clearly  $\leq$  is a right-ordering of  $\mathcal{B}_n$ .

If  $n \geq 2$  and  $\phi \in \mathcal{B}_n$  is  $\sigma_1$ -negative, then, as in the proof of Theorem 6.6(b2) $\Rightarrow$ (c2), we have  $(t_1 \blacktriangleleft)^{\phi} \subset (t_1 \bigstar)$ . Since  $\max(t_1 \bigstar) = t_1^{\infty}$ , we see that  $(t_1^{\infty})^{\phi} < t_1^{\infty}$ . Hence  $\phi < 1$  and  $1 < \phi$ . Similar arguments with  $(t_i \bigstar)$ ,  $i \in [2\uparrow n]$ , show that, if  $\phi \in \mathcal{B}_n$  is  $\sigma$ -positive (resp.  $\sigma$ -negative), then  $1 < \phi$ (resp.  $1 > \phi$ ). Hence the right-ordering we have obtained from  $(\mathfrak{E}^n, \leq)$  coincides with the Dehornoy right-ordering. However, the study of ends does not seem to readily yield the  $\sigma_1$ -trichotomy.

The following will be useful in the study of squarefreeness.

**7.4 Lemma.** Let  $n \ge 1$ , let  $i \in [1 \uparrow n]$ , and let  $w \in \Sigma_{0,1,n} - (\star t_i) - (\star \overline{t}_i)$ . Then, in  $\mathfrak{E}(\Sigma_{0,1,n})$ , the following hold:

- (i).  $wt_i\overline{w}((\Pi t_{[1\uparrow n]})^{\infty}) \leq wt_i((\Pi t_{[i\uparrow n]}\Pi t_{[1\uparrow i-1]})^{\infty}) = \min(wt_it_i\blacktriangleleft);$
- (ii).  $\min(wt_it_i \blacktriangleleft) < \max(w\overline{t}_i\overline{t}_i \bigstar);$

(iii).  $\max(w\overline{t}_i\overline{t}_i\blacktriangleleft) = w\overline{t}_i((\Pi\overline{t}_{[i\downarrow 1]}\Pi\overline{t}_{[n\downarrow i+1]})^\infty) \le w\overline{t}_i\overline{w}((\Pi\overline{t}_{[n\downarrow 1]})^\infty);$ 

(iv).  $(wt_it_i \blacktriangleleft) \cup (w\overline{t}_i\overline{t}_i \blacktriangleleft) \subseteq [wt_i\overline{w}((\Pi t_{[1\uparrow n]})^\infty), w\overline{t}_i\overline{w}((\Pi \overline{t}_{[n\downarrow 1]})^\infty)].$ 

(v). If  $n \ge 3$ , then one of the following holds:

(a). 
$$t_1((\Pi \overline{t}_{[n\downarrow 1]})^{\infty}) < w t_i \overline{w}((\Pi t_{[1\uparrow n]})^{\infty});$$
  
(b).  $t_1((\Pi \overline{t}_{[n\downarrow 1]})^{\infty}) > w \overline{t}_i \overline{w}((\Pi \overline{t}_{[n\downarrow 1]})^{\infty});$ 

and, hence,  $t_1((\Pi \overline{t}_{[n\downarrow 1]})^{\infty}) \notin [wt_i \overline{w}((\Pi t_{[1\uparrow n]})^{\infty}), w\overline{t}_i \overline{w}((\Pi \overline{t}_{[n\downarrow 1]})^{\infty})]$ , that is,  $t_1(z_1^{\infty}) \notin [wt_i \overline{w}(\overline{z}_1^{\infty}), w\overline{t}_i \overline{w}(z_1^{\infty})]$ 

*Proof.* Recall that:

$$(t_1 \blacktriangleleft) < (\overline{t}_1 \bigstar) < (t_2 \bigstar) < \dots < (t_n \bigstar) < (\overline{t}_n \bigstar),$$

$$(t_i t_{i+1} \bigstar) < (t_i \overline{t}_{i+1} \bigstar) < \dots < (t_i \overline{t}_n \bigstar) < (t_i t_1 \bigstar) < \dots < (t_i \overline{t}_{i-1} \bigstar) < (t_i t_i \bigstar),$$

$$(\overline{t}_i \overline{t}_i \bigstar) < (\overline{t}_i \overline{t}_{i+1} \bigstar) < \dots < (\overline{t}_i \overline{t}_n \bigstar) < (\overline{t}_i t_1 \bigstar) < \dots < (\overline{t}_i \overline{t}_{i-1} \bigstar) < (\overline{t}_i \overline{t}_{i-1} \bigstar).$$

(i). It is straightforward to see that  $wt_i((\Pi t_{[i\uparrow n]}\Pi t_{[1\uparrow i-1]})^{\infty}) = \min(wt_it_i \blacktriangleleft).$ 

Let x denote the element of  $t_{[1\uparrow n]} \cup \overline{t}_{[1\uparrow n]}$  such that  $\overline{w}((\Pi t_{[1\uparrow n]})^{\infty}) \in (x\blacktriangleleft)$ ; notice that  $x \neq \overline{t}_i$ .

If  $x \neq t_i$ , then  $(wt_i x \blacktriangleleft) < (wt_i t_i \blacktriangleleft)$ , and we have

$$wt_i\overline{w}((\Pi t_{[1\uparrow n]})^{\infty}) \in (wt_ix \blacktriangleleft) < (wt_it_i\bigstar) \ni \min(wt_it_i\bigstar).$$

If  $x = t_i$ , then  $\overline{w}$  is completely cancelled in  $\overline{w}((\Pi t_{[1\uparrow n]})^{\infty})$ , and, moreover,

$$wt_i\overline{w}((\Pi t_{[1\uparrow n]})^{\infty}) = wt_i((\Pi t_{[i\uparrow n]}\Pi t_{[1\uparrow i-1]})^{\infty}) = \min(wt_it_i\blacktriangleleft)$$

Thus, (i) holds.

(ii) is clear.

(iii). It is straightforward to see that  $w\overline{t}_i((\Pi\overline{t}_{[i\downarrow 1]}\Pi\overline{t}_{[n\downarrow i+1]})^{\infty}) = \max(w\overline{t}_i\overline{t}_i\blacktriangleleft)$ . Let x denote the element of  $t_{[1\uparrow n]} \cup \overline{t}_{[1\uparrow n]}$  such that  $\overline{w}((\Pi\overline{t}_{[n\downarrow 1]})^{\infty}) \in (x\blacktriangleleft)$ ; notice that  $x \neq t_i$ . If  $x \neq \overline{t}$ , then  $(w\overline{t},\overline{t}, \blacktriangleleft) = (w\overline{t}, x \dashv)$  and we have

If  $x \neq \overline{t}_i$ , then  $(w\overline{t}_i\overline{t}_i\blacktriangleleft) < (w\overline{t}_ix\blacktriangleleft)$ , and we have

$$\max(w\overline{t}_i\overline{t}_i\blacktriangleleft) \in (w\overline{t}_i\overline{t}_i\blacktriangleleft) < (w\overline{t}_ix\blacktriangleleft) \ni w\overline{t}_i\overline{w}((\Pi\overline{t}_{[n\downarrow 1]})^\infty).$$

If  $x = \overline{t}_i$ , then  $\overline{w}$  is completely cancelled in  $\overline{w}(\Pi \overline{t}_{[n \downarrow 1]})^{\infty}$ , and, moreover,

$$w\overline{t}_i\overline{w}((\Pi\overline{t}_{[n\downarrow 1]})^{\infty}) = w\overline{t}_i((\Pi\overline{t}_{[i\downarrow 1]}\Pi\overline{t}_{[n\downarrow i+1]})^{\infty}) = \max(w\overline{t}_i\overline{t}_i\blacktriangleleft)$$

Thus, (iii) holds.

(v). It is not difficult to see that

 $wt_i\overline{w}((\Pi t_{[1\uparrow n]})^\infty) \in (wt_i\blacktriangleleft)$  and  $w\overline{t}_i\overline{w}((\Pi \overline{t}_{[n\downarrow 1]})^\infty) \in (w\overline{t}_i\blacktriangleleft).$ 

Case 1.  $w \notin (t_1 \star)$ .

If w = 1, then

$$t_1((\Pi \overline{t}_{[n \downarrow 1]})^{\infty}) \in (t_1 \overline{t}_n \blacktriangleleft) < (t_i t_1 \blacktriangleleft) \ni t_i((\Pi t_{[1 \uparrow n]})^{\infty}) = w t_i \overline{w}((\Pi t_{[1 \uparrow n]})^{\infty}).$$

If  $w \neq 1$ , then  $t_1((\Pi \overline{t}_{[n \downarrow 1]})^{\infty}) \in (t_1 \blacktriangleleft) < (w \blacktriangleleft) \ni w t_i \overline{w}((\Pi t_{[1\uparrow n]})^{\infty})$ . In both subcases, (a) holds.

Case 2.  $w \in (t_1 \star)$ .

Here,  $w\overline{t}_i\overline{w}((\Pi\overline{t}_{[n\downarrow 1]})^{\infty}) \in (w\blacktriangleleft) \subseteq (t_1\blacktriangleleft)$ . Hence,

 $w\overline{t}_i\overline{w}((\Pi\overline{t}_{[n\downarrow 1]})^\infty) \leq \max(t_1 \blacktriangleleft) = t_1((\Pi\overline{t}_{[n\downarrow 1]})^\infty).$ 

To prove that (b) holds, it remains to show that

$$w\overline{t}_i\overline{w}((\Pi\overline{t}_{[n\downarrow 1]})^\infty) \neq t_1((\Pi\overline{t}_{[n\downarrow 1]})^\infty)$$

that is,  $\overline{t}_1 w \overline{t}_i \overline{w} ((\Pi \overline{t}_{[n \downarrow 1]})^{\infty}) \neq (\Pi \overline{t}_{[n \downarrow 1]})^{\infty}$ , that is,  $\overline{t}_1 w \overline{t}_i \overline{w} \notin \langle \Pi \overline{t}_{[n \downarrow 1]} \rangle$ . We can write  $w = t_1 u$  where  $u \notin (\overline{t}_1 \star)$ . Then  $\overline{t}_1 w \overline{t}_i \overline{w} = u \overline{t}_i \overline{u} \overline{t}_1$ , in normal form. Thus it suffices to show that  $u \overline{t}_i \overline{u} \overline{t}_1 \notin \langle \Pi \overline{t}_{[n \downarrow 1]} \rangle$ .

If u = 1, then  $u\overline{t}_{i}\overline{u}\overline{t}_{1} = \overline{t}_{i}\overline{t}_{1} \notin \langle \Pi\overline{t}_{[n\downarrow 1]} \rangle$ , since  $n \geq 3$ .

If  $u \neq 1$ , then  $u\overline{t}_i\overline{u}\overline{t}_1 \notin \langle \Pi\overline{t}_{[n\downarrow 1]} \rangle$ , since  $u\overline{t}_i\overline{u}\overline{t}_1$  does not lie in the submonoid of  $\Sigma_{0,1,n}$  generated by  $t_{[1\uparrow n]}$ , nor in the submonoid generated by  $\overline{t}_{[1\uparrow n]}$ .

In both subcases, (b) holds.

In both cases, (v) holds.

The following appeared as [5, Lema 2.2.17].

**7.5 Theorem.** If  $n \ge 1$  then, for each  $\phi \in \mathfrak{B}_n$ ,  $t_1^{\phi}((\Pi \overline{t}_{[n \downarrow 1]})^{\infty})$  is a squarefree end.

*Proof.* This is clear if n = 1.

For n = 2,  $\mathcal{B}_2 = \langle \sigma_1 \rangle$ , and

$$t_1^{\mathcal{B}_2} = \{ t_1^{\sigma_1^{2m}}, t_1^{\sigma_1^{1+2m}} \mid m \in \mathbb{Z} \} = \{ t_1^{(t_1 t_2)^m}, t_2^{(t_1 t_2)^m} \mid m \in \mathbb{Z} \}.$$

Thus, every word in  $t_1^{\mathcal{B}_2}$  is squarefree and does not end in  $\overline{t}_2$ . Hence, every end in  $t_1^{\mathcal{B}_2}((\Pi \overline{t}_{[n \downarrow 1]})^{\infty})$  is squarefree.

Thus, we may assume that  $n \geq 3$ .

Recall that  $z_1 = \Pi \overline{t}_{[n\uparrow 1]}$ , and, hence,  $\overline{z}_1 = \Pi t_{[1\uparrow n]}$ . Let  $\cup [t]_{[1\uparrow n]}$  denote  $\bigcup_{i\in[1\uparrow n]} [t_i]$ . By Lemma 7.4(v),  $t_1(z_1^{\infty})$  does not lie in

$$\bigcup_{x \in \cup[t]_{[1\uparrow n]}} [x(\overline{z}_1^{\infty}), \, \overline{x}(z_1^{\infty})] \quad (= \bigcup_{i=1}^n \bigcup_{w \in \Sigma_{0,1,n} - (\star t_i) - (\star \overline{t}_i)} [wt_i \overline{w}(\overline{z}_1^{\infty}), \, w\overline{t}_i \overline{w}(z_1^{\infty})]).$$

Notice that  $\phi$  permutes the elements of each of the following sets:  $\cup [t]_{[1\uparrow n]}$ ;  $\{\overline{z}_1^{\infty}\}$ ;  $\{z_1^{\infty}\}$ ; and,  $\bigcup_{x \in \cup [t]_{[1\uparrow n]}} [x(\overline{z}_1^{\infty}), \overline{x}(z_1^{\infty})]$ . Hence  $(t_1(z_1^{\infty}))^{\phi}$  does not lie in  $\bigcup_{x \in \cup [t]_{[1\uparrow n]}} [x(\overline{z}_1^{\infty}), \overline{x}(z_1^{\infty})]$ . By Lemma 7.4(iv),

$$\bigcup_{x \in \cup[t]_{[1\uparrow n]}} [x(\overline{z}_1^{\infty}), \, \overline{x}(z_1^{\infty})] \quad \supseteq \quad \bigcup_{i=1}^n \, \bigcup_{w \in \Sigma_{0,1,n} - (\star t_i) - (\star \overline{t}_i)} ((wt_i t_i \blacktriangleleft) \cup (w\overline{t}_i \overline{t}_i \blacktriangleleft))$$

Hence,  $(t_1(z_1^{\infty}))^{\phi}$  does not lie in the latter set either, and, hence,  $(t_1(z_1^{\infty}))^{\phi}$  is a squarefree end. Since  $(t_1(z_1^{\infty}))^{\phi} = t_1^{\phi}(z_1^{\infty})$ , the desired result holds.

We now obtain new information about the  $\mathcal{B}_n$ -orbit of  $t_1$  in  $\Sigma_{0,1,n}$ .

**7.6 Corollary.** Let  $n \ge 1$ , let  $\phi \in \mathcal{B}_n$ , and let  $k \in [1 \uparrow n]$ .

- (i).  $t_1^{\phi}$  is a squarefree word in  $\Sigma_{0,1,n}$ .
- (ii).  $t_1^{\phi} \not\in (\Pi \overline{t}_{[n \downarrow k+1]} t_k \star) \{ t_k^{\Pi t_{[k+1\uparrow n]}} \}.$
- (iii).  $t_1^{\phi} \not\in (\Pi t_{[1\uparrow k-1]} \overline{t}_k \star).$

*Proof.* Recall from Notation 3.1 that we write  $t_1^{\phi} = t_{1^{\pi(\phi)}}^{w_1(\phi)}$ . Let  $\pi = \pi(\phi)$  and  $w_1 = w_1(\phi)$ .

It is not difficult to see that

$$t_1^{\phi}(z_1^{\infty}) = \overline{w}_1 t_{1^{\pi}} w_1((\Pi \overline{t}_{[n \downarrow 1]})^{\infty}) \in (\overline{w}_1 \blacktriangleleft).$$

By Theorem 7.5,  $t_1^{\phi}(z_1^{\infty})$  is a squarefree end. Hence,  $\overline{w}_1$  is a squarefree word, and  $w_1 \notin (\star \overline{t}_k \Pi t_{[k+1 \downarrow n]})$ .

Since  $\overline{w}_1$  is a squarefree word,  $t_1^{\phi}$  is also a squarefree word. Hence (i) holds. Also,  $w_1 \notin (\star \overline{t}_k \Pi t_{[k+1\uparrow n]})$  implies that  $\overline{w}_1 \notin (\Pi \overline{t}_{[n\downarrow k+1]} t_k \star)$  and, hence,  $t_1^{\phi} \notin (\Pi \overline{t}_{[n\downarrow k+1]} t_k \star) - \{t_k^{\Pi t_{[k+1\uparrow n]}}\}$  and, also,  $\overline{t}_1^{\phi} \notin (\Pi \overline{t}_{[n\downarrow k+1]} t_k \star)$ . In particular, (ii) holds.

 $\frac{j\in[1\uparrow n]}{(t_j)^{\xi}} \quad . \quad \text{Then}$ Let  $\xi$  be the automorphism of  $\Sigma_{0,1,n}$  determined by  $=(\overline{t}_{n+1-i})$  $\xi^2 = 1$  and  $\xi \in Out_{0,1,n}^- := Out_{0,1,n} - Out_{0,1,n}^+$ . Also,

$$t_n^{\phi^{\xi}} = t_n^{\xi\phi\xi} = \overline{t}_1^{\phi\xi} \notin (\Pi \overline{t}_{[n\downarrow k+1]} t_k \star)^{\xi} = (\Pi t_{[1\uparrow n-k]} \overline{t}_{n+1-k} \star).$$

It follows that  $t_n^{\mathfrak{B}_n^{\xi}} \cap (\Pi t_{[1\uparrow n-k]} \overline{t}_{n+1-k} \star) = \emptyset$ . Since  $\mathfrak{B}_n^{\xi} = \mathfrak{B}_n$  and  $t_n^{\mathfrak{B}_n} = t_1^{\mathfrak{B}_n}$ , we see that  $t_1^{\phi} \notin (\Pi t_{[1\uparrow n-k]} \overline{t}_{n+1-k} \star)$ . Now replacing k with n+1-k gives (iii).  $\Box$ 

In Remark IV.3, we shall give a second proof of Corollary 7.6 using Larue-Whitehead diagrams.

#### Actions on free products of cyclic groups 8

**8.1 Notation.** Throughout this section, we assume that  $n \ge 1$  and we fix a positive integer N.

Let  $p_{([1\uparrow N])}$  be a partition of n. Thus,  $p_{([1\uparrow N])}$  is an N-tuple for  $[1\uparrow\infty]$  such that  $p_1 + \cdots + p_N = n$ .

Let  $m_{([1\uparrow N])}$  be an N-tuple for  $\mathbb{N} - \{1\}$ .

We let  $\Sigma_{0,1,p_1^{(m_1)} \perp p_2^{(m_2)} \perp \cdots \perp p_N^{(m_n)}}$  denote the group with presentation

$$\langle z, \tau_{[1\uparrow n]} \mid z \Pi \tau_{[1\uparrow n]}, \{\tau_{j+\sum p_{[1\uparrow i-1]}}^{m_i}\}_{i \in [1\uparrow N], j \in [1\uparrow p_i]} \rangle.$$

Thus,  $\Sigma_{0,1,p_1^{(m_1)} \perp p_2^{(m_2)} \perp \cdots \perp p_N^{(m_N)}}$  is isomorphic to a free product of cyclic groups,  $C_{m_1}^{*p_1} * C_{m_2}^{*p_2} * \cdots C_{m_N}^{*p_N}$ , where  $C_0$  is interpreted as  $C_{\infty}$ , and  $p_i^{(0)}$  is also written  $p_i$ with no exponent.

We let  $\operatorname{Out}_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}$  denote the group of all automorphisms of  $\Sigma_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_n)}}$  which map  $\{z,\overline{z}\}$  and

$$\{\{\{[\tau_i], [\overline{\tau}_i]\} \mid i \in [p_1 + \ldots + p_{j-1} + 1 \uparrow p_1 + \ldots + p_j]\} \mid j \in [1 \uparrow N]\}$$

to themselves.

We let  $\operatorname{Out}_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_n)}}^+$  denote the group of all automorphisms of  $\Sigma_{0,1,p_1^{(m_1)} \perp p_2^{(m_2)} \perp \dots \perp p_N^{(m_n)}}$  which map  $\{z\}$  and

$$\{\{[\tau_i] \mid i \in [p_1 + \dots + p_{j-1} + 1 \uparrow p_1 + \dots + p_j]\} \mid j \in [1 \uparrow N]\}$$

to themselves.

In the case where all the  $m_i$  are 0, we get groups denoted  $\operatorname{Out}_{0,1,p_1\perp p_2\perp\cdots\perp p_N}$ and  $\operatorname{Out}_{0,1,p_1 \perp p_2 \perp \cdots \perp p_N}^+$ . Notice that  $\operatorname{Out}_{0,1,p_1 \perp p_2 \perp \cdots \perp p_N}$  is the subgroup of  $\operatorname{Out}_{0,1,n}$ consisting of those elements such that the permutation in  $Sym_n$ , arising from the permutation of  $\{\{[t_1], [\overline{t}_1]\}, \ldots, \{[t_n], [\overline{t}_n]\}\}$ , lies in the natural image of  $\operatorname{Sym}_{p_1} \times \operatorname{Sym}_{p_2} \times \cdots \times \operatorname{Sym}_{p_N}$  in  $\operatorname{Sym}_n$ .

There are natural maps

(8.1.1) 
$$\operatorname{Out}_{0,1,p_1 \perp p_2 \perp \dots \perp p_N} \to \operatorname{Out}_{0,1,p_1^{(m_1)} \perp p_2^{(m_2)} \perp \dots \perp p_N^{(m_n)}},$$

(8.1.2) 
$$\operatorname{Out}_{0,1,p_1 \perp p_2 \perp \cdots \perp p_N}^+ \to \operatorname{Out}_{0,1,p_1^{(m_1)} \perp p_0^{(m_2)} \perp \cdots \perp p_N^{(m_n)}}^+$$

Since (8.1.2) is of index two in (8.1.1), we see that (8.1.1) is injective, resp. surjective, resp. bijective, if and only if (8.1.2) is.

For topological reasons, we suspect that (8.1.1) and (8.1.2) are isomorphisms. In this section, we shall prove that this holds in the case where all the  $m_i$  are equal or N = 1. We begin by proving that (8.1.1) and (8.1.2) are injective, which seems to be new.

8.2 Theorem. With Notation 8.1, the maps

(8.1.1) 
$$\operatorname{Out}_{0,1,p_1 \perp p_2 \perp \dots \perp p_N} \to \operatorname{Out}_{0,1,p_1^{(m_1)} \perp p_2^{(m_2)} \perp \dots \perp p_N^{(m_n)}},$$

(8.1.2) 
$$\operatorname{Out}_{0,1,p_1 \perp p_2 \perp \dots \perp p_N}^+ \to \operatorname{Out}_{0,1,p_1^{(m_1)} \perp p_2^{(m_2)} \perp \dots \perp p_N^{(m_n)}}^+$$

are injective.

Proof. Suppose that  $\phi$  is an element of the kernel of (8.1.1) or (8.1.2). Clearly,  $\phi \in \operatorname{Out}_{0,1,n}^+$ , and  $t_{([1\uparrow n])}^{\phi}$ ,  $t_{([1\uparrow n])}$  have the same image in  $\Sigma_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}$ . By Theorem 7.5,  $(t_{([1\uparrow n])})^{\phi}$  is an *n*-tuple of squarefree words in  $\Sigma_{0,1,n}$ , and, hence, has the same normal form in  $\Sigma_{0,1,n}$  and in  $\Sigma_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}$ . Hence  $t_{([1\uparrow n])}^{\phi} = t_{([1\uparrow n])}$  as *n*-tuples for  $\Sigma_{0,1,n}$ . Thus  $\phi = 1$ , and the result is proved.  $\Box$ 

**8.3 Historical Remarks.** Let us now restrict to the classic case where N = 1. Here, for an integer  $m \ge 2$ , we are considering the action of  $\operatorname{Out}_{0,1,n}$  on  $C_m^{*n}$ , and it induces maps

$$(8.3.1) \qquad \qquad \operatorname{Out}_{0,1,n} \to \operatorname{Out}_{0,1,n^{(m)}},$$

$$(8.3.2) \qquad \qquad \operatorname{Out}_{0,1,n}^+ \to \operatorname{Out}_{0,1,n^{(m)}}^+$$

Theorem 8.2 shows that these maps are injective. Birman-Hilden [6, Theorem 7] gave a topological proof that (8.3.2) is injective, thus answering a question of Magnus. Crisp-Paris [11] gave an elegant algebraic proof of the injectivity of (8.3.2) using the trichotomy argument of Larue [22] and Shpilrain [28]. The Crisp-Paris argument can be summarized as follows.

For each  $i \in [1\uparrow n]$ , let  $(\langle \tau_i \rangle \star)$  denote the set of elements of  $\Sigma_{0,1,n^{(m)}}$  whose free-product normal form begins with an element of  $\langle \tau_i \rangle - \{1\}$ .

Suppose that  $\phi$  is a non-trivial element of  $\mathcal{B}_n = \operatorname{Out}_{0,1,n}^+$ . We will show that  $\phi$  acts non-trivially on  $\Sigma_{0,1,n^{(m)}}$ .

We may assume that  $n \geq 3$ . By Theorem 6.7, by replacing  $\phi$  with  $\overline{\phi}$  if necessary, we may assume that  $\phi$  is  $\sigma$ -negative. Thus there exists some  $i \in [1\uparrow n - 1]$  such that  $\phi$  is the product of a finite sequence of elements of  $\sigma_{[i+1\uparrow n-1]} \cup \overline{\sigma}_{[i\uparrow n-1]}$ , and  $\overline{\sigma}_i$  appears at least once in the sequence.

With Notation 6.5,

$$(\langle \tau_i \rangle \star)^{\overline{\sigma}_i} = (\langle \tau_i \rangle \star)^{\overline{\sigma}'_i \overline{\sigma}'_i} = (\langle \tau_{i+1} \rangle \star)^{\overline{\sigma}'_i} \subseteq (\tau_i (\langle \tau_{i+1} \rangle \star)) \subset (\langle \tau_i \rangle \star),$$

since  $n \geq 3$ . Because the elements of  $\sigma_{[i+1\uparrow n-1]} \cup \overline{\sigma}_{[i\uparrow n-1]}$  act as injective self-maps on  $(\langle \tau_i \rangle \star)$ , it follows that  $(\langle \tau_i \rangle \star)^{\phi} \subset (\langle \tau_i \rangle \star)$ , and, hence,  $\phi$  acts non-trivially on  $\Sigma_{0,1,n^{(m)}}$ , as desired.

Let us now verify the surjectivity of the maps (8.3.1) and (8.3.2). The case where m = 2 was verified by Stephen Humphries [2, Lemma 2.1.7].

**8.4 Notation.** Let  $m, n \in \mathbb{N}$  with  $n \geq 1$  and  $m \geq 2$ . Let  $\lfloor \frac{m}{2} \rfloor$  denote the greatest integer not exceeding  $\frac{m}{2}$ . Then  $[0\uparrow\lfloor\frac{m}{2}\rfloor] \cup [-1\downarrow(-\lfloor\frac{m-1}{2}\rfloor)]$  is a set of representatives for the integers modulo m. For  $\tau^k \in \langle \tau \mid \tau^m = 1 \rangle$ , we define  $|\tau^k|_{k \in [0\uparrow\lfloor\frac{m}{2}\rfloor]} k \in [-1\downarrow-\lfloor\frac{m-1}{2}\rfloor]$ 

by  $\begin{array}{c|c} \frac{k \in [0\uparrow \lfloor \frac{m}{2} \rfloor]}{\left( \begin{array}{c|c} |\tau^k| \end{array} & \frac{k \in [-1\downarrow - \lfloor \frac{m-1}{2} \rfloor]}{|\tau^k| \end{array} \right)} \\ = \left( \begin{array}{cc} 2k \end{array} & -2k-1 \right) \end{array}; \text{ we extend } |-| \text{ to all of } \Sigma_{0,1,n^{(m)}} \text{ by using normal} \end{array}$ 

forms for the free product  $C_m^*$ .

Let  $\phi \in \operatorname{Out}_{0,1,n^{(m)}}^+$ . There exists a unique permutation  $\pi \in \operatorname{Sym}_n$ , and a unique (n+2)-tuple  $(w_{([0\uparrow n+1])})$  for  $\Sigma_{0,1,n^{(m)}}$  such that  $w_0 = 1$  and  $w_{n+1} = 1$ , and, for each  $i \in [1\uparrow n]$ ,  $w_i \notin (t_{i\pi}\star) \cup (\overline{t}_{i\pi}\star)$  and  $t_i^{\phi} = t_{i\pi}^{w_i}$ . For each  $i \in [0\uparrow n]$ , let  $u_i = w_i \overline{w}_{i+1}$ . We define  $\pi(\phi) := \pi$ ,  $w_i(\phi) := w_i$ ,  $i \in [0\uparrow n+1]$ , and  $u_i(\phi) := u_i$ ,  $i \in [0\uparrow n]$ . We write  $\|\phi\| := n+2\sum_{i\in [1\uparrow n]} |w_i(\phi)|$ .

The following is similar to Artin's Lemma 3.2.

**8.5 Lemma.** Let  $n \ge 1$ ,  $m \ge 2$  and let  $\phi \in \operatorname{Out}_{0,1,n^{(m)}}$ . Let  $\pi = \pi(\phi)$ . For each  $i \in [0\uparrow n]$ , let  $u_i = u_i(\phi)$ . For each  $i \in [1\uparrow n]$ , let  $a_i$ ,  $b_i$  denote the elements of [0, m-1] determined by the following: there exists some  $u'_i \in \Sigma_{0,1,n^{(m)}} - (\star \langle \tau_i \pi \rangle)$  such that  $u_{i-1} = u'_i \tau_{i\pi}^{a_i}$ ; there exists some  $u''_i \in \Sigma_{0,1,n^{(m)}} - (\langle \tau_i \pi \rangle \star)$  such that  $u_i = \tau_{i\pi}^{b_i} u''_i$ . In particular,  $a_1 = b_n = 0$ .

- (i). Suppose that there exists some  $i \in [2\uparrow n]$  such that  $a_i \in [\lfloor \frac{m}{2} \rfloor \uparrow m-1]$ . Then  $\|\sigma_{i-1}\phi\| < \|\phi\|$ .
- (ii). Suppose that there exists some  $i \in [1\uparrow n-1]$  such that  $b_i \in [\lfloor \frac{m+1}{2} \rfloor \uparrow m-1]$ . Then  $\|\overline{\sigma}_i \phi\| < \|\phi\|$ .

(iii). If  $\phi \neq 1$ , there exists some  $\sigma_i^{\epsilon} \in \sigma_{[1\uparrow n-1]} \cup \overline{\sigma}_{[1\uparrow n-1]}$  such that  $\|\sigma_i^{\epsilon}\phi\| < \|\phi\|$ .

*Proof.* (i). Let  $a = a_i$ . There exists some  $v \in \Sigma_{0,1,n^{(m)}} - (\star \langle \tau_{i^{\pi}} \rangle)$  such that  $u_{i-1} = v \tau_{i^{\pi}}^a$ . Since  $w_{i-1}(\phi) = u_{i-1} w_i(\phi)$ , we have

(8.5.1) 
$$w_{i-1}(\phi) = v\tau_{i^{\pi}}^{a}w_{i}(\phi);$$

since  $w_i(\phi) \notin (\langle \tau_{i^\pi} \rangle \star)$  and  $v \notin (\star \langle \tau_{i^\pi} \rangle), v \tau_{i^\pi}^a w_i(\phi)$  is a free-product normal form for  $w_{i-1}(\phi)$ .

Claim.  $|\tau_{i^{\pi}}^{a+1}| < |\tau_{i^{\pi}}^{a}|.$ 

*Proof.* If  $a \in \lfloor \lfloor \frac{m}{2} \rfloor + 1 \uparrow m - 1 \rfloor$ , then  $a - m \in \lfloor - \lfloor \frac{m-1}{2} \rfloor \uparrow - 1 \rfloor$ , and, hence,

$$|\tau_{i^{\pi}}^{a}| = |\tau_{i^{\pi}}^{a-m}| = -2(a-m) - 1 = 2m - 2a - 1.$$

Therefore, if  $a \in [\lfloor \frac{m}{2} \rfloor \uparrow m - 2]$ ,  $|\tau_{i^{\pi}}^{a+1}| = 2m - 2(a+1) - 1 = 2m - 2a - 3$ . Thus,  $|\tau_{i^{\pi}}^{a+1}| < |\tau_{i^{\pi}}^{a}|$  if  $a \in [\lfloor \frac{m}{2} \rfloor + 1 \uparrow m - 2]$ . For  $a = \lfloor \frac{m}{2} \rfloor$ ,  $a \ge \frac{m-1}{2}$ , and  $|\tau_{i^{\pi}}^{a}| = 2a > 2m - 2a - 3 = |\tau_{i^{\pi}}^{a+1}|$ . For a = m - 1,  $|\tau_{i^{\pi}}^{a}| = 1$  and  $|\tau_{i^{\pi}}^{a+1}| = 0$ . 

Thus,

$$|w_{i-1}(\phi)| = |v| + |\tau_{i^{\pi}}^{a}| + |w_{i}(\phi)| > |v| + |\tau_{i^{\pi}}^{a+1}| + |w_{i}(\phi)|.$$

By (8.5.1),  $w_{i-1}(\phi)\overline{w}_{i}(\phi)\tau_{i^{\pi}} = v\tau_{i^{\pi}}^{a+1}$ ; hence

$$\tau_i^{\sigma_{i-1}\phi} = (\tau_{i-1}^{\tau_i})^{\phi} = (\tau_{(i-1)^{\pi}}^{w_{i-1}(\phi)})^{(\tau_{i^{\pi}}^{w_i(\phi)})} = \tau_{(i-1)^{\pi}}^{v\tau_{i^{\pi}}^{a+1}w_i(\phi)}.$$

Hence,  $|w_i(\sigma_{i-1}\phi)| = |v\tau_{i\pi}^{a+1}w_i(\phi)| \le |v| + |\tau_{i\pi}^{a+1}| + |w_i(\phi)| < |w_{i-1}(\phi)|.$ For each  $j \in [1\uparrow i-2] \cup [i+1\uparrow n], \ \tau_j^{\sigma_{i-1}\phi} = \tau_j^{\phi}$ , and, hence,  $|w_j(\sigma_{i-1}\phi)| = |w_i(\phi)|.$ 

 $|w_j(\phi)|.$ 

Also,  $\tau_{i-1}^{\sigma_{i-1}\phi} = \tau_i^{\phi}$ ; in particular,  $|w_{i-1}(\sigma_{i-1}\phi)| = |w_i(\phi)|$ . It now follows that  $||\sigma_{i-1}\phi|| < ||\phi||$ .

(ii). Let  $b = b_i$ . There exists some  $v \in \sum_{0,1,n^{(m)}} - (\langle \tau_{i\pi} \rangle \star)$  such that  $u_i = \tau_{i\pi}^b v$ . Since  $w_{i+1}(\phi) = \overline{u}_i w_i(\phi)$ , we have

(8.5.2) 
$$w_{i+1}(\phi) = \overline{v} \ \overline{\tau}_{i^{\pi}}^{b} w_{i}(\phi),$$

Since  $w_i(\phi) \notin (\langle \tau_{i^{\pi}} \rangle \star)$  and  $\overline{v} \notin (\star \langle \tau_{i^{\pi}} \rangle)$ ,  $\overline{v} \ \overline{\tau}_{i^{\pi}}^b w_i(\phi)$  is a free-product normal form for  $w_{i+1}(\phi)$ . Hence,  $|w_{i+1}(\phi)| = |\overline{v}| + |\overline{\tau}_{i^{\pi}}^b| + |w_i(\phi)|$ . Claim.  $|\overline{\tau}_{i^{\pi}}^{b+1}| < |\overline{\tau}_{i^{\pi}}^{b}|.$ 

*Proof.* For any  $b \in \lfloor \lfloor \frac{m+1}{2} \rfloor \uparrow m \rfloor$ , then  $m - b \in \lfloor \lfloor \frac{m}{2} \rfloor \downarrow 0 \rfloor$ , and, hence,

$$|\overline{\tau}_{i^{\pi}}^{b}| = |\tau_{i^{\pi}}^{m-b}| = 2(m-b) = 2m - 2b$$

Therefore, since  $b \in \left[ \lfloor \frac{m+1}{2} \rfloor \uparrow m - 1 \right]$ ,

$$|\overline{\tau}_{i^{\pi}}^{b+1}| = 2m - 2(b+1) = 2m - 2b - 2 < |\overline{\tau}_{i^{\pi}}^{b}|,$$

as claimed.

Hence  $|w_{i+1}(\phi)| > |\overline{v}| + |\overline{\tau}_{i\pi}^{b+1}| + |w_i(\phi)|$ . For all  $j \in [1\uparrow i-1] \cup [i+2\uparrow n], \ \tau_j^{\overline{\sigma}_i\phi} = \tau_j^{\phi}$ ; hence,  $|w_j(\overline{\sigma}_i\phi)| = |w_j(\phi)|$ . Since  $\tau_{i+1}^{\overline{\sigma}_i\phi} = \tau_i^{\phi}$ , we see that  $|w_{i+1}(\overline{\sigma}_i\phi)| = |w_i(\phi)|$ . By (8.5.2),  $w_{i+1}(\phi)\overline{w}_i(\phi)\overline{\tau}_{i^{\pi}} = \overline{v} \ \overline{\tau}_{i^{\pi}}^{b+1}$ ; hence

$$\tau_i^{\overline{\sigma}_i \phi} = (\tau_{i+1}^{\overline{\tau}_i})^{\phi} = (\tau_{(i+1)^{\pi}}^{w_{i+1}(\phi)})^{(\overline{\tau}_{i^{\pi}}^{w_i(\phi)})} = \tau_{i^{\pi}}^{\overline{v} \, \overline{\tau}_{i^{\pi}}^{b+1} w_i(\phi)}.$$

Hence,  $|w_i(\overline{\sigma}_i\phi)| = |\overline{v} \ \overline{\tau}_{i^{\pi}}^{b+1}w_i(\phi)| \le |\overline{v}| + |\overline{\tau}_{i^{\pi}}^{b+1}| + |w_i(\phi)| < |w_{i+1}(\phi)|.$ It now follows that  $\|\overline{\sigma}_i\phi\| < \|\phi\|$ , and (ii) is proved.

(iii). If  $\phi \neq 1$ , we choose a distinguished element of  $[1\uparrow n]$  as follows.

If, for some  $i \in [1 \uparrow n]$ ,  $\tau_{i\pi}^{a_i+1+b_i} = 1$ , we take any such i to be our distinguished element of  $[1\uparrow n]$ .

Consider then the case where, for all  $i \in [1 \uparrow n]$ ,  $\tau_{i^{\pi}}^{a_i+1+b_i} \neq 1$ . Thus, there is no further cancellation in  $\Pi \tau^{\phi}_{[1\uparrow n]}$ . Since  $\phi$  fixes  $\Pi \tau_{[1\uparrow n]}$ , it is not difficult to see that, for all  $i \in [1\uparrow n]$ ,  $\tau^{a_i+1+b_i}_{i\pi} = \tau_i$ . Since  $\phi \neq 1$ , it is then not difficult to show that there exists some  $i \in [1 \uparrow n]$  such that  $(a_i, b_i) \neq (0, 0)$ . We take any such i to be our distinguished element of  $[1\uparrow n]$ .

Let *i* denote our distinguished element of  $[1\uparrow n]$ .

Notice that  $(a_i, b_i) \neq (0, 0)$  and that  $\tau_{i^{\pi}}^{a_i+1+b_i} \in \{1, \tau_{i^{\pi}}\}$ . Hence,  $a_i + 1 + b_i \in$  $\{m, m+1\}$ , and, hence,  $b_i \in \{m-a_i-1, m-a_i\}$ .

Case 1.  $a_i \in [\lfloor \frac{m}{2} \rfloor \uparrow m - 1].$ Here,  $i \in [2 \uparrow n]$  and, by (i),  $\|\sigma_{i-1}\phi\| < \|\phi\|.$ 

Case 2.  $a_i \in [0\uparrow\lfloor\frac{m-2}{2}\rfloor]$ Here,  $m - a_i - 1 \in [m - 1\downarrow\lfloor\frac{m+1}{2}\rfloor]$ , and, hence,  $b_i \in [\lfloor\frac{m+1}{2}\rfloor\uparrow m - 1]$ . Here,  $i \in [1\uparrow n - 1]$  and, by (ii),  $\|\overline{\sigma}_i\phi\| < \|\phi\|$ .

**8.6 Theorem.** Let  $n \geq 1$ ,  $m \geq 2$ . The natural map  $\operatorname{Out}_{0,1,n}^+ \to \operatorname{Out}_{0,1,n^{(m)}}^+$  is an isomorphism, and, hence, the natural map  $\operatorname{Out}_{0,1,n} \to \operatorname{Out}_{0,1,n^{(m)}}$  is an isomorphism.

With Notation 8.1, the maps  $\operatorname{Out}_{0,1,p_1 \perp p_2 \perp \cdots \perp p_N} \to \operatorname{Out}_{0,1,p_1^{(m)} \perp p_2^{(m)} \perp \cdots \perp p_N^{(m)}}$ , and  $\operatorname{Out}_{0,1,p_1\perp p_2\perp\cdots\perp p_N}^+ \to \operatorname{Out}_{0,1,p_1^{(m)}\perp p_2^{(m)}\perp\cdots\perp p_N^{(m)}}^+$  are isomorphisms.

The following is essentially an algebraic translation of a part of a topological argument in [26, Section 3].

**8.7 Proposition.** With Notation 8.1, let H be a subgroup of

 $\Sigma_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_n)}}$ 

of finite index, and let A be the subgroup of

$$\operatorname{Out}_{0,1,p_1^{(m_1)} \perp p_2^{(m_2)} \perp \dots \perp p_N^{(m_n)}}$$

consisting of elements which map H to itself. Then, either the induced map  $A \rightarrow \operatorname{Aut}(H)$  is injective or  $(n, N, m_1) = (2, 1, 2)$ .

*Proof.* Suppose that  $\phi \in \text{Out}_{0,1,p_1^{(m_1)} \perp p_2^{(m_2)} \perp \cdots \perp p_N^{(m_n)}}$ , and that  $\phi$  acts as the identity on H. We shall show that  $\phi = 1$  or  $(n, N, m_1) = (2, 1, 2)$ .

Let  $G = \sum_{0,1,p_1^{(m_1)} \perp p_2^{(m_2)} \perp \dots \perp p_N^{(m_n)}}$ .

For any  $g \in G$ , right multiplication by g permutes the elements of the finite set  $H \setminus G$ , so there exists some positive integer k such that  $g^k$  acts trivially on  $H \setminus G$ . In particular,  $Hg^k = H$  and, hence,  $g^k \in H$ .

Hence, there exists some positive integer k such that  $(\Pi \tau_{[1\uparrow n]})^k \in H$ . Now  $(\Pi \tau_{[1\uparrow n]})^{\phi} = (\Pi \tau_{[1\uparrow n]})^{\epsilon}$  for some  $\epsilon \in \{1, -1\}$ , and, hence,

$$(\Pi\tau_{[1\uparrow n]})^k = (\Pi\tau_{[1\uparrow n]})^{k\phi} = (\Pi\tau_{[1\uparrow n]})^{\phi k} = (\Pi\tau_{[1\uparrow n]})^{\epsilon k} = (\Pi\tau_{[1\uparrow n]})^{k\epsilon}.$$

Since  $\Pi \tau_{[1\uparrow n]}$  has infinite order in G, we see that  $\epsilon = 1$ . Thus  $\phi$  fixes  $\Pi \tau_{[1\uparrow n]}$ .

Consider any  $i \in [1\uparrow n]$ . Since  $(\Pi \tau_{[1\uparrow n]})^{\tau_i} \in G$ , there exists some positive integer k such that  $(\Pi \tau_{[1\uparrow n]})^{\tau_i k} \in H$ . Hence,

$$(\Pi\tau_{[1\uparrow n]})^{k\tau_i} = (\Pi\tau_{[1\uparrow n]})^{\tau_i k} = (\Pi\tau_{[1\uparrow n]})^{\tau_i k\phi} = (\Pi\tau_{[1\uparrow n]})^{k\phi\tau_i^{\phi}} = (\Pi\tau_{[1\uparrow n]})^{k\tau_i^{\phi}}.$$

Hence  $\tau_i^{\phi} \overline{\tau}_i$  commutes with  $(\Pi \tau_{[1\uparrow n]})^k$ . A straightforward normal-form argument shows that  $\tau_i^{\phi} \overline{\tau}_i \in \langle \Pi \tau_{[1\uparrow n]} \rangle$ .

Hence there exists an integer j such that  $\tau_i^{\phi} = (\Pi \tau_{[1\uparrow n]})^j \tau_i$ . Since  $\tau_i^{\phi}$  is a conjugate of  $\tau_{i^{\pi(\phi)}}$ , the cyclically-reduced form of  $(\tau_{[1,n]})^j \tau_i$  is  $\tau_{i^{\pi(\phi)}}$ . Either j = 0, or there must be cyclic cancellation, and a straightforward analysis then shows that  $(n, N, m_1) = (2, 1, 2)$ . Since i was arbitrary, this completes the proof.  $\Box$ 

## 9 The $\mathcal{B}_{n+1}$ -group $\Phi_n$

**9.1 Notation.** Recall that  $\Sigma_{0,1,(n+1)^{(2)}} = C_2^{*(n+1)} = \langle \tau_{[1\uparrow n+1]} \mid \tau_{[1\uparrow n+1]}^2 = 1 \rangle$ . We define  $\Phi_n$  to be the  $\mathcal{B}_{n+1}$ -group consisting of the set of elements of  $\Sigma_{0,1,(n+1)^{(2)}}$  which have even exponent sum in the  $\tau_i$ . It is not difficult to see that  $\Phi_n$  is a free group of rank n, and that there is induced a map from  $\operatorname{Out}_{0,1,n+1} = \operatorname{Out}_{0,1,(n+1)^{(2)}}$  to  $\operatorname{Aut} \Phi_n$ . Since  $\mathcal{B}_{n+1} = \operatorname{Out}_{0,1,n+1}^+ = \operatorname{Out}_{0,1,(n+1)^{(2)}}^+$ ,  $\Phi_n$  has a  $\mathcal{B}_{n+1}$ -action; we say that  $\Phi_n$  is a  $\mathcal{B}_{n+1}$ -group, and that  $\Phi_n$  is a  $\mathcal{B}_{n+1}$ -subgroup of  $\Sigma_{0,1,(n+1)^{(2)}}$ .

Proposition 8.7 shows that, if  $n \neq 1$ , then the map from  $\operatorname{Out}_{0,1,n+1} = \operatorname{Out}_{0,1,(n+1)^{(2)}}$  to  $\operatorname{Aut} \Phi_n$  is injective, and we say that the  $\mathcal{B}_{n+1}$ -action is faithful, and that  $\Phi_n$  is a faithful  $\mathcal{B}_{n+1}$ -group.

Over the course of this section, we shall choose various free generating sets of  $\Phi_n$  to obtain interesting actions. In the next two examples, we identify  $\Sigma_{g,1,0}$ with  $\Phi_{2g}$  and  $\Sigma_{g,2,0}$  with  $\Phi_{2g+1}$ .

**9.2 Example.** Now that algebraic proofs of the requisite theorems are known to us, let us review [18, Example 15.6] which was an algebraic approximation of results in [26, Section 3].

Let  $g \in \mathbb{N}$ . Let  $\Sigma_{g,1,0} := \langle x_1, y_1, \dots, x_g, y_g, z_1 \mid [x_1, y_1] \cdots [x_g, y_g] z_1 = 1 \rangle$ ,

where the commutator [x, y] of group elements x, y is  $\overline{x} \ \overline{y}xy$ . Let  $\operatorname{Out}_{g,1,0}^+$  denote the group of all automorphisms of  $\Sigma_{g,1,0}$  which fix  $z_1$ . Then  $\Sigma_{g,1,0}$  is free of rank 2g with ordered free generating set  $(x_1, y_1, \ldots, x_g, y_g)$ , and  $\operatorname{Out}_{g,1,0}^+$  is the group of all automorphisms of  $\Sigma_{g,1,0}$  which fix  $[x_1, y_1] \cdots [x_g, y_g]$ .

We now recall some Dehn-twist elements of  $\operatorname{Out}_{g,1,0}^+$  from Definitions 3.10 and Remarks 5.1 of [18].

For each  $i \in [1\uparrow g]$ , we define  $\alpha_i, \beta_i \in \text{Out}_{g,1,0}^+$  by

$k \in [1 \uparrow$	i - 1]			$k \in [i +$	$1\uparrow g]$		$k \in [1 \uparrow$	i - 1]			$k{\in}[i+$	$1\uparrow g]$
					$(y_k)^{\alpha_i}$	and	$(x_k)$	$y_k$	$x_i$	$y_i$	$x_k$	$(y_k)^{\beta_i}$
$=(x_k)$	$y_k$	$\overline{y}_i x_i$	$y_i$	$x_k$	$y_k),$		$=(x_k)$	$y_k$	$x_i$	$x_i y_i$	$x_k$	$y_k).$

For each  $i \in [1 \uparrow g - 1]$ , we define  $\gamma_i \in \text{Out}_{q,1,0}^+$  by

$$\frac{k \in [1\uparrow i-1]}{(x_k \ y_k \ x_i \ y_i \ x_{i+1} \ y_{i+1} \ x_k \ y_k)^{\gamma}} = (x_k \ y_k \ y_{i+1}^{x_{i+1}} \overline{y}_i x_i \ y_i^{\overline{y}_{i+1}^{x_{i+1}}} \ x_{i+1} y_i \overline{y}_{i+1}^{x_{i+1}} \ y_{i+1} \ x_k \ y_k).$$

Let us identify  $\Sigma_{g,1,0}$  with  $\Phi_{2g}$  via

$$\begin{array}{c|c} & & \\ \hline & & \\ \hline (x_k & y_k & z_1)^{\Sigma_{g,1,0} \stackrel{\sim}{\to} \Phi_{2g}} \\ = (\Pi \tau_{[2k+1\downarrow 2k]} & \tau_{2k+1} \Pi \tau_{[1\uparrow 2k+1]} & z_1^2). \end{array}$$

Notice that  $[x_k, y_k] = \overline{x}_k \overline{y}_k x_k y_k$  is then identified with

$$\Pi \tau_{[2k\uparrow 2k+1]} \Pi \tau_{[2k+1\downarrow 1]} \tau_{2k+1} \Pi \tau_{[2k+1\downarrow 2k]} \tau_{2k+1} \Pi \tau_{[1\uparrow 2k+1]}$$

which equals  $\Pi \tau_{[2k-1\downarrow 1]} \Pi \tau_{[2k\uparrow 2k+1]} \Pi \tau_{[1\uparrow 2k+1]}$ . Hence  $\prod_{k \in [1\uparrow g]} [x_k, y_k]$  is identified with  $(\Pi \tau_{[1\uparrow 2g+1]})^2$ .

This corresponds to the surface of genus g with one boundary component arising as a two-sheeted branched cover of a sphere with one boundary component and 2g + 1 double points. Then  $\mathcal{B}_{2g+1} = \operatorname{Out}_{0,1,2g+1}^+ = \operatorname{Out}_{0,1,(2g+1)^{(2)}}^+$ becomes embedded in  $\operatorname{Out}_{g,1,0}^+$  via the homomorphism represented as

$$\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \cdots & \sigma_{2g-2} & \sigma_{2g-1} & \sigma_{2g} \\ \alpha_1 & \beta_1 & \gamma_1 & \beta_2 & \gamma_2 & \cdots & \beta_{g-1} & \gamma_{g-1} & \beta_g \end{pmatrix}.$$

Clearly, in the preceding example, the subgroup  $\mathcal{B}_{2g}$  of  $\mathcal{B}_{2g+1}$  is also embedded in  $\operatorname{Out}_{g,1,0}$ , but it is more natural to remove from the surface a handle containing the boundary component (a sphere with three boundary components or a 'pair of pants'), and embed  $\mathcal{B}_{2g}$  in  $\operatorname{Out}_{g-1,2,0}$ , as follows.

**9.3 Example.** Now that algebraic proofs of the requisite theorems are known to us, let us review [18, Example 15.7] which was an algebraic approximation of results in [26, Section 3].

Let  $g \in \mathbb{N}$ . Let

$$\Sigma_{g,2,0} := \langle x_{[1\uparrow g]}, y_{[1\uparrow g]}, z_{[1\uparrow 2]} \mid (\prod_{i \in [1\uparrow g]} [x_i, y_i]) \prod z_{[1\uparrow 2]} = 1 \rangle.$$

Recall that  $[x, y] := \overline{x} \ \overline{y}xy$ . Then  $\Sigma_{g,2,0}$  is free of rank 2g+1 with free generating set  $(x_{[1\uparrow g]}, y_{[1\uparrow g]}, z_1)$  and distinguished element  $z_2$  such that  $\overline{z}_2 = (\prod_{i \in [1\uparrow g]} [x_i, y_i])z_1$ . Let  $\operatorname{Out}_{g,1\perp 1,0}^+$  denote the group of all automorphisms of  $\Sigma_{g,2,0} * \langle e_1 | \rangle$  which map  $\Sigma_{g,2,0}$  to itself, and fix  $z_1^{e_1}$  and  $z_2$ . It can be shown that  $\operatorname{Out}_{g,1\perp 1,0}^+$  acts faithfully on the subset  $\Sigma_{g,2,0} \cup \Sigma_{g,2,0}e_1$  of  $\Sigma_{g,2,0} * \langle e_1 | \rangle$ .

Here,  $e_1$  represents an arc from the base-point of one boundary component, to the base-point of the other boundary component. Karen Vogtmann calls such an arc a 'tether joining the basepoint to the second boundary component'. For any surface-with-boundaries, A'Campo [1, Section 4, Remarque 6], [26, p.232] identifies basepoints of all the boundary components, which makes tethers into loops, to obtain a topological quotient space whose fundamental group is acted on, faithfully, by the mapping-class group of the surface-with-boundaries.

We now recall some Dehn-twist elements of  $\operatorname{Out}_{g,1\perp 1,0}^+$  from Definitions 3.10 and Remarks 5.1 of [18].

For each  $i \in [1\uparrow g]$ , we define  $\alpha_i, \beta_i \in \text{Out}_{q,1\perp 1,0}^+$  by

$$\frac{k \in [1\uparrow i-1]}{(x_{k} \quad y_{k} \quad x_{i} \quad y_{i} \quad x_{k} \quad y_{k} \quad z_{1} \quad e_{1})^{\alpha_{i}}}{= (x_{k} \quad y_{k} \quad \overline{y}_{i}x_{i} \quad y_{i} \quad x_{k} \quad y_{k} \quad z_{1} \quad e_{1}),}$$

$$\frac{k \in [1\uparrow i-1]}{(x_{k} \quad y_{k} \quad x_{i} \quad y_{i} \quad x_{k} \quad y_{k} \quad z_{1} \quad e_{1})^{\beta_{i}}}{= (x_{k} \quad y_{k} \quad x_{i} \quad x_{i}y_{i} \quad x_{k} \quad y_{k} \quad z_{1} \quad e_{1}).}$$

For each  $i \in [1\uparrow g - 1]$ , we define  $\gamma_i \in \text{Out}_{g,1\perp 1,0}^+$  by

$$\frac{k \in [1\uparrow i-1]}{(x_k \ y_k \ x_i \ y_i \ x_{i+1}} \underbrace{x_{i+1} \ y_i \ x_{i+1} \ y_{i+1} \ y_{i+1} \ x_k \ y_k \ z_1 \ e_1)^{\gamma_i}}_{= (x_k \ y_k \ y_{i+1}^{x_{i+1}} \overline{y}_i x_i \ y_i^{\overline{y}_{i+1}^{x_{i+1}}} \ x_{i+1} y_i \overline{y}_{i+1}^{x_{i+1}} \ y_{i+1} \ x_k \ y_k \ z_1 \ e_1),$$

and we define  $\gamma_g \in \operatorname{Out}_{g,1\perp 1,0}^+$ 

$$\frac{k \in [1\uparrow i-1]}{(x_k \quad y_k \quad x_g \quad y_g \quad z_1 \quad e_1)^{\gamma_g}} = (x_k \quad y_k \quad \overline{z}_1 \overline{y}_g x_g \quad y_g^{z_1} \quad z_1^{y_g z_1} \quad \overline{z}_1 \overline{y}_g e_1).$$

Let us identify  $\Sigma_{g,2,0}$  with  $\Phi_{2g+1}$  and  $\Sigma_{g,2,0} \cup \Sigma_{g,2,0}e_1$  with  $\Sigma_{0,1,(2g+2)^{(2)}}$  via the map  $\Sigma_{g,2,0} * \langle e_1 \rangle \to \Sigma_{0,1,(2g+2)^{(2)}}$  determined by

$$\begin{array}{c|c} & & \\ & & \\ ( \begin{array}{ccc} x_k & y_k & z_1 & e_1 & z_2 \end{array})^{\sum_{g,2,0} * \langle e_1 \rangle \to \sum_{0,1,(2g+2)^{(2)}}} \\ & = (\Pi \tau_{[2k+1\downarrow 2k]} & \tau_{2k+1} \Pi \tau_{[1\uparrow 2k+1]} & z_1^{\tau_{2g+2}} & \tau_{2g+2} & z_1 ). \end{array}$$

This corresponds to the surface of genus g with two boundary components arising as a two-sheeted branched cover of a sphere with one boundary component and 2g + 2 double points. Now  $\mathcal{B}_{2g+2} = \operatorname{Out}_{0,1,2g+2}^+ = \operatorname{Out}_{0,1,(2g+2)^{(2)}}^+$  is embedded in  $\operatorname{Out}_{g,1\perp 1,0}^+$  via a homomorphism represented as

$$\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \cdots & \sigma_{2g-2} & \sigma_{2g-1} & \sigma_{2g} & \sigma_{2g+1} \\ \alpha_1 & \beta_1 & \gamma_1 & \beta_2 & \gamma_2 & \cdots & \beta_{g-1} & \gamma_{g-1} & \beta_g & \gamma_g \end{pmatrix}.$$

For  $g \ge 1$ , Proposition 8.7 shows that this is an embedding. In the case where g = 0, the interpretation of the notation is as follows:  $\sigma_1$  is mapped to  $\gamma_0$ ;  $\gamma_0$  fixes  $z_1$  and sends  $e_1$  to  $\overline{z}_1 e_1$ .

Clearly, in the preceding example, the subgroup  $\mathcal{B}_{2g+1}$  of  $\mathcal{B}_{2g+2}$  is also embedded in  $\operatorname{Out}_{g,1\perp,0}^+$ , but it is more natural to remove from the surface a disc containg the two boundary components (a sphere with three boundary components or a 'pair of pants'), and embed  $\mathcal{B}_{2g+1}$  in  $\operatorname{Out}_{g,1,0}^+$ , as in Example 9.2.

We next discuss the Perron-Vannier isomorphism  $\mathcal{B}_{n+1} \ltimes \Phi_n \simeq \operatorname{Artin} \langle D_{n+1} \rangle$ for  $n \geq 1$ . The following was shown to us by Mladen Bestvina.

**9.4 Lemma.** Let  $n \geq 2$ . Then,  $\operatorname{Artin}\langle D_n \rangle$  has a unique automorphism v of order two which fixes  $d_1, \ldots, d_{n-2}$  and interchanges  $d_{n-1}$  and  $d_n$ . The semidirect product  $\operatorname{Artin}\langle D_n \rangle \rtimes \langle v \rangle$  has presentation

Artin
$$\langle d_1 - d_2 - \cdots - d_{n-3} - d_{n-2} - d_{n-1} = v \mid v^2 = 1 \rangle$$

*Proof.* Notice that  $\langle d_{n-1}, d_n, v \mid v^2 = 1, d_{n-1}^v = d_n, d_{n-1}d_n = d_nd_{n-1}\rangle$  is isomorphic to  $\langle d_{n-1}, v \mid v^2 = 1, d_{n-1}d_{n-1}^v = d_{n-1}^v d_{n-1}\rangle$ , and the latter is Artin $\langle d_{n-1} = v \mid v^2 = 1 \rangle$ . The result now follows easily.

Part of the following appears in [26] and [10].

**9.5 Theorem** (Perron-Vannier [26]). Let  $n \ge 2$ . The semidirect product  $\mathcal{B}_n \ltimes \Phi_{n-1}$  has presentation

Artin
$$\langle \sigma_1 - \sigma_2 - \cdots - \sigma_{n-3} - \sigma_{n-2} - \sigma_{n-1} \rangle \simeq \operatorname{Artin} \langle D_n \rangle.$$

Hence,  $\mathfrak{B}_n \ltimes \Phi_{n-1}$  has a unique automorphism  $\upsilon$  of order two which fixes  $\sigma_1, \ldots, \sigma_{n-2}$  and interchanges  $\sigma_{n-1}$  and  $\sigma_{n-1}\tau_n\tau_{n-1}$ . The double semidirect product  $(\mathfrak{B}_n \ltimes \Phi_{n-1}) \rtimes \langle \upsilon \rangle$  has presentation

Artin
$$\langle \sigma_1 - \sigma_2 - \cdots - \sigma_{n-3} - \sigma_{n-2} - \sigma_{n-1} = v \mid v^2 = 1 \rangle.$$

*Proof.* By Corollary 5.5, we have a presentation

$$\mathcal{B}_n \ltimes \Sigma_{0,1,n} = \operatorname{Artin} \langle \sigma_1 - \cdots - \sigma_{n-1} = \overline{t}_n \rangle$$

If we impose the relation  $t_n^2 = 1$ , we transform  $\mathcal{B}_n \ltimes \Sigma_{0,1,n}$  into  $\mathcal{B}_n \ltimes \Sigma_{0,1,n^{(2)}}$ , and we have

$$\mathcal{B}_n \ltimes \Sigma_{0,1,n^{(2)}} = \operatorname{Artin} \langle \sigma_1 - \cdots - \sigma_{n-1} = \tau_n \mid \tau_n^2 = 1 \rangle.$$

Here, there exists a retraction to  $\langle \tau_n \rangle$  with kernel the normal subgroup generated by  $\sigma_{[1\uparrow n-1]}$ . This normal subgroup contains  $\sigma_i^{\tau_{i+1}} = \sigma_i \tau_{i+1} \tau_i$  for all  $i \in [1\uparrow n-1]$ . By Lemma 9.4, the normal subgroup has presentation

$$\mathcal{B}_{n} \ltimes \Phi_{n-1} = \operatorname{Artin} \langle \sigma_{1} - \cdots - \sigma_{n-3} - \sigma_{n-2} - \sigma_{n-1} \rangle,$$

and this agrees with the desired presentation.

**9.6 Remarks.** Corollary 5.5 says that, for  $n \ge 1$ , we can go down by index n+1 from  $\operatorname{Artin}\langle A_n \rangle$  by squaring the last generator, and arrive at  $\operatorname{Artin}\langle B_n \rangle \simeq \operatorname{Artin}\langle A_{n-1} \rangle \ltimes \Sigma_{0,1,n}$ .

Theorem 9.5 says that, for  $n \geq 2$ , we can kill the square of the new last generator, go down by index 2, and arrive at  $\operatorname{Artin}\langle D_n \rangle \simeq \operatorname{Artin}\langle A_{n-1} \rangle \ltimes \Phi_{n-1}$ .

We now record some other free generating sets of  $\Phi_n$  which appear in the literature.

**9.7 Examples.** Recall Notation 9.1. In particular, the  $\mathcal{B}_{n+1}$ -action on  $\Phi_n$  is faithful if  $n \neq 1$ .

(1). For each  $k \in [1\uparrow n]$ , set  $x_k = \tau_k \tau_{k+1}$  in  $\Phi_n$ . Then  $x_{[1\uparrow n]}$  is a free generating set for  $\Phi_n$ , and, for each  $i \in [1\uparrow n]$ , the action of  $\sigma_i$  on  $\Phi_n$  is determined by

$k{\in}[1{\uparrow}i{-}2]$			$\frac{k}{k}$	$\in [i+2\uparrow n]$
$(x_k$	$x_{i-1}$	$x_i$	$x_{i+1}$	$(x_k)^{\sigma_i}$
$=(x_k)$	$x_{i-1}x_i$	$x_i$	$\overline{x}_i x_{i+1}$	$x_k),$

interpretated appropriately for i = 1 and i = n.

(2). For each  $k \in [1\uparrow n]$ , set  $x_k = \tau_{n+1}\tau_k$  in  $\Phi_n$ , Then  $x_{[1\uparrow n]}$  is a free generating set for  $\Phi_n$ , and, for each  $i \in [1\uparrow n-1]$ ,  $\sigma_i$  acts on  $x_{[1\uparrow n]}$  as follows.

$k{\in}[1{\uparrow}i{-}1]$			$k{\in}[i{+}2{\uparrow}n]$	$k{\in}[1{\uparrow}n{-}1]$	
$(x_k$	$x_i$	$x_{i+1}$	$(x_k)^{\sigma_i}$	$(x_k)$	$(x_n)^{\sigma_n}$
$=(x_k)$	$x_{i+1}$	$x_{i+1}\overline{x}_i x_{i+1}$	$x_k).$	$=(x_{n-1}x_k)$	$x_n).$

(3). We next consider the free generating set used in the proof of [11, Proposition A.1(2)].

For each  $k \in [1\uparrow n]$ , set  $x_k = \tau_{n+1}^{\Pi\tau_{[1\uparrow k]}}\tau_{k+1}$  in  $\Phi_n$ . Then  $x_{[1\uparrow n]}$  is a free generating set for  $\Phi_n$ , and, for each  $i \in [1\uparrow n-1]$ ,  $\sigma_i$  acts on  $x_{[1\uparrow n]}$  as follows,

$k{\in}[1{\uparrow}i{-}1]$			$k{\in}[i{+}2{\uparrow}n]$
$(x_k$	$x_i$	$x_{i+1}$	$(x_k)^{\sigma_i}$
$=(x_k$	$x_i \prod x_{[i \uparrow i+1]}$	$\prod \overline{x}_{[i+1 \downarrow i]} x_{i+1}$	$x_k).$

Let  $w = (\prod x_{[1\uparrow n-1]}^2 x_n)^{-1}$ ; then  $\sigma_n$  acts as follows.

$$\begin{array}{c} \underbrace{k \in [1\uparrow n-1]}_{(x_k} & x_n \\ = (w^{(-1)^k \prod x_{[1\uparrow k-1]}} x_k & w^{(-1)^n \prod x_{[1\uparrow n-1]}} x_n w). \end{array}$$

(4). By reflecting the previous example, we can invert the elements of  $\sigma_{[1\uparrow n]}$ .

For each  $k \in [1\uparrow n]$ , set  $x_k = (\tau_{n+1}^{\Pi\tau_{[n\downarrow 1]}}\tau_k)^{\Pi\tau_{[k\uparrow n+1]}}$  in  $\Phi_n$ . Then  $x_{[1\uparrow n]}$  is a free generating set for  $\Phi_n$ , and, for each  $i \in [1\uparrow n-1]$ ,  $\sigma_i$  acts on  $x_{[1\uparrow n]}$  as follows.

$$\frac{k \in [1\uparrow i-1]}{(x_k \quad x_i \quad x_{i+1} \quad x_k)^{\sigma_i}} = (x_k \quad x_i \Pi \overline{x}_{[i+1\downarrow i]} \quad \Pi x_{[i\uparrow i+1]} x_{i+1} \quad x_k).$$

Let  $w = \prod x_{[1\uparrow n-1]}^2 x_n$ ; then  $\sigma_n$  acts as follows.

$$\frac{k \in [1\uparrow n-1]}{(x_k \quad x_n \ )^{\sigma_n}} = (w^{(-1)^k \Pi x_{[1\uparrow k-1]}} x_k \quad w^{(-1)^n \Pi x_{[1\uparrow n-1]}} x_n w). \qquad \Box$$

**9.8 Historical Remarks.** Let us view  $\mathcal{B}_n$  as a subgroup of  $\mathcal{B}_{n+1}$  by suppressing  $\sigma_n$ . Then the  $\mathcal{B}_{n+1}$ -group  $\Phi_n$  becomes a faithful  $\mathcal{B}_n$ -group, even if n = 1.

Wada [29] defined various left actions of  $\mathcal{B}_n$  on a free group of rank n. All but four of them are obviously non-faithful, and two of the remaining four actions are obviously equivalent up to changing the free generating set, leaving three actions to be studied for faithfulness. Shpilrain [28] ingeniously used the  $\sigma_1$ -trichotomy to prove that these three are all faithful. Crisp-Paris [11, Proposition A.1(2)] showed that the second and third of these three Wada actions are equivalent up to changing the free generating set. They correspond to Examples 9.7(2), (4), above, with  $\sigma_n$  suppressed. Notice that our actions on the right are the inverses of their actions on the left. In summary, the *second* and *third* Wada actions are obtained by choosing suitable free generating sets of the Perron-Vannier  $\mathcal{B}_{n+1}$ -group  $\Phi_n$ .

The first Wada action is constructed by choosing a non-zero integer m, and, for each  $1 \in [1\uparrow n - 1]$ , letting  $\sigma_i$  act on  $\langle x_{[1\uparrow n]} | \rangle$  via

$$\frac{k \in [1\uparrow i-1]}{(x_k \quad x_i \quad x_{i+1} \quad x_k)^{\sigma_i}} = (x_k \quad x_{i+1} \quad x_i^{x_{i+1}^m} \quad x_k).$$

Edward Formanek has pointed out that  $x_{[1,n]}^m$  is then a free generating set of a faithful  $\mathcal{B}_n$ -subgroup of  $\langle x_{[1,n]} | \rangle$ , where faithfulness can be seen from the fact that the  $\mathcal{B}_n$ -action is the standard Artin action with respect to this free generating set. This gives a transparent proof that the first Wada action is faithful.

# Appendix. Larue-Whitehead diagrams

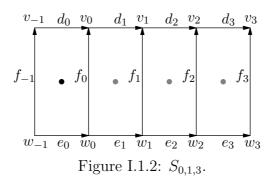
In this appendix, we rework ideas from Larue's thesis [21, Chapter 2 and Appendix A], using combinatorial arguments to obtain a description of the  $\mathcal{B}_n$ -orbit of  $t_1$  when  $n \geq 1$ . A topological treatment of similar ideas was given in [19], and it was arrived at independently of [21]. See [16, Chapters 5, 6].

## I Self-homeomorphisms

This section is purely motivational. We shall briefly indicate the mapping-class viewpoint of the braid group, and the Jordan-curve nature of the Whitehead graphs of the elements in the  $\mathcal{B}_n$ -orbit of  $t_1$  if  $n \geq 1$ .

Let  $\mathbb{C}$  denote the complex plane, and  $\mathbb{C}$  the Riemann sphere, or projective complex line,  $\mathbb{C} \cup \{\infty\}$ . For each  $z \in \mathbb{C}$  and each non-negative real number r, let  $\mathbf{D}(z,r)$ , resp.  $\mathbf{D}^{\circ}(z,r)$ , denote the closed, resp. open, disc in  $\mathbb{C}$  with centre z and radius r.

Let  $S_{0,1,n}$  denote the surface formed by deleting from a sphere an open disc and n points. We shall think of the discs and points as being distinguished rather than deleted; for example, it is then meaningful to speak of the self-homeomorphisms of  $S_{0,1,n}$  as permuting the points. We take as our model of  $S_{0,1,n}$  the sphere  $\widehat{\mathbb{C}}$  having  $[1\uparrow n]$  as its set of n distinguished points, and  $\mathbf{D}^{\circ}(0, \frac{1}{2})$ as its distinguished open disc. We are particularly interested in the set  $[0\uparrow n]$ , and, in our diagrams, we shall mark these points out by drawing discs of small radii around them.



For each distinguished point  $k \in [0\uparrow n]$ , we have a distinguished oriented tether, or arc,  $\{k - r\mathbf{i} \mid -\infty \leq r \leq 0\}$ , joining  $\infty$  to k. We label the right flank of this oriented arc  $t_k$ , and label the left flank  $\overline{t}_k$ ; we then cut  $\widehat{\mathbb{C}}$  open along these arcs and obtain a (2n + 2)-gon, with clockwise boundary label  $\prod_{k \in [0\uparrow n]} (t_k \overline{t}_k)$ ; see Fig. I.1.4. We shall use  $t_0$  and  $z_1$  interchangeably in this section. Performing the boundary identifications then gives back  $\widehat{\mathbb{C}}$ .

The self-homeomorphism  $\lambda$  of  $\mathbf{D}(0,1)$  given by  $\lambda(re^{\mathbf{i}\theta}) := re^{\mathbf{i}(\theta-2\pi r)}$  fixes the boundary of  $\mathbf{D}(0,1)$  and interchanges  $\frac{1}{2}$  and  $-\frac{1}{2}$ ; see Fig. I.1.1. For each

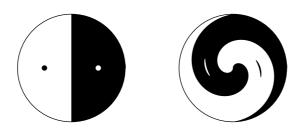


Figure I.1.1: The map  $\lambda: \mathbf{D}(0,1) \to \mathbf{D}(0,1), re^{\mathbf{i}\theta} \mapsto re^{\mathbf{i}(\theta-2\pi r)}$ 

 $i \in [1\uparrow n-1]$ , let  $\phi_i$  denote the self-homeomorphism of  $\widehat{\mathbb{C}}$  which, on  $\widehat{\mathbb{C}} - \mathbf{D}(i + \frac{1}{2}, 1)$ , acts as the identity map, and, on  $\mathbf{D}(i + \frac{1}{2}, 1)$ , acts by  $z \mapsto \lambda(z - i - \frac{1}{2}) + i + \frac{1}{2}$ . Then  $\phi_{[1\uparrow n-1]}$  generates a group  $\langle \phi_{[1\uparrow n-1]} \rangle$  of self-homeomorphisms of  $\widehat{\mathbb{C}}$ , which will shed light on the  $\mathcal{B}_n$ -orbit of  $t_1$ . To describe the induced action of  $\langle \phi_{[1\uparrow n-1]} \rangle$ on the fundamental group of  $S_{0,1,n}$ , we first give  $\widehat{\mathbb{C}}$  a CW-structure by specifying a graph  $S_{0,1,n}^{(1)}$  embedded in  $\mathbb{C} \subset \widehat{\mathbb{C}}$ .

a graph  $S_{0,1,n}^{(1)}$  embedded in  $\mathbb{C} \subset \widehat{\mathbb{C}}$ . For each  $k \in [-1\uparrow n]$ , we have vertices  $w_k := k + \frac{1}{2} - \mathbf{i}$  and  $v_k := k + \frac{1}{2} + \mathbf{i}$ , and an oriented straight edge  $f_k$  joining  $w_k$  to  $v_k$ . For each  $k \in [0\uparrow n]$ , we have an oriented straight edge  $e_k$  joining  $w_{k-1}$  to  $w_k$ , and an oriented straight edge  $d_k$  joining  $v_{k-1}$  to  $v_k$ . This completes the description of the graph  $S_{0,1,n}^{(1)}$ . For  $n = 3, S_{0,1,3}^{(1)}$  can be seen in Fig. I.1.2. Each distinguished point  $k \in [0\uparrow n]$  is the midpoint of the rectangle in  $\mathbb{C}$  cut out by the path  $f_{k-1}d_k\overline{f}_k\overline{e}_k$ .

Let  $\langle S_{0,1,n}^{(1)} | \rangle$  denote the (free) fundamental groupoid of  $S_{0,1,n}^{(1)}$ , and let  $\langle S_{0,1,n}^{(1)} | \rangle (w_{-1}, w_{-1})$  denote the (free) fundamental group of  $S_{0,1,n}^{(1)}$  at  $w_{-1}$ . The

subgraph of  $S_{0,1,n}^{(1)}$  spanned by  $e_{[0\uparrow n]} \cup f_{[-1\uparrow n]}$  is a maximal subtree of  $S_{0,1,n}^{(1)}$ , and  $d_{[0\uparrow n]}$  then determines a free generating set  $t_{[0\uparrow n]}$  of  $\langle S_{0,1,n}^{(1)} | \rangle (w_{-1}, w_{-1})$ ; explicitly, for each  $k \in [0\uparrow n]$ ,  $t_k = \prod e_{[0\uparrow k-1]} f_{k-1} d_k \overline{f}_k \prod \overline{e}_{[k\downarrow 0]}$ .

The path  $f_{-1}\Pi d_{[0\uparrow n]}f_n\Pi \overline{e}_{[n\downarrow 0]}$  cuts out a rectangle in  $\mathbb{C}$ ; the complementary region in  $\widehat{\mathbb{C}}$  together with the graph  $S_{0,1,n}^{(1)}$  is then a retract of  $\widehat{\mathbb{C}} - [0\uparrow n]$ . Let ~ denote homotopy for closed paths at  $w_{-1}$  in  $\widehat{\mathbb{C}} - [0\uparrow n]$ . We can identify the fundamental groupoid of  $S_{0,1,n}$  with  $\langle S_{0,1,n}^{(1)} | f_{-1}\Pi d_{[0\uparrow n]}\overline{f_n}\Pi \overline{e}_{[n\downarrow 0]} \sim w_{-1} \rangle$ . We then identify  $\Sigma_{0,1,n}$  with the fundamental group of  $S_{0,1,n}$  at  $w_{-1}$ ,

$$\Sigma_{0,1,n} = \langle S_{0,1,n}^{(1)} \mid f_{-1} \Pi d_{[0\uparrow n]} \overline{f}_n \Pi \overline{e}_{[n\downarrow 0]} \sim w_{-1} \rangle (w_{-1}, w_{-1}) = \langle t_{[0\uparrow n]} \mid \Pi t_{[0\uparrow n]} = 1 \rangle.$$

Consider the action of  $\phi_1$  on the graph  $S_{0,1,n}^{(1)}$ . For n = 3, the result can be

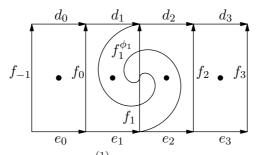


Figure I.1.3:  $S_{0,1,3}^{(1)}$  and its image under  $\phi_1$ .

seen in Fig. I.1.3. The crucial point is that  $f_1^{\phi_1} \sim e_2 f_2 \overline{d}_2 \overline{f}_1 \overline{e}_1 f_0 d_1$ , and all the other elements of  $S_{0,1,3}^{(1)}$  are fixed by  $\phi_1$ ; this makes the action quite simple algebraically. Then,  $\overline{f}_1^{\phi_1} \sim \overline{d}_1 \overline{f}_0 e_1 f_1 d_2 \overline{f}_2 \overline{e}_2$ , and, for the free generator  $t_1 = e_0 f_0 d_1 \overline{f}_1 \overline{e}_{[1,0]}$ , we have

$$t_1^{\phi_1} \sim e_0 f_0 d_1 (\overline{d_1} \overline{f_0} e_1 f_1 d_2 \overline{f_2} \overline{e_2}) \overline{e_{[1,0]}} \sim e_{[0,1]} f_1 d_2 \overline{f_2} \overline{e_{[2,0]}} = t_2.$$

Similarly, for this element,  $t_2$ , we have

$$t_{2}^{\phi_{1}} \sim e_{[0,1]}(e_{2}f_{2}\overline{d}_{2}\overline{f}_{1}\overline{e}_{1}f_{0}d_{1})d_{2}\overline{f}_{2}\overline{e}_{[2,0]} \sim e_{[0,2]}f_{2}\overline{d}_{2}\overline{f}_{1}\overline{e}_{1}f_{0}d_{[1,2]}\overline{f}_{2}\overline{e}_{[2,0]} \sim \overline{t}_{2}t_{1}t_{2},$$

where the latter homotopy can be seen directly by collapsing the elements of  $e_{[0,2]} \cup f_{[0,2]}$ , which lie in the maximal subtree. Thus, we see that  $\phi_1$  acts on  $\Sigma_{0,1,n}$  as the automorphism  $\sigma_1$ .

It follows that the action of any given element of  $\mathcal{B}_n$  on  $\Sigma_{0,1,n}$  is induced by some self-homeomorphism  $\phi \in \langle \phi_{[1\uparrow n-1]} \rangle$ . The interesting feature now is that  $\phi$ carries the oriented Jordan curve  $f_{-1}d_{[0\uparrow 1]}\overline{f_1}\overline{e}_{[1\downarrow 0]}$  ( $\sim t_0t_1$ ) to an oriented Jordan curve  $f_{-1}d_{[0\uparrow 1]}\overline{f_1}^{\phi}\overline{e}_{[1\downarrow 0]}$  ( $\sim (t_0t_1)^{\phi} \sim t_0t_1^{\phi}$ ). Recall that  $\widehat{\mathbb{C}}$  is obtained by edge

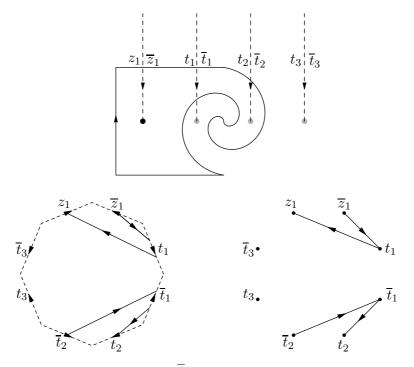


Figure I.1.4: Jordan curves for  $z_1 t_1^{\overline{\phi}_1}$  and a Whitehead graph for  $t_1^{\overline{\sigma}_1} = t_1 t_2 \overline{t}_1$ .

identification from the (2n+2)-gon with clockwise, boundary label  $\prod_{i \in [0\uparrow n]} (t_i \overline{t}_i)$ .

The Jordan curve  $f_{-1}d_{[0\uparrow 1]}\overline{f}_{1}^{\phi}\overline{e}_{[1\downarrow 0]}$  has as its preimage, in the (2n + 2)-gon, the union of a family of (disjoint) oriented arcs. These arcs can be used to reconstruct  $t_{1}^{\phi}$ , since the Jordan curve cyclically reads off  $t_{0}t_{1}^{\phi}$  from its meetings with the labelled oriented tethers; notice that the set of tethers is now dual to the set of generators  $t_{[0\uparrow n]}$ . The purpose of this appendix is to define and study a combinatorial representation of the family of arcs, and recover Larue's characterization of the elements of  $t_{1}^{\mathcal{B}_{n}}$ .

Although it will not be used in our arguments, let us mention the fact that, on collapsing the interior of each labelled edge of the (2n + 2)-gon to a labelled vertex, each oriented arc in the family becomes an oriented edge, and we recover the (directed, multi-edge, non-cyclic) Whitehead graph of  $t_1^{\phi}$ ; see Fig. I.1.4.

#### II Nested sets

We now introduce some formal definitions that will allow us to associate a combinatorial Jordan curve to each element of  $t_1^{\mathcal{B}_n}$ .

**II.1 Definitions.** Let  $(A, \leq)$  be a finite ordered set, and let  $m \in \mathbb{N}$ .

Let N denote the number of elements of A. Then A is order-isomorphic to  $[1\uparrow N]$  in a unique way, and we assign to A the induced metric, denoted  $d_A$ . Thus

 $d_A(a_1, a_2) = 1$  if and only if  $a_1 \neq a_2$  and no element of A lies strictly between  $a_1$ and  $a_2$ .

Recall that the elements of  $A^m$  are called *m*-tuples for A.

Let  $a_1, a_2, b_1, b_2$  be elements of A. We say that  $\{a_1, b_1\}$  is nested with  $\{a_2, b_2\}$ (for  $(A, \leq)$ ) if  $a_1, a_2, b_1, b_2$  are distinct elements of A, and either both of, or neither of,  $a_2$  and  $b_2$  lie between  $a_1$  and  $b_1$  in  $(A, \leq)$ . It is not difficult to see that, in this event,  $\{a_2, b_2\}$  is nested with  $\{a_1, b_1\}$ .

Let  $a_{([1\uparrow m])}$  and  $b_{([1\uparrow m])}$  be *m*-tuples of A.

We say that  $a_{([1\uparrow m])}$  is an *m*-tuple without repetitions if  $a_i \neq a_j$  for all  $i \neq j$ in  $|1\uparrow m|$ .

We say that  $(a_{[1\uparrow m]})$  is an ascending m-tuple (for  $(A, \leq)$ ) if  $a_1 \leq a_2 \leq \cdots \leq a_n$  $a_m$  in  $(A, \leq)$ .

We say that  $\{\{a_i, b_i\}\}_{i \in [1\uparrow m]}$  is nested (for  $(A, \leq)$ ) if, for all  $i \neq j$  in  $[1\uparrow m]$ ,  $\{a_i, b_i\}$  is nested with  $\{a_j, b_j\}$  for  $(A, \leq)$ .

We let  $\operatorname{Sym}_m$  act on  $A^m$ , on the left, by  $\pi(a_{([1\uparrow m])}) := a_{([1\uparrow m])}\overline{\pi}$ . For example,  $^{(1,2,3)}(a_1, a_2, a_3) = (a_3, a_1, a_2)$ , and, hence,  $^{(1,2,3)}(a, b, c) = (c, a, b)$ . The ascending rearrangement of  $a_{([1\uparrow m])}$  is the unique ascending *m*-tuple for  $(A, \leq)$  that lies in the Sym<sub>m</sub>-orbit of  $a_{([1\uparrow m])}$ .

Let  $a_{([1\uparrow 2m])}$  be a 2*m*-tuple for *A*.

A permutation  $\pi \in \text{Sym}_{2m}$  is said to embed  $a_{([1\uparrow 2m])}$  in a plane if  $\pi a_{([1\uparrow 2m])}$ is ascending for  $(A, \leq)$ , and both  $\{[2i-1\uparrow 2i]^{\pi}\}_{i\in[1\uparrow m]}$  and  $\{[2i\uparrow 2i+1]^{\pi}\}_{i\in[1\uparrow m-1]}$ are nested in  $(\mathbb{N}, \leq)$ .

We say that  $a_{([1\uparrow 2m])}$  is a planar 2*m*-tuple (for  $(A, \leq)$ ) if there exists some  $\pi \in \text{Sym}_{2m}$  which embeds  $a_{([1\uparrow 2m])}$  in a plane. (If no two consecutive terms of  $a_{([1\uparrow 2m])}$  are equal,  $\pi$  is then unique, but we shall not need this fact.) There is then an associated diagram formed as follows. We assign, to each point  $i \in [1 \uparrow 2m] \subset \mathbb{R} \subset \mathbb{C}$ , the label  $a_{i\pi}$ ; notice that this means that the label of  $i^{\pi}$ is  $a_i$ . For each  $i \in [1 \uparrow m]$ , we join  $(2i-1)^{\pi}$  (labelled  $a_{2i-1}$ ) to  $(2i)^{\pi}$  (labelled  $a_{2i}$ ) by an oriented semi-circle in the upper half-plane, and for each  $i \in [1\uparrow m - 1]$ , we join  $(2i)^{\pi}$  (labelled  $a_{2i}$ ) to  $(2i+1)^{\pi}$  (labelled  $a_{2i+1}$ ) by an oriented semi-circle in the lower half-plane. These oriented semi-circles form an oriented arc with no crossings which traces out the 2*m*-tuple  $a_{([1\uparrow 2m])}$ 

**II.2 Example.** Suppose that  $a_{([1\uparrow 8])} = (\overline{z}_1, t_1, \overline{t}_1, t_2, \overline{t}_2, \overline{t}_1, t_1, z_1)$  is an 8-tuple for some ordered set  $(A, \leq)$ , and that the ascending rearrangement of  $a_{([1\uparrow 8])}$  is

 $(\overline{z}_1, t_1, \overline{t}_1, \overline{t}_1, \overline{t}_2, \overline{t}_2, z_1).$ The permutation  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 5 & 6 & 7 & 4 & 3 & 8 \end{pmatrix} = (3, 5, 7)(4, 6)$  embeds  $a_{([1\uparrow 8])}$  are in a plane since both  $\{\{1,2\},\{5,6\},\{7,4\},\{3,8\}\}$  and  $\{\{2,5\},\{6,7\},\{4,3\}\}$  are nested in  $(\mathbb{N}, \leq)$ , and  $(\overline{z}_1, \overline{t}_1, \overline{t}_1, \overline{t}_2, \overline{t}_2, \overline{t}_1, \overline{t}_1, z_1) = (\overline{z}_1, \overline{t}_1, \overline{t}_1, \overline{t}_1, \overline{t}_2, \overline{t}_2, z_1).$ The associated diagram can be seen in Fig. II.2.1.

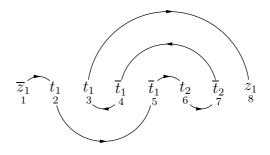


Figure II.2.1:  $(\overline{z}_1, t_1, \overline{t}_1, t_2, \overline{t}_2, \overline{t}_1, t_1, z_1)$ .

Let us record two results which will be useful later.

**II.3 Lemma.** Let  $(A, \leq)$  be an ordered set, and let m be a positive integer. Let  $c_{[1\uparrow m]}$  and  $\overline{c}_{[1\uparrow m]}$  be m-tuples without repetitions for  $(A, \leq)$  such that  $\{\{c_i, \overline{c}_i\}\}_{i\in[1\uparrow m]}$  is nested, and  $\max(c_{[1\uparrow m]}) < \min(\overline{c}_{[1\uparrow m]})$ . If  $c_{([1\uparrow m])}$  is ascending, then  $\overline{c}_{([m\downarrow 1])}$  is also ascending.

Proof. We argue by induction on m. If m = 1, the conclusion is trivial. Now, assume that  $m \ge 2$  and that the implication holds with m - 1 in place of m. We see that  $c_1 < c_2 \le \max(c_{[1\uparrow m]}) < \min(\overline{c}_{[1\uparrow m]}) \le \overline{c}_1$ . Since  $\{c_1, \overline{c}_1\}$  is nested with  $\{c_2, \overline{c}_2\}$ , we also see that  $c_1 < \overline{c}_2 < \overline{c}_1$ . By the induction hypothesis,  $\overline{c}_{([m\downarrow 2])}$ is ascending, and hence  $\overline{c}_{([m\downarrow 1])}$  is ascending. Hence, the result is proved.

**II.4 Lemma.** Let  $(A, \leq)$  be an ordered set, let  $m \in \mathbb{N}$ , and let  $a_{([1\uparrow 2m])}$  be a 2*m*-tuple for A.

Then  $a_{([1\uparrow 2m])}$  is planar for  $(A, \leq)$  if and only if there exists an ordered set  $(B, \leq)$ , and a 2m-tuple  $b_{([1\uparrow 2m])}$  for B, without repetitions, and an ordered-set map  $B \to A$ ,  $b \mapsto \text{label}(b)$ , such that  $b_{[1\uparrow 2m]} = B$ ,  $\text{label}(b_{([1\uparrow 2m])}) = a_{([1\uparrow 2m])}$ , and  $\{b_{[2i\uparrow 2i+1]}\}_{i\in[1\uparrow m-1]}$  and  $\{b_{[2i-1\uparrow 2i]}\}_{i\in[1\uparrow m]}$  are nested for  $(B, \leq)$ .

*Proof.* Suppose first that  $a_{([1\uparrow 2m])}$  is planar for  $(A, \leq)$ , and let  $\pi$  be an element of  $\operatorname{Sym}_{2m}$  that embeds  $a_{([1\uparrow 2m])}$  in a plane. We take B to be  $[1\uparrow 2m]$  with the usual ordering. For each  $i \in [1\uparrow 2m]$ , let  $\operatorname{label}(i) = a_{i\pi}$  and let  $b_i = i^{\pi}$ ; thus,  $\operatorname{label}(b_i) = \operatorname{label}(i^{\pi}) = a_i$ . All the conditions are satisfied.

Conversely, if *B* exists, we can identify *B* with  $[1\uparrow 2m]$  with the usual ordering, in a unique way. Then the map  $i \mapsto b_i$  is an element  $\pi$  of  $\operatorname{Sym}_{2m}$  that embeds  $a_{([1\uparrow 2m])}$  in a plane.

# III Planar words in $\Sigma_{0,1,n}$

**III.1 Definitions.** Let A be the monoid generating set  $\{z_1, \overline{z}_1\} \cup t_{[1\uparrow n]} \cup \overline{t}_{[1\uparrow n]}$ of  $\Sigma_{0,1,n}$ . We form the ordered set  $(A, \leq)$  with

$$\overline{z}_1 < t_1 < \overline{t}_1 < \dots < t_n < \overline{t}_n < z_1.$$

We remark that, for  $n \neq 1$ , the ordering on A is reminiscent of the ordering of the ends of  $\Sigma_{0,1,n}$  in Section 7. We emphasize that, even if n = 1,  $z_1 \neq \overline{t}_1$  in A.

Let  $m \in \mathbb{N}$ . Consider an *m*-tuple  $a_{([1\uparrow m])}$  for  $t_{[1\uparrow n]} \cup \overline{t}_{[1\uparrow n]}$ , and let  $w = \prod a_{[1\uparrow m]} \in \Sigma_{0,1,n}$ ; thus  $a_{([1\uparrow m])}$  is an expression for w. We define the Whitehead expansion of  $a_{([1\uparrow m])}$  to be the (2m+2)-tuple

$$(\overline{z}_1, a_1, \overline{a}_1, a_2, \overline{a}_2, \dots, a_m, \overline{a}_m, z_1)$$

for A, and we shall express it in the format  $(\overline{z}_1, ((a_i, \overline{a}_i))_{i \in [1\uparrow m]}, z_1)$ . We say that  $a_{([1\uparrow m])}$  is a *planar* expression for w if the Whitehead expansion of  $a_{([1\uparrow m])}$  is planar for  $(A, \leq)$ . If the unique reduced expression for w is a planar expression for w, then we say that w is a *planar* word in  $\Sigma_{0,1,n}$ .

**III.2 Examples.** (i). The word  $t_1\overline{t}_2\overline{t}_1$  is planar, since the Whitehead expansion of the reduced expression is  $(\overline{z}_1, t_1, \overline{t}_1, t_2, \overline{t}_2, \overline{t}_1, t_1, z_1)$ , and, by Example II.2,  $(\overline{z}_1, t_1, \overline{t}_1, t_2, \overline{t}_2, \overline{t}_1, t_1, z_1)$  is planar for  $(A, \leq)$ ; in a sense, Fig. II.2.1 reflects Fig. I.1.4. We call Fig. II.2.1 the Larue-Whitehead diagram of  $t_1\overline{t}_2\overline{t}_1$ .

(ii). The word  $t_1 \overline{t}_2$  is not planar; there is only one permutation to consider.

(iii). The word  $t_1^2$  is not planar; there are four permutations to consider.

(iv). The word  $t_3^{\overline{t_1}\overline{t_2}\overline{t_1}}$  is planar, while the word  $t_3^{t_1t_2\overline{t_1}}$  is not planar, and these two words have the same Whitehead graph.

**III.3 Proposition.** Let  $w \in \Sigma_{0,1,n}$ . If there exists some planar expression for w, then (the reduced expression for) w is planar.

*Proof.* Suppose that  $a_{([1\uparrow m])}$  is a planar expression for w, as in Definitions III.1.

By Lemma II.4, there exists an ordered set  $(B, \leq)$ , and a planar (2m + 2)-tuple  $b_{([1\uparrow 2m+2])}$  for  $(B, \leq)$ , without repetitions, and a labelling  $B \to A, b \mapsto label(b)$ , such that the labelling respects the orderings and  $label(b_{([1\uparrow 2m+2])})$  is the Whitehead expansion of  $a_{([1\uparrow m])}$ . Moreover,  $B = b_{[1\uparrow 2m+2]}$ .

Suppose that the given planar expression  $a_{([1\uparrow m])}$  is not reduced. We shall find a shorter planar expression for w.

There exists some  $j \in [1\uparrow m - 1]$  such that  $a_{j+1} = \overline{a}_j$  in  $t_{[1\uparrow n]} \cup \overline{t}_{[1\uparrow n]}$ , and we may suppose that we have chosen this j in such a way that  $d_B(b_{2j+1}, b_{2j+2})$  has the minimum possible value. Notice that  $label(b_{([2j\uparrow 2j+3])}) = (a_j, \overline{a}_j, \overline{a}_j, a_j)$ .

Clearly,  $w = \prod a_{[1\uparrow j-1]} \prod a_{[j+1\uparrow m]}$ , and  $label(b_{([1\uparrow 2j-1])}, b_{([2j+4\uparrow 2m+2])})$  is

 $(\overline{z}_1, ((a_i, \overline{a}_i))_{i \in [1 \uparrow j - 1]}, ((a_i, \overline{a}_i))_{i \in [j + 2 \uparrow m]}, z_1)$ 

 $(\overline{z}_1, a_1, \overline{a}_1, \ldots, a_{j-1}, \overline{a}_{j-1}, a_{j+2}, \overline{a}_{j+2}, \ldots, a_m, \overline{a}_m, z_1).$ 

It suffices to show that  $(b_{([1\uparrow 2j-1])}, b_{([2j+4\uparrow 2m+2])})$  is planar for  $(B, \leq)$ . Claim.  $d_B(b_{2j}, b_{2j+3}) = 1$ . *Proof.* Consider any  $k \in [1\uparrow 2m - 1]$  such that  $b_k$  lies between  $b_{2j}$  and  $b_{2j+3}$ . Let  $\eta$  denote  $(-1)^k$ .

Since  $label(b_{2j}) = label(b_{2j+3}) = a_j$ , we see that  $label(b_k) = a_j$ . Hence  $label(b_{k+\eta}) = \overline{a_j} = label(b_{2j+1}) = label(b_{2j+2})$ .

Either  $a_j < \overline{a}_j$  or  $a_j > \overline{a}_j$  in  $(A, \leq)$ . Hence,

either  $\max\{b_{2j}, b_k, b_{2j+3}\} < \min\{b_{2j+1}, b_{k+\eta}, b_{2j+2}\}$  in  $(B, \leq)$ ,

or 
$$\min\{b_{2j}, b_k, b_{2j+3}\} > \max\{b_{2j+1}, b_{k+\eta}, b_{2j+2}\}$$
 in  $(B, \leq)$ ,

respectively.

Since  $\{\{b_{2j}, b_{2j+1}\}, \{b_{2j+2}, b_{2j+3}\}, \{b_k, b_{k+\eta}\}\}$  is nested, and  $b_k$  lies between  $b_{2j}$  and  $b_{2j+3}$ , we see, from Lemma II.3, that  $b_{k+\eta}$  lies between  $b_{2j+1}$  and  $b_{2j+2}$ .

Since  $\{b_{2j+1}, b_{2j+2}\}$  is nested with  $\{b_{k+\eta}, b_{k+2\eta}\}$  and  $b_{k+\eta}$  lies between  $b_{2j+1}$  and  $b_{2j+2}$ , we see that  $b_{k+2\eta}$  lies between  $b_{2j+1}$  and  $b_{2j+2}$ . Hence,

$$d_B(b_{k+2\eta}, b_{k+\eta}) \le d_B(b_{2j+1}, b_{2j+2}),$$

with equality holding only if  $\{b_{k+2\eta}, b_{k+\eta}\} = \{b_{2j+1}, b_{2j+2}\}$ . Also,  $label(b_{k+2\eta}) = \overline{a}_j$ , and, hence,  $label(b_{k+3\eta}) = a_j$ . Thus

$$label(b_k, b_{k+\eta}, b_{k+2\eta}, b_{k+3\eta}) = (a_j, \overline{a}_j, \overline{a}_j, a_j).$$

By the minimality of  $d_B(b_{2j+1}, b_{2j+2})$ , we see that k = 2j or k = 2j + 3. This proves the claim.

Now consider the passage from  $b_{([1\uparrow 2m+2])}$  to  $b_{([1\uparrow 2j-1])}, b_{([2j+4\uparrow 2m+2])}$ . On the odd-to-even steps, we pass from  $\{b_{[2i-1\uparrow 2i]}\}_{i\in[1\uparrow m+1]}$  to

$$\{b_{[2i-1\uparrow 2i]}\}_{i\in[1\uparrow j-1]\cup[j+3\uparrow m+1]}\cup\{\{b_{2j-1},b_{2j+4}\}\}.$$

Thus, we remove  $\{b_{2j-1}, b_{2j}\}$ ,  $\{b_{2j+1}, b_{2j+2}\}$ ,  $\{b_{2j+3}, b_{2j+4}\}$ , and we add only  $\{b_{2j-1}, b_{2j+4}\}$ . To see that, for all  $k \in [1\uparrow j - 1] \cup [j + 3\uparrow m + 1]$ ,  $\{b_{2k-1}, b_{2k}\}$  is nested with  $\{b_{2j-1}, b_{2j+4}\}$ , we note the following:

$$\begin{array}{l} (b_{2j-1} \text{ lies between } b_{2k-1} \text{ and } b_{2k}) \\ \Leftrightarrow (b_{2j} \text{ lies between } b_{2k-1} \text{ and } b_{2k}) \\ & \text{ since } \{b_{2j-1}, b_{2j}\} \text{ is nested with } \{b_{2k-1}, b_{2k}\} \\ \Leftrightarrow (b_{2j+3} \text{ lies between } b_{2k-1} \text{ and } b_{2k}) \\ & \text{ since } d_B(b_{2j}, b_{2j+3}) = 1 \\ \Leftrightarrow (b_{2j+4} \text{ lies between } b_{2k-1} \text{ and } b_{2k}) \\ & \text{ since } \{b_{2j+3}, b_{2j+4}\} \text{ is nested with } \{b_{2k-1}, b_{2k}\}. \end{array}$$

On the even-to-odd steps, we pass from  $\{b_{[2i\uparrow 2i+1]}\}_{i\in[1\uparrow m]}$  to

$$\{b_{[2i\uparrow 2i+1]}\}_{i\in[1\uparrow j-1]\cup[j+2\uparrow m]}$$

Thus, we remove  $\{b_{2j}, b_{2j+1}\}$  and  $\{b_{2j+2}, b_{2j+3}\}$ , and we add nothing. Hence this remains nested. This completes the proof.

**III.4 Proposition.** Let w be a planar word in  $\Sigma_{0,1,n}$ , and let  $k \in [1\uparrow n]$ .

- (i). w is a squarefree word in  $\Sigma_{0,1,n}$ .
- (ii).  $w \notin (\Pi \overline{t}_{[n \downarrow k+1]} t_k \star) \{ t_k^{\Pi t_{[k+1\uparrow n]}} \}.$
- (iii).  $w \notin (\Pi t_{[1\uparrow k-1]} \overline{t}_k \star)$ .

*Proof.* For some  $m \in \mathbb{N}$ , there exists a reduced expression  $a_{([1\uparrow m])}$  for w.

(i). Suppose that w is not squarefree, say  $t_i, t_i$  occurs in  $a_{([1\uparrow m])}$ , then  $t_i, \overline{t_i}, t_i, \overline{t_i}$  occurs in

$$(\overline{z}_1, ((a_i, \overline{a}_i))_{i \in [1 \uparrow m]}, z_1).$$

Let  $m_i$  be the number of occurrences of  $t_i^{\pm 1}$  in  $a_{([1\uparrow m])}$ .

Suppose  $c_{([1\uparrow m_i])}$  are labelled  $t_i$  and  $\overline{c}_{([m_i\downarrow 1]}$  are such that the even-to-odd pairing contains  $\{\{c_k, \overline{c}_k\}\}_{k\in[1\uparrow m_i]}$ . The odd-to-even pairing contains  $\{c_k, \overline{c}_j\}$  for some  $k, j \in [1\uparrow m_i]$ . Let us choose (k, j) so that k+j is as large as possible. Then  $c_k < c_{k+1} < \overline{c}_j$ . Whatever  $c_{k+1}$  is paired with in the odd-to-even pairing must lie in the interval  $[c_k, \overline{c}_j]$  and cannot have label  $t_i$  since the signs alternate, so  $c_{j+1}$  is paired with  $\overline{c}_k$  for some k > j. This contradicts the maximality of k + j. Hence  $k = m_i$ . Similarly,  $j = m_i$ . Thus  $\{c_{m_i}, \overline{c}_{m_i}\}$  lies in both the even-to-odd pairings and the odd-to-even pairings. This gives a single component, which is a contradiction.

(ii). Suppose that  $w \in (\Pi \overline{t}_{[n \downarrow k+1]} t_k \star)$ . Thus  $(\overline{z}_1, ((a_i, \overline{a}_i))_{i \in [1\uparrow n-k+2]})$  is

$$(\overline{z}_1, \overline{t}_n, t_n, \overline{t}_{n-1}, t_{n-1}, \dots, \overline{t}_{k+1}, t_{k+1}, t_k, \overline{t}_k, a_{n-k+2}, \overline{a}_{n-k+2})$$

Notice that  $\{\overline{t}_k, a_{n-k+2}\}$  must be nested with  $\{t_{k+1}, t_k\}$ , and, hence  $a_{n-k+2}$  must lie in  $\{t_k, \overline{t}_k, t_{k+1}\}$ . By (i),  $a_{n-k+2} \neq t_k$ . Since  $a_{([1\uparrow m])}$  is a reduced expression,  $a_{n-k+2} \neq \overline{t}_k$ . Hence  $a_{n-k+2} = t_{k+1}$ . Let us denote this term  $t'_{k+1}$  to distinguish it from the preceding occurrence of  $t_{k+1}$ .  $\{\overline{t}_k, t'_{k+1}\}$  is nested with  $\{t_{k+1}, t_k\}$ . Hence, Then  $t'_{k+1} < t_{k+1}$ . By Lemma II.3,  $\overline{t}'_{k+1} > \overline{t}_{k+1}$ .

Thus  $(\overline{z}_1, ((a_i, \overline{a}_i))_{i \in [1 \uparrow n - k + 3]})$  is

$$(\overline{z}_1, \overline{t}_n, t_n, \overline{t}_{n-1}, t_{n-1}, \dots, \overline{t}_{k+1}, t_{k+1}, t_k, \overline{t}_k, t'_{k+1}, \overline{t}'_{k+1}, a_{n-k+3}, \overline{a}_{n-k+3})$$

Notice that  $\{\overline{t}'_{k+1}, a_{n-k+3}\}$  must be nested with  $\{t_{k+2}, \overline{t}_{k+1}\}$ , and, hence,  $a_{n-k+3}$  must lie in  $\{\overline{t}_{k+1}, t_{k+2}\}$ . Since  $a_{([1\uparrow m])}$  is a reduced respression,  $a_{n-k+3} \neq \overline{t}_{k+1}$ . Hence  $a_{n-k+3} = t_{k+2}$ , and we denote this by  $t'_{k+2}$ . Then  $t'_{k+2} < t_{k+2}$ , and, by Lemma II.3,  $\overline{t}'_{k+2} > \overline{t}_{k+2}$ .

By repeating the argument in the last paragraph, we eventually find that  $w = t_k^{\Pi t_{[k+1\uparrow n]}}$ .

(iii). Suppose that  $w \in (\Pi t_{[1\uparrow k-1]} \overline{t}_k \star)$ .

Then  $(\overline{z}_1, ((a_i, \overline{a}_i))_{i \in [1\uparrow 2k]}) = (\overline{z}_1, t_1, \overline{t}_1, t_2, \overline{t}_2, \dots, t_{k-1}, \overline{t}_{k-1}, \overline{t}_k, t_k)$ , and by an argument similar to that in (ii), we find that this is impossible.

### IV $\mathcal{B}_n$ permutes the planar words in $\Sigma_{0,1,n}$

**IV.1 Proposition.** Let  $w \in \Sigma_{0,1,n}$  and let  $i \in [1 \uparrow n - 1]$ . If w is a planar word in  $\Sigma_{0,1,n}$ , then  $w^{\sigma_i}$  is a planar word in  $\Sigma_{0,1,n}$ .

*Proof.* Suppose that  $r_{([1\uparrow m])}$  is any planar expression for w, as in Definitions III.1. In applying  $\sigma_i$  to  $(\overline{z}_1, ((r_i, \overline{r}_i))_{i \in [1\uparrow m]}, z_1)$ , we

replace each  $t_i, \overline{t}_i$  with  $t_{i+1}, \overline{t}_{i+1},$ 

replace each  $\overline{t}_i, t_i$  with  $\overline{t}_{i+1}, t_{i+1}$ ,

replace each  $t_{i+1}, \overline{t}_{i+1}$  with  $\overline{t}_{i+1}, t_{i+1}, \overline{t}_i, \overline{t}_i, t_{i+1}, \overline{t}_{i+1}, \overline{$ 

replace each  $\overline{t}_{i+1}, t_{i+1}$  with  $\overline{t}_{i+1}, t_{i+1}, \overline{t}_i, t_i, t_{i+1}, \overline{t}_{i+1}$ .

We will not perform any cancellations in the resulting sequence.

Let  $\pi \in \text{Sym}_{2m+2}$  be a permutation which embeds  $(\overline{z}_1, ((r_i, \overline{r}_i))_{i \in [1\uparrow m]}, z_1)$  in a plane. By Lemma II.4, there exists an ordered set  $(B, \leq)$ , and a (2m+2)-tuple  $p_{([1\uparrow 2m+2])}$  without repetitions, for  $(B, \leq)$ , such that  $\pi$  embeds  $p_{([1\uparrow 2m+2])}$  in a plane. Moreover, there exists a labelling  $B \to A, b \mapsto \text{label}(b)$ , such that the labelling respects the orderings and

$$\operatorname{label}(p_{([1\uparrow 2m+2])}) = (\overline{z}_1, ((r_i, \overline{r}_i))_{i \in [1\uparrow m]}, z_1).$$

Moreover,  $B = p_{[1\uparrow 2m+2]}$ .

Let  $m_i$  denote the number of elements of B with label  $t_i$ , and let  $m_{i+1}$  denote the number of elements of B with label  $t_{i+1}$ . To begin, we have to add  $4m_{i+1}$ elements to B, and we have to specify the ordering on the expanded set.

Let  $c_{[1\uparrow m_i]}$  denote the set, in ascending order, of those elements of B which have the label  $t_i$ . Let  $\overline{c}_{[m_i\downarrow 1]}$  denote the set, in ascending order, of those elements of B which have the label  $\overline{t}_i$ . Let  $d_{[1\uparrow m_{i+1}]}$  denote the set, in ascending order, of those elements of B which have the label  $t_{i+1}$ . Let  $\overline{d}_{[m_{i+1}\downarrow 1]}$  denote the set, in ascending order, of those elements of B which have the label  $\overline{t}_{i+1}$ . Thus we have

$$c_1 < \ldots < c_{m_i} < \overline{c}_{m_i} < \ldots < \overline{c}_1 < d_1 < \ldots < d_{m_{i+1}} < \overline{d}_{m_{i+1}} < \ldots < \overline{d}_1$$

and no other element of B lies in the interval  $[c_1 \uparrow \overline{d}_1]$ . We write

$$[c_1 \uparrow d_1] = (c_{([1 \uparrow m_i])}, \overline{c}_{([m_i \downarrow 1])}, d_{([1 \uparrow m_{i+1}])}, d_{([m_{i+1} \downarrow 1])})$$

to express this.

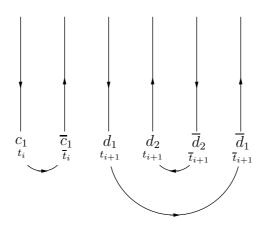


Figure IV.1.1:  $(c_{([1\uparrow 1])}, \overline{c}_{([1\downarrow 1])}, d_{([1\uparrow 2])}, \overline{d}_{([2\downarrow 1])}).$ 

With the preceding notation, we create an interval of  $4m_{i+1}$  new elements denoted

$$[a_1 \uparrow \overline{b}_1] = (a_{([1 \uparrow m_{i+1}])}, \overline{a}_{([m_{i+1} \downarrow 1])}, b_{([1 \uparrow m_{i+1}])}, \overline{b}_{([m_{i+1} \downarrow 1])}).$$

We expand B by inserting this interval just before  $c_1$ , that is, just before the interval  $[c_1 \uparrow \overline{d}_1]$ . We now have a new ordered set B' with  $2m+2+4m_{i+1}$  elements.

We have to specify the new labelling  $B' \to A$ . On  $c_{[1\uparrow m_i]}$ , we change the labels from  $t_i$  to  $t_{i+1}$ . On  $\overline{c}_{[m_i\downarrow 1]}$ , we change the labels from  $\overline{t}_i$  to  $\overline{t}_{i+1}$ . On  $d_{[1\uparrow m_{i+1}]}$ , we change the labels from  $t_{i+1}$  to  $\overline{t}_{i+1}$ . On  $\overline{d}_{[m_{i+1}\downarrow 1]}$ , we keep the same labels,  $\overline{t}_{i+1}$ . On the rest of  $B - [c_1\uparrow \overline{d}_1]$ , we keep the same labels. We give all the elements of  $a_{[1\uparrow m_{i+1}]}$  the label  $t_i$ ; we give all the elements of  $\overline{a}_{[m_{i+1}\downarrow 1]}$  the label  $\overline{t}_i$ ; we give all the elements of  $b_{[1\uparrow m_{i+1}]}$  and  $\overline{b}_{[m_{i+1}\downarrow 1]}$  the label  $t_{i+1}$ . The labelling clearly respects the orderings of B' and A.

For the even-to-odd steps, it follows from Lemma II.3 that

$$\{p_{[2k\uparrow 2k+1]}\}_{k\in[1\uparrow m]} \supseteq \{\{c_i,\overline{c}_i\}\}_{i\in[1\uparrow r]} \cup \{\{d_j,d_j\}\}_{j\in[1\uparrow s]}.$$

Let  $q_{([1\uparrow 2m+4m_{i+1}])}$  be the 2m + 4s-tuple obtained from  $p_{([1\uparrow 2m+2])}$  as follows. For each  $j \in [1\uparrow m_{i+1}]$ , there exists a unique  $i \in [1\uparrow m]$  such that  $p_{[2i-1\uparrow 2i]} = \{d_j, \overline{d}_j\}$ . If  $p_{([2i-1\uparrow 2i])} = (d_j, \overline{d}_j)$ , then it is to be expanded to  $(d_j, \overline{b}_j, a_j, \overline{a}_j, b_j, \overline{d}_j)$ . If  $p_{([2i-1\uparrow 2i])} = (\overline{d}_j, d_j)$ , then it is to be expanded to  $(\overline{d}_j, b_j, \overline{a}_j, a_j, \overline{b}_j, d_j)$ . This completes the definition of  $q_{([1\uparrow 2m+4m_{i+1}])}$ .

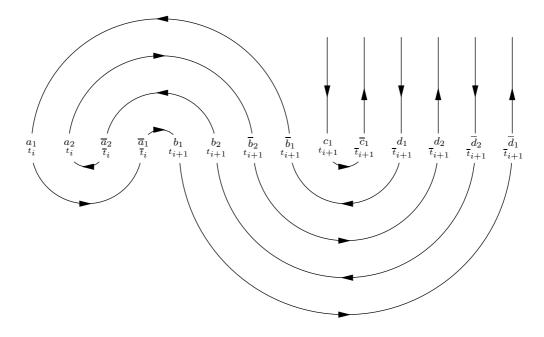


Figure IV.1.2:  $(a_{([1\uparrow 2])}, \overline{a}_{([2\downarrow 1])}, b_{([1\uparrow 2])}, \overline{b}_{([2\downarrow 1])}, c_{([1\uparrow 1])}, \overline{c}_{([1\downarrow 1])}, d_{([1\uparrow 2])}, \overline{d}_{([2\downarrow 1])})$ . In passing from  $\{p_{[2k\uparrow 2k+1]}\}_{k\in[1\uparrow m-1]}$  to  $\{q_{[2k\uparrow 2k+1]}\}_{k\in[1\uparrow m+2m_{i+1}-1]}$ , we add  $\{\{\overline{b}_j, a_j\}\}_{j\in[1\uparrow s]} \cup \{\{\overline{a}_j, b_j\}\}_{j\in[1\uparrow s]}$ . In B', for each  $j \in [1\uparrow s]$ ,

$$\begin{split} [\overline{a}_{j}\uparrow b_{j}] &= (\overline{a}_{([j\downarrow 1])}, b_{([1\uparrow j])}) \\ \text{and the underlying set is } \cup \{\overline{a}_{k}, b_{k}\}_{k\in[1\uparrow j]}, \\ [a_{j}\uparrow \overline{b}_{j}] &= (a_{[j\uparrow s]}, \overline{a}_{[s\downarrow 1]}, b_{[1\uparrow s]}, \overline{b}_{[s\downarrow j]}) \\ \text{and the underlying set is } \cup \{\overline{a}_{k}, b_{k}\}_{k\in[1\uparrow s]} \cup \cup \{\overline{b}_{k}, a_{k}\}_{k\in[j\uparrow s]}. \end{split}$$

Both of these intervals are closed under the pairing-off of

 $\{q_{[2k\uparrow 2k+1]}\}_{k\in [1\uparrow m+2m_{i+1}-1]}.$ 

Thus,  $\{q_{[2k\uparrow 2k+1]}\}_{k\in[1\uparrow m+2m_{i+1}-1]}$  is also nested.

In passing from  $\{p_{[2k-1\uparrow 2k]}\}_{k\in[1\uparrow m]}$  to  $\{q_{[2k-1\uparrow 2k]}\}_{k\in[1\uparrow m+m_{i+1}]}$ , we delete  $\{\{d_j, \overline{d}_j\}\}_{j\in[1\uparrow s]}$ , and add  $\{\{d_j, \overline{b}_j\}\}_{j\in[1\uparrow s]} \cup \{\{a_j, \overline{a}_j\}\}_{j\in[1\uparrow s]} \cup \{\{b_j, \overline{d}_j\}\}_{j\in[1\uparrow s]}$ . In B', for each  $j \in [1\uparrow s]$ ,

$$\begin{split} [a_j,\overline{a}_j] &= (\overline{a}_{([j\downarrow1])},a_{([1\uparrow j])}) \\ & \text{and the underlying set is } \bigcup_{k\in[1\uparrow j]} \{a_k,\overline{a}_k\}, \\ [\overline{b}_j,d_j] &= (\overline{b}_{([j\downarrow1])},c_{([1\uparrow r])},\overline{c}_{([r\downarrow1])},d_{([1\uparrow j])}) \\ & \text{and the underlying set is } \bigcup_{k\in[1\uparrow j]} \{d_k,\overline{b}_k\} \cup \bigcup_{i\in[1\uparrow r]} \{c_i,\overline{c}_i\}, \\ [b_j,\overline{d}_j] &= (b_{([j,s])},\overline{b}_{([s\downarrow1])},c_{([1\uparrow r])},\overline{c}_{([r\downarrow1])},d_{([1\uparrow s])},\overline{d}_{([s\downarrow j])}) \\ & \text{and the underlying set is } \bigcup_{k\in[1\uparrow s]} \{d_k,\overline{b}_k\} \cup \bigcup_{k\in[j\uparrow s]} \{b_k,\overline{d}_k\} \cup \bigcup_{i\in[1\uparrow r]} \{c_i,\overline{c}_i\}. \end{split}$$

Each of these intervals is closed under the pairing-off of  $\{q_{[2k-1\uparrow 2k]}\}_{k\in[1\uparrow m+2m_{i+1}]}$ . Thus,  $\{q_{[2k-1\uparrow 2k]}\}_{k\in[1\uparrow m+2m_{i+1}]}$  is nested.

A similar argument shows that  $\overline{\sigma}_i$  carries planar words to planar words.

**IV.2 Theorem.** The group  $\mathcal{B}_n$  acts on the set of planar words in  $\Sigma_{0,1,n}$ , and, hence, if  $n \geq 1$ , then every element of  $t_1^{\mathcal{B}_n}$  is a planar word.

**IV.3 Remark.** By combining Theorem IV.2 and Proposition III.4, we get another proof of Corollary 7.6.

## V The $\mathcal{B}_n$ -orbits of the planar words in $\Sigma_{0,1,n}$

In this section we rework [21, Lemma 2.3.12] and in this case our argument seems to be longer than Larue's. The object is to show that the number of  $\mathcal{B}_n$ -orbits in the set of all planar words in  $\Sigma_{0,1,n}$  is n+1, and that  $\{\Pi t_{[1\uparrow k]}\}_{k\in[0\uparrow n]}$  is a complete set of representatives.

**V.1 Lemma.** Let *i*, *j* be elements of  $[1\uparrow n]$  such that  $j \leq i-1$ , let  $\phi = \Pi \sigma_{[j\uparrow i-1]}$ , and let *w* be a planar word in  $\Sigma_{0,1,n}$ .

- (i) If  $w \in (\Pi t_{[1\uparrow i]}t_j\star)$ , then  $|w^{\phi}| < |w|$ .
- (ii) If  $w \in (\Pi t_{[1\uparrow i]}\overline{t}_{j\star})$ , then  $|w^{\phi}| < |w|$ .

*Proof.* It is straightforward to show that  $\phi$  acts as

$k \in [1, j-1]$		$k \in [j+1,i]$	$k \in [i+1,n]$
$(t_k$	$t_j$	$t_k$	$t_k)^{\phi}$
$=(t_k$	$t_i$	$t_{k-1}^{t_i}$	$t_k).$

(i). Suppose that  $w \in (\Pi t_{[1\uparrow i]}t_j\star)$ .

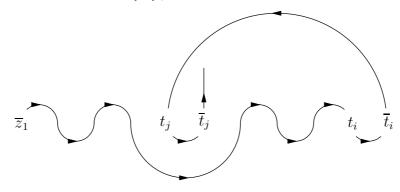


Figure V.1.1:  $w \in (\prod t_{[1\uparrow i]}t_j\star), j \leq i-1.$ 

Since  $t_i t_j$  is a subword of w, every letter occurring in w that belongs to  $t_{[j\uparrow i]} \cup \overline{t}_{[j\uparrow i]}$  belongs to a (reduced) subword of w of the form  $av\overline{b}$ , where  $a, b \in$ 

 $\{\overline{t}_i, t_j\}$  and  $v \in \langle t_{[j\uparrow i]} \rangle$ . Since, moreover, w begins with  $\prod t_{[1\uparrow i]}$ , it can be shown that it is not possible to have  $a = \overline{t}_i$  or  $b = \overline{t}_i$ . Thus  $a = b = t_j$ . Here,  $|(av\overline{b})^{\phi}| = |avb| - 2$ .

We factor w into syllables consisting of all such subwords together with the individual remaining letters, all of which lie in  $t_{[1\uparrow j-1]} \cup t_{[i+1\uparrow n]}$ , and all of which are mapped to single letters by  $\phi$ .

Since  $t_i$  occurs in w, we see that  $|w^{\phi}| < |w|$ .

(ii). Suppose that  $w \in (\Pi t_{[1\uparrow i]} \overline{t}_j \star)$ .

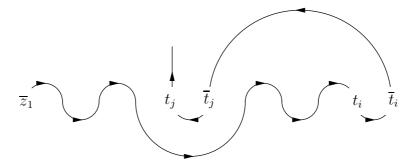


Figure V.1.2:  $w \in (\Pi t_{[1\uparrow i]} \overline{t}_j \star), j \leq i-1.$ 

Since  $t_i \overline{t}_j$  is a subword of w, every letter occurring in w that belongs to  $t_{[j+1\uparrow i]} \cup \overline{t}_{[j+1\uparrow i]}$  belongs to a (reduced) subword of w of the form  $av\overline{b}$ , where  $a, b \in \{t_j, \overline{t}_i\}$  and  $v \in \langle t_{[j+1\uparrow i]} \rangle$ . Since, moreover, w begins with  $\Pi t_{[1\uparrow i]}$ , it can be shown that it is not possible to have  $a = \overline{t}_i$  or  $b = \overline{t}_i$ . Thus  $a = b = t_j$ . Here,  $|(av\overline{b})^{\phi}| = |avb| - 2$ .

We factor w into syllables consisting of all such subwords together with the individual remaining letters, all of which lie in  $t_{[1\uparrow j]} \cup t_{[i+1\uparrow n]}$ , and all of which are mapped to single letters by  $\phi$ .

Since  $t_i$  occurs in w, it is then clear that  $|w^{\phi}| \leq |w| - 2$ .

**V.2 Lemma.** Let *i*, *j* be elements of  $[1\uparrow n]$  such that  $j \ge i+2$ , let  $\phi = \Pi \overline{\sigma}_{[j-1\downarrow i+1]}$ , and let *w* be a planar word in  $\Sigma_{0,1,n}$ .

- (i) If  $w \in (\Pi t_{[1\uparrow i]}t_j\star)$ , then  $|w^{\phi}| \leq |w|$ , and, moreover, if  $|w^{\phi}| = |w|$  then  $w^{\phi} \in (\Pi t_{[1\uparrow i+1]}\star)$ .
- (ii) If  $w \in (\Pi t_{[1\uparrow i]}\overline{t}_j\star)$ , then  $|w^{\phi}| < |w|$ .

*Proof.* It is straightforward to show that  $\phi$  acts as

$k \in [1,i]$	$\scriptstyle k \in [i+1,j-1]$		$k \in [j+1,n]$
$(t_k$	$t_k$	$t_{j}$	$t_k)^{\phi}$
$=(t_k$	$t_{k+1}^{\overline{t}_{i+1}}$	$t_{i+1}$	$t_k).$

(i). Suppose that  $w \in (\Pi t_{[1\uparrow i]}t_j\star)$ .

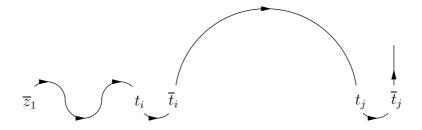


Figure V.2.1:  $w \in (\Pi t_{[1\uparrow i]}t_j\star), j \geq i+2.$ 

Since  $t_i t_j$  is a subword of w, every letter occurring in w that belongs to  $t_{[i+1\uparrow j-1]} \cup \overline{t}_{[i+1\uparrow j-1]}$  belongs to a (reduced) subword of w of the form  $av\overline{b}$ , where  $a, b \in \{t_i, \overline{t}_j\}$  and  $v \in \langle t_{[i+1\uparrow j-1]} \rangle$ . Since, moreover, w begins with  $\Pi t_{[1\uparrow i]}$ , it can be shown that it is not possible to have  $a = t_i$  or  $b = t_i$ . Thus  $a = b = \overline{t}_j$ . Here,  $|(av\overline{b})^{\phi}| = |avb| - 2$ .

We factor w into syllables consisting of all such subwords together with the individual remaining letters, all of which lie in  $t_{[1\uparrow i]} \cup t_{[j\uparrow n]}$ , and all of which are mapped to single letters by  $\phi$ .

It is then clear that  $|w^{\phi}| \leq |w|$ .

Moreover, if  $|w^{\phi}| = |w|$ , then  $w \in \langle t_{[1\uparrow i]} \cup t_{[j\uparrow n]} \rangle$ , and  $w^{\phi} \in (\Pi t_{[1\uparrow i+1]}\star)$ . (ii). Suppose that  $w \in (\Pi t_{[1\uparrow i]}\overline{t}_{j}\star)$ .

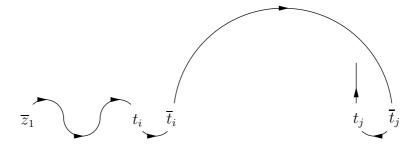


Figure V.2.2:  $w \in (\Pi t_{[1\uparrow i]} \overline{t}_j \star), j \ge i+2.$ 

Since  $t_i \overline{t}_j$  is a subword of w, every letter occurring in w that belongs to  $t_{[i+1\uparrow j]} \cup \overline{t}_{[i+1\uparrow j]}$  belongs to a (reduced) subword of w of the form  $av\overline{b}$ , where  $a, b \in \{t_i, \overline{t}_j\}$  and  $v \in \langle t_{[i+1\uparrow j]} \rangle$ . Since, moreover, w begins with  $\Pi t_{[1\uparrow i]}$ , it can be shown that it is not possible to have  $a = t_i$  or  $b = t_i$ . Thus  $a = b = \overline{t}_j$ . Here,  $|(av\overline{b})^{\phi}| = |avb| - 2$ .

We factor w into syllables consisting of all such subwords together with the individual remaining letters, all of which lie in  $t_{[1\uparrow i]} \cup t_{[j+1\uparrow n]}$ , and all of which are mapped to single letters by  $\phi$ .

Since  $\overline{t}_j$  occurs in w, it is then clear that  $|w^{\phi}| \leq |w| - 2$ .

**V.3 Theorem** (Larue). The set  $\{\Pi t_{[1\uparrow k]}\}_{k\in[0\uparrow n]}$  is a complete set of representatives of the  $\mathcal{B}_n$ -orbits in the set of all planar words in  $\Sigma_{0,1,n}$ .

*Proof.* Let w be a planar word in  $\Sigma_{0,1,n}$ . We wish to show that there exists some  $k \in [0\uparrow n]$  such that  $t_{[1\uparrow k]} \in w^{\mathcal{B}_n}$ .

Let *i* be the largest integer such that  $w \in (\Pi t_{[1\uparrow i]}\star)$ .

We may assume that, for all  $v \in w^{\mathcal{B}_n}$ ,  $|v| \geq |w|$ , and if |v| = |w|, then  $v \notin (\prod t_{[1\uparrow i+1]}\star)$ .

By Lemma V.1, for all  $j \in [1\uparrow i-1], w \notin (\Pi t_{[1\uparrow i]}t_j\star) \cup (\Pi t_{[1\uparrow i]}\overline{t}_j\star)$ . By Proposition III.4(i),  $w \notin (\Pi t_{[1\uparrow i]}t_i\star)$ . By the maximality of  $i, w \notin (\Pi t_{[1\uparrow i]}t_{i+1}\star)$ . By Proposition III.4(iii),  $w \notin (\Pi t_{[1\uparrow i]}\overline{t}_{i+1}\star)$ . By Lemma V.2, for all  $j \in [i+2\uparrow n], w \notin (\Pi t_{[1\uparrow i]}t_j\star) \cup (\Pi t_{[1\uparrow i]}\overline{t}_j\star)$ . Hence,  $w = \Pi t_{[1\uparrow i]}$ , as desired.

#### **V.4 Remarks.** (i). Let w be a planar word in $\Sigma_{0,1,n}$ .

Lemmas V.1 and V.2 give an effective procedure for finding  $\phi \in \mathcal{B}_n$  first to minimize  $|w^{\phi}|$ , and then to obtain the form  $w^{\phi} = \prod t_{[1\uparrow k]}$  for some  $k \in [0\uparrow n]$ .

(ii). Let  $n \ge 1$  and let w be a word in  $\Sigma_{0,1,n}$ .

Theorem V.3 shows that w lies in the  $\mathcal{B}_n$ -orbit of  $t_1$  if and only if the cyclically-reduced form of w lies in  $t_{[1\uparrow n]}$  and w is planar. Moreover, in this event, Lemmas V.1 and V.2 effectively produce a  $\phi \in \mathcal{B}_n$  such that  $w^{\phi} = t_1$ .

(iii). There is then an algorithm which, for any  $k \in [1\uparrow n]$ , and any k-tuple  $w_{([1\uparrow k])}$  for  $\Sigma_{0,1,n}$ , decides if there exists some  $\phi \in \mathcal{B}_n$  such that  $w_{([1\uparrow k])}^{\phi} = t_{([1\uparrow k])}$ , and effectively finds such a  $\phi$ . We proceed as follows. We first convert  $w_1$  to  $t_1$  if possible, and then we restrict to  $\langle \sigma_{[2\uparrow n-1]} \rangle$ .

It is interesting to compare this algorithm for  $\mathcal{B}_n$  with the Whitehead algorithm for the much larger group  $\operatorname{Aut}(\Sigma_{0,1,n})$ . The information provided by planarity is more detailed then the information carried by the Whitehead graph used in the Whitehead algorithm.

We record the following.

**V.5 Theorem** (Larue). Let  $n \ge 1$  and let  $w \in \Sigma_{0,1,n}$ . Then w lies in the  $\mathcal{B}_n$ -orbit of  $t_1$  if and only if the cyclically-reduced form of w lies in  $t_{[1\uparrow n]}$  and w is planar.

#### Acknowledgments

The research of both authors was funded by the MEC (Spain) and the EFRD (EU) through Projects BFM2003-06613 and MTM2006-13544.

We are grateful to Patrick Dehornov for encouraging us to study the work of Larue, to Bert Wiest for supplying us with photocopies of many pages of Larue's thesis, and to David Larue for kindly making his thesis available online.

We thank Mladen Bestvina and Edward Formanek for many interesting observations.

#### References

 Norbert A'Campo, Le groupe de monodromie du déploiement des singularités isolées de courbes planes I. Math. Ann. 213(1975), 1–32.

- [2] Hirotaka Akiyoshi, Makoto Sakuma, Masaaki Wada and Yasushi Yamashita, Punctured torus groups and 2-bridge knot groups (I). Lecture Notes in Mathematics 1909, Springer, Berlin, 2007. xliii+252pp.
- [3] Emil Artin, Theorie der Zöpfe, Abh. Math. Sem. Univ. Hamburg 4(1925), 47–72.
- [4] E. Artin, *Theory of braids*, Ann. of Math. **48**(1947), 101–126.
- [5] Lluís Bacardit Carrasco, Representació de grups de trenes per automorfismes de grups, MSc thesis, Univ. Autònoma de Barcelona, 2006, 74 pp.
- [6] Joan S. Birman and Hugh M. Hilden, On isotopies of homeomorphisms of Riemann sufaces, Ann. of Math. 97(1973), 424–439.
- [7] F. Bohnenblust, The algebraical braid group, Ann. of Math. 48(1947), 127–136.
- [8] Wei-Liang Chow, On the algebraical braid group, Ann. of Math. 49(1948), 654–658.
- [9] Daryl Cooper, Automorphisms of free groups have finitely generated fixed point sets, J. Algebra 111(1987), 453–456.
- [10] John Crisp and Luis Paris, Artin groups of type B and D, Adv. Geom. 5(2005), 607–636.
- [11] John Crisp and Luis Paris, Representations of the braid group by automorphisms of groups, invariants of links, and Garside groups, Pacific J. Math. 221(2005), 1–27.
- [12] Patrick Dehornoy, Sur la structure des gerbes libres, C. R. Acad. Sci. Paris Sér. I Math. 309(1989), 143–148.
- [13] Patrick Dehornoy, Free distributive groupoids, J. Pure Appl. Algebra **61**(1989), 123–146.
- [14] Patrick Dehornoy, Braid groups and left distributive operations, Trans. Amer. Math. Soc. 345(1994), 115–150.
- [15] Patrick Dehornoy, Braids and self-distributivity, Progress in Mathematics 192, Birkhäuser Verlag, Basel, 2000.
- [16] Patrick Dehornoy, Ivan Dynnikov, Dale Rolfsen and Bert Wiest, Why are braids orderable?, Panoramas et Synthèses 14, Soc. Math. France, Paris, 2002.
- [17] Warren Dicks and M. J. Dunwoody, Groups acting on graphs, Cambridge Stud. Adv. Math. 17, CUP, Cambridge, 1989
   Errets et al. https://www.studies.com/combridge.com/studies.com/combridge.com/com/combridge.com/combridge.com/com/combridge.com/com/com/combr
  - Errata at: http://mat.uab.cat/~dicks/DDerr.html
- [18] Warren Dicks and Edward Formanek, Algebraic mapping-class groups of orientable surfaces with boundaries, pp. 57–116, in: Infinite groups: geometric, combinatorial and dynamical aspects (eds. Laurent Bartholdi, Tullio Ceccherini-Silberstein, Tatiana Smirnova-Nagnibeda, Andrzej Zuk), Progress in Mathematics 248, Birkhäuser Verlag, Basel, 2005.

Errata and addenda at: http://mat.uab.cat/~dicks/Boundaries.html

- [19] R. Fenn, M. T. Greene, D. Rolfsen, C. Rourke and B. Wiest, Ordering the braid groups, Pacific J. Math. 191(1999), 49–74.
- [20] Jonathon Funk, The Hurwitz action and braid group orderings, Theory Appl. Categ. 9(2001/02), 121–150.
- [21] David Maurice Larue, Left-distributive and left-distributive idempotent algebras, PhD thesis, University of Colorado, Boulder, 1994, ix + 138 pp. http://www.mines.edu/fs\_home/dlarue/papers/dml.pdf
- [22] David M. Larue, On braid words and irreflexivity, Algebra Universalis **31**(1994), 104–112.
- [23] Wilhelm Magnus, Uber Automorphismen von Fundamentalgruppen berandeter Flächen, Math. Ann. 109(1934), 617–646.
- [24] Sandro Manfredini, Some subgroups of Artin's braid groups, Topology Appl. 78(1997), 123–142.
- [25] A. Markov, Foundations of the algebraic theory of braids (Russian), Trav. Inst. Math. Steklov 16(1945), 53 pp.
- [26] B. Perron and J. P. Vannier, Groupe de monodromie géométrique des singularités simples (Russian), Math. Ann. 306(1996), 231–245.

- [27] Hamish Short and Bert Wiest, Orderings of mapping class groups after Thurston, Enseign. Math. 46(2000), 279–312.
- [28] Vladimir Shpilrain, Representing braids by automorphisms, Internat. J. Algebra Comput. 11(2001), 773–777.
- [29] Masaaki Wada, Group invariants of links, Topology 31(1992), 399–406.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra (Barcelona), Spain

*E-mail addresses*: lluisbc@mat.uab.cat, dicks@mat.uab.cat

URL: http://mat.uab.cat/~dicks/