Multigrid methods for H^{div}-conforming discontinuous Galerkin methods for the Stokes equations

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Abstract

A multigrid method for the Stokes system discretized with an H^{div} -conforming discontinuous Galerkin method is presented. It acts on the combined velocity and pressure spaces and thus does not need a Schur complement approximation. The smoothers used are of overlapping Schwarz type and employ a local Helmholtz decomposition. Additionally, we use the fact that the discretization provides nested divergence free subspaces. We present the convergence analysis and numerical evidence that convergence rates are not only independent of mesh size, but also reasonably small.

1 Introduction

The efficient solution of the Stokes equations is an important step in the development of fast flow solvers. In this paper we present analysis and numerical results for a multigrid method with subspace correction smoother, which performs very efficiently on divergence-conforming discretizations with interior penalty. We obtain convergence rates for the Stokes problem which are comparable to those for the Laplacian.

Multigrid methods are known to be the most efficient preconditioners and solvers for diffusion problems. Nevertheless, for Stokes equations, the divergence constraint makes the solution process more complicated. A typical solution employs the use of block preconditioners, e. g. [13, 22, 27, 28], but their disadvantage is, that their performance is limited by the inf-sup constant of the problem. This could be avoided, if the multigrid method operated on the divergence free subspace directly, and thus would not have to deal with the saddle point problem at all. Such methods have been developed in different context and have proven very successful as reported for instance by Hiptmair [18] for Maxwell equations and by Schöberl [33] for incompressible elasticity with reduced integration.

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The main ingredients into such a method are a smoother which operates on the divergence free subspace and a grid transfer operator from coarse to fine mesh which maps the coarse divergence free subspace into the fine one. The second objective can be achieved by using a mixed finite element discretization for which the weakly divergence free functions are point-wise divergence free. For such a discretization, the natural finite element embedding operator from coarse to fine mesh does not increase the divergence of a function. Discretizations of this type are available, such as for instance in Scott and Vogelius [36, 39], Neilan and coauthors [14,16] and Zhang [41,42]. Here, we focus on the divergence conforming discontinuous Galerkin (DivDG) method of Cockburn, Kanschat, and Schötzau [11] due to its simplicity.

Following the approach by Schöberl [33], in order to study smoothers for the Stokes equations, we first consider a problem on the velocity space only with penalty for the divergence. This leads to a singularly perturbed problem with an operator with a large kernel. When it comes to smoothers for such operators, there are two basic options. One approach is to smooth the kernel space explicitly, as proposed for instance by Hiptmair [18] and Xu in [19]. The other option was presented by Arnold, Falk, and Winther in [2, 3] and smoothens the kernel implicitly, while never employing an explicit description of it.

We follow the implicit approach and use the same domain decomposition principle (i.e additive and multiplicative Schwarz methods and vertex patches), but instead of the Maxwell or divergence dominated mass matrix as in [2,3] apply it to the DivDG Stokes discretization. Then, we prove the convergence of the multigrid method with variable V-cycle algorithm for the singularly perturbed problem. The second pillar we rest on is the equivalence between singularly perturbed, divergence dominated elliptic forms and mixed formulations established by Schöberl in [32, 33]. This equivalence allows us to apply the smoother to a mixed formulation of nearly incompressible elasticity and then to proceed to the Stokes limit. As far as we know, the combination of these techniques has not been applied the DivDG method in [11]. Since our analysis is based on domain decomposition, fundamental results are also drawn from the seminal paper by Feng and Karakashian [15] on domain decomposition for discontinuous Galerkin methods for elliptic problems.

There is a close relation between our technique and the smoother suggested by Vanka in [38] for the MAC scheme: the MAC scheme can be considered the lowest order case of the DivDG methods (see [23]). In this case, the subspace decomposition structure of Vanka smoother corresponds to Neumann problems on cells, while our smoother is based on Dirichlet problems for vertex patches. Generalizations of the Vanka smoother have been applied successfully to different other discretizations albeit their velocity-pressure spaces are not matched in the sense of (2) (see for instance [37, 40] and literature cited there).

Recently, an alternative preconditioning method for Stokes discretizations of the same type as here has been introduced in [5] by Ayuso et al. Their method is based on auxiliary spaces introduced by Hiptmair and Xu in [19]. The exact sequence property of the divergence-conforming velocity element plays a crucial role as in our scheme, but their preconditioner uses a multigrid method for the biharmonic problem to solve the Stokes problem. As a consequence, it is not possible to use the preconditioning method for no-slip boundary conditions. On the other hand, it has been demonstrated in [26] that the multigrid method here can be lifted to the biharmonic problem, providing an efficient method for clamped boundary conditions.

The paper is organized as follows. In Section 2 we present the model problem and the DG discretization. The multigrid method and domain decomposition smoother are derived in Section 3. Section 4 is devoted to the convergence analysis of our preconditioning technique with the man result in Theorem 1 on page 9. The paper concludes with numerical experiments in Section 5.

2 The Stokes problem and its discretization

We consider discretizations of the Stokes equations

$$\begin{array}{ll}
-\Delta u + \nabla p = f & \text{in } \Omega, \\
\nabla \cdot u &= 0 & \text{in } \Omega, \\
u &= u^B & \text{on } \partial\Omega,
\end{array} \tag{1}$$

with no-slip boundary conditions on a bounded and convex domain $\Omega \subset \mathbb{R}^d$ with dimension d = 2, 3. The natural solution spaces for this problem are $V = H_0^1(\Omega; \mathbb{R}^d)$ for the velocity u and the space of mean value free square integrable functions $Q = L_0^2(\Omega)$ for the pressure p, although we point out that other wellposed boundary conditions do not pose a problem.

In order to obtain a finite element discretization, we partition the domain Ω into a hierarchy of meshes $\{\mathbb{T}_{\ell}\}_{\ell=0,...,L}$ of parallelogram and parallelepiped cells in two and three dimensions, respectively. In view of multilevel methods, the index ℓ refers to the mesh level defined as follows: let a coarse mesh \mathbb{T}_0 be given. The mesh hierarchy is defined recursively, such that the cells of $\mathbb{T}_{\ell+1}$ are obtained by splitting each cell of \mathbb{T}_{ℓ} into 2^d congruent children (refinement). These meshes are nested in the sense that every cell of \mathbb{T}_{ℓ} is equal to the union of its four children. We define the mesh size h_{ℓ} as the maximum of the diameters of the cells of \mathbb{T}_{ℓ} . Due to the refinement process, we have $h_{\ell} = 2^{-\ell} h_0$.

By construction, these meshes are conforming in the sense that every face of a cell is either at the boundary or a whole face of another cell; nevertheless, local refinement and hanging nodes do not pose a particular problem, since they can be treated following [20, 21]. By \mathbb{F}_{ℓ} we denote the set of all faces of the mesh \mathbb{T}_{ℓ} , which is composed of the set of interior faces \mathbb{F}_{ℓ}^{i} and the set of all boundary faces \mathbb{F}_{ℓ}^{j} .

We introduce a short hand notation for integral forms on \mathbb{T}_{ℓ} and on \mathbb{F}_{ℓ} by

$$\begin{split} (\phi,\psi)_{\mathbb{T}_{\ell}} &= \sum_{T \in \mathbb{T}_{\ell}} \int_{T} \phi \odot \psi \, dx, \qquad \left\langle \phi,\psi \right\rangle_{\mathbb{F}_{\ell}} = \sum_{F \in \mathbb{F}_{\ell}} \int_{F} \phi \odot \psi \, ds, \\ \left\| \phi \right\|_{\mathbb{T}_{\ell}} &= \left(\sum_{T \in \mathbb{T}_{\ell}} \int_{T} |\phi|^{2} \, dx \right)^{\frac{1}{2}}, \qquad \left\| \phi \right\|_{\mathbb{F}_{\ell}} = \left(\sum_{F \in \mathbb{F}_{\ell}} \int_{F} |\phi|^{2} \, ds \right)^{\frac{1}{2}}, \end{split}$$

The point-wise multiplication operator $\phi \odot \psi$ refers to the product $\phi \psi$, the scalar product $\phi \cdot \psi$ and the double contraction $\phi : \psi$ for scalar, vector and tensor arguments, respectively. The modulus $|\phi| = \sqrt{\phi \odot \phi}$ is defined accordingly.

In order to discretize (1) on the mesh \mathbb{T}_{ℓ} , we choose discrete subspaces $X_{\ell} = V_{\ell} \times Q_{\ell}$, where $Q_{\ell} \subset Q$. Following [11], we employ discrete subspaces V_{ℓ} of the

space $H_0^{\text{div}}(\Omega)$, where

$$H^{\operatorname{div}}(\Omega) = \left\{ v \in L^2(\Omega; \mathbb{R}^d) \big| \nabla \cdot v \in L^2(\Omega) \right\},\$$

$$H^{\operatorname{div}}_0(\Omega) = \left\{ v \in H^{\operatorname{div}}(\Omega) \big| v \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \right\}.$$

Here, we choose the well-known Raviart–Thomas space [30], but we point out that any pair of velocity spaces V_{ℓ} and pressure spaces Q_{ℓ} is admissible, if the key relation

$$\nabla \cdot V_{\ell} = Q_{\ell} \tag{2}$$

holds. The details of constructing the Raviart-Thomas space follow.

Each cell $T \in \mathbb{T}_{\ell}$ can be obtained as the image of a linear mapping Ψ_T of the reference cell $\hat{T} = [0, 1]^d$. On the reference cell, we define two polynomial spaces: first, \hat{Q}_k , the space of polynomials in d variables, such that the degree with respect to each variable does not exceed k. Second, we consider the vector valued space of Raviart–Thomas polynomials $\hat{V}_k = \hat{Q}_k^d + x\hat{Q}_k$. Polynomial spaces V_T and Q_T on the mesh cell T are obtained by the pull-back under the mapping Ψ_T (see for instance [4]). The polynomial degree k is arbitrary, but chosen uniformly on the whole mesh. Thus, we will omit the index k from now on. Concluding this construction, we obtain the finite element spaces

$$V_{\ell} = \left\{ v \in H_0^{\operatorname{div}}(\Omega) \middle| \forall T \in \mathbb{T}_{\ell} : v_{|T} \in V_T \right\},\ Q_{\ell} = \left\{ q \in L_0^2(\Omega) \middle| \forall T \in \mathbb{T}_{\ell} : q_{|T} \in Q_T \right\}.$$

2.1 Discontinuous Galerkin discretization

While the fact that V_{ℓ} is a subspace of $H_0^{\text{div}}(\Omega)$ implies continuity of the normal component of its functions across interfaces between cells, this is not true for tangential components. Thus, $V_{\ell} \not\subset H^1(\Omega; \mathbb{R}^d)$, and it cannot be used immediately to discretize (1). We follow the example in for instance [11, 24, 25] and apply a DG formulation to the discretization of the elliptic operator. Here, we focus on the interior penalty method [1,29]. Let T_1 and T_2 be two mesh cells with a joint face F, and let u_1 and u_2 be the traces of a function u on F from T_1 and T_2 , respectively. On this face F, we introduce the averaging operator

$$\{\!\!\{u\}\!\!\} = \frac{u_1 + u_2}{2}.\tag{3}$$

In this notation, the interior penalty bilinear form reads

$$a_{\ell}(u,v) = (\nabla u, \nabla v)_{\mathbb{T}_{\ell}} + 4 \langle \sigma_L \{\!\!\{ u \otimes \mathbf{n} \}\!\!\}, \{\!\!\{ v \otimes \mathbf{n} \}\!\!\} \rangle_{\mathbb{F}_{\ell}^i} - 2 \langle \{\!\!\{ \nabla u \}\!\!\}, \{\!\!\{ \mathbf{n} \otimes v \}\!\!\} \rangle_{\mathbb{F}_{\ell}^i} - 2 \langle \{\!\!\{ \nabla v \}\!\!\}, \{\!\!\{ \mathbf{n} \otimes u \}\!\!\} \rangle_{\mathbb{F}_{\ell}^i} + 2 \langle \sigma_L u, v \rangle_{\mathbb{F}_{\ell}^{\partial}} - \langle \partial_n u, v \rangle_{\mathbb{F}_{\ell}^{\partial}} - \langle \partial_n v, u \rangle_{\mathbb{F}_{\ell}^{\partial}}.$$

$$(4)$$

The operator " \otimes " denotes the Kronecker product of two vectors. We note that the term $4\{\!\{u \otimes \mathbf{n}\}\!\}: \{\!\{v \otimes \mathbf{n}\}\!\}$ actually denotes the product of the jumps of u and v.

The discrete weak formulation of (1) reads now: find $(u_\ell, p_\ell) \in V_\ell \times Q_\ell$, such that for all test functions $v_\ell \in V_\ell$ and $q_\ell \in Q_\ell$ there holds

$$\mathcal{A}_{\ell}\left(\binom{u_{\ell}}{p_{\ell}}, \binom{v_{\ell}}{q_{\ell}}\right) \equiv a_{\ell}(u_{\ell}, v_{\ell}) + (p_{\ell}, \nabla \cdot v_{\ell}) - (q_{\ell}, \nabla \cdot u_{\ell}) = \mathcal{F}(v_{\ell}, q_{\ell}) \equiv (f, v_{\ell})$$
(5)

Discussion on the existence and uniqueness of such solutions can be found for instance in [11, 12, 17, 24]. Here, we summarize, that is symmetric. If σ_L is sufficiently large, the form $a_\ell(.,.)$ is positive definite independently of the multigrid level $\ell \in [0, L]$, and that thus we can define a norm on V_ℓ by

$$\left\|v_{\ell}\right\|_{V_{\ell}} = \sqrt{a_{\ell}(v_{\ell}, v_{\ell})}.$$
(6)

In order to obtain optimal convergence results and to satisfy Proposition 2.2 below, σ_L is chosen as $\overline{\sigma}/h_L$, where h_L is mesh size on the finest level L and $\overline{\sigma}$ is a positive constant depending on the polynomial degree. By this choice, the bilinear forms on lower levels are inherited from finer levels in the sense, that

$$a_{\ell}(u_{\ell}, v_{\ell}) = a_L(u_{\ell}, v_{\ell}), \quad \forall \ u_{\ell}, v_{\ell} \in V_{\ell}.$$

$$\tag{7}$$

A particular feature of this method is (see [10, 11]), that the solution u_{ℓ} is in the divergence free subspace

$$V_{\ell}^{0} = \left\{ v_{\ell} \in V_{\ell} \middle| \nabla \cdot v_{\ell} = 0 \right\},\tag{8}$$

where the divergence condition is to be understood in the strong sense.

Proposition 0.1 (Inf-sup condition). For any pressure function $q \in Q_{\ell}$, there exists a velocity function $v \in V_{\ell}$, satisfying

$$\inf_{q \in Q_{\ell}} \sup_{v \in V_{\ell}} \frac{(q, \nabla \cdot v)}{\|v\|_{V_{\ell}} \|q\|_{Q_{\ell}}} \ge \gamma_{\ell} > 0 \tag{9}$$

where $\gamma_{\ell} = c \sqrt{\frac{h_L}{h_{\ell}}} = c \sqrt{2^{\ell-L}}$ and *c* is a constant independent of the multigrid level ℓ .

Proof. The proof of this proposition can be found in [35, Section 6.4]. Indeed, a different result is proven there, with $\gamma_{\ell} \approx 1/k$, where k is the polynomial degree in the hp-method. Thorough study of the proof though reveals, that this k-dependence is due to the penalty parameter of the form $\sigma_{\ell} \approx k^2/h_{\ell}$. In our case, the penalty parameter depends on the fine mesh, not on h_{ℓ} , such that $\sigma_{\ell} \approx (h_{\ell}/h_L)/h_{\ell}$, and that the role of the k^2 in the penalty is taken by the factor h_{ℓ}/h_L .

For any $u \in V_{\ell}$, we consider the following discrete Helmholtz decomposition:

$$u = u^0 + u^\perp \tag{10}$$

where $u^0 \in V^0_{\ell}$ is the divergence free part and u^{\perp} belongs to its $a_{\ell}(.,.)$ -orthogonal complement. For functions from this complement holds the estimate:

Lemma 0.1. Let $u^{\perp} \in V_{\ell}$ be $a_{\ell}(.,.)$ -orthogonal to V_{ℓ}^{0} , that is,

$$a_{\ell}(u^{\perp}, v) = 0 \quad \forall \ v \in V_{\ell}^0.$$

Then, there is a constant $\alpha > 0$ such that

$$\frac{\alpha}{d^2} \left\| \nabla \cdot u^{\perp} \right\|^2 \le a_{\ell}(u^{\perp}, u^{\perp}) \le \frac{1}{\gamma_{\ell}} \left\| \nabla \cdot u^{\perp} \right\|^2, \tag{11}$$

 γ_{ℓ} is the inf-sup constant from inequality (9).

Proof. On the left side, we already argued above that σ_L is chosen large enough such that $a_\ell(.,.)$ is uniformly positive definite. Thus, we have with a positive constant α

$$\alpha \|\nabla u^{\perp}\|_{\mathbb{T}_{\ell}}^2 \le a_{\ell}(u^{\perp}, u^{\perp}).$$

But then,

$$\left(\nabla \cdot u^{\perp}, \nabla \cdot u^{\perp}\right)_{\Omega} \leq d^2 \left(\nabla u^{\perp}, \nabla u^{\perp}\right)_{\mathbb{T}_{\ell}} \leq \frac{d^2}{\alpha} a_{\ell}(u^{\perp}, u^{\perp}),$$

On the right side, let $q = \nabla \cdot u^{\perp}$. Then $q \in Q_{\ell}$ due to (2). From (9), we conclude that there is $u \in V_{\ell}$ such that $\nabla \cdot u = q$ and $||u||_{V_{\ell}} \leq 1/\gamma_{\ell}||q||$. On the other hand, u^{\perp} is the error of the orthogonal projection into V_{ℓ}^{0} . Thus, u^{\perp} must be the element with minimal norm, and in particular $||u^{\perp}||_{V_{\ell}} \leq ||u||_{V_{\ell}}$.

2.2 The nearly incompressible problem

We are going to prove convergence uniform with respect to the refinement level ℓ of the proposed multigrid method for the Stokes problem by deviating twice. First, we provide estimates robust with respect to the parameter ε of the nearly incompressible problem: find $(u_{\ell}, p_{\ell}) \in V_{\ell} \times Q_{\ell}$ such that for all $(v_{\ell}, q_{\ell}) \in V_{\ell} \times Q_{\ell}$ there holds

$$\mathcal{A}_{\ell}\left(\begin{pmatrix}u_{\ell}\\p_{\ell}\end{pmatrix},\begin{pmatrix}v_{\ell}\\q_{\ell}\end{pmatrix}\right)+\varepsilon\left(p_{\ell},q_{\ell}\right)=\mathcal{F}(v_{\ell},q_{\ell}).$$
(12)

This problem is connected with the simpler penalty bilinear form (see for instance also [17])

$$A_{\ell,\varepsilon}(u_{\ell}, v_{\ell}) \equiv a_{\ell}(u_{\ell}, v_{\ell}) + \varepsilon^{-1} \left(\nabla \cdot u_{\ell}, \nabla \cdot v_{\ell} \right)$$
(13)

and the singularly perturbed, elliptic problem: find $u_{\ell} \in V_{\ell}$ such that for all $v_{\ell} \in V_{\ell}$ there holds

$$A_{\ell,\varepsilon}(u_\ell, v_\ell) = (f, v_\ell).$$
(14)

Lemma 0.2. Let (u_m, p_m) be the solution to (12) and u_e be the solution to (14). Then, if (2) holds, the following equations hold true:

$$u_m = u_e, \quad and \quad \varepsilon p_m = \nabla \cdot u_m = \nabla \cdot u_e.$$

Proof. Testing (12) with $v_{\ell} = 0$ and $q_{\ell} \in Q_{\ell}$ yields

$$-\left(\nabla \cdot u_m, q_\ell\right) + \varepsilon(p_m, q_\ell) = 0 \quad \forall \ q_\ell \in Q_\ell.$$

Due to (2), this translates to the point-wise equality $\varepsilon p_m = \nabla \cdot u_m$. Substituting p_m in (12) and testing with the pair $(v_\ell, \nabla \cdot v_\ell)$, which is possible again due to (2), we obtain that u_m solves (14).

If on the other hand u_e solves (14), we introduce $p_e = \frac{1}{\varepsilon} \nabla u_e$, which translates to

$$-(\nabla \cdot u_e, q_\ell) + \varepsilon(p_e, q_\ell) = 0 \quad \forall \ q_\ell \in Q_\ell,$$

corresponding to (12) tested with $(0, q_{\ell})$. On the other hand, (12) tested with $(v_{\ell}, 0)$ is obtained directly from (14) substituting p_e . Thus, the equivalence is proven.

In order to help keeping the notation separate, we adopt the following convention: the subscript ε is dropped wherever possible. Furthermore, curly letters refer to the mixed form, while straight capitals refer to operators on the velocity space only. Thus:

 $a_{\ell}(u, v)$ the vector valued interior penalty form

 $A_{\ell}(u, v)$ the form of the singularly perturbed, elliptic problem (14)

$$\mathcal{A}_{\ell}\left(\begin{pmatrix}u\\p\end{pmatrix}, \begin{pmatrix}v\\q\end{pmatrix}\right)$$
 the mixed bilinear form (12)

Similarly, capital letters like in R_{ℓ} for the smoother (26) refer to the singularly perturbed, elliptic problem, while \mathcal{R}_{ℓ} is the corresponding symbol for the Stokes smoother (24). Additionally, we associate operators with bilinear forms using the same symbol:

$$\begin{aligned} A_{\ell,\varepsilon} : V_{\ell} \to V_{\ell} & (A_{\ell,\varepsilon}u, v) = A_{\ell,\varepsilon}(u, v) = A_{\ell}(u, v) = A_{L}(u, v) \quad \forall u, v \in V_{\ell} \\ \mathcal{A}_{\ell,\varepsilon} : X_{\ell} \to X_{\ell} & (\mathcal{A}_{\ell,\varepsilon}x, y) = \mathcal{A}_{\ell,\varepsilon}(x, y) = \mathcal{A}_{\ell}(x, y) = \mathcal{A}_{L}(x, y) \quad \forall x, y \in X_{\ell} \end{aligned}$$

3 Multigrid method

In Section 2, we introduced hierarchies of meshes $\{\mathbb{T}_{\ell}\}$. Due to the nestedness of mesh cells, the finite element spaces associated with these meshes are nested as well:

This relation also extends to the divergence free subspaces, see for instance [26]:

$$V_0^0 \subset V_1^0 \subset \dots \subset V_L^0.$$
(15)

The nestedness of the spaces implies that there is a sequence of natural injections $\mathcal{I}_{\ell}: X_{\ell} \to X_{\ell+1}$ of the form $\mathcal{I}_{\ell}(v_{\ell}, q_{\ell}) = (I_{\ell,u}v_{\ell}, I_{\ell,p}q_{\ell})$, such that

$$I_{\ell,u}: V_{\ell} \to V_{\ell+1}, \qquad I_{\ell,p}: Q_{\ell} \to Q_{\ell+1}, \tag{16}$$

$$I_{\ell,u}: V^0_{\ell} \to V^0_{\ell+1}.$$
 (17)

The L^2 -projection from $X_{\ell+1} \to X_\ell$ is defined by $\mathcal{I}^t_\ell(v_\ell, q_\ell) = (I^t_{\ell,u}v_\ell, I^t_{\ell,p}q_\ell)$ with

$$(v_{\ell+1} - I_{\ell,u}^t v_{\ell+1}, w_\ell) = 0 \ \forall w_\ell \in V_\ell \quad (q_{\ell+1} - I_{\ell,p}^t q_{\ell+1}, r_\ell) = 0 \ \forall r_\ell \in Q_\ell.$$
(18)

The \mathcal{A} -orthogonal projection \mathcal{P}_{ℓ} from $(V_L \times Q_L) \to (V_{\ell} \times Q_{\ell})$ is defined by

$$\mathcal{A}_{L}\left(\mathcal{P}_{\ell}\begin{pmatrix}u\\p\end{pmatrix}, \begin{pmatrix}v_{\ell}\\q_{\ell}\end{pmatrix}\right) = \mathcal{A}_{L}\left(\begin{pmatrix}u\\p\end{pmatrix}, \begin{pmatrix}v_{\ell}\\q_{\ell}\end{pmatrix}\right)$$
(19)

for all $(u, p) \in (V_L \times Q_L), (v_\ell, q_\ell) \in V_\ell \times Q_\ell$. Similarly, The A-orthogonal projection P_ℓ from $V_L \to V_\ell$ is defined by

$$A_L(P_\ell u, v_\ell) = A_L(u, v_\ell) \tag{20}$$

for all $u \in V_L, v_\ell \in V_\ell$.

3.1 The V-cycle algorithm

In this subsection we define V-cycle multigrid preconditioners $\mathcal{B}_{\ell,\varepsilon}$ and $B_{\ell,\varepsilon}$ for the operators $\mathcal{A}_{\ell,\varepsilon}$ and $A_{\ell,\varepsilon}$, respectively. For simplicity of the presentation, we drop the index ε .

First, we define the action of the multigrid preconditioner $\mathcal{B}_{\ell} : X_{\ell} \to X_{\ell}$ recursively as the multigrid V-cycle with $m(\ell) \ge 1$ pre- and post-smoothing steps. Let \mathcal{R}_{ℓ} be a suitable smoother. Let $\mathcal{B}_0 = \mathcal{A}_0^{-1}$. For $\ell \ge 1$, define the action of \mathcal{B}_{ℓ} on a vector $\mathcal{L}_{\ell} = (f_{\ell}, g_{\ell})$ by

1. Pre-smoothing: begin with $(u_0, p_0) = (0, 0)$ and let

$$\binom{u_i}{p_i} = \binom{u_{i-1}}{p_{i-1}} + \mathcal{R}_\ell \left(\mathcal{L}_\ell - \mathcal{A}_\ell \binom{u_{i-1}}{p_{i-1}} \right) \quad i = 1, \dots, m(\ell),$$
(21a)

2. Coarse grid correction:

$$\begin{pmatrix} u_{m(\ell)+1} \\ p_{m(\ell)+1} \end{pmatrix} = \begin{pmatrix} u_{m(\ell)} \\ p_{m(\ell)} \end{pmatrix} + \mathcal{B}_{\ell-1} \mathcal{I}_{\ell-1}^t \left(\mathcal{L}_{\ell} - \mathcal{A}_{\ell} \begin{pmatrix} u_{m(\ell)} \\ p_{m(\ell)} \end{pmatrix} \right), \quad (21b)$$

3. Post-smoothing:

$$\begin{pmatrix} u_i \\ p_i \end{pmatrix} = \begin{pmatrix} u_{i-1} \\ p_{i-1} \end{pmatrix} + \mathcal{R}_{\ell} \left(\mathcal{L}_{\ell} - \mathcal{A}_{\ell} \begin{pmatrix} u_{i-1} \\ p_{i-1} \end{pmatrix} \right), \quad i = m(\ell) + 2, \dots, 2m(\ell) + 1$$
(21c)

4. Assign:

$$\mathcal{B}_{\ell}\mathcal{L}_{\ell} = \begin{pmatrix} u_{2m(\ell)+1} \\ p_{2m(\ell)+1} \end{pmatrix}$$
(21d)

We distinguish between the standard and variable V-cycle algorithms by the choice

$$m(\ell) = \begin{cases} m(L) & \text{standard V-cycle,} \\ m(L)2^{L-\ell} & \text{variable V-cycle,} \end{cases}$$

where the number m(L) of smoothing steps on the finest level is a free parameter. We refer to \mathcal{B}_L as the V-cycle preconditioner of \mathcal{A}_L . The iteration

$$\begin{pmatrix} u_{k+1} \\ p_{k+1} \end{pmatrix} = \begin{pmatrix} u_k \\ p_k \end{pmatrix} + \mathcal{B}_L \left(\mathcal{L}_L - \mathcal{A}_L \begin{pmatrix} u_k \\ p_k \end{pmatrix} \right)$$
(22)

is the V-cycle iteration.

The definition of the preconditioner $B_{\ell}: V_{\ell} \to V_{\ell}$ for the elliptic operator A_{ℓ} follows the same concept, but dropping the pressure variables.

3.2 Overlapping Schwarz smoothers

In this subsection, we define a class of smoothing operators \mathcal{R}_{ℓ} based on a subspace decomposition of the space X_{ℓ} . Let \mathcal{N}_{ℓ} be the set of vertices in the triangulation \mathbb{T}_{ℓ} , and let $\mathbb{T}_{\ell,\upsilon}$ be the set of cells in \mathbb{T}_{ℓ} sharing the vertex υ . They form a triangulation with N(N > 0) subdomains or patches which we denote by $\{\Omega_{\ell,\upsilon}\}_{\upsilon=1}^N$.

The subspace $X_{\ell,\upsilon} = V_{\ell,\upsilon} \times Q_{\ell,\upsilon}$ consists of the functions in X_ℓ with support in $\Omega_{\ell,\upsilon}$. Note that this implies homogeneous slip boundary conditions on $\partial\Omega_{\ell,\upsilon}$ for the velocity subspace $V_{\ell,\upsilon}$ and zero mean value on $\Omega_{\ell,\upsilon}$ for the pressure subspace $Q_{\ell,\upsilon}$. The Ritz projection $\mathcal{P}_{\ell,\upsilon} : X_\ell \to X_{\ell,\upsilon}$ is defined by the equation

$$\mathcal{A}_{\ell}\left(\mathcal{P}_{\ell,\upsilon}\begin{pmatrix}u_{\ell}\\p_{\ell}\end{pmatrix},\begin{pmatrix}v_{\ell,\upsilon}\\q_{\ell,\upsilon}\end{pmatrix}\right) = \mathcal{A}_{\ell}\left(\begin{pmatrix}u_{\ell}\\p_{\ell}\end{pmatrix},\begin{pmatrix}v_{\ell,\upsilon}\\q_{\ell,\upsilon}\end{pmatrix}\right) \qquad \forall \begin{pmatrix}v_{\ell,\upsilon}\\q_{\ell,\upsilon}\end{pmatrix} \in X_{\ell,\upsilon}.$$
(23)

Note that each cell belongs to not more than four (eight in 3D) patches $\mathbb{T}_{\ell,\upsilon}$, one for each of its vertices.

Then we define the additive Schwarz smoother

$$\mathcal{R}_{\ell} = \eta \sum_{\upsilon \in \mathcal{N}_{\ell}} \mathcal{P}_{\ell,\upsilon} \mathcal{A}_{\ell}^{-1}$$
(24)

where $\eta \in (0, 1]$ is a scaling factor, \mathcal{R}_{ℓ} is L^2 symmetric and positive definite.

Similarly, we define smoothers of the singularly perturbed elliptic operator A_{ℓ} , namely, $P_{\ell,v}: V_{\ell} \to V_{\ell,v}$ is defined as

$$A_{\ell}\left(P_{\ell,\upsilon}u_{\ell}, v_{\ell,\upsilon}\right) = A_{\ell}\left(u_{\ell}, v_{\ell,\upsilon}\right) \qquad \forall v_{\ell,\upsilon} \in V_{\ell,\upsilon}, \tag{25}$$

and the additive Schwarz smoother is

$$R_{\ell} = \eta \sum_{\upsilon \in \mathcal{N}_{\ell}} P_{\ell,\upsilon} A_{\ell}^{-1}.$$
 (26)

4 Convergence analysis

In this section, we provide a proof of the convergence for the variable V-cycle iteration with additive Schwarz preconditioning method. Our proof is based on the assumption that the domain Ω is bounded and convex, which will be omitted for simplicity in the statement of following theorems and propositions. Our main result is:

Theorem 1. The multilevel iteration $\mathcal{I} - \mathcal{B}_L \mathcal{A}_L$ for the Stokes problem (5) with the variable V-cycle operator \mathcal{B}_L defined in Section 3.1 employing the smoother \mathcal{R}_ℓ defined in equation (24) with suitably small scaling factor η is a contraction with contraction number independent of the mesh level L.

Proof. First, we consider the nearly incompressible problem (12). For this weak formulation, we have by Theorem 3, that the multigrid method $\mathcal{I} - \mathcal{B}_{L,\varepsilon}\mathcal{A}_{L,\varepsilon}$ is equivalent to the method $I - B_{L,\varepsilon}\mathcal{A}_{L,\varepsilon}$ applied to the singularly perturbed problem (14) in the velocity space.

Convergence of the multilevel iteration $I - B_{L,\varepsilon}A_{L,\varepsilon}$ is shown in Theorem 2 for all $\varepsilon > 0$ with a contraction number $\delta < 1$ independent of L and ε . Thus, by Theorem 3, the same holds for $\mathcal{I} - \mathcal{B}_{L,\varepsilon}\mathcal{A}_{L,\varepsilon}$ with positive ε .

Finally, in (12) we can let ε converge to zero. The limit yields the well-posed Stokes problem (5), and since the contraction number δ is independent of ε , we obtain uniform convergence with respect to the mesh level L in the limit $\varepsilon \rightarrow 0$.

The theorems and lemmas of the following subsections serve to establish the building blocks of the proof of Theorem 1.

4.1 The singularly perturbed problem

Theorem 2. Let R_{ℓ} be the smoother defined in (26) with suitably small scaling factor η . Then, the multilevel iteration $I - B_L A_L$ with the variable V-cycle operator B_L defined in Section 3.1 is a contraction with contraction number independent of the mesh level L and the parameter ε .

The proof of this theorem is postponed to page 12 and relies on

Proposition 2.1. If R_{ℓ} satisfies the conditions:

A

$$A_L((I - R_\ell A_\ell)w, w) \ge 0, \quad \forall w \in V_\ell$$
(27)

and

$$(R_{\ell}^{-1}[I - P_{\ell-1}]w, [I - P_{\ell-1}]w) \le \beta_{\ell} A_L([I - P_{\ell-1}]w, [I - P_{\ell-1}]w), \quad \forall w \in V_{\ell}$$
(28)

where $\beta_{\ell} = O(\frac{1}{\gamma_{\ell}})$ is defined in equation (55) below. Then

$$0 \le A_L \big((I - B_\ell A_\ell) w, w \big) \le \delta A_L (w, w), \quad \forall w \in V_\ell$$
⁽²⁹⁾

where $\delta = \frac{\hat{C}}{1+\hat{C}}$ and \hat{C} are defined in Lemma 2.3.

Proof. In the case of self-adjoint operators A_{ℓ} which are inherited from a common bilinear form a(.,.), this proposition would be part of the standard multigrid theory if β_{ℓ} were constant. Its proof can be adapted from similar theorems in [2,8,9]. We will prove the version needed here in the appendix.

In the remainder of this section we use several propositions and lemmas to establish our smoother R_{ℓ} satisfies the assumptions of Proposition 2.1. For $u \in (I - P_{\ell-1})w$ with arbitrary $w \in V_{\ell}$, it follows from the discrete Helmholtz decomposition in Section 2 and the projection operator $P_{\ell,v}$ in Section 3.2 that u admits a local discrete Helmholtz decomposition

$$u_{\upsilon} = u_{\upsilon}^0 + u_{\upsilon}^\perp \tag{30}$$

Lemma 2.1. Given L^2 -symmetric positive definite R_ℓ defined in 3.2 and symmetric positive definite $A_L(\cdot, \cdot)$ defined in (13), there exists a constant $\eta \in (0, 1]$ independent of ℓ such that

$$\eta(R_{\ell}^{-1}u, u) = \inf_{\substack{u_{\upsilon} \in V_{\ell,\upsilon} \\ \Sigma_{\upsilon} u_{\upsilon} = u}} \sum_{\upsilon \in \mathcal{N}_{\ell}} A_L(u_{\upsilon}, u_{\upsilon})$$
(31)

Proof. The following proof can be found in [2] for the L^2 -inner product instead of $a_\ell(.,.)$. We copy it here to ascertain that it does not depend on the actual structure of the operator A_L since it is purely algebraic. Thus, it applies to the operator A_L in this paper as it applies to the different operator applied there. Recall that

$$R_{\ell} = \eta \sum_{\upsilon \in \mathcal{N}_{\ell}} P_{\ell,\upsilon} A_{\ell}^{-1} = \eta \sum_{\upsilon \in \mathcal{N}_{\ell}} P_{\ell,\upsilon} A_{L}^{-1}.$$
 (32)

From

$$u = \sum_{\upsilon \in \mathcal{N}_{\ell}} u_{\upsilon} \tag{33}$$

we get

$$\eta(R_{\ell}^{-1}u, u) = \eta \sum_{v \in \mathcal{N}_{\ell}} (R_{\ell}^{-1}u, u_{v})$$

$$= \eta \sum_{v \in \mathcal{N}_{\ell}} (A_{L}P_{\ell,v}A_{L}^{-1}R_{\ell}^{-1}u, u_{v})$$
(34)
(35)

$$\leq \eta^{\frac{1}{2}} \left\{ \sum_{v \in \mathcal{N}_{\ell}} (A_L \eta P_{\ell,v} A_L^{-1} R_{\ell}^{-1} u, A_L^{-1} R_{\ell}^{-1} u) \right\}^{\frac{1}{2}} \left\{ \sum_{v \in \mathcal{N}_{\ell}} (A_L u_v, u_v) \right\}^{\frac{1}{2}}$$
(36)

$$= \eta^{\frac{1}{2}} \left\{ (A_L u, A_L^{-1} R_\ell^{-1} u) \right\}^{\frac{1}{2}} \left\{ \sum_{\upsilon \in \mathcal{N}_\ell} (A_L u_\upsilon, u_\upsilon) \right\}^2$$
(37)
$$= \eta^{\frac{1}{2}} \left\{ (u, R_\ell^{-1} u) \right\}^{\frac{1}{2}} \left\{ \sum_{\upsilon \in \mathcal{N}_\ell} (A_L u_\upsilon, u_\upsilon) \right\}^{\frac{1}{2}}$$
(38)

The above inequality works for arbitrary splitting, hence we have

$$\eta(R_{\ell}^{-1}u, u) \le \sum_{\upsilon \in \mathcal{N}_{\ell}} A_L(u_{\upsilon}, u_{\upsilon})$$
(39)

For the choice $u_{\upsilon} = P_{\ell,\upsilon} P_{\ell} A_L^{-1} R^{-1} u$ we get

$$\eta(R_{\ell}^{-1}u, u) = \inf_{\substack{u_{\upsilon} \in V_{\ell,\upsilon} \\ \Sigma_{\upsilon}u_{\upsilon} = u}} \sum_{\upsilon \in \mathcal{N}_{\ell}} A_L(u_{\upsilon}, u_{\upsilon})$$
(40)

Lemma 2.2. Given the local Helmholtz decomposition in (30). For any $u_{\upsilon}^{\perp} \in V_{\ell,\upsilon}$, there exists constant C_1 independent of multigrid level satisfying:

$$\sum_{\upsilon \in \mathcal{N}_{\ell}} \left\| \nabla \cdot u_{\upsilon}^{\perp} \right\|^2 \le C_1 \sum_{\upsilon \in \mathcal{N}_{\ell}} a_{\ell}(u_{\upsilon}^{\perp}, u_{\upsilon}^{\perp}) \tag{41}$$

Proof. It follows from Lemma 0.1 that the estimate $\|\nabla \cdot u^{\perp}\|^2 \leq Ca_{\ell}(u^{\perp}, u^{\perp})$ hold for all $u^{\perp} \in V_{\ell}$. It is easy to see that $V_{\ell,v}$ is a subspace of V_{ℓ} for any v, so the estimate are also valid on any patch. In 2-D case, one cell could at most be sharing by four patches(eight patches in 3D). Hence there exists a constant C_1 independent of multigrid level such that the estimates holds for the summation of local estimates.

Proposition 2.2. Given the overlapping subspace decomposition of V_{ℓ} in 3.2 and the interior penalty bilinear form $a_{\ell}(u, v)$ in (4). Assume σ_{ℓ} is chosen sufficiently large, the following estimate holds on each level ℓ . Then, there is a constant C_2 which is independent of multigrid level such that for any $u \in V_{\ell}$ holds

$$\sum_{\upsilon \in \mathcal{N}_{\ell}} a_{\ell}(u_{\upsilon}, u_{\upsilon}) \le C_2 a_{\ell}(u, u) \tag{42}$$

Proof. For a fixed L, the penalty constant σ_{ℓ} is $\overline{\sigma}/h_L$ which is greater than $\overline{\sigma}/h_{\ell}$. For the latter, this is a standard result: the proof and details on the choice of $\overline{\sigma}$ can be found in [15, p. 1361].

Proof of Theorem 2. Recall the definition of A_L -orthogonal projections P_ℓ and $P_{\ell,\nu}$ which restrict the projection on $\Omega_{\ell,\nu}$ (zero elsewhere). Following [2], we show that if $0 < \eta \leq 1/4$, the smoother R_ℓ satisfies the first condition in Theorem 2. For $w \in V_\ell$

$$A_L([I - R_\ell A_\ell]w, w) = A_L(w, w) - \eta \sum_{v \in \mathcal{N}_\ell} A_L(P_{\ell, v}w, w)$$
(43)

but

$$A_L(P_{\ell,\upsilon}w,w) = A_L(P_{\ell,\upsilon}w,P_{\ell,\upsilon}w) \le A_L(w,w)^{\frac{1}{2}}A_L(P_{\ell,\upsilon}w,P_{\ell,\upsilon}w)^{\frac{1}{2}}$$
(44)

so

$$\sum_{v \in \mathcal{N}_{\ell}} A_L(P_{\ell,v}w, w) \le \sum_{v \in \mathcal{N}_{\ell}} A_L(w, w) \le 4A_L(w, w)$$
(45)

Hence the first hypothesis holds.

Thus, it remains to check the second condition which could be reduced to the following problem: for $u = (I - P_{\ell-1})w$ (where $w \in V_{\ell}$) with the decomposition $u = \sum_{v} u_{v}$, there is a constant C such that

$$\sum_{\upsilon \in \mathcal{N}_{\ell}} A_L(u_{\upsilon}, u_{\upsilon}) \le C A_L(u, u) \tag{46}$$

Following Lemmas 0.1, 2.1, 2.2 and Proposition 2.2, we get:

$$\sum_{\upsilon \in \mathcal{N}_{\ell}} A_L(u_{\upsilon}, u_{\upsilon}) = \sum_{\upsilon \in \mathcal{N}_{\ell}} \left\{ a_\ell(u_{\upsilon}, u_{\upsilon}) + \varepsilon^{-1} (\nabla \cdot u_{\upsilon}, \nabla \cdot v_{\upsilon}) \right\}$$
(47)

$$= \sum_{\upsilon \in \mathcal{N}_{\ell}} \left\{ a_{\ell}(u_{\upsilon}, u_{\upsilon}) + \varepsilon^{-1} (\nabla \cdot u_{\upsilon}^{\perp}, \nabla \cdot u_{\upsilon}^{\perp}) \right\}$$
(48)

$$\leq C_2 a_\ell(u, u) + \sum_{\upsilon \in \mathcal{N}_\ell} C_1 \frac{1}{\alpha} \varepsilon^{-1} a_\ell(u_\upsilon^\perp, u_\upsilon^\perp)$$
(49)

$$\leq C_2 a_\ell(u, u) + \varepsilon^{-1} C_1 \frac{1}{\alpha} a_\ell(u^\perp, u^\perp)$$
(50)

$$\leq C_2 a_\ell(u, u) + \varepsilon^{-1} C_1 \frac{1}{\alpha} \frac{1}{\gamma_\ell} (\nabla \cdot u^\perp, \nabla \cdot u^\perp)$$
(51)

$$= C_2 a_\ell(u, u) + \varepsilon^{-1} C_1 \frac{1}{\alpha} \frac{1}{\gamma_\ell} (\nabla \cdot u, \nabla \cdot u)$$
(52)

$$\leq \max\left\{C_2, C_1 \frac{1}{\alpha} \frac{1}{\gamma_\ell}\right\} A_L(u, u) \tag{53}$$

$$= C_{\ell} A_L(u, u) \tag{54}$$

where
$$C_{\ell} = \max\left\{C_2, C_1 \frac{1}{\alpha} \frac{1}{\gamma_{\ell}}\right\}.$$

Now set

$$\beta_{\ell} = \frac{1}{\eta} C_{\ell} \tag{55}$$

We have verified the two conditions in Proposition 2.1.

Lemma 2.3. Given β_{ℓ} above and $m(\ell)$ defined in 3.1, there is a constant \hat{C} such that

$$\frac{\beta_{\ell}}{2m(\ell)} \le \hat{C} \tag{56}$$

Proof. We will discuss this inequality in two cases: first, if $\beta_{\ell} = \frac{1}{\eta}C_2$, then

$$\frac{\beta_{\ell}}{2m(\ell)} = \frac{\frac{1}{\eta}C_2}{2m_02^{L-\ell}} \le \frac{\frac{1}{\eta}C_2}{2m_0} =: \hat{C}$$
(57)

On the other hand, if $\beta = \frac{1}{\eta} C_1 \frac{1}{\alpha} \frac{1}{\gamma}$

$$\frac{\beta_{\ell}}{2m(\ell)} = \frac{C_1 \frac{1}{\alpha} \frac{1}{c} \sqrt{2^{L-\ell}}}{2m_0 2^{L-\ell}} = \frac{C_1 \frac{1}{\alpha} \frac{1}{c}}{2m_0 \sqrt{2^{L-\ell}}} \le \frac{C_1 \frac{1}{\alpha} \frac{1}{c}}{2m_0} =: \hat{C}$$
(58)

4.2 The mixed problem

Secondly, we will discuss the Stokes equation in mixed variables. Set $X_{\ell,\varepsilon} := \{(u_\ell, p_\ell) \in X_\ell : \nabla \cdot u_\ell = \varepsilon p_\ell\}$. Now, it remains to show the equivalence between the multigrid algorithms.

Proposition 2.3. The multigrid components fulfill the following properties:

1. The smoother \mathcal{R}_{ℓ} for the mixed problem defined in (24) preserves $X_{\ell,\varepsilon}$. On the subspace it is equivalent to the smoother R_{ℓ} in primal variables. This means for $(u_{\ell}, p_{\ell}) \in X_{\ell,\varepsilon}$ and

$$\begin{pmatrix} \hat{u}_{\ell} \\ \hat{p}_{\ell} \end{pmatrix} = \mathcal{R}_{\ell} \begin{pmatrix} u_{\ell} \\ p_{\ell} \end{pmatrix}$$
(59)

there holds $(\hat{u}_{\ell}, \hat{p}_{\ell}) \in X_{\ell, \varepsilon}$ and

$$\hat{u}_{\ell} = R_{\ell} u_{\ell} \tag{60}$$

2. The prolongation $\mathcal{I}_{\ell-1}$ maps $X_{\ell-1,\varepsilon}$ into $X_{\ell,\varepsilon}$. On the subspace it is equivalent to the prolongation I_{ℓ} in primal variables. This means for $(u_{\ell-1}, p_{\ell-1}) \in X_{\ell-1,\varepsilon}$ and

$$\begin{pmatrix} \hat{u}_{\ell} \\ \hat{p}_{\ell} \end{pmatrix} = \mathcal{I}_{\ell} \begin{pmatrix} u_{\ell-1} \\ p_{\ell-1} \end{pmatrix}$$
(61)

there holds $(\hat{u}_{\ell}, \hat{p}_{\ell}) \in X_{\ell, \varepsilon}$ and

$$\hat{u}_{\ell} = I_{\ell}(u_{\ell-1})$$
 (62)

 The coarse grid solution operator maps X_{ℓ−1,ε} into X_{ℓ,ε}. On the subspace it is equivalent to the coarse grid solution operator in primal variables. This means for (u_ℓ, p_ℓ) ∈ X_{ℓ,ε} and

$$\begin{pmatrix} \hat{u}_{\ell-1} \\ \hat{p}_{\ell-1} \end{pmatrix} = \mathcal{A}_{\ell-1}^{-1} [\mathcal{I}_{\ell-1}]^t \mathcal{A}_\ell \begin{pmatrix} u_\ell \\ p_\ell \end{pmatrix}$$
 (63)

there holds $(\hat{u}_{\ell-1}, \hat{p}_{\ell-1}) \in X_{\ell-1,\varepsilon}$ and

$$\hat{u}_{\ell-1} = A_{\ell-1}^{-1} [I_{\ell-1}]^t A_\ell u_\ell \tag{64}$$

Proof. The proof of this proposition can be found for the operators there in [33, p. 93]. We do not provide it here since the arguments are purely linear algebra, and thus apply independent of the actual bilinear form.

Theorem 3. The multigrid algorithm in mixed variables preserves the space $X_{\ell,\varepsilon}$. On this subspace it is equivalent to the multigrid algorithm in primal variables. This means for $(u_\ell, p_\ell) \in X_{\ell,\varepsilon}$ and $(\hat{u}_\ell, \hat{p}_\ell) = \mathcal{B}_\ell(u_\ell, p_\ell)$ there holds $(\hat{u}_\ell, \hat{p}_\ell) \in X_{\ell,\varepsilon}$ and

$$\hat{u_\ell} = B_\ell u_\ell \tag{65}$$

where \mathcal{B}_{ℓ} and B_{ℓ} are the corresponding multigrid operators for each algorithm.

Proof. The multigrid operator B_{ℓ} fulfills the recursion

$$B_0 = A_0^{-1}, (66)$$

$$B_{\ell} = (R_{\ell})^{m_{\ell}} (I - I_{\ell} (I - (B_{\ell-1})) A_{\ell-1}^{-1} [I_{\ell-1}]^{t} A_{\ell}) (R_{\ell})^{m_{\ell}}, \qquad (67)$$

and the mixed operator \mathcal{B}_{ℓ} fulfills a corresponding one. Then we apply the above proposition, and the theorem is proved by induction.

5 Numerical results

We test the additive Schwarz method which we have analyzed in the preceding section in order to ascertain that the contraction numbers are not only bounded away from one, but are actually small enough to make this method interesting. Furthermore, we conduct experiments, which go beyond our analysis, in particular regarding the choice of the penalty parameter and the number of smoothing steps on lower levels.

The experimental setup for most of the tables is as follows: the domain is $\Omega = [-1, 1]^2$, the coarsest mesh \mathbb{T}_0 consists of a single cell $T = \Omega$. The mesh \mathbb{T}_ℓ on level ℓ is obtained by dividing all cells in $\mathbb{T}_{\ell-1}$ into four quadrilaterals by connecting the edge midpoints. Thus, a mesh on level ℓ has 4^ℓ cells, and the length of their edges is $2^{1-\ell}$. The right hand side is f = (1, 1). For the relaxation parameter in the additive Schwarz method, we found that 0.5 is the value which provides the best results for all experimental setups, hence we keep it there in all the following experimental setups.

level	RT_1	RT_2
3	4	4
4	4	4
5	4	4
6	4	4
7	4	4
8	4	5

Table 1: Number of iterations n_6 to reduce the residual by 10^{-6} with the variable V-cycle algorithm with penalty parameter dependent of the finest level mesh size.

	$m(\ell) = 1$		$m(\ell) = 2$	
level	RT_1	RT_2	RT_1	RT_2
3	7	7	4	4
4	7	7	4	4
5	7	7	4	4
6	7	7	4	4
7	8	8	4	4
8	8	8	4	4

Table 2: Number of iterations n_6 to reduce the residual by 10^{-6} with the standard V-cycle iteration with one and two pre- and post-smoothing steps. Penalty parameter dependent of the finest level mesh size.

In Table 1, we first test the additive Schwarz smoother using variable V-cycle algorithm on a square domain with no-slip boundary condition. For the penalty constant in the DG form (4), we choose the penalty parameter as $\bar{\sigma}/h_L$, where $\bar{\sigma} = (k+1)(k+2)$, on the finest level L and all lower levels ℓ . Results for different pairs of RT_k/Q_k are reported in the table which show the fast and uniform convergence.

In Table 2, we keep the same experimental setup and present iteration counts for the standard V-cycle algorithm with one and two pre- and post-smoothing steps, respectively. Although our analysis does not apply, we still observe uniform convergence results. We also see that the variable V-cycle with a single smoothing step on the finest level is as fast as the standard V-cycle with two smoothing steps, and thus the variable V-cycle is more efficient.

In Table 3, we test the variable and standard V-cycles with penalty parameters depending on the mesh level ℓ , namely $\bar{\sigma}/h_{\ell}$ (where $\bar{\sigma}$ is a positive constant depending on the polynomial degree) in the DG form (4). While our convergence analysis does not cover this case either, we observe convergence rates equal to the case with inherited forms.

In Table 4, we provide results with GMRES solver and \mathcal{B}_L as preconditioner for different experimental setups as in Tables 1, 2 and 3 respectively. The second and third columns are results for variable V-cycle with penalty parameter dependent of the finest level mesh size. The fourth and fifth columns are the results for standard V-cycle with penalty parameter dependent of the finest level mesh size. The last two columns are the results for standard V-cycle with penalty parameter depend on the mesh size of each level. From this table, we see that the GMRES

	variable		standard	
level	RT_1	RT_2	RT_1	RT_2
3	4	4	7	7
4	4	4	7	7
5	4	4	7	7
6	4	4	7	7
7	4	4	7	8
8	4	5	8	8

Table 3: Penalty parameter dependent on the mesh size of each level. Number of iterations n_6 to reduce the residual by 10^{-6} with variable and standard V-cycle iterations with m(L) = 1.

	vari	able	standard		noninherited	
level	RT_1	RT_2	RT_1	RT_2	RT_1	RT_2
3	2	2	2	2	2	2
4	3	3	3	3	3	3
5	3	3	4	3	4	4
6	3	3	5	4	5	5
7	3	3	5	5	5	5
8	5	4	6	6	8	6

Table 4: Number of iterations n_6 to reduce the residual by 10^{-6} with GMRES solver and preconditioner \mathcal{B}_L ; variable and standard V-cycle with inherited forms, variable V-cycle with noninherited forms. One pre- and post-smoothing step on the finest level.

method, as expected, is faster in every case.

6 Conclusions

In this paper, we have investigated smoothers based on the ones introduced by Arnold, Falk, and Winther for problems in H^{div} in a variable V-cycle preconditioner for the Stokes system. We presented the convergence analysis and showed uniform contraction independent on the mesh level. In numerical experiments we showed that the contraction is not only uniform, but also very fast, thus making our method a feasible solver or preconditioner.

In theory, the performance of the smoother relies on an exact sequence property of finite element spaces, in particular an H^{div} -conforming discontinuous Galerkin discretization of the Stokes problem. Our experiments with the Taylor–Hood elements, where the method fails, demonstrate that this is not an artifact of the analysis, but that the technique does not work due to the lack of an exact Hodge decomposition and nested divergence free subspaces.

Proof for Proposition 2.1 A

Following the proof in [2], we want to show by induction on i that

$$0 \le A_L((I - B_i A_i)u, u) \le \delta A_L(u, u), \quad \forall u \in V_\ell$$
(68)

For i = 1 is obvious since $B_1 = A_1^{-1}$. Now check if the above inequality hold for $i = \ell - 1$. Recall the relaxation operator $K_{\ell} = I - R_{\ell}A_{\ell}$ and the recurrence relation :

$$I - B_{\ell}A_{\ell} = K_{\ell}^{m(\ell)} \left[(I - P_{\ell-1}) + (I - B_{\ell-1}A_{\ell-1})P_{\ell-1} \right] K_{\ell}^{m(\ell)}$$
(69)

The lower bound easily follows from the inductive hypothesis and the above identity. For the upper bound, we use the induction hypothesis to obtain

$$\begin{aligned} A_{L}((I - B_{\ell}A_{\ell})u, u) &\leq A_{L}([I - P_{\ell-1}]K_{\ell}^{m(\ell)}u, K_{\ell}^{m(\ell)}u) + \delta A_{L}(P_{j-1}K_{\ell}^{m(\ell)}u, K_{\ell}^{m(\ell)}u) \\ & (70) \\ &= (1 - \delta)A_{L}([I - P_{\ell-1}]K_{\ell}^{m(\ell)}u, K_{\ell}^{m(\ell)}u) + \delta A_{L}(K_{\ell}^{m(\ell)}u, K_{\ell}^{m(\ell)}u). \end{aligned}$$

Now by the orthogonality from (20)

$$A_{L}([I - P_{\ell-1}]K_{\ell}^{m(\ell)}u, [I - P_{\ell-1}]K_{\ell}^{m(\ell)}u)$$
(72)

$$= A_L([I - P_{\ell-1}]K_{\ell}^{m(\ell)}u, K_{\ell}^{m(\ell)}u)$$
(73)

$$= ([I - P_{\ell-1}]K_{\ell}^{m(\ell)}u, A_{\ell}K_{\ell}^{m(\ell)}u)$$
(74)

$$= (R_{\ell}^{-1}[I - P_{\ell-1}]K_{\ell}^{m(\ell)}u, R_{\ell}A_{\ell}K_{\ell}^{m(\ell)}u)$$
(75)

$$\leq (R_{\ell}^{-1}[I - P_{\ell-1}]K_{\ell}^{m(\ell)}u, [I - P_{\ell-1}]K_{\ell}^{m(\ell)}u)^{\frac{1}{2}}(R_{\ell}A_{\ell}K_{\ell}^{m(\ell)}u, A_{\ell}K_{\ell}^{m(\ell)}u)^{\frac{1}{2}}$$
(76)

$$\leq \sqrt{\beta_{\ell}} ([I - P_{\ell-1}] K_{\ell}^{m(\ell)} u, [I - P_{\ell-1}] K_{\ell}^{m(\ell)} u)^{\frac{1}{2}} (R_{\ell} A_{\ell} K_{\ell}^{m(\ell)} u, A_{\ell} K_{\ell}^{m(\ell)} u)^{\frac{1}{2}}$$
(77)

Hence, we get

$$A_{L}([I - P_{\ell-1}]K_{\ell}^{m(\ell)}u, K_{\ell}^{m(\ell)}u) \leq \beta_{\ell}(R_{\ell}A_{\ell}K_{\ell}^{m(\ell)}u, A_{\ell}K_{\ell}^{m(\ell)}u)$$
(78)
= $\beta_{\ell}A_{L}([I - K_{\ell-1}]K_{\ell}^{2m(\ell)}u, u)$ (79)

$$=\beta_{\ell}A_{L}([I-K_{\ell-1}]K_{\ell}^{2m(\ell)}u,u)$$
(79)

It follows from the positive semi-definiteness and (27) that the spectrum of K_{ℓ} is contained in the interval [0, 1]. Therefore, we have

$$A_L([I - K_{\ell-1}]K_{\ell}^{2m(\ell)}u, u) \le A_L([I - K_{\ell-1}]K_{\ell}^iu, u), \quad for \quad i \le 2m(\ell)$$
(80)

whence

$$A_L([I - K_{\ell-1}]K_{\ell}^{2m(\ell)}u, u) \le \frac{1}{2m(\ell)} \sum_{i=0}^{2m(\ell)-1} A_L([I - K_{\ell}]K_{\ell}^iu, u)$$
(81)

$$=\frac{1}{2m(\ell)}A_L([I-K_\ell]K_\ell^{2m(\ell)}u,u)$$
(82)

Combining (70) and (80) and following Lemma 2.3, we get

$$A_{L}((I - B_{\ell}A_{\ell})u, u) \leq (1 - \delta)\frac{\beta_{\ell}}{2m(\ell)}A_{L}([I - K^{2m(\ell)}]u, u) + \delta A_{L}(K_{\ell}^{m(\ell)}u, K_{\ell}^{m(\ell)}u)$$
(83)

$$\leq (1-\delta)\hat{C}A_{L}([I-K^{2m(\ell)}]u,u) + \delta A_{L}(K_{\ell}^{m(\ell)}u,K_{\ell}^{m(\ell)}u)$$
(84)

$$= (1 - \delta)\hat{C}A_L(u, u) + [\delta - (1 - \delta)\hat{C}]A_L(K_{\ell}^{m(\ell)}u, K_{\ell}^{m(\ell)}u)$$
(85)

The results now follows by choosing :

$$\delta = (1 - \delta)\hat{C}, \quad \text{i. e.,} \quad \delta = \frac{\hat{C}}{1 + \hat{C}}$$
(86)

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