Overcoming the curse of dimensionality in the numerical approximation of backward stochastic differential equations

Martin Hutzenthaler¹, Arnulf Jentzen^{2,3}, Thomas Kruse⁴, and Tuan Anh Nguyen⁵

¹ Faculty of Mathematics, University of Duisburg-Essen,

Essen, Germany; e-mail: martin.hutzenthaler@uni-due.de

 2 Applied Mathematics Münster, Faculty of Mathematics and Computer Science,

University of Münster, Münster, Germany; e-mail: ajentzen@uni-muenster.de

³School of Data Science and Shenzhen Research Institute of Big Data,

The Chinese University of Hong Kong,

Shenzhen, China; e-mail: ajentzen@cuhk.edu.ch

⁴ Institute of Mathematics, University of Gießen,

Gießen, Germany; e-mail: thomas.kruse@math.uni-giessen.de

⁵ Faculty of Mathematics, University of Duisburg-Essen,

Essen, Germany; e-mail: tuan.nguyen@uni-due.de

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Abstract

Backward stochastic differential equations (BSDEs) belong nowadays to the most frequently studied equations in stochastic analysis and computational stochastics. BSDEs in applications are often nonlinear and high-dimensional. In nearly all cases such nonlinear high-dimensional BSDEs cannot be solved explicitly and it has been and still is a very active topic of research to design and analyze numerical approximation methods to approximatively solve nonlinear high-dimensional BSDEs. Although there are a large number of research articles in the scientific literature which analyze numerical approximation methods for nonlinear BSDEs, until today there has been no numerical approximation method in the scientific literature which has been proven to overcome the curse of dimensionality in the numerical approximation of nonlinear BSDEs in the sense that the number of computational operations of the numerical approximation method to approximatively compute one sample path of the BSDE solution grows at most polynomially in both the reciprocal $1/\varepsilon$ of the prescribed approximation accuracy $\varepsilon \in (0,\infty)$ and the dimension $d \in \mathbb{N} = \{1, 2, 3, \ldots\}$ of the BSDE. It is the key contribution of this article to overcome this obstacle by introducing a new Monte Carlo-type numerical approximation method for high-dimensional BSDEs and by proving that this Monte Carlo-type numerical approximation method does indeed overcome the curse of dimensionality in the approximative computation of solution paths of BSDEs.

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1 Introduction

Backward stochastic differential equations (BSDEs) have been introduced by Pardoux & Peng in 1990 (see [93]) and belong nowadays to the most frequently studied equations in stochastic analysis and computational stochastics. One central reason for the high interest in studying BSDEs is their numerous occurrence in relevant real life problems. In particular, BSDEs appear in the approximative valuation of financial products such as financial derivative contracts (see, e.g., [33, 42, 48]), BSDEs arise in the solution of stochastic optimal control problems (see, e.g., [97, 106, 110]), and BSDEs are strongly linked to nonlinear partial differential equations (PDEs) which themselves arise naturally in many applications (see, e.g., [92, 94–96]).

BSDEs in applications are often nonlinear and high-dimensional where, e.g., in the approximative valuation of financial products the dimension of the BSDE essentially corresponds to the number of financial assets in the associated hedging portfolio, where, e.g., in stochastic optimal control problems the dimension of the BSDE is determined by the dimension of the state space of the stochastic control problem, and where, e.g., in the case of the connection of BSDEs and PDEs the dimension of the BSDE coincides with the dimension of the associated nonlinear PDE.

In nearly all cases nonlinear high-dimensional BSDEs cannot be solved explicitly and it has been and still is a very active topic of research to design and analyze numerical approximation methods to approximatively solve nonlinear high-dimensional BSDEs. Standard numerical approximation methods for nonlinear BSDEs in the scientific literature suffer under the so-called curse of dimensionality (cf., e.g., Bellman [13], Novak & Wozniakowski [90, Chapter 1], and Novak & Ritter [89]) in the sense that the number of computational operations of the numerical approximation method to approximatively compute one sample path of the BSDE solution grows at least exponentially in the reciprocal $1/\varepsilon$ of the prescribed approximation accuracy $\varepsilon \in (0, \infty)$ or the dimension $d \in \mathbb{N} = \{1, 2, 3, ...\}$ of the BSDE and it is a key objective in computational stochastics to design and analyze numerical approximation methods which overcome the curse of dimensionality in the numerical approximation of BSDEs.

Since BSDEs have been introduced by Pardoux & Peng in 1990 (see [93]), a large number of numerical approximation methods for nonlinear BSDEs have been proposed and analyzed in the scientific literature. In particular, we refer, for example, to [20, 28, 34, 55, 82, 98, 107, 112]for numerical approximation methods for BSDEs based on one-step temporal discretizations of BSDEs, we refer, for example, to [25, 26, 105, 115, 118] for numerical approximation methods for BSDEs based on multi-step temporal discretizations of BSDEs, we refer, for example, to [18, 57-61, 81, 102] for numerical approximation methods for BSDEs based on suitable projections on function spaces, we refer, for example, to [27, 35, 36, 39] for cubature-based numerical approximation methods for BSDEs, we refer, for example, to [20, 37, 69] for numerical approximation methods for BSDEs based on Malliavin calculus, we refer, for example, to [4, 5, 40, 41]for numerical approximation methods for BSDEs based on quantization algorithms, we refer, for example, to [24, 78–80] for numerical approximation methods for BSDEs based on density representations of particle systems, we refer, for example, to [29, 76, 100, 101] for numerical approximation methods for quadratic BSDEs, we refer, for example, to [14, 19] for numerical approximation methods for BSDEs based on Picard iterations and the least squares Monte Carlo method, we refer, for example, to [56, 77] for numerical approximation methods for BS-DEs based on Picard iterations and adaptive control variates, we refer, for example, to [49,111] for numerical approximation methods for BSDEs based on sparse grid approximations, we refer, for example, to [23, 51] for numerical approximation methods for BSDEs based on Wiener chaos expansions, we refer, for example, to [114, 116, 117] for numerical approximation methods for BSDEs based on the theta-scheme, we refer, for example, to [38] for numerical approximation methods for BSDEs based on steepest descent algorithms, we refer, for example, to [21, 22, 31, 52, 53, 83] for numerical approximation methods for BSDEs based on discrete time approximations of Brownian motions, we refer, for example, to [70, 103] for numerical approximation methods for BSDEs based on Fourier expansions, we refer, for example, to [15-17] for numerical approximation methods for BSDEs based on the primal-dual method, we refer, for example, to [3,65,67,68,86,99,104,108,109] for numerical approximation methods for BSDEs based on branching diffusion representations of PDEs, we refer, for example, to [43,84,85,87,88] for numerical approximation methods for BSDEs based on the four-step-scheme, we refer, for example, to [1,2] for numerical approximation methods for BSDEs based on conditional Monte Carlo learning for diffusion processes, and we refer, for example, to [30, 44, 50, 66] and the references mentioned in the overview articles [10, 45] for deep learning-based approximation methods for BSDEs.

Although there are a large number of research articles in the scientific literature which analyze numerical approximation methods for nonlinear BSDEs, until today there has been no numerical approximation method in the scientific literature which has been proven to overcome the curse of dimensionality in the numerical approximation of nonlinear BSDEs in the sense that the number of computational operations of the numerical approximation method to approximatively compute one sample path of the BSDE solution grows at most polynomially in both the reciprocal $1/\varepsilon$ of the prescribed approximation accuracy $\varepsilon \in (0, \infty)$ and the dimension $d \in \mathbb{N} = \{1, 2, 3, ...\}$ of the BSDE. This concept is also referred to as *polynomial tractability* in the scientific literature (see, e.g., Novak & Wozniakowski [90, Definition 4.44]).

It is the key contribution of this article to overcome this obstacle by introducing a new Monte Carlo-type numerical approximation method for high-dimensional BSDEs and by proving that this Monte Carlo-type numerical approximation method does indeed overcome the curse of dimensionality in the approximative computation of solution paths of BSDEs. Remarkably, this article even demonstrates that the introduced Monte Carlo-type numerical approximation method approximates solution paths of BSDEs with essentially the same computational complexity that is used by standard Monte Carlo methods for the approximative computation of integrals. More specifically, the main result of this article, Theorem 5.1 in Section 5 below, proves that the introduced Monte Carlo-type numerical approximates solution paths of BSDEs with a computational effort which grows at most polynomially in the dimension $d \in \mathbb{N}$ of the driving Brownian motion and essentially at most quadratically in the reciprocal of the prescribed approximation accuracy.

The Monte Carlo-type numerical approximation method for BSDEs proposed in this article (see (2) below) is based on full-history recursive multilevel Picard approximation methods [46,47,73] (in the following we abbreviate *full-history recursive multilevel Picard* by MLP) and on the multilevel approach in Heinrich [62, 63]. MLP approximations have previously been shown to overcome the curse of dimensionality in the case of a number of semilinear PDE problems (cf. [7, 8, 11, 12, 46, 47, 54, 71-75]) and this is also the key ingredient in this article to overcome the curse of dimensionality in the numerical approximation of solution paths of

BSDEs.

To briefly sketch the contribution of this article within this introductory section, we now present in the following result, Theorem 1.1 below, a special case of Theorem 5.1, the main result of this article. Below Theorem 1.1 we explain in words the statement of Theorem 1.1 as well as the mathematical objects appearing in Theorem 1.1.

Theorem 1.1. Let $T, \delta \in (0, \infty)$, $\Theta = \bigcup_{n \in \mathbb{N}} \mathbb{Z}^n$, $f \in C^2(\mathbb{R}, \mathbb{R})$, let $g_d \in C^1(\mathbb{R}^d, \mathbb{R})$, $d \in \mathbb{N}$, satisfy $\sup_{d \in \mathbb{N}} \sup_{x=(x_1, x_2, \dots, x_d) \in \mathbb{R}^d} (|f(x_1)| + |f'(x_1)| + |f''(x_1)| + |g_d(x)| + \sum_{i=1}^d |\frac{\partial g_d}{\partial x_i}(x)|^2) < \infty$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0,T]})$ be a filtered probability space, let $\mathfrak{r}^{\theta} \colon \Omega \to [0,1]$, $\theta \in \Theta$, be i.i.d. random variables, assume for all $t \in (0,1)$ that $\mathbb{P}(\mathfrak{r}^0 \leq t) = t$, let $W^{d,\theta} = (W^{d,\theta,1}, W^{d,\theta,2}, \dots, W^{d,\theta,d}) \colon [0,T] \times$ $\Omega \to \mathbb{R}^d$, $d \in \mathbb{N}$, $\theta \in \Theta$, be independent standard $(\mathbb{F}_t)_{t \in [0,T]}$ -Brownian motions, assume that $(\mathfrak{r}^{\theta})_{\theta \in \Theta}$ and $(W^{d,\theta})_{(d,\theta) \in \mathbb{N} \times \Theta}$ are independent, let $U^{d,\theta}_{n,M} \colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$, $d, M, n \in \mathbb{N}_0$, $\theta \in \Theta$, satisfy for all $d, M \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\theta \in \Theta$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$U_{n,M}^{d,\theta}(t,x) = (T-t)f(0)\mathbb{1}_{\mathbb{N}}(n) + \frac{\mathbb{1}_{\mathbb{N}}(n)}{M^{n}} \sum_{i=1}^{M^{n}} g_{d}\left(x + W_{T-t}^{d,(\theta,0,-i)}\right) + \sum_{\ell=1}^{n-1} \left[\frac{(T-t)}{M^{n-\ell}} \sum_{i=1}^{M^{n-\ell}} \left(f \circ U_{\ell,M}^{d,(\theta,\ell,i)} - f \circ U_{\ell-1,M}^{d,(\theta,-\ell,i)}\right) \left(t + (T-t)\mathfrak{r}^{(\theta,\ell,i)}, x + W_{(T-t)\mathfrak{r}^{(\theta,\ell,i)}}^{d,(\theta,\ell,i)}\right)\right],$$
(1)

let $\lfloor \cdot \rfloor_M \colon \mathbb{R} \to \mathbb{R}, \ M \in \mathbb{N}, \ and \ \lceil \cdot \rceil_M \colon \mathbb{R} \to \mathbb{R}, \ M \in \mathbb{N}, \ satisfy \ for \ all \ M \in \mathbb{N}, \ t \in [0, T] \ that$ $\lfloor t \rfloor_M = \max(([0, t] \setminus \{T\}) \cap \{0, \frac{T}{M}, \frac{2T}{M}, \ldots\}) \ and \ \lceil t \rceil_M = \min(((t, \infty) \cup \{T\}) \cap \{0, \frac{T}{M}, \frac{2T}{M}, \ldots\}), \ let \ \mathcal{Y}^{d,n,M} \colon [0, T] \times \Omega \to \mathbb{R}, \ d, n, M \in \mathbb{N}, \ satisfy \ for \ all \ d, n, M \in \mathbb{N}, \ t \in [0, T] \ that$

$$\mathscr{Y}_{t}^{d,n,M} = \sum_{\ell=0}^{n-1} \left[\left[\frac{[t]_{M^{l+1}-t}}{(T/M^{l+1})} \right] U_{n-\ell,M}^{d,\ell} (\lfloor t \rfloor_{M^{l+1}}, W_{\lfloor t \rfloor_{M^{l+1}}}^{d,0}) + \left[\frac{t-\lfloor t \rfloor_{M^{l+1}}}{(T/M^{l+1})} \right] U_{n-\ell,M}^{d,\ell} (\lceil t \rceil_{M^{l+1}}, W_{\lceil t \rceil_{M^{l+1}}}^{d,0}) - \mathbb{1}_{\mathbb{N}} (\ell) \left(\left[\frac{[t]_{M^{l-t}}}{(T/M^{l})} \right] U_{n-\ell,M}^{d,\ell} (\lfloor t \rfloor_{M^{l}}, W_{\lfloor t \rfloor_{M^{l}}}^{d,0}) + \left[\frac{t-\lceil t \rceil_{M^{l}}}{(T/M^{l})} \right] U_{n-\ell,M}^{d,\ell} (\lceil t \rceil_{M^{l}}, W_{\lceil t \rceil_{M^{l}}}^{d,0}) \right], \quad (2)$$

and for every $d, n, M \in \mathbb{N}$ let $\mathfrak{C}_{d,n,M} \in \mathbb{N}_0$ be the number of realizations of scalar random variables, the number of function evaluations of f, and the number of function evaluations of g_d which are used to compute one realization of $(\mathscr{Y}_{kT/M^n}^{d,n,M})_{k\in\{0,1,\dots,M^n\}}$ (cf. (107) for a precise definition), let $\mathbf{Y}^d = (Y^d, Z^{d,1}, Z^{2,d}, \dots, Z^{d,d}) \colon [0,T] \times \Omega \to \mathbb{R}^{d+1}, d \in \mathbb{N}, be (\mathbb{F}_t)_{t\in[0,T]}$ -predictable stochastic processes, assume for all $d \in \mathbb{N}$ that $\int_0^T \mathbb{E}\left[|Y_s^d| + \sum_{j=1}^d |Z_s^{d,j}|^2\right] ds < \infty$, and assume that for all $d \in \mathbb{N}, t \in [0,T]$ it holds \mathbb{P} -a.s. that

$$Y_t^d = g_d(W_T^{d,0}) + \int_t^T f(Y_s^d) \, ds - \sum_{j=1}^d \int_t^T Z_s^{d,j} \, dW_s^{d,0,j}.$$
(3)

Then there exist $c \in \mathbb{R}$ and $\mathbf{n} \colon \mathbb{N} \times (0,1] \to \mathbb{N}$ such that for all $d \in \mathbb{N}$, $\varepsilon \in (0,1]$ it holds that $\sup_{t \in [0,T]} (\mathbb{E}[|\mathcal{Y}_t^{d,\mathbf{n}(d,\varepsilon),\mathbf{n}(d,\varepsilon)} - Y_t^d|^2])^{1/2} \le \varepsilon$ and $\mathfrak{C}_{d,\mathbf{n}(d,\varepsilon),\mathbf{n}(d,\varepsilon)} \le cd^c \varepsilon^{-(2+\delta)}$.

Theorem 1.1 is an immediate consequence from Corollary 5.3 in Section 5 below. Corollary 5.3, in turn, follows from Theorem 5.1, which is the main result of this article. In the following we add some comments on the mathematical objects appearing in Theorem 1.1 above.

In (3) in Theorem 1.1 we specify the BSDEs whose solution processes we intend to approximate in Theorem 1.1. The strictly positive real number $T \in (0, \infty)$ in the first line of Theorem 1.1 describes the time horizon of the BSDEs in (3). The function $f \colon \mathbb{R} \to \mathbb{R}$ in the first line of Theorem 1.1 specifies the driver (the nonlinearity) of the BSDEs in (3). The quadrupel $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0,T]})$ in the third line of Theorem 1.1 is the filtered probability space on which the BSDEs in (3) are formulated. In Theorem 1.1 we do not assume that the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0,T]})$ satisfies the usual conditions in the sense that for

all $t \in [0,T)$ it holds that $\{A \in \mathcal{F} : \mathbb{P}(A) = 0\} \subseteq \mathbb{F}_t = \bigcap_{s \in (t,T]} \mathbb{F}_s$. The $(\mathbb{F}_t)_{t \in [0,T]}$ -predictable stochastic processes $Y^d : [0,T] \times \Omega \to \mathbb{R}, d \in \mathbb{N}$, in the last but fifth line of Theorem 1.1 are the solution processes of the BSDEs in (3).

In (1)–(2) in Theorem 1.1 we specify the Monte Carlo-type approximation algorithm which we propose to approximate the solution processes of the BSDEs in (3). To formulate the proposed Monte Carlo-type approximation algorithm in (1)–(2) we need, roughly speaking, sufficiently many independent random quantities which are indexed over a sufficiently large index set. This sufficiently large index set is provided through the set $\Theta = \bigcup_{n \in \mathbb{N}} \mathbb{Z}^n$ in the first line of Theorem 1.1. The i.i.d. random variables $\mathfrak{r}^{\theta} \colon \Omega \to [0,1], \theta \in \Theta$, in the third line of Theorem 1.1 and the independent standard $(\mathbb{F}_t)_{t \in [0,T]}$ -Brownian motions $W^{d,\theta} \colon [0,T] \times \Omega \to \mathbb{R}^d$, $d \in \mathbb{N}, \theta \in \Theta$, in the fourth line of Theorem 1.1 provide the random quantities which we employ to formulate the BSDEs in (3) and the proposed Monte Carlo-type approximation algorithm in (1)–(2).

More formally, observe that the independent standard Brownian motions $W^{d,0}: [0,T] \times \Omega \to \mathbb{R}^d$, $d \in \mathbb{N}$, in the fourth line of Theorem 1.1 are the driving standard Brownian motions in the BSDEs in (3) and observe that the i.i.d. random variables $\mathfrak{r}^{\theta}: \Omega \to [0,1], \theta \in \Theta$, in the third line of Theorem 1.1 and the independent standard Brownian motions $W^{d,\theta}: [0,T] \times \Omega \to \mathbb{R}^d$, $d \in \mathbb{N}, \theta \in (\Theta \setminus \{0\})$, in the fourth line of Theorem 1.1 are the random quantities which we use as random input sources to formulate the proposed Monte Carlo-type approximation algorithm in (1)-(2). Note that the assumption in third line of Theorem 1.1 that for all $t \in (0, 1)$ it holds that $\mathbb{P}(\mathfrak{r}^0 \leq t) = t$ ensures that for all $\theta \in \Theta$ it holds that \mathfrak{r}^{θ} is an on [0, 1] continuous uniformly distributed random variable.

The functions $g_d \colon \mathbb{R}^d \to \mathbb{R}, d \in \mathbb{N}$, in the first line of Theorem 1.1 and the independent standard Brownian motions $W^{d,0} \colon [0,T] \times \Omega \to \mathbb{R}^d, d \in \mathbb{N}$, in the fourth line of Theorem 1.1 determine the terminal conditions of the BSDEs in (3). More precisely, note that (3) in Theorem 1.1 ensures that for all $d \in \mathbb{N}$ it holds \mathbb{P} -a.s. that $Y_T^d = g_d(W_T^{d,0})$. In Theorem 1.1 we assume that the driver $f \colon \mathbb{R} \to \mathbb{R}$ in the first line of Theorem 1.1 and the functions $g_d \colon \mathbb{R}^d \to \mathbb{R}, d \in \mathbb{N}$, in the first line of Theorem 1.1 satisfy some regularity hypotheses. More formally, observe that the assumption $\sup_{d \in \mathbb{N}} \sup_{x=(x_1,x_2,\dots,x_d)\in\mathbb{R}^d} \left(|f(x_1)|+|f'(x_1)|+|f''(x_1)|+|g_d(x)|+\sum_{i=1}^d |\frac{\partial g_d}{\partial x_i}(x)|^2\right) < \infty$ in the second line of Theorem 1.1 assures that there exists a real number $\kappa \in \mathbb{R}$ such that for all $d \in \mathbb{N}, v \in \mathbb{R}, x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ it holds that $|f(v)| \leq \kappa, |f'(v)| \leq \kappa, |f''(v)| \leq \kappa,$ $|g_d(x)| \leq \kappa, \text{ and } \sum_{i=1}^d |\frac{\partial g_d}{\partial x_i}(x)|^2 \leq \kappa.$

The numbers $\mathfrak{C}_{d,n,M} \in \mathbb{N}_0$, $d, n, m \in \mathbb{N}$, in the first line below (2) in Theorem 1.1 model the computational cost of the Monte Carlo-type approximation algorithm in (1)–(2). More specifically, for every $d, n, M \in \mathbb{N}$ we have that $\mathfrak{C}_{d,n,M}$ specifies the sum of the number of realizations of one-dimensional random variables, of the number of function evaluations of $f: \mathbb{R} \to \mathbb{R}$, and of the number of function evaluations of $g_d: \mathbb{R}^d \to \mathbb{R}$ which are used to compute one realization of $(\mathscr{Y}_{kT/M^n}^{d,n,M})_{k\in\{0,1,\dots,M^n\}}$ (cf. (107) for a precise definition). Observe that (2) in Theorem 1.1 ensures that for every $d, n, M \in \mathbb{N}$ we have that $(\mathscr{Y}_t^{d,n,M})_{t\in[0,T]}$ is the piecewise affine linear interpolation associated to $(\mathscr{Y}_{kT/M^n}^{d,n,M})_{k\in\{0,1,\dots,M^n\}}$ in sense that for all $k \in \{1, 2, \dots, M^n\}, t \in [\frac{(k-1)T}{M^n}, \frac{kT}{M^n}]$ it holds that $\mathscr{Y}_t^{d,n,M} = \frac{M^n}{T} [(\frac{kT}{M^n} - t)\mathscr{Y}_{(k-1)T/M^n}^{d,n,M} + (t - \frac{(k-1)T}{M^n})\mathscr{Y}_{kT/M^n}^{d,n,M}].$

Theorem 1.1 proves that the solution processes $Y^d : [0, T] \times \Omega \to \mathbb{R}, d \in \mathbb{N}$, of the BSDEs in (3) can be approximated by means of the Monte Carlo-type approximation algorithm in (1)–(2) with a computational cost which grows at most polynomially in the dimension $d \in \mathbb{N}$ of the BSDE and up to an arbitrarily small polynomial order at most quadratically in the reciprocal $1/\varepsilon$ of the prescribed approximation accuracy $\varepsilon > 0$. The arbitrarily small polynomial order is described through the real number $\delta \in (0, \infty)$ in the first line of Theorem 1.1.

In the following we also add some comments on shortcomings and possible generalizations of Theorem 1.1. In particular, we observe that the driver $f \colon \mathbb{R} \to \mathbb{R}$ in the BSDEs in (3) does only depend on the solution processes $Y^d \colon [0,T] \times \Omega \to \mathbb{R}, d \in \mathbb{N}$, but not on the time variable $s \in [0, T]$, not on the driving Brownian motions $W^{d,0}: [0, T] \times \Omega \to \mathbb{R}^d$, $d \in \mathbb{N}$, and also not on the stochastic processes $Z^{d,j}: [0,T] \times \Omega \to \mathbb{R}$, $j \in \{1, 2, \ldots, d\}$, $d \in \mathbb{N}$. However, in the more general result in Theorem 5.1 in Section 4 below the drivers of the BSDEs under consideration do additionally also depend on the time variable $s \in [0,T]$ and on the driving Bronwnian motions $W^{d,0}: [0,T] \times \Omega \to \mathbb{R}^d$, $d \in \mathbb{N}$. We refer to (80) in Theorem 5.1 below for details. The dependence of the drivers of the BSDEs under considerations on the stochastic processes $Z^{d,j}: [0,T] \times \Omega \to \mathbb{R}$, $j \in \{1,2,\ldots,d\}$, $d \in \mathbb{N}$, is not covered within this article and the numerical approximation of the stochastic processes $Z^{d,j}: [0,T] \times \Omega \to \mathbb{R}$, $j \in \{1,2,\ldots,d\}$, $d \in \mathbb{N}$, is also not covered within this article but the arguments revealed in this article together with the arguments in the article [71] allow also to overcome the curse of dimensionality in these more general cases of BSDEs.

In Theorem 1.1 we also use a rather restrictive regularity hypothesis on the driver $f: \mathbb{R} \to \mathbb{R}$ and the functions $g_d: \mathbb{R}^d \to \mathbb{R}, d \in \mathbb{N}$, in the sense that there exists $\kappa \in \mathbb{R}$ such that for all $d \in \mathbb{N}, v \in \mathbb{R}, x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ it holds that $|f(v)| \leq \kappa, |f'(v)| \leq \kappa, |f''(v)| \leq \kappa, |g_d(x)| \leq \kappa$, and $\sum_{i=1}^d |\frac{\partial g_d}{\partial x_i}(x)|^2 \leq \kappa$. In the more general result in Theorem 5.1 in Section 4 below this hypothesis is replaced by suitable more general Lipschitz-type assumptions. We refer to (42)-(43) in Theorem 5.1 below for details.

The remainder of this article is organized as follows. In Section 2 below we establish upper bounds for a generalized norm of the difference between a vector space valued process and appropriate multi-grid approximations for this process. A key aspect in the derivation of the Monte Carlo-type approximation algorithm in (1)-(2) in Theorem 1.1 is, roughly speaking, to reformulate the solutions of the BSDEs in (3) as solutions of appropriate stochastic fixed-point equations (SFPEs) associated to the BSDEs in (3) and in Section 3 below we establish existence, uniqueness, and Hölder continuity properties for solutions of precisely such SFPEs. In Section 4 below we establish upper bounds for appropriate Hölder seminorms of the difference between the solutions of such SFPEs and suitable MLP approximations for such SFPEs. In Section 5 below we combine the findings from Sections 2 and 4 to provide a computational complexity analysis for the Monte Carlo-type approximation algorithm in (1)-(2) and, thereby, we also prove Theorem 1.1 above.

2 Error analysis for multi-grid approximations

A central aspect in the derivation of the Monte Carlo-type approximation algorithm for BSDEs in (1)-(2) in Theorem 1.1 in Section 1 above is, roughly speaking, to approximate the exact solution of the BSDE under consideration by means of appropriate multi-grid approximations on coarser and coarser time grids and, then, to exploit suitable uniform temporal regularity properties for the employed multi-grid approximations.

In Lemma 2.3 in this section we formulate this approach in an abstract setting and in Lemma 2.3 we also establish explicit upper bounds for a generalized error norm of the difference between a vector space valued process (which we think of as the solution process of the considered BSDE) and appropriate multi-grid approximations for this process.

Our approach is based on the multilevel method in the articles Heinrich [62,63]. In these references Heinrich proposed and formulated the multilevel method in the context of Monte Carlo approximations of certain parameter-dependent integrals (see also Heinrich & Sindambiwe [64]).

Our proof of Lemma 2.3 employs the essentially well-known error estimate for piecewise affine linear interpolation functions in Lemma 2.1 and the essentially well-known Hölder continuity result for piecewise affine linear interpolation functions in Lemma 2.2. Lemma 2.1 is, e.g., a slight extension of Cox et al. [32, Lemma 2.2] and Lemma 2.2 is, e.g., a slight extension of Cox et al. [32, Lemma 2.3].

Lemma 2.1. Let V be an \mathbb{R} -vector space, let $\|\cdot\|: V \to [0,\infty]$ satisfy for all $v, w \in V$, $v, w \in \mathbb{R}$ with $\|v\| + \|w\| < \infty$ that $\|vv + ww\| \le |v|\|v\| + |w|\|w\|$, let $T, \alpha \in (0,\infty)$, $m \in \mathbb{N}$,

 $\tau_0, \tau_1, \ldots, \tau_m \in \mathbb{R} \text{ satisfy } 0 = \tau_0 < \tau_1 < \ldots < \tau_m = T, \text{ and let } x = (x_t)_{t \in [0,T]} : [0,T] \to V$ and $X = (X_t)_{t \in [0,T]} : [0,T] \to V \text{ satisfy for all } k \in \{1, 2, \ldots, m\}, t \in [\tau_{k-1}, \tau_k] \text{ that } X_t = (\tau_k - \tau_{k-1})^{-1} [(\tau_k - t)x_{\tau_{k-1}} + (t - \tau_{k-1})x_{\tau_k}].$ Then

$$\sup_{t \in [0,T]} \left\| X_t - x_t \right\| \le 2^{-\min\{3,\alpha\}} \left[\max_{k \in \{1,2,\dots,m\}} |\tau_k - \tau_{k-1}|^{\alpha} \right] \left[\sup_{r,s \in [0,T], r \neq s} \frac{\|x_r - x_s\|}{|r - s|^{\alpha}} \right].$$
(4)

Proof of Lemma 2.1. Throughout this proof assume without loss of generality that for all $s, t \in [0, T]$ with $s \neq t$ it holds that $||x_s - x_t|| < \infty$. Note that for all $k \in \{1, 2, \ldots, m\}, t \in (\tau_{k-1}, \tau_k)$ it holds that

$$X_{t} - x_{t} = \left[\frac{(\tau_{k} - t)x_{\tau_{k-1}} + (t - \tau_{k-1})x_{\tau_{k}}}{\tau_{k} - \tau_{k-1}}\right] - x_{t}$$

$$= \frac{(\tau_{k} - t)(x_{\tau_{k-1}} - x_{t}) + (t - \tau_{k-1})(x_{\tau_{k}} - x_{t})}{\tau_{k} - \tau_{k-1}}$$

$$= \left[\frac{(t - \tau_{k-1})(\tau_{k} - t)^{\alpha}}{\tau_{k} - \tau_{k-1}}\right] \left[\frac{x_{\tau_{k}} - x_{t}}{(\tau_{k} - t)^{\alpha}}\right] - \left[\frac{(\tau_{k} - t)(t - \tau_{k-1})^{\alpha}}{\tau_{k} - \tau_{k-1}}\right] \left[\frac{x_{t} - x_{\tau_{k-1}}}{(t - \tau_{k-1})^{\alpha}}\right].$$
(5)

The assumption that for all $v, w \in V$, $v, w \in \mathbb{R}$ with $||v|| + ||w|| < \infty$ it holds that $||vv + ww|| \le ||v|| ||v|| + |w|| ||w||$ hence ensures that for all $k \in \{1, 2, ..., m\}$, $t \in [\tau_{k-1}, \tau_k]$ it holds that

$$\begin{split} \|X_{t} - x_{t}\| &\leq \left(\left[\frac{(t - \tau_{k-1})(\tau_{k} - t)^{\alpha}}{\tau_{k} - \tau_{k-1}} \right] + \left[\frac{(\tau_{k} - t)(t - \tau_{k-1})^{\alpha}}{\tau_{k} - \tau_{k-1}} \right] \right) \left[\sup_{r,s \in [0,T], r \neq s} \frac{\|x_{r} - x_{s}\|}{|r - s|^{\alpha}} \right] \\ &= \left(\left[\frac{t - \tau_{k-1}}{\tau_{k} - \tau_{k-1}} \right] \left[\frac{\tau_{k} - t}{\tau_{k} - \tau_{k-1}} \right]^{\alpha} + \left[\frac{\tau_{k} - t}{\tau_{k} - \tau_{k-1}} \right] \left[\frac{t - \tau_{k-1}}{\tau_{k} - \tau_{k-1}} \right]^{\alpha} \right) \\ &\quad \cdot [\tau_{k} - \tau_{k-1}]^{\alpha} \left[\sup_{r,s \in [0,T], r \neq s} \frac{\|x_{r} - x_{s}\|}{|r - s|^{\alpha}} \right] \\ &= \left(\left[\frac{t - \tau_{k-1}}{\tau_{k} - \tau_{k-1}} \right] \left[1 - \left(\frac{t - \tau_{k-1}}{\tau_{k} - \tau_{k-1}} \right) \right]^{\alpha} + \left[1 - \left(\frac{t - \tau_{k-1}}{\tau_{k} - \tau_{k-1}} \right) \right] \left[\frac{t - \tau_{k-1}}{\tau_{k} - \tau_{k-1}} \right]^{\alpha} \right) \\ &\quad \cdot [\tau_{k} - \tau_{k-1}]^{\alpha} \left[\sup_{r,s \in [0,T], r \neq s} \frac{\|x_{r} - x_{s}\|}{|r - s|^{\alpha}} \right] \\ &\leq \left[\max_{l \in \{1,2,\dots,m\}} |\tau_{l} - \tau_{l-1}|^{\alpha} \right] \left[\sup_{z \in [0,1]} (z(1 - z)^{\alpha} + (1 - z)z^{\alpha}) \right] \left[\sup_{r,s \in [0,T], r \neq s} \frac{\|x_{r} - x_{s}\|}{|r - s|^{\alpha}} \right]. \end{split}$$

Next observe that the fact that for all $z \in [0,1]$ it holds that $z(1-z) \leq 2^{-2}$ and Jensen's inequality imply that for all $z \in [0,1]$ it holds that

$$\mathbb{1}_{(0,1]}(\alpha) \left(z(1-z)^{\alpha} + (1-z)z^{\alpha} \right) \le \mathbb{1}_{(0,1]}(\alpha) \left(z(1-z) + (1-z)z \right)^{\alpha} \\
= \mathbb{1}_{(0,1]}(\alpha) 2^{\alpha} \left(z(1-z) \right)^{\alpha} \le \mathbb{1}_{(0,1]}(\alpha) 2^{-\alpha}.$$
(7)

In addition, note the fact that for all $z \in [0,1]$ it holds that $z(1-z) \leq 2^{-2}$ and Jensen's inequality imply that for all $z \in [0,1]$ it holds that

$$\mathbb{1}_{(1,2]}(\alpha) \left(z(1-z)^{\alpha} + (1-z)z^{\alpha} \right) = \mathbb{1}_{(1,2]}(\alpha) (2z(1-z)) \left(\frac{(1-z)^{\alpha-1}}{2} + \frac{z^{\alpha-1}}{2} \right) \\
\leq \mathbb{1}_{(1,2]}(\alpha) \left(\frac{1}{2} \right) \left[\frac{(1-z)}{2} + \frac{z}{2} \right]^{\alpha-1} = \mathbb{1}_{(1,2]}(\alpha) 2^{-\alpha}.$$
(8)

Next observe that the fact that for all $z \in [0, 1]$ it holds that $z(1 - z) \leq 2^{-2}$, and the fact that for all $c \in [0, 1]$ it holds that $[0, 1/4] \ni y \mapsto y(1 - 2y)^c \in \mathbb{R}$ is non-decreasing, and Jensen's

inequality imply that for all $z \in [0, 1]$ it holds that

$$\begin{aligned} \mathbb{1}_{(2,\infty)}(\alpha) \left(z(1-z)^{\alpha} + (1-z)z^{\alpha} \right) &\leq \mathbb{1}_{(2,\infty)}(\alpha) \left(z(1-z)^{\min\{\alpha,3\}} + (1-z)z^{\min\{\alpha,3\}} \right) \end{aligned} \tag{9} \\ &= \mathbb{1}_{(2,\infty)}(\alpha) (z(1-z)) \left((1-z)[(1-z)^{\min\{\alpha,3\}-2}] + z[z^{\min\{\alpha,3\}-2}] \right) \\ &\leq \mathbb{1}_{(2,\infty)}(\alpha) (z(1-z)) \left((1-z)^2 + z^2 \right)^{\min\{\alpha,3\}-2} = \mathbb{1}_{(2,\infty)}(\alpha) (z(1-z)) \left(1 - 2z(1-z) \right)^{\min\{\alpha,3\}-2} \\ &\leq \mathbb{1}_{(2,\infty)}(\alpha) \max_{y \in [0,1/4]} \left[y \left(1 - 2y \right)^{\min\{\alpha,3\}-2} \right] = \mathbb{1}_{(2,\infty)}(\alpha) \left[\frac{1}{4} \left(1 - \frac{1}{2} \right)^{\min\{\alpha,3\}-2} \right] = \mathbb{1}_{(2,\infty)}(\alpha) 2^{-\min\{\alpha,3\}}. \end{aligned}$$

Combining this with (7) and (8) demonstrates that $\sup_{z \in [0,1]} (z(1-z)^{\alpha} + (1-z)z^{\alpha}) \leq 2^{-\min\{3,\alpha\}}$. This and (6) show that

$$\sup_{t \in [0,T]} \left\| X_t - x_t \right\| \le 2^{-\min\{3,\alpha\}} \left[\max_{k \in \{1,2,\dots,m\}} |\tau_k - \tau_{k-1}|^{\alpha} \right] \left[\sup_{r,s \in [0,T], r \neq s} \frac{\|x_r - x_s\|}{|r - s|^{\alpha}} \right].$$
(10)

The proof of Lemma 2.1 is thus complete.

Lemma 2.2. Let V be an \mathbb{R} -vector space, let $\|\cdot\|: V \to [0, \infty]$ satisfy for all $v, w \in V, v, w \in \mathbb{R}$ with $\|v\| + \|w\| < \infty$ that $\|vv + ww\| \le |v| \|v\| + |w| \|w\|$, let $T \in (0, \infty)$, $\alpha \in (0, 1]$, $m \in \mathbb{N}$, $\tau_0, \tau_1, \ldots, \tau_m \in \mathbb{R}$ satisfy $0 = \tau_0 < \tau_1 < \ldots < \tau_m = T$, and let $x = (x_t)_{t \in [0,T]}: [0,T] \to V$ and $X = (X_t)_{t \in [0,T]}: [0,T] \to V$ satisfy for all $k \in \{1, 2, \ldots, m\}$, $t \in [\tau_{k-1}, \tau_k]$ that $X_t = (\tau_k - \tau_{k-1})^{-1}[(\tau_k - t)x_{\tau_{k-1}} + (t - \tau_{k-1})x_{\tau_k}]$. Then

$$\left[\sup_{s,t\in[0,T],\,s\neq t}\frac{\|X_s - X_t\|}{|s-t|^{\alpha}}\right] \le \left[\sup_{s,t\in[0,T],\,s\neq t}\frac{\|x_s - x_t\|}{|s-t|^{\alpha}}\right].$$
(11)

Proof of Lemma 2.2. Throughout this proof assume without loss of generality that for all $s, t \in [0, T]$ with $s \neq t$ it holds that $||x_s - x_t|| < \infty$ and let $n: [0, T] \to \mathbb{N}$ and $\rho: [0, T] \to [0, 1]$ satisfy for all $t \in [0, T]$ that

$$n(t) = \min\{k \in \{1, 2, \dots, m\} \colon \tau_k \ge t\} \quad \text{and} \quad \rho(t) = \frac{t - \tau_{n(t)-1}}{\tau_{n(t)} - \tau_{n(t)-1}}.$$
 (12)

Note that (12) ensures that for all $t \in [0, T]$ it holds that

$$X_t = (1 - \rho(t))x_{\tau_{n(t)-1}} + \rho(t)x_{\tau_{n(t)}} = x_{\tau_{n(t)-1}} + \rho(t)(x_{\tau_{n(t)}} - x_{\tau_{n(t)-1}}).$$
(13)

The fact that for all $v, w \in V$, $v, w \in \mathbb{R}$ with $||v|| + ||w|| < \infty$ it holds that $||vv + ww|| \le ||v|| ||v|| + |w|| ||w||$ hence ensures that for all $t_1, t_2 \in [0, T]$ with $t_1 < t_2$ and $n(t_1) = n(t_2)$ it holds that

$$\begin{split} \|X_{t_{1}} - X_{t_{2}}\| &= \left\| (\rho(t_{1}) - \rho(t_{2}))(x_{\tau_{n(t_{1})}} - x_{\tau_{n(t_{1})-1}}) \right\| \\ &= \left\| (\rho(t_{1}) - \rho(t_{2}))(x_{\tau_{n(t_{1})}} - x_{\tau_{n(t_{1})-1}}) + 0(x_{T} - x_{0}) \right\| \leq |\rho(t_{1}) - \rho(t_{2})| \left\| x_{\tau_{n(t_{1})}} - x_{\tau_{n(t_{1})-1}} \right\| \\ &\leq \left[|\rho(t_{1}) - \rho(t_{2})| \right] \left[|\tau_{n(t_{1})} - \tau_{n(t_{1})-1}|^{\alpha} \right] \left[\sup_{s,t \in [0,T], s \neq t} \frac{\|x_{s} - x_{t}\|}{|s - t|^{\alpha}} \right] \\ &= |\rho(t_{1}) - \rho(t_{2})|^{1-\alpha} \left[|\rho(t_{1}) - \rho(t_{2})| |\tau_{n(t_{1})} - \tau_{n(t_{1})-1}| \right]^{\alpha} \left[\sup_{s,t \in [0,T], s \neq t} \frac{\|x_{s} - x_{t}\|}{|s - t|^{\alpha}} \right] \\ &\leq \left[|\rho(t_{1}) - \rho(t_{2})| |\tau_{n(t_{1})} - \tau_{n(t_{1})-1}| \right]^{\alpha} \left[\sup_{s,t \in [0,T], s \neq t} \frac{\|x_{s} - x_{t}\|}{|s - t|^{\alpha}} \right] = |t_{1} - t_{2}|^{\alpha} \left[\sup_{s,t \in [0,T], s \neq t} \frac{\|x_{s} - x_{t}\|}{|s - t|^{\alpha}} \right]. \end{split}$$

Moreover, observe that (13) and the fact that for all $v, w \in V$, $v, w \in \mathbb{R}$ with $||v|| + ||w|| < \infty$ it holds that $||vv + ww|| \le |v|||v|| + |w|||w||$ ensure that for all $t_1, t_2 \in [0, T]$ with $n(t_1) < n(t_2)$ it holds that

$$\begin{aligned} \|X_{t_{1}} - X_{t_{2}}\| &= \left\| \left[(1 - \rho(t_{1})) x_{\tau_{n(t_{1})-1}} + \rho(t_{1}) x_{\tau_{n(t_{1})}} \right] - \left[(1 - \rho(t_{2})) x_{\tau_{n(t_{2})-1}} + \rho(t_{2}) x_{\tau_{n(t_{2})}} \right] \right\| \\ &\leq (1 - \rho(t_{1})) (1 - \rho(t_{2})) \left\| x_{\tau_{n(t_{1})-1}} - x_{\tau_{n(t_{2})-1}} \right\| + \rho(t_{1}) \rho(t_{2}) \left\| x_{\tau_{n(t_{1})}} - x_{\tau_{n(t_{2})}} \right\| \\ &+ (1 - \rho(t_{1})) \rho(t_{2}) \left\| x_{\tau_{n(t_{1})-1}} - x_{\tau_{n(t_{2})}} \right\| + \rho(t_{1}) (1 - \rho(t_{2})) \left\| x_{\tau_{n(t_{1})}} - x_{\tau_{n(t_{2})-1}} \right\| \\ &\leq \left[\sup_{s,t \in [0,T], s \neq t} \frac{\|x_{s} - x_{t}\|}{|s - t|^{\alpha}} \right] \left[(1 - \rho(t_{1})) (1 - \rho(t_{2})) |\tau_{n(t_{1})-1} - \tau_{n(t_{2})-1}|^{\alpha} + \rho(t_{1}) \rho(t_{2}) |\tau_{n(t_{1})} - \tau_{n(t_{2})}|^{\alpha} \\ &+ (1 - \rho(t_{1})) \rho(t_{2}) |\tau_{n(t_{1})-1} - \tau_{n(t_{2})}|^{\alpha} + \rho(t_{1}) (1 - \rho(t_{2})) |\tau_{n(t_{1})} - \tau_{n(t_{2})-1}|^{\alpha} \right]. \end{aligned}$$

The fact that the function $(-\infty, 0] \ni z \mapsto |z|^{\alpha} \in \mathbb{R}$ is concave hence shows that for all $t_1, t_2 \in [0, T]$ with $n(t_1) < n(t_2)$ it holds that

$$\begin{aligned} \|X_{t_{1}} - X_{t_{2}}\| \\ &\leq \left[\sup_{s,t \in [0,T], s \neq t} \frac{\|x_{s} - x_{t}\|}{|s - t|^{\alpha}}\right] \left| (1 - \rho(t_{1}))(1 - \rho(t_{2}))(\tau_{n(t_{1}) - 1} - \tau_{n(t_{2}) - 1}) + \rho(t_{1})\rho(t_{2})(\tau_{n(t_{1})} - \tau_{n(t_{2})}) \right. \\ &+ (1 - \rho(t_{1}))\rho(t_{2})(\tau_{n(t_{1}) - 1} - \tau_{n(t_{2})}) + \rho(t_{1})(1 - \rho(t_{2}))(\tau_{n(t_{1})} - \tau_{n(t_{2}) - 1}) \right|^{\alpha} (16) \\ &= \left[\sup_{s,t \in [0,T], s \neq t} \frac{\|x_{s} - x_{t}\|}{|s - t|^{\alpha}}\right] \left| [\tau_{n(t_{1}) - 1} + \rho(t_{1})(\tau_{n(t_{1})} - \tau_{n(t_{1}) - 1})] - [\tau_{n(t_{2}) - 1} + \rho(t_{2})(\tau_{n(t_{2})} - \tau_{n(t_{2}) - 1})] \right|^{\alpha} \\ &= \left[\sup_{s,t \in [0,T], s \neq t} \frac{\|x_{s} - x_{t}\|}{|s - t|^{\alpha}}\right] |t_{1} - t_{2}|^{\alpha}. \end{aligned}$$

Combining this and (14) proves (11). The proof of Lemma 2.2 is thus complete.

Lemma 2.3. Let V be an \mathbb{R} -vector space, let $\|\cdot\|: V \to [0,\infty]$ satisfy for all $v, w \in V$, $v, w \in \mathbb{R}$ with $\|v\| + \|w\| < \infty$ that $\|vv + ww\| \le |v| \|v\| + |w| \|w\|$, let $T \in (0,\infty)$, $\alpha \in (0,1]$, $n \in \mathbb{N}, m_1, m_2, \ldots, m_n \in \mathbb{N}$, let $\tau_{l,k} \in \mathbb{R}$, $k \in \{0, 1, \ldots, m_l\}$, $l \in \{1, 2, \ldots, n\}$, satisfy for all $l \in \{1, 2, \ldots, n\}$ that $0 = \tau_{l,0} < \tau_{l,1} < \ldots < \tau_{l,m_l} = T$ and $(\bigcup_{i=0}^{m_l} \{\tau_{l,i}\}) \subseteq (\bigcup_{i=0}^{m_{l+1}} \{\tau_{l+1,i}\})$, let $\mathcal{L}_l: V^{[0,T]} \to V^{[0,T]}$, $l \in \mathbb{N}$, satisfy for all $l \in \{1, 2, \ldots, n\}$, $k \in \{1, 2, \ldots, m_l\}$, $t \in [\tau_{l,k-1}, \tau_{l,k}]$, $y = (y_t)_{t \in [0,T]}: [0,T] \to V$ that $(\mathcal{L}_l(y))(t) = (\tau_{l,k} - \tau_{l,k-1})^{-1}[(\tau_{l,k} - t)y_{\tau_{l,k-1}} + (t - \tau_{l,k-1})y_{\tau_{l,k}}]$, and let $Y^{\ell} = (Y_{\ell}^{\ell})_{t \in [0,T]}: [0,T] \to V$, $\ell \in \mathbb{N}_0$, and $\mathcal{Y} = (\mathcal{Y}_t)_{t \in [0,T]}: [0,T] \to V$ satisfy

$$\mathscr{Y} = \mathscr{L}_1(Y^n) + \sum_{\ell=1}^{n-1} \left[\mathscr{L}_{l+1}(Y^{n-\ell}) - \mathscr{L}_l(Y^{n-\ell}) \right].$$
(17)

Then

$$\sup_{t \in [0,T]} \left\| \mathscr{Y}_{t} - Y_{t}^{0} \right\| \leq \max_{k \in \{0,1,\dots,m_{1}\}} \left\| Y_{\tau_{1,k}}^{n} - Y_{\tau_{1,k}}^{0} \right\| + \left[\max_{k \in \{1,2,\dots,m_{n}\}} \frac{|\tau_{n,k} - \tau_{n,k-1}|^{\alpha}}{2^{\alpha}} \right] \left[\sup_{t,s \in [0,T], t \neq s} \frac{\left\| Y_{t}^{0} - Y_{s}^{0} \right\|}{|t-s|^{\alpha}} \right] + \sum_{l=1}^{n-1} \left[\max_{k \in \{1,2,\dots,m_{l}\}} \frac{|\tau_{l,k} - \tau_{l,k-1}|^{\alpha}}{2^{\alpha}} \right] \left[\sup_{t,s \in [0,T], t \neq s} \frac{\left\| (Y_{t}^{n-l} - Y_{t}^{0}) - (Y_{s}^{n-l} - Y_{s}^{0}) \right\|}{|t-s|^{\alpha}} \right].$$
(18)

Proof of Lemma 2.3. Throughout this proof let $\varepsilon_l \in \mathbb{R}$, $l \in \{1, 2, ..., n\}$, satisfy for all $l \in \{1, 2, ..., n\}$ that $\varepsilon_l = \max_{k \in \{1, 2, ..., m_l\}} |\tau_{l,k} - \tau_{l,k-1}|$. Observe that for all $l \in \{0, 1, ..., n\}$, $y = (y_t)_{t \in [0,T]} \colon [0,T] \to V$ it holds that

$$\sup_{t \in [0,T]} \|(\mathscr{L}_{l}(y))(t)\| \le \max_{k \in \{0,1,\dots,m_{l}\}} \|y_{\tau_{l,k}}\|.$$
(19)

Next note that (17) and the fact that $\mathscr{L}_n(Y^0) = \mathscr{L}_1(Y^0) + \sum_{l=1}^{n-1} [\mathscr{L}_{l+1}(Y^0) - \mathscr{L}_l(Y^0)]$ demonstrate that

$$\begin{aligned} \mathscr{Y} - \mathscr{L}_{n}(Y^{0}) \\ &= \left[\mathscr{L}_{1}(Y^{n}) + \sum_{l=1}^{n-1} \left[\mathscr{L}_{l+1}(Y^{n-l}) - \mathscr{L}_{l}(Y^{n-l})\right]\right] - \left[\mathscr{L}_{1}(Y^{0}) + \sum_{l=1}^{n-1} \left[\mathscr{L}_{l+1}(Y^{0}) - \mathscr{L}_{l}(Y^{0})\right]\right] \\ &= \mathscr{L}_{1}(Y^{n} - Y^{0}) + \sum_{l=1}^{n-1} \left[\mathscr{L}_{l+1}(Y^{n-l} - Y^{0}) - \mathscr{L}_{l}(Y^{n-l} - Y^{0})\right]. \end{aligned}$$
(20)

This, the fact that for all $v, w \in V$ it holds that $||v + w|| \le ||v|| + ||v||$, and (19) ensure that

$$\sup_{t \in [0,T]} \left\| \mathscr{Y}_t - (\mathscr{L}_n(Y^0))(t) \right\|$$
(21)

$$\leq \sup_{t \in [0,T]} \left\| (\mathscr{L}_{1}(Y^{n} - Y^{0}))(t) \right\| + \sum_{l=1}^{n-1} \left[\sup_{t \in [0,T]} \left\| (\mathscr{L}_{l+1}(Y^{n-l} - Y^{0}))(t) - (\mathscr{L}_{l}(Y^{n-l} - Y^{0}))(t) \right\| \right]$$

$$\leq \max_{k \in \{0,1,\dots,m_{1}\}} \left\| Y^{n}_{\tau_{1,k}} - Y^{0}_{\tau_{1,k}} \right\| + \sum_{l=1}^{n-1} \left[\sup_{t \in [0,T]} \left\| (\mathscr{L}_{l+1}(Y^{n-l} - Y^{0}))(t) - (\mathscr{L}_{l}(Y^{n-l} - Y^{0}))(t) \right\| \right].$$

Moreover, note that for all $l \in \{1, 2, \ldots, n-1\}$, $y = (y_t)_{t \in [0,T]} \colon [0,T] \to V$, $i \in \{0, 1, \ldots, m_{l+1}\}$ it holds that $(\mathscr{L}_{l+1}(y))(\tau_{l+1,i}) = y_{\tau_{l+1,i}}$. The assumption that for all $l \in \{1, 2, \ldots, n-1\}$ it holds that $(\bigcup_{i=0}^{m_l} \{\tau_{l,i}\}) \subseteq (\bigcup_{i=0}^{m_{l+1}} \{\tau_{l+1,i}\})$ therefore implies that for all $l \in \{1, 2, \ldots, n-1\}$, $y = (y_t)_{t \in [0,T]} \colon [0,T] \to V$, $i \in \{0, 1, \ldots, m_l\}$ it holds that $(\mathscr{L}_{l+1}(y))(\tau_{l,i}) = y_{\tau_{l,i}}$. This proves that for all $l \in \{1, 2, \ldots, n-1\}$, $y = (y_t)_{t \in [0,T]} \colon [0,T] \to V$ it holds that $\mathscr{L}_l(y) = \mathscr{L}_l(\mathscr{L}_{l+1}(y))$. This and (21) ensure that

$$\sup_{t \in [0,T]} \left\| \mathscr{Y}_{t} - (\mathscr{L}_{n}(Y^{0}))(t) \right\| \leq \max_{k \in \{0,1,\dots,m_{1}\}} \left\| Y^{n}_{\tau_{1,k}} - Y^{0}_{\tau_{1,k}} \right\| + \sum_{l=1}^{n-1} \left[\sup_{t \in [0,T]} \left\| \left(\mathscr{L}_{l+1}(Y^{n-l} - Y^{0}) \right)(t) - \left(\mathscr{L}_{l}(\mathscr{L}_{l+1}(Y^{n-l} - Y^{0})) \right)(t) \right\| \right].$$

$$(22)$$

Lemma 2.1 hence proves that

$$\sup_{t \in [0,T]} \left\| \mathscr{Y}_{t} - (\mathscr{L}_{n}(Y^{0}))(t) \right\| \leq \max_{k \in \{0,1,\dots,m_{1}\}} \left\| Y_{\tau_{1,k}}^{n} - Y_{\tau_{1,k}}^{0} \right\| \\ + \sum_{l=1}^{n-1} \frac{|\varepsilon_{l}|^{\alpha}}{2^{\alpha}} \left[\sup_{r,s \in [0,T], r \neq s} \frac{\left\| (\mathscr{L}_{l+1}(Y^{n-l} - Y^{0}))(r) - (\mathscr{L}_{l+1}(Y^{n-l} - Y^{0}))(s) \right\|}{|r-s|^{\alpha}} \right].$$

$$(23)$$

Lemma 2.2 hence ensures that

$$\sup_{t \in [0,T]} \left\| \mathscr{Y}_{t} - (\mathscr{L}_{n}(Y^{0}))(t) \right\| \leq \max_{k \in \{0,1,\dots,m_{1}\}} \left\| Y_{\tau_{1,k}}^{n} - Y_{\tau_{1,k}}^{0} \right\| \\ + \sum_{l=1}^{n-1} \frac{|\varepsilon_{l}|^{\alpha}}{2^{\alpha}} \left[\sup_{r,s \in [0,T], r \neq s} \frac{\left\| (Y_{s}^{n-l} - Y_{s}^{0}) - (Y_{r}^{n-l} - Y_{r}^{0}) \right\|}{|r-s|^{\alpha}} \right].$$

$$(24)$$

Moreover, observe that Lemma 2.1 ensures that

$$\sup_{t \in [0,T]} \left\| (\mathscr{L}_n(Y^0))(t) - Y_t^0 \right\| \le \frac{|\varepsilon_n|^{\alpha}}{2^{\alpha}} \left[\sup_{t,s \in [0,T], t \ne s} \frac{\|Y_t^0 - Y_s^0\|}{|t-s|^{\alpha}} \right].$$
(25)

The fact that for all $v, w \in V$ it holds that $||v + w|| \le ||v|| + ||v||$ and (24) hence demonstrate that

$$\sup_{t \in [0,T]} \left\| \mathscr{Y}_{t} - Y_{t}^{0} \right\| \leq \sup_{t \in [0,T]} \left[\left\| \mathscr{Y}_{t} - (\mathscr{L}_{n}(Y^{0}))(t) \right\| + \left\| (\mathscr{L}_{n}(Y^{0})(t) - Y_{t}^{0} \right\| \right] \\ \leq \max_{k \in \{0,1,\dots,m_{1}\}} \left\| Y_{\tau_{1,k}}^{n} - Y_{\tau_{1,k}}^{0} \right\| + \frac{|\varepsilon_{n}|^{\alpha}}{2^{\alpha}} \left[\sup_{t,s \in [0,T], t \neq s} \frac{\|Y_{t}^{0} - Y_{s}^{0}\|}{|t-s|^{\alpha}} \right]$$

$$+ \sum_{l=1}^{n-1} \frac{|\varepsilon_{l}|^{\alpha}}{2^{\alpha}} \left[\sup_{r,s \in [0,T], r \neq s} \frac{\|(Y_{s}^{n-l} - Y_{s}^{0}) - (Y_{r}^{n-l} - Y_{r}^{0})\|}{|r-s|^{\alpha}} \right].$$

$$(26)$$

The proof of Lemma 2.3 is thus complete.

3 Existence, uniqueness, and Hölder continuity properties for solutions of stochastic fixed-point equations

An important aspect in the derivation of the Monte Carlo-type approximation algorithm for BSDEs in (1)-(2) in Theorem 1.1 in Section 1 above is, loosely speaking, to reformulate the solutions of the BSDEs in (3) as solutions of appropriate SFPEs associated to the BSDEs in (3) and in this section we establish in Lemma 3.1 below existence, uniqueness, and Hölder continuity properties for solutions of such SFPEs.

In particular, under suitable assumptions, item (i) in Lemma 3.1 proves that the SFPE in (32) below has a unique solution within the set of functions which grow at most like a certain Lyapunov-type function (see the function $V: [0,T] \times \mathbb{R}^d \to [1,\infty)$ above (32) in Lemma 3.1 for details), item (ii) in Lemma 3.1 establishes a suitable explicit a priori growth bound for the unique solution of the SFPE in (32), and item (iii) in Lemma 3.1 proves that the unique solution of the SFPE in (32) is 1/2-Hölder-continuous in the time variable $t \in [0,T]$ and locally 1-Hölder continuous (locally Lipschitz continuous) in the space variable $x \in \mathbb{R}^d$.

Further existence, uniqueness, and regularity results for SFPEs can, e.g., be found in [71, Section 4], [9, Section 2 and Section 3], [6, Section 2 and Section 3], and [72, Section 2].

Lemma 3.1. Let $d \in \mathbb{N}$, $L \in [0, \infty)$, $T \in (0, \infty)$, $p_1, p_2, p_3 \in [1, \infty]$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \leq 1$, let $\|\cdot\| : \mathbb{R}^d \to [0, \infty)$ be a norm, let $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}^d \to \mathbb{R}$, $\phi : [0, T] \times \mathbb{R}^d \to [1, \infty)$, $V : [0, T] \times \mathbb{R}^d \to [1, \infty)$, and $\psi : [0, T] \times \mathbb{R}^d \to [1, \infty)$ be measurable, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, for every random variable $\mathfrak{X} : \Omega \to \mathbb{R}$ let $\|\mathfrak{X}\|_p \in [0, \infty]$, $p \in [1, \infty]$, satisfy for all $p \in [1, \infty)$ that $\|\mathfrak{X}\|_p = (\mathbb{E}[|\mathfrak{X}|^p])^{1/p}$ and $\|\mathfrak{X}\|_{\infty} = \inf(\{r \in [0, \infty) : \mathbb{P}(|\mathfrak{X}| > r) = 0\} \cup \{\infty\})$, for every $s \in [0, T]$, $x \in \mathbb{R}^d$ let $X_{s,(\cdot)}^x = (X_{s,t}^x(\omega))_{(t,\omega)\in[s,T]\times\Omega} : [s,T] \times \Omega \to \mathbb{R}^d$ be measurable, assume for all measurable $h : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ that $\{(\mathfrak{s},\mathfrak{t}) \in [0,T]^2 : \mathfrak{s} \leq \mathfrak{t}\} \times \mathbb{R}^d \times \mathbb{R}^d \ni (\mathfrak{s},\mathfrak{t},x,y) \mapsto \mathbb{E}[h(\mathfrak{t}, X_{\mathfrak{s},\mathfrak{t}}^x, X_{\mathfrak{s},\mathfrak{t}}^y)] \in [0,\infty]$ is measurable, and assume for all $s \in [0,T]$, $t \in [s,T]$, $r \in [t,T]$, $x, y \in \mathbb{R}^d$, $v, w \in \mathbb{R}$ and all measurable $h : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ that

$$\|\phi(t, X_{s,t}^x)\|_{p_3} \le \phi(s, x), \qquad \max\{|g(x)|\mathbb{1}_{\{T\}}(s), |Tf(s, x, 0)|, \|V(t, X_{s,t}^x)\|_{p_1}\} \le V(s, x)$$
(27)

$$|g(x) - g(y)| \le \frac{1}{2\sqrt{T}} (V(T, x) + V(T, y)) ||x - y||,$$
(28)

$$|f(t,x,v) - f(t,y,w)| \le L|v - w| + \frac{1}{2T^{3/2}}(V(t,x) + V(t,y))||x - y||,$$
(29)

$$\|\|X_{s,t}^{x} - x\|\|_{p_{2}} \le \psi(s,x)|s - t|^{1/2}, \qquad \|\|X_{s,t}^{x} - X_{s,t}^{y}\|\|_{p_{2}} \le \frac{1}{2}(\phi(s,x) + \phi(s,y))\|x - y\|, \quad (30)$$

and
$$\mathbb{E}\left[\mathbb{E}\left[h\left(r, X_{t,r}^{a}, X_{t,r}^{b}\right)\right]\right|_{(a,b)=(X_{s,t}^{x}, X_{s,t}^{y})}\right] = \mathbb{E}\left[h\left(r, X_{s,r}^{x}, X_{s,r}^{y}\right)\right],\tag{31}$$

Then

(i) there exists a unique measurable $u: [0,T] \times \mathbb{R}^d \to \mathbb{R}$ which satisfies for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that $\mathbb{E}\left[|g(X_{t,T}^x)|\right] + \int_t^T \mathbb{E}\left[|f(r, X_{t,r}^x, u(r, X_{t,r}^x))|\right] dr + \sup_{r \in [0,T], \xi \in \mathbb{R}^d} \left(\frac{|u(r,\xi)|}{V(r,\xi)}\right) < \infty$ and

$$u(t,x) = \mathbb{E}\left[g(X_{t,T}^x)\right] + \int_t^T \mathbb{E}\left[f(r, X_{t,r}^x, u(r, X_{t,r}^x))\right] dr,$$
(32)

(ii) it holds for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that $|u(t,x)| \leq 2e^{L(T-t)}V(t,x)$, and

 $\begin{array}{l} (iii) \ it \ holds \ for \ all \ s \in [0,T], \ t \in [s,T], \ x,y \in \mathbb{R}^d \ that \ |u(s,x)-u(t,y)| \leq T^{-1/2}e^{2LT}(V(s,x)+V(t,y))(\phi(s,x)+\phi(t,y)) \big[\psi(s,x)|s-t|^{1/2}+\|x-y\|\big]. \end{array}$

Proof of Lemma 3.1. Observe that [72, Proposition 2.2] (applied with $\mathcal{O} \curvearrowright \mathbb{R}^d$ in the notation of [72, Proposition 2.2]) and (27) prove items (i) and (ii). Next note that (32), the triangle inequality, and (31) show that for all $s \in [0, T]$, $t \in [s, T]$, $x, y \in \mathbb{R}^d$ it holds that

$$\mathbb{E}\left[\left|u(t, X_{s,t}^{x}) - u(t, X_{s,t}^{y})\right|\right] = \mathbb{E}\left[\left|u(t, a) - u(t, b)\right|\right|_{(a,b)=(X_{s,t}^{x}, X_{s,t}^{y})}\right]
= \mathbb{E}\left[\left|\mathbb{E}\left[g(X_{t,T}^{a}) - g(X_{t,T}^{b})\right] + \int_{t}^{T} \mathbb{E}\left[f(r, X_{t,r}^{a}, u(r, X_{t,r}^{a})) - f(r, X_{t,r}^{b}, u(r, X_{t,r}^{b}))\right]dr\right|\right|_{(a,b)=(X_{s,t}^{x}, X_{s,t}^{y})} \\
\leq \mathbb{E}\left[\mathbb{E}\left[\left|g(X_{t,T}^{a}) - g(X_{t,T}^{b})\right|\right]\right|_{(a,b)=(X_{s,t}^{x}, X_{s,t}^{y})}\right] \\
+ \int_{t}^{T} \mathbb{E}\left[\mathbb{E}\left[\left|f(r, X_{t,r}^{a}, u(r, X_{t,r}^{a})) - f(r, X_{t,r}^{b}, u(r, X_{t,r}^{b}))\right|\right]\right|_{(a,b)=(X_{s,t}^{x}, X_{s,t}^{y})}\right]dr \\
= \mathbb{E}\left[\left|g(X_{s,T}^{x}) - g(X_{s,T}^{y})\right|\right] + \int_{t}^{T} \mathbb{E}\left[\left|f(r, X_{s,r}^{x}, u(r, X_{s,r}^{x})) - f(r, X_{s,r}^{x}, u(r, X_{s,r}^{y}))\right|\right]dr. \tag{33}$$

Hölder's inequality, (28), (29), the fact that $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$, (30), and (27) hence demonstrate that for all $s \in [0, T]$, $t \in [s, T]$, $x, y \in \mathbb{R}^d$ it holds that

$$\mathbb{E}\left[\left|u(t, X_{s,t}^{x}) - u(t, X_{s,t}^{y})\right|\right] \\
\leq \frac{1}{2\sqrt{T}} \mathbb{E}\left[\left(V(T, X_{s,T}^{x}) + V(T, X_{s,T}^{y})\right) \|X_{s,T}^{x} - X_{s,T}^{y}\|\right] + \int_{t}^{T} L \mathbb{E}\left[\left|u(r, X_{s,r}^{x}) - u(r, X_{s,r}^{y})\right|\right] dr \\
+ \frac{1}{2T\sqrt{T}} \int_{t}^{T} \mathbb{E}\left[\left(V(r, X_{s,r}^{x}) + V(r, X_{s,r}^{y})\right) \|X_{s,r}^{x} - X_{s,r}^{y}\|\right] dr \\
\leq \sup_{r \in [t,T]} \left[\frac{1}{\sqrt{T}} \|V(r, X_{s,r}^{x}) + V(r, X_{s,r}^{y})\|_{p_{1}} \|\|X_{s,r}^{x} - X_{s,r}^{y}\|\|_{p_{2}}\right] \\
+ \int_{t}^{T} L \mathbb{E}\left[\left|u(r, X_{s,r}^{x}) - u(r, X_{s,r}^{y})\right|\right] dr \\
\leq \left[\frac{V(s,x) + V(s,y)}{\sqrt{T}}\right] \left[\frac{\phi(s,x) + \phi(s,y)}{2}\right] \|x - y\| + \int_{t}^{T} L \mathbb{E}\left[\left|u(r, X_{s,r}^{x}) - u(r, X_{s,r}^{y})\right|\right] dr.$$
(34)

This, item (ii), (27), and Gronwall's lemma (see, e.g., [74, Lemma 3.2]) show that for all $s \in [0, T], t \in [s, T], x, y \in \mathbb{R}^d$ it holds that

$$\mathbb{E}\left[\left|u(t, X_{s,t}^{x}) - u(t, X_{s,t}^{y})\right|\right] \le \frac{1}{2\sqrt{T}} (V(s, x) + V(s, y))(\phi(s, x) + \phi(s, y)) \|x - y\|e^{L(T-t)}.$$
 (35)

Moreover, observe that (30) ensures that for all $t \in [0, T]$, $x \in \mathbb{R}^d$ it holds that $\mathbb{P}(||X_{t,t}^x - x|| = 0) = 1$. Hence, we obtain that for all $t \in [0, T]$, $x \in \mathbb{R}^d$ it holds that $\mathbb{P}(X_{t,t}^x = x) = 1$. Combining this with (35) establishes that for all $t \in [0, T]$, $x, y \in \mathbb{R}^d$ it holds that

$$|u(t,x) - u(t,y)| \le \frac{1}{2\sqrt{T}} (V(t,x) + V(t,y)) (\phi(t,x) + \phi(t,y)) ||x - y|| e^{L(T-t)}.$$
(36)

Next note that (32), Fubini's theorem, and (31) show that for all $s \in [0, T]$, $t \in [s, T]$, $x \in \mathbb{R}^d$ it holds that

$$u(s,x) - \mathbb{E}\left[u(t,X_{s,t}^{x})\right] = \mathbb{E}\left[g(X_{s,T}^{x})\right] + \int_{s}^{T} \mathbb{E}\left[f(r,X_{s,r}^{x},u(r,X_{s,r}^{x}))\right]dr$$
$$- \mathbb{E}\left[\left[\mathbb{E}\left[g(X_{t,T}^{\tilde{x}})\right] + \int_{t}^{T} \mathbb{E}\left[f(r,X_{t,r}^{a},u(r,X_{t,r}^{\tilde{x}}))\right]dr\right]\Big|_{a=X_{s,t}^{x}}\right]$$
$$= \int_{s}^{t} \mathbb{E}\left[f(r,X_{s,r}^{x},u(r,X_{s,r}^{x}))\right]dr.$$
(37)

This, the triangle inequality, (29), (27), and item (ii) demonstrate that for all $s \in [0, T]$, $t \in [s, T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned} \left| u(s,x) - \mathbb{E} \left[u(t,X_{s,t}^{x}) \right] \right| &\leq |t-s| \left[\sup_{r \in [s,t]} \left(\mathbb{E} \left[|f(r,X_{s,r}^{x},0)| \right] + L\mathbb{E} \left[|u(r,X_{s,r}^{x})| \right] \right) \right] \\ &\leq |t-s| \left[\sup_{r \in [s,t]} \left(\left(\frac{1}{T} + 2Le^{LT} \right) \mathbb{E} \left[V(r,X_{s,r}^{x}) \right] \right) \right] \leq \left[\frac{1+2TLe^{LT}}{T} \right] |t-s|V(s,x) \\ &\leq \frac{1}{\sqrt{T}} (2 + 4LTe^{LT}) \frac{1}{2} (V(s,x) + V(t,y)) |t-s|^{1/2}. \end{aligned}$$
(38)

Next observe that (36), Hölder's inequality, the fact that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \le 1$, the triangle inequality, (27), and (30) prove that for all $s \in [0, T]$, $t \in [s, T]$, $x, y \in \mathbb{R}^d$ it holds that

$$\begin{aligned} \left| \mathbb{E} \left[u(t, X_{s,t}^{x}) \right] - u(t,y) \right| &\leq \mathbb{E} \left[\left| u(t, X_{s,t}^{x}) - u(t,y) \right| \right] \\ &\leq \mathbb{E} \left[\frac{2}{\sqrt{T}} \frac{1}{2} (V(t, X_{s,t}^{x}) + V(t,y)) \frac{1}{2} (\phi(t, X_{s,t}^{x}) + \phi(t,y)) \|X_{s,t}^{x} - y\| \right] e^{L(T-t)} \\ &\leq \frac{2e^{LT}}{\sqrt{T}} \frac{1}{2} \left(\|V(t, X_{s,t}^{x})\|_{p_{1}} + V(t,y) \right) \frac{1}{2} \left(\|\phi(t, X_{s,t}^{x})\|_{p_{3}} + \phi(t,y) \right) \|\|X_{s,t}^{x} - y\|\|_{p_{2}} \\ &\leq \frac{1}{\sqrt{T}} 2e^{LT} \frac{1}{2} (V(s,x) + V(t,y)) \frac{1}{2} (\phi(s,x) + \phi(t,y)) \left[\psi(s,x)|s - t|^{1/2} + \|x - y\| \right]. \end{aligned}$$
(39)

This, the triangle inequality, (38), the fact that $\phi \ge 1$, the fact that $\psi \ge 1$, and the fact that $2 + 4LTe^{LT} + 2e^{LT} \le 4e^{LT}(1 + LT) \le 4e^{2LT}$ show that for all $s \in [0, T], t \in [s, T], x, y \in \mathbb{R}^d$ it holds that

$$\begin{aligned} |u(s,x) - u(t,y)| &\leq \left| u(s,x) - \mathbb{E} \left[u(t,X_{s,t}^x) \right] \right| + \left| \mathbb{E} \left[u(t,X_{s,t}^x) \right] - u(t,y) \right| \\ &\leq \frac{1}{\sqrt{T}} 4e^{2LT} \frac{1}{2} (V(s,x) + V(t,y)) \frac{1}{2} (\phi(s,x) + \phi(t,y)) \left[\psi(s,x) |s-t|^{1/2} + \|x-y\| \right]. \end{aligned}$$

$$\tag{40}$$

This proves item (iii). The proof of Lemma 3.1 is thus complete.

4 Error analysis in Hölder seminorms for full-history recursive multilevel Picard (MLP) approximations

In Theorem 5.1 in Section 5 below we supply a computational complexity analysis for the Monte Carlo-type approximation algorithm for BSDEs in (1)–(2) in Theorem 1.1 in Section 1 above. Our proof of Theorem 5.1 exploits the multi-grid approximation result in Lemma 2.3 in Section 2 above as well as the error analysis for appropriate MLP approximations in Proposition 4.2 in this section. Specifically, in Proposition 4.2 below we establish upper bounds for appropriate Hölder seminorms of the difference between solutions of SFPEs and suitable MLP approximations for such SFPEs.

Setting 4.1. Let $T \in (0,\infty)$, $L, \rho \in [0,\infty)$, $\beta \in (0, 1/12]$, $d \in \mathbb{N}$, $f \in C([0,T] \times \mathbb{R}^d \times \mathbb{R}, \mathbb{R})$, $g \in C(\mathbb{R}^d, \mathbb{R})$, let $V \colon \mathbb{R}^d \to [1,\infty)$ be measurable, let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ be a norm, let $\Theta = (0,\infty)$

 $\bigcup_{n\in\mathbb{N}}\mathbb{Z}^n, \ let \ (\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]}) \ be \ a \ filtered \ probability \ space, \ let \ \mathfrak{r}^{\theta} \colon \Omega \to [0,1], \ \theta \in \Theta, \ be \ i.i.d. \ random \ variables, \ assume \ for \ all \ t \in (0,1) \ that \ \mathbb{P}(\mathfrak{r}^0 \leq t) = t, \ let \ \mathfrak{z}^{\theta} \colon \Omega \to \mathbb{R}^d, \ \theta \in \Theta, \ be \ i.i.d. \ standard \ normal \ random \ vectors, \ let \ W = (W^1, W^2, \ldots, W^d) \colon [0,T] \times \Omega \to \mathbb{R}^d \ be \ a \ standard \ (\mathbb{F}_t)_{t\in[0,T]} \text{-}Brownian \ motion, \ assume \ that \ (\mathfrak{r}^{\theta})_{\theta\in\Theta}, \ (\mathfrak{z}^{\theta})_{\theta\in\Theta}, \ and \ W \ are \ independent, \ let \ X^{\theta}_{s,t} \colon \mathbb{R}^d \times \Omega \to \mathbb{R}^d, \ s,t \in [0,T], \ \theta \in \Theta, \ satisfy \ for \ all \ s,t \in [0,T], \ x \in \mathbb{R}^d, \ \theta \in \Theta \ that \ X^{\theta}_{s,t}(x) = x + |t - s|^{1/2} \mathfrak{z}^{\theta}, \ let \ F \colon \mathbb{R}^{[0,T] \times \mathbb{R}^d} \to \mathbb{R}^{[0,T] \times \mathbb{R}^d} \ satisfy \ for \ all \ t \in [0,T], \ x \in \mathbb{R}^d, \ v \in \mathbb{R}^{[0,T] \times \mathbb{R}^d} \ that \ (F(v))(t,x) = f(t,x,v(t,x)), \ let \ U^{\theta}_{n,M} \colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}, \ n, M \in \mathbb{Z}, \ \theta \in \Theta, \ satisfy \ for \ all \ M \in \mathbb{N}, \ n \in \mathbb{N}_0, \ \theta \in \Theta, \ t \in [0,T], \ x \in \mathbb{R}^d \ that$

$$U_{n,M}^{\theta}(t,x) = \frac{\mathbb{1}_{\mathbb{N}}(n)}{M^{n}} \sum_{i=1}^{M^{n}} g\left(X_{t,T}^{(\theta,0,-i)}(x)\right)$$

$$+ \sum_{\ell=0}^{n-1} \left[\frac{(T-t)}{M^{n-\ell}} \sum_{i=1}^{M^{n-\ell}} \left(F\left(U_{\ell,M}^{(\theta,\ell,i)}\right) - \mathbb{1}_{\mathbb{N}}(\ell)F\left(U_{\ell-1,M}^{(\theta,-\ell,i)}\right)\right) \left(t + (T-t)\mathfrak{r}^{(\theta,\ell,i)}, X_{t,t+(T-t)\mathfrak{r}^{(\theta,\ell,i)}}^{(\theta,\ell,i)}(x)\right) \right],$$
(41)

and assume for all $s, t \in [0, T]$, $x, y \in \mathbb{R}^d$, $v_1, v_2, w_1, w_2 \in \mathbb{R}$ that

$$\max\{|f(s, x, v_1) - f(t, y, v_2)|, T^{-1}|g(x) - g(y)|\} \le T^{-3/2} |V(x) + V(y)|^{\beta} (|s - t|^{1/2} + ||x - y||) + L|v_1 - v_2|,$$
(42)

$$|[f(s, x, v_1) - f(s, x, w_1)] - [f(t, y, v_2) - f(t, y, w_2)]| \le L |(v_1 - w_1) - (v_2 - w_2)| + T^{-3/2} |V(x) + V(y)|^{\beta} [(\max \{ \mathbb{E} [\| \mathbf{z}^0 \|^4], 1 \})^{1/4} |s - t|^{1/2} + \| x - y \|] |v_1 - w_1| + T^{-1} |V(x) + V(y)|^{\beta} (|v_1 - w_1| + |v_2 - w_2|) |w_1 - w_2|,$$
(43)

and $\max\{|Tf(t,x,0)|^{1/\beta}, |g(x)|^{1/\beta}, \mathbb{E}[V(X^0_{s,t}(x))]\} \le e^{\rho|t-s|}V(x).$

Proposition 4.2. Assume Setting 4.1, let $c \in \mathbb{R}$ satisfy $c = (\max\{\mathbb{E}[||\mathbf{z}^0||^4], 1\})^{1/4}$, and let $\mathbb{V}: [0,T] \times \mathbb{R}^d \to \mathbb{R}$ satisfy for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that $\mathbb{V}(t,x) = e^{\rho(T-t)}V(x)$. Then

(i) there exists a unique measurable $u: [0,T] \times \mathbb{R}^d \to \mathbb{R}$ which satisfies for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that $\mathbb{E}\left[|g(X^0_{t,T}(x))|\right] + \int_t^T \mathbb{E}\left[|(F(u))(s,X^0_{t,s}(x))|\right] ds + \sup_{r \in [0,T], y \in \mathbb{R}^d} \left(\frac{|u(r,y)|}{|V(y)|^{\beta}}\right) < \infty$ and

$$u(t,x) = \mathbb{E}\left[g(X_{t,T}^{0}(x))\right] + \int_{t}^{T} \mathbb{E}\left[(F(u))(s, X_{t,s}^{0}(x))\right] ds,$$
(44)

- (ii) it holds for all $\theta \in \Theta$, $n \in \mathbb{N}_0$, $M \in \mathbb{N}$ that $U_{n,M}^{\theta}$ is measurable,
- (iii) it holds for all $M \in \mathbb{N}$, $N \in \mathbb{N}_0$ that

$$\sup_{s \in [0,T]} \sup_{x \in \mathbb{R}^d} \left[\frac{\left(\mathbb{E} \left[|U_{N,M}^0(s,x) - u(s,x)|^2 \right] \right)^{1/2}}{(\mathbb{V}(s,x))^{\beta}} \right] \le e^{M/2} M^{-N/2} \left(50e^{2LT} \right)^{N+1}, \quad (45)$$

and

(iv) it holds for all $M \in \mathbb{N}$, $N \in \mathbb{N}_0$ that

$$\sup_{\substack{s,t\in[0,T],\ x,y\in\mathbb{R}^{d},\\s\neq t}} \sup_{\substack{x\neq y}} \left[\frac{T^{1/2} \left(\mathbb{E}\left[\left| \left[U_{N,M}^{0}(s,x) - u(s,x) \right] - \left[U_{N,M}^{0}(t,y) - u(t,y) \right] \right|^{2} \right] \right)^{1/2}}{\left[c|s-t|^{1/2} + \|x-y\| \right] \left(\mathbb{V}(s,x) + \mathbb{V}(t,y)\right)^{1/4}} \right] \le e^{M/2} M^{-N/2} \left(50e^{2LT} \right)^{N+1}.$$
(46)

Proof of Proposition 4.2. Throughout this proof let $\Lambda: [0,1] \times [0,T] \to [0,T]$ satisfy for all $t \in [0,T], \lambda \in [0,1]$ that $\Lambda(\lambda,t) = t + \lambda(T-t)$, for every $q \in [1,\infty)$ and every random variable $\mathfrak{X}: \Omega \to \mathbb{R}$ let $\|\mathfrak{X}\|_q \in [0,\infty]$ satisfy that $\|X\|_q = (\mathbb{E}[|\mathfrak{X}|^q])^{1/q}$, and for every $r \in [0,T]$ and every random field $H: [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ let $|||H|||_{k,r} \in [0,\infty], k \in \{0,1,2\}$, satisfy

$$\|\|H\|_{0,r} = \max_{j \in \{1,2\}} \|\|H\|_{j,r}, \qquad \|\|H\|_{1,r} = \sup_{\substack{x \in \mathbb{R}^d, s \in [r,T]}} \left[\frac{\left(\mathbb{E}[|H(s,x)|^2]\right)^{1/2}}{\left(\mathbb{V}(s,x)\right)^{\beta}} \right],$$

d
$$\|\|H\|_{2,r} = \sup_{\substack{s,t \in [r,T], x, y \in \mathbb{R}^d \\ (s,x) \neq (t,y)}} \left[\frac{T^{1/2} \left(\mathbb{E}[|H(s,x) - H(t,y)|^2]\right)^{1/2}}{\left[c|s-t|^{1/2} + \|x-y\|\right] \left(\mathbb{V}(s,x) + \mathbb{V}(t,y)\right)^{1/4}} \right].$$
 (47)

an

Observe that (47) ensures that for all $j \in \{0, 1, 2\}, r \in [0, T], \lambda, \mu \in \mathbb{R}$ and all random fields $H_k: [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}, k \in \{1,2\}, \text{ it holds that}$

$$\||\lambda H_1 + \mu H_2||_{j,r} \le |\lambda| \||H_1||_{j,r} + |\mu| \||H_2||_{j,r}.$$
(48)

Moreover, note that (47) assures that for all $j \in \{0, 1, 2\}$ and all random fields $H: [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ $\Omega \to \mathbb{R}$ it holds that $[0,T] \ni r \mapsto |||H|||_{i,r} \in [0,\infty]$ is non-increasing. This shows that for all $j \in \{0, 1, 2\}$ and all random fields $H: [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ it holds that $[0, T] \ni r \mapsto ||H||_{i,r} \in \mathbb{R}^d$ $[0,\infty]$ is measurable. Next observe that Jensen's inequality and the fact that for all $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $\mathbb{V}(t,x) = e^{\rho(T-t)}V(x)$ show that for all $\gamma \in [0,1], s \in [0,T], t \in [s,T],$ $x \in \mathbb{R}^d$ it holds that

$$\mathbb{E}\left[\left(\mathbb{V}(t, X_{s,t}^{0}(x))\right)^{\gamma}\right] \leq \left(\mathbb{E}\left[\mathbb{V}(t, X_{s,t}^{0}(x))\right]\right)^{\gamma} \leq \left(e^{\rho(T-t)}e^{\rho(t-s)}V(x)\right)^{\gamma} \leq (\mathbb{V}(s, x))^{\gamma}.$$
(49)

Combining this, the fact that $0 < \beta \leq 3/4$, the fact that $c \geq 1$, (42), and Lemma 3.1 (applied with $p_1 \curvearrowleft 1/\beta$, $p_2 \curvearrowleft 4$, $p_3 \curvearrowleft \infty$, $\phi \curvearrowleft (\mathbb{R}^d \ni x \mapsto 1 \in [1,\infty))$, $V \curvearrowleft 2\mathbb{V}^\beta$, $\psi \curvearrowleft (\mathbb{R}^d \ni x \mapsto 1 \in [1,\infty))$ $c \in [1,\infty)), \ (X^x_{s,t}(\omega))_{(s,t,x,\omega) \in \{(\mathfrak{s},\mathfrak{t}) \in [0,T]^2: \mathfrak{s} \leq \mathfrak{t}\} \times \mathbb{R}^d \times \Omega} \ \curvearrowleft \ (X^0_{s,t}(x,\omega))_{(s,t,x,\omega) \in \{(\mathfrak{s},\mathfrak{t}) \in [0,T]^2: \mathfrak{s} \leq \mathfrak{t}\} \times \mathbb{R}^d \times \Omega} \ \text{in}$ the notation of Lemma 3.1) implies that

(a) there exists a unique measurable $u: [0, T] \times \mathbb{R}^d \to \mathbb{R}$ which satisfies for all $s \in [0, T], x \in \mathbb{R}^d$ that $\mathbb{E}[|g(X_{s,T}^{0}(x))|] + \int_{s}^{T} \mathbb{E}[|(F(u))(t, X_{s,t}^{0}(x))|] dt + \sup_{r \in [0,T], y \in \mathbb{R}^{d}} \left(\frac{|u(r,y)|}{|\mathbb{V}(r,y)|^{\beta}}\right) < \infty$ and

$$u(s,x) = \mathbb{E}\left[g(X_{s,T}^0(x))\right] + \int_s^T \mathbb{E}\left[(F(u))\left(t, X_{s,t}^0(x)\right)\right] ds,\tag{50}$$

(b) it holds for all $t \in [0, T], x \in \mathbb{R}^d$ that

$$|u(t,x)| \le 2e^{LT} 2(\mathbb{V}(t,x))^{\beta} = 4e^{LT} (\mathbb{V}(t,x))^{\beta},$$
(51)

and

(c) it holds for all $s, t \in [0, T], x, y \in \mathbb{R}^d$ that

$$\begin{aligned} |u(s,x) - u(t,y)| &\leq \frac{1}{\sqrt{T}} 4e^{2LT} \frac{1}{2} \left(2(\mathbb{V}(s,x))^{\beta} + 2(\mathbb{V}(t,y))^{\beta} \right) \left[c|s-t|^{1/2} + ||x-y|| \right] \\ &\leq \frac{1}{\sqrt{T}} 8e^{2LT} (\mathbb{V}(s,x) + \mathbb{V}(t,y))^{\beta} \left[c|s-t|^{1/2} + ||x-y|| \right]. \end{aligned}$$
(52)

This establishes item (i). Next observe that, e.g., [72, Lemmas 3.2–3.4], the fact that $\mathbf{z}^{\theta}, \theta \in \Theta$, are independent, and item (i) show that

- (A) it holds for all $n \in \mathbb{N}_0$, $M \in \mathbb{N}$, $\theta \in \Theta$ that $U_{n,M}^{\theta}$ and $F(U_{n,M}^{\theta})$ are measurable,
- (B) it holds for all $n, m \in \mathbb{N}_0, M \in \mathbb{N}, i, j, k, \ell \in \mathbb{Z}, \theta \in \Theta$ with $(i, j) \neq (k, l)$ that $U_{n,M}^{(\theta, i, j)}$, $U_{m,M}^{(\theta,k,\ell)}, \mathfrak{r}^{(\theta,i,j)}, \text{ and } X^{(\theta,i,j)} \text{ are independent},$

- (C) it holds for all $M, n \in \mathbb{N}, t \in [0,T], x \in \mathbb{R}^d$ that $U^{\theta}_{n,M}(t,x), \theta \in \Theta$, are identically distributed,
- (D) it holds for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that $g(X_{t,T}^{\theta}(x)), \theta \in \Theta$, are i.i.d.,
- (E) it holds for all $M, n \in \mathbb{N}, t \in [0, T], x \in \mathbb{R}^d$ that $\mathbb{E}[|U_{n,M}^0(t, x)| + |g(X_{t,T}^0(x))|] < \infty$,
- (F) it holds for all $\ell \in \mathbb{N}_0$, $M \in \mathbb{N}$ that $((T-t)(F(U_{\ell,M}^{(0,\ell,i)}) \mathbb{1}_{\mathbb{N}}(\ell)F(U_{\ell-1,M}^{(0,-\ell,i)}))(t+(T-t)\mathfrak{r}^{(0,\ell,i)}, X_{t,t+(T-t)\mathfrak{r}^{(0,\ell,i)}}^{(0,\ell,i)}(x)))_{(t,x)\in[0,T]\times\mathbb{R}^d}$, $i \in \mathbb{N}$, are i.i.d.,
- (G) it holds for all $\ell \in \mathbb{N}_0$, $M \in \mathbb{N}$, $t \in [0, T]$, $x \in \mathbb{R}^d$ that

$$\mathbb{E}\Big[\big|(F(U_{\ell,M}^{(0,\ell,1)}) - \mathbb{1}_{\mathbb{N}}(\ell)F(U_{\ell-1,M}^{(0,-\ell,1)}))(t + (T-t)\mathfrak{r}^{(0,\ell,1)}, X_{t,t+(T-t)\mathfrak{r}^{(0,\ell,1)}}^{(0,\ell,1)}(x))\big|\Big] < \infty, \quad (53)$$

(H) it holds for all $M, n \in \mathbb{N}, t \in [0, T], x \in \mathbb{R}^d$ that

$$\mathbb{E}\Big[\left|(T-t)(F(U_{n-1,M}^{0}-F(u))(t+\mathfrak{r}^{0}(T-t),X_{t,t+\mathfrak{r}^{0}(T-t)}^{0}(x))\right|\Big]<\infty$$
(54)

(I) it holds for all $M, n \in \mathbb{N}, t \in [0, T], x \in \mathbb{R}^d$ that

$$\mathbb{E}\left[U_{n,M}^{0}(t,x)\right] - u(t,x) \\ = \mathbb{E}\left[(T-t)(F(U_{n-1,M}^{0}) - F(u))(t + \mathfrak{r}^{0}(T-t), X_{t,t+\mathfrak{r}^{0}(T-t)}^{0}(x)))\right],$$
(55)

and

(J) it holds for all $M \in \mathbb{N}$, $n \in \mathbb{N}_0$ that $U_{n,M}^0$, X^0 , and \mathfrak{r}^0 are independent.

This establishes (ii). Next note that (47), (51), (52), and the fact that for all $M \in \mathbb{N}$ it holds that $U_{0,M}^0 = 0$ show that for all $r \in [0,T]$, $M \in \mathbb{N}$ it holds that

$$\left\| \left\| U_{0,M}^{0} - u \right\| \right\|_{0,r} = \left\| u \right\|_{0,r} = \max_{j \in \{1,2\}} \left\| u \right\|_{j,r} \le \max\left\{ 4e^{LT}, 8e^{2LT} \right\} = 8e^{2LT}.$$
(56)

Furthermore, observe that (42), Hölder's inequality, the triangle inequality, (49), the fact that $0 \le 4\beta \le 1$, and the fact that $V \ge 1$ imply that for all $r \in [0, T]$, $s, t \in [r, T]$, $x, y \in \mathbb{R}^d$ it holds that

$$\begin{aligned} \|g(X_{s,T}^{0}(x)) - g(X_{t,T}^{0}(y))\|_{2} &\leq \frac{1}{\sqrt{T}} \left\| \left(\mathbb{V}(T, X_{s,T}^{0}(x)) + \mathbb{V}(T, X_{t,T}^{0}(y)) \right)^{\beta} \left\| X_{s,T}^{0}(x) - X_{t,T}^{0}(y) \right\| \right\|_{2} \\ &\leq \frac{1}{\sqrt{T}} \left(\mathbb{E} \left[\left\| \mathbb{V}(T, X_{s,T}^{0}(x)) + \mathbb{V}(T, X_{t,T}^{0}(y)) \right\|^{4\beta} \right] \right)^{1/4} \left\| \left\| X_{s,T}^{0}(x) - X_{t,T}^{0}(y) \right\| \right\|_{4} \\ &\leq \frac{1}{\sqrt{T}} \left(\mathbb{V}(s, x) + \mathbb{V}(t, y) \right)^{1/4} \left[c |t - s|^{1/2} + \|x - y\| \right]. \end{aligned}$$
(57)

This and (47) assure that for all $r \in [0, T]$ it holds that

$$\left\| \left[[0,T] \times \mathbb{R}^d \times \Omega \ni (s,x,\omega) \mapsto g(X^0_{s,T}(x,\omega)) \in \mathbb{R} \right] \right\|_{2,r} \le 1.$$
(58)

Next note that (42) and (47) show that for all $r \in [0, T]$, $n \in \mathbb{N}_0$, $M \in \mathbb{N}$ it holds that

$$\left\| \left\| F(U_{n,M}^{0}) - F(u) \right\| \right\|_{1,r} \le L \left\| \left\| U_{n,M}^{0} - u \right\| \right\|_{1,r}.$$
(59)

In addition, observe that (43), the triangle inequality, (47), (52), the fact that $V \ge 1$, and the fact that $0 \le 3\beta \le 1/4$ imply that for all $r \in [0, T]$, $s, t \in [r, T]$, $x, y \in \mathbb{R}^d$, $n \in \mathbb{N}_0$, $M \in \mathbb{N}$ it holds that

$$\left\| [F(U_{n,M}^0) - F(u)](s,x) - [F(U_{n,M}^0) - F(u)](t,y) \right\|_2$$

$$= \left\| \left[f(s, x, U_{n,M}^{0}(s, x)) - f(s, x, u(s, x)) \right] - \left[f(t, y, U_{n,M}^{0}(t, y)) - f(t, y, u(t, y)) \right] \right\|_{2} \\ \leq L \left\| (U_{n,M}^{0}(s, x) - u(s, x)) - (U_{n,M}^{0}(t, y) - u(t, y)) \right\|_{2} \\ + \frac{1}{T\sqrt{T}} (\mathbb{V}(s, x) + \mathbb{V}(t, y))^{\beta} \left[c|t - s|^{1/2} + ||x - y|| \right] \| U_{n,M}^{0}(s, x) - u(s, x) \|_{2} \\ + \frac{1}{T} (\mathbb{V}(s, x) + \mathbb{V}(t, y))^{\beta} \left[\| U_{n,M}^{0}(s, x) - u(s, x) \|_{2} + \| U_{n,M}^{0}(t, y) - u(t, y) \|_{2} \right] |u(s, x) - u(t, y)| \\ \leq L \left\| \| U_{n,M}^{0} - u \| \right\|_{2,r} \frac{1}{\sqrt{T}} \left[c|t - s|^{1/2} + ||x - y|| \right] (\mathbb{V}(s, x) + \mathbb{V}(t, y))^{1/4} \\ + \frac{1}{T\sqrt{T}} (\mathbb{V}(s, x) + \mathbb{V}(t, y))^{\beta} \left[c|t - s|^{1/2} + ||x - y|| \right] \left\| U_{n,M}^{0} - u \| \right\|_{1,r} (\mathbb{V}(s, x))^{\beta} \\ + \frac{1}{T} (\mathbb{V}(s, x) + \mathbb{V}(t, y))^{\beta} \left\| \| U_{n,M}^{0} - u \| \right\|_{1,r} \left[(\mathbb{V}(s, x))^{\beta} + (\mathbb{V}(t, y))^{\beta} \right] \\ \cdot \frac{8e^{2LT}}{\sqrt{T}} \left[c|t - s|^{1/2} + ||x - y|| \right] (\mathbb{V}(s, x) + \mathbb{V}(t, y))^{\beta} \\ \leq \left[\frac{1}{\sqrt{T}} L \left\| \| U_{n,M}^{0} - u \| \right\|_{2,r} + \frac{1}{T\sqrt{T}} (1 + 16e^{2LT}) \left\| \| U_{n,M}^{0} - u \| \right\|_{1,r} \right] \\ \cdot \left[c|t - s|^{1/2} + ||x - y|| \right] (\mathbb{V}(s, x) + \mathbb{V}(t, y))^{1/4}. \tag{60}$$

Combining this and (47) shows for all $r \in [0, T]$, $n \in \mathbb{N}_0$, $M \in \mathbb{N}$ that

$$\left\| \left\| F(U_{n,M}^{0}) - F(u) \right\| \right\|_{2,r} \le L \left\| \left\| U_{n,M}^{0} - u \right\| \right\|_{2,r} + \frac{1}{T} (1 + 16e^{2LT}) \left\| \left\| U_{n,M}^{0} - u \right\| \right\|_{1,r}.$$
 (61)

Next observe that the fact that $\forall t \in [0,T], \lambda \in [0,1]: \Lambda(\lambda,t) = t + \lambda(T-t)$, the fact that $\forall a, b \in [0,\infty): |a^{1/2} - b^{1/2}| \leq |a - b|^{1/2}$, and the fact that $\forall \lambda \in [0,1]: |1 - \lambda|^{1/2} + |\lambda|^{1/2} \leq \sqrt{2}$ demonstrate that for all $\lambda \in [0,1], r \in [0,T], s, t \in [r,T]$ it holds that

$$\begin{aligned} |\Lambda(\lambda, s) - \Lambda(\lambda, t)|^{1/2} + \left| |\Lambda(\lambda, s) - s|^{1/2} - |\Lambda(\lambda, t) - t|^{1/2} \right| \\ &= |(1 - \lambda)(s - t)|^{1/2} + \left| |\lambda(T - t)|^{1/2} - |\lambda(T - s)|^{1/2} \right| \\ &\leq |(1 - \lambda)(s - t)|^{1/2} + |\lambda(s - t)|^{1/2} \leq |2(s - t)|^{1/2} \,. \end{aligned}$$
(62)

This, (47), Hölder's inequality, the triangle inequality, and (49) show that for all $\lambda \in [0, 1]$, $r \in [0, T]$, $s, t \in [r, T]$, $x, y \in \mathbb{R}^d$, $n \in \mathbb{N}_0$, $M \in \mathbb{N}$, $H \in \{\lambda_1 F(U^0_{n,M}) + \lambda_2 F(u) + \lambda_3 F(0) \colon \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\}$ it holds that

$$\begin{aligned}
\sqrt{T} \left\| \left\| H\left(\Lambda(\lambda,s),a\right) - H\left(\Lambda(\lambda,t),b\right) \right\|_{2} \right|_{(a,b)=(X_{s,\Lambda(\lambda,s)}^{0}(x),X_{t,\Lambda(\lambda,t)}^{0}(y))} \right\|_{2} \\
\leq \left\| H \right\|_{2,\Lambda(\lambda,r)} \left(c \left| \Lambda(\lambda,s) - \Lambda(\lambda,t) \right|^{1/2} + \left\| \left\| X_{s,\Lambda(\lambda,s)}^{0}(x) - X_{t,\Lambda(\lambda,t)}^{0}(y) \right\| \right\|_{4} \right) \\
\cdot \left\| \left(\mathbb{V}(\Lambda(\lambda,s),X_{s,\Lambda(\lambda,s)}^{0}(x)) + \mathbb{V}(\Lambda(\lambda,t),X_{t,\Lambda(\lambda,t)}^{0}(y)) \right)^{1/4} \right\|_{4} \\
\leq \left\| H \right\|_{2,\Lambda(\lambda,r)} \left[c \left| \Lambda(\lambda,s) - \Lambda(\lambda,t) \right|^{1/2} + c \left| \left| \Lambda(\lambda,s) - s \right|^{1/2} - \left| \Lambda(\lambda,t) - t \right|^{1/2} \right| + \left\| x - y \right\| \right] \\
\cdot \left(\mathbb{E} \left[\mathbb{V}(\Lambda(\lambda,s),X_{s,\Lambda(\lambda,s)}^{0}(x)) + \mathbb{V}(\Lambda(\lambda,t),X_{t,\Lambda(\lambda,t)}^{0}(y)) \right] \right)^{1/4} \\
\leq \left\| H \right\|_{2,\Lambda(\lambda,r)} \left(c \left| 2(s-t) \right|^{1/2} + \left\| x - y \right\| \right) \left(\mathbb{V}(s,x) + \mathbb{V}(t,y) \right)^{1/4}.
\end{aligned}$$
(63)

This, e.g., the disintegration-type result in [73, Lemma 2.2], and the independence property in item (**J**) imply that for all $r \in [0,T]$, $s,t \in [r,T]$, $x,y \in \mathbb{R}^d$, $n \in \mathbb{N}_0$, $M \in \mathbb{N}$, $H \in \{\lambda_1 F(U^0_{n,M}) + \lambda_2 F(u) + \lambda_3 F(0) : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\}$ it holds that

$$\begin{aligned} &\sqrt{T} \|H\left(\Lambda(\mathfrak{r}^{0},s), X^{0}_{s,\Lambda(\mathfrak{r}^{0},s)}(x)\right) - H\left(\Lambda(\mathfrak{r}^{0},t), X^{0}_{t,\Lambda(\mathfrak{r}^{0},t)}(y)\right)\|_{2} \\ &= \|\sqrt{T}\|\|H\left(\Lambda(\lambda,s),a\right) - H\left(\Lambda(\lambda,t),b\right)\|_{2}\Big|_{(a,b)=(X^{0}_{s,\Lambda(\lambda,s)}(x),X^{0}_{t,\Lambda(\lambda,t)}(y))}\|_{2}\Big|_{\lambda=\mathfrak{r}^{0}}\|_{2} \\ &\leq \|\|\|H\|_{2,\Lambda(\mathfrak{r}^{0},r)}\|_{2} \left(c |2(s-t)|^{1/2} + \|x-y\|\right) \left(\mathbb{V}(s,x) + \mathbb{V}(t,y)\right)^{1/4}.
\end{aligned}$$
(64)

Next note that, e.g., the disintegration-type result in [73, Lemma 2.2], the independence property in item (J), and (47) imply that for all $r \in [0, T]$, $t \in [r, T]$, $y \in \mathbb{R}^d$, $n \in \mathbb{N}_0$, $M \in \mathbb{N}$, $H \in \{\lambda_1 F(U_{n,M}^0) + \lambda_2 F(u) + \lambda_3 F(0) \colon \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\}$ it holds that

$$\begin{aligned} \left\| H\left(\Lambda(\mathfrak{r}^{0},t),X_{t,\Lambda(\mathfrak{r}^{0},t)}^{0}(y)\right)\right\|_{2} &= \left\| \left\| \left\| H\left(\Lambda(\lambda,t),b\right) \right\|_{2} \right|_{b=X_{t,\Lambda(\lambda,t)}^{0}(y)} \right\|_{2} \right|_{\lambda=\mathfrak{r}^{0}} \right\|_{2} \\ &\leq \left\| \left\| \left[\left\| H \right\|_{1,\Lambda(\lambda,r)} (\mathbb{V}(\Lambda(\lambda,t),b))^{\beta} \right] \right\|_{b=X_{t,\Lambda(\lambda,t)}^{0}(y)} \right\|_{2} \right|_{\lambda=\mathfrak{r}^{0}} \right\|_{2} \\ &= \left\| \left\| \left\| \left\| H \right\|_{1,\Lambda(\lambda,r)} (\mathbb{V}(\Lambda(\lambda,t),X_{t,\Lambda(\lambda,t)}^{0}(y)))^{\beta} \right\|_{2} \right|_{\lambda=\mathfrak{r}^{0}} \right\|_{2} \leq \left\| \left\| H \right\|_{1,\Lambda(\mathfrak{r}^{0},r)} \right\|_{2} (\mathbb{V}(t,y))^{\beta}. \end{aligned}$$
(65)

This, the triangle inequality, (64), the fact that $c \ge 1$ the fact that $V \ge 1$, and the fact that $0 \le \beta \le 1/4$ imply that for all $\lambda \in [0, 1]$, $r \in [0, T]$, $s, t \in [r, T]$, $x, y \in \mathbb{R}^d$, $n \in \mathbb{N}_0$, $M \in \mathbb{N}$, $H \in \{\lambda_1 F(U_{n,M}^0) + \lambda_2 F(u) + \lambda_3 F(0) : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\}$ it holds that

$$\begin{aligned}
\sqrt{T} \| (T-s)H(\Lambda(\mathfrak{r}^{0},s), X_{s,\Lambda(\mathfrak{r}^{0},s)}^{0}(x)) - (T-t)H(\Lambda(\mathfrak{r}^{0},t), X_{t,\Lambda(\mathfrak{r}^{0},t)}^{0}(y)) \|_{2} \\
&\leq \sqrt{T}(T-s) \| H(\Lambda(\mathfrak{r}^{0},s), X_{s,\Lambda(\mathfrak{r}^{0},s)}^{0}(x)) - H(\Lambda(\mathfrak{r}^{0},t), X_{t,\Lambda(\mathfrak{r}^{0},t)}^{0}(y)) \|_{2} \\
&+ \sqrt{T} |s-t| \| H(\Lambda(\mathfrak{r}^{0},t), X_{t,\Lambda(\mathfrak{r}^{0},t)}^{0}(y)) \|_{2} \\
&\leq (T-s) \| \| H \|_{2,\Lambda(\mathfrak{r}^{0},r)} \|_{2} \left(c |2(s-t)|^{1/2} + \|x-y\| \right) (\mathbb{V}(s,x) + \mathbb{V}(t,y))^{1/4} \\
&+ \sqrt{T} |s-t| \| \| H \|_{1,\Lambda(\mathfrak{r}^{0},r)} \|_{2} (\mathbb{V}(t,y))^{\beta} \\
&\leq |T(T-r)|^{1/2} \left[\| \| H \|_{1,\Lambda(\mathfrak{r}^{0},r)} \|_{2} + \sqrt{2} \| \| H \|_{2,\Lambda(\mathfrak{r}^{0},r)} \|_{2} \right] \\
&\cdot \left[c |s-t|^{1/2} + \|x-y\| \right] (\mathbb{V}(s,x) + \mathbb{V}(t,y))^{1/4}.
\end{aligned}$$
(66)

Next observe that (47), (42), the fact that $\forall x \in \mathbb{R}^d, t \in [0,T] : |Tf(t,x,0)| \leq (V(x))^{\beta}$, the fact that $V \geq 1$, and the fact that $0 \leq \beta \leq 1/4$ imply that for all $r \in [0,T]$ it holds that $\max_{j \in \{1,2\}} ||TF(0)||_{j,r} \leq 1$. This, (47), (66), and the fact that $\mathbb{P}(0 \leq \mathfrak{r}^0 \leq 1) = 1$ show that for all $r \in [0,T]$ it holds that

$$\begin{aligned} \left\| \left\| [0,T] \times \mathbb{R}^{d} \times \Omega \ni (s,x,\omega) \mapsto \left[(T-s)(F(0)) \left(s + \mathfrak{r}^{0}(T-s), X^{0}_{s,s+\mathfrak{r}^{0}(T-s)}(x) \right) \right] (\omega) \in \mathbb{R} \right\|_{2,r} \\ &\leq |(T-r)T|^{1/2} \max_{\zeta \in [r,T]} \left[\left\| F(0) \right\|_{1,\zeta} + \sqrt{2} \left\| F(0) \right\|_{2,\zeta} \right] \leq 3. \end{aligned}$$
(67)

Moreover, note that the integral transformation theorem and the fact that \mathfrak{r}^0 is continuous uniformly distributed on [0, 1] imply that for all $r \in [0, T]$ and all measurable $h: [0, T] \to \mathbb{R}$ it holds that

$$|T - r|^{1/2} ||h(\Lambda(\mathfrak{r}^{0}, r))||_{2} = \left[\int_{0}^{1} (T - r) |h(r + (T - r)\lambda)|^{2} d\lambda \right]^{1/2} = \left[\int_{r}^{T} |h(\zeta)|^{2} d\zeta \right]^{1/2}.$$
 (68)

This, (66), (47), (59), (61), and the fact that $TL + \sqrt{2}(16e^{2LT} + 1) + \sqrt{2}TL \leq 16\sqrt{2}e^{2LT} + \sqrt{2}(1 + 2TL) \leq 17\sqrt{2}e^{2LT} \leq 24.5e^{2LT}$ show for all $r \in [0, T]$, $n \in \mathbb{N}_0$, $M \in \mathbb{N}$ that

$$\begin{split} & \left\| \begin{bmatrix} 0,T] \times \mathbb{R}^{d} \times \Omega \ni (s,x,\omega) \\ & \mapsto (T-s) \left[(F(U_{n,M}^{0}) - F(u)) \left(s + \mathfrak{r}^{0}(T-s), X_{s,s+\mathfrak{r}^{0}(T-s)}^{0}(x) \right) \right] (\omega) \in \mathbb{R} \right\|_{2,r} \\ & \leq |T(T-r)|^{1/2} \left[\left\| \left\| F(U_{n,M}^{0}) - F(u) \right\|_{1,\Lambda(\mathfrak{r}^{0},r)} \right\|_{2} + \sqrt{2} \| \left\| F(U_{n,M}^{0}) - F(u) \right\|_{2,\Lambda(\mathfrak{r}^{0},r)} \|_{2} \right] \\ & \leq |T(T-r)|^{1/2} \left[\left(L + \sqrt{2} \frac{16e^{2LT} + 1}{T} \right) \| \left\| U_{n,M}^{0} - u \right\|_{1,\Lambda(\mathfrak{r}^{0},r)} \|_{2} + \sqrt{2} L \| \left\| U_{n,M}^{0} - u \right\|_{2,\Lambda(\mathfrak{r}^{0},r)} \|_{2} \right] \end{split}$$

$$\leq \frac{1}{\sqrt{T}} T \left[\left(L + \sqrt{2} \frac{16e^{2LT} + 1}{T} \right) + \sqrt{2}L \right] \max_{j \in \{1,2\}} \left[|T - r|^{1/2} \| \left\| U_{n,M}^0 - u \right\| \right]_{j,\Lambda(\mathfrak{r}^0,r)} \|_2 \right]$$

$$\leq \frac{24.5}{\sqrt{T}} e^{2LT} \left[\int_r^T \left\| \left\| U_{n,M}^0 - u \right\| \right\|_{0,\zeta}^2 d\zeta \right]^{1/2}.$$
(69)

This, (41), (48), (47), items (A)–(J), Bienaymé's identity, (58), and (67) imply for all $n, M \in \mathbb{N}$, $r \in [0, T]$ that

$$\begin{split} \left\| \left\| U_{n,M}^{0} - \mathbb{E}[U_{n,M}^{0}] \right\| \right\|_{2,r} &\leq \left\| \left\| [0,T] \times \mathbb{R}^{d} \times \Omega \ni (t,x,\omega) \mapsto \frac{1}{M^{n}} \sum_{i=1}^{M^{n}} \left[g(X_{t,T}^{(0,0,-i)}(x)) \right](\omega) \in \mathbb{R} \right\| \right\|_{2,r} \\ &+ \sum_{\ell=0}^{n-1} \left\| \left\| \left\| [0,T] \times \mathbb{R}^{d} \times \Omega \ni (t,x,\omega) \mapsto \frac{T-t}{M^{n-\ell}} \sum_{i=1}^{M^{n-\ell}} \left[\left(F(U_{\ell,M}^{(0,\ell,i)}) - \mathbb{1}_{\mathbb{N}}(\ell) F(U_{\ell-1,M}^{(0,-\ell,i)}) \right) \right\| \right\|_{2,r} \right] \\ &= \frac{1}{\sqrt{M^{n}}} \left\| \left\| [0,T] \times \mathbb{R}^{d} \times \Omega \ni (t,x,\omega) \mapsto \left[g(X_{t,T}^{0}(x)) \right](\omega) \in \mathbb{R} \right\| \\ &+ \sum_{\ell=0}^{n-1} \left[\frac{1}{\sqrt{M^{n-\ell}}} \right\| \left\| [0,T] \times \mathbb{R}^{d} \times \Omega \ni (t,x,\omega) \mapsto \left[g(X_{t,T}^{0}(x)) \right](\omega) \in \mathbb{R} \right\| \\ &+ \sum_{\ell=0}^{n-1} \left[\frac{1}{\sqrt{M^{n-\ell}}} \right\| \left\| [0,T] \times \mathbb{R}^{d} \times \Omega \ni (t,x,\omega) \mapsto \left[g(X_{t,T}^{0,\ell,1}) (x) \right](\omega) \in \mathbb{R} \right\| \\ &= \frac{1}{\sqrt{M^{n}}} \left\| \left\| [0,T] \times \mathbb{R}^{d} \times \Omega \ni (t,x,\omega) \mapsto \left[g(X_{t,T}^{0,\ell,1}) (x) \right](\omega) \in \mathbb{R} \right\| \\ &= \frac{1}{\sqrt{M^{n}}} \left\| \left\| [0,T] \times \mathbb{R}^{d} \times \Omega \ni (t,x,\omega) \mapsto \left[g(X_{t,T}^{0,\ell,1}) (x) \right](\omega) \in \mathbb{R} \right\| \\ &= \frac{1}{\sqrt{M^{n}}} \left\| \left\| [0,T] \times \mathbb{R}^{d} \times \Omega \ni (t,x,\omega) \mapsto \left[g(X_{t,T}^{0,\ell,1}) (x) \right](\omega) \in \mathbb{R} \right\| \\ &+ \frac{1}{\sqrt{M^{n}}} \left\| \left\| [0,T] \times \mathbb{R}^{d} \times \Omega \ni (t,x,\omega) \mapsto (T-t) \left[\left(F(U_{\ell,M}^{0}) - F(u) \right) \right] \\ &+ \sum_{\ell=1}^{n-1} \left[\frac{1}{\sqrt{M^{n-\ell}}} \right\| \left\| [0,T] \times \mathbb{R}^{d} \times \Omega \ni (t,x,\omega) \mapsto (T-t) \left[\left(F(U_{\ell,M}^{0}) - F(u) \right) \right\| \\ &+ \sum_{\ell=1}^{n-1} \left[\frac{1}{\sqrt{M^{n-\ell}}} \right\| \left\| [0,T] \times \mathbb{R}^{d} \times \Omega \ni (t,x,\omega) \mapsto (T-t) \left[\left(F(U_{\ell,M}^{0}) - F(u) \right) \right\| \\ &= 2,r \right] \\ &+ \sum_{\ell=1}^{n-1} \left[\frac{1}{\sqrt{M^{n-\ell}}} \right\| \left\| [0,T] \times \mathbb{R}^{d} \times \Omega \ni (t,x,\omega) \mapsto (T-t) \left[\left(F(U_{\ell,M}^{0}) - F(u) \right) \right\| \\ &= 2,r \right] \\ &+ \sum_{\ell=1}^{n-1} \left[\frac{1}{\sqrt{M^{n-\ell}}} \left\| \left\| [0,T] \times \mathbb{R}^{d} \times \Omega \ni (t,x,\omega) \mapsto (T-t) \left[\left(F(U_{\ell-1,M}^{0}) - F(u) \right) \right\| \\ &= 2,r \right] \\ &\leq \frac{4}{\sqrt{M^{n}}} + \sum_{\ell=0}^{n-1} \left[\frac{(2-1_{\{n-1\}(\ell)\} 24.5T^{-1/2}e^{2LT}}{\sqrt{M^{n-\ell-1}}} \left[\int_{r}^{T} \left\| \left\| U_{\ell,M}^{0} - u \right\| \right\|_{0,\zeta}^{2} d\zeta \right]^{1/2} \right]. \end{split}$$

Next observe that (55), (47), Jensen's inequality, and (69) assure that for all $n, M \in \mathbb{N}$, $r \in [0, T]$ it holds that

$$\begin{split} \left\| \left\| \mathbb{E}[U_{n,M}^{0}] - u \right\| \right\|_{2,r} &= \left\| \left\| [0,T] \times \mathbb{R}^{d} \times \Omega \ni (t,x,\omega) \mapsto \\ \mathbb{E}\left[(T-t)(F(U_{n-1.M}^{0}) - F(u))(t + \mathfrak{r}^{0}(T-t), X_{t,t+\mathfrak{r}^{0}(T-t)}^{0}(x)) \right] \in \mathbb{R} \right\| \\ &\leq \left\| \left\| [0,T] \times \mathbb{R}^{d} \times \Omega \ni (t,x,\omega) \mapsto \\ (T-t)\left[(F(U_{n-1.M}^{0} - F(u))(t + \mathfrak{r}^{0}(T-t), X_{t,t+\mathfrak{r}^{0}(T-t)}^{0}(x)) \right] (\omega) \in \mathbb{R} \right\| \\ &\leq 24.5T^{-1/2}e^{2LT} \left[\int_{r}^{T} \left\| \left\| U_{\ell,M}^{0} - u \right\| \right\|_{0,\zeta}^{2} d\zeta \right]^{1/2}. \end{split}$$

$$(71)$$

This, (70), and the triangle inequality show for all $n, M \in \mathbb{N}, r \in [0, T]$ that

$$\left\| \left\| U_{n,M}^{0} - u \right\| \right\|_{2,r} \le \frac{4}{\sqrt{M^{n}}} + \sum_{\ell=0}^{n-1} \left[\frac{49T^{-1/2}e^{2LT}}{\sqrt{M^{n-\ell-1}}} \left[\int_{r}^{T} \left\| \left\| U_{\ell,M}^{0} - u \right\| \right\|_{0,\zeta}^{2} d\zeta \right]^{1/2} \right].$$
(72)

Moreover, note that [72, Lemma 3.5] (applied for every $s \in [0, T]$ with $\rho \curvearrowleft 2\beta\rho$, $\varphi \curvearrowleft V^{2\beta}$, $Y \curvearrowleft X$, $t \backsim s$ in the notation of [72, Lemma 3.5]), (42), the fact that $\forall M \in \mathbb{N} \colon U_{0,M}^0 = 0$, (41), (50), (51), and (49) prove that for all $s \in [0, T]$, $M, n \in \mathbb{N}$ it holds that

$$\sup_{x \in \mathbb{R}^{d}} \left[\frac{e^{\beta \rho s} \|U_{n,M}^{0}(s,x) - u(s,x)\|_{2}}{(V(x))^{\beta}} \right] \\
\leq \frac{2e^{\beta \rho T}}{\sqrt{M^{n}}} + \sum_{\ell=0}^{n-1} \left[\frac{2L(T-s)^{1/2}}{\sqrt{M^{n-\ell-1}}} \left(\int_{s}^{T} \sup_{\eta \in [\zeta,T]} \sup_{x \in \mathbb{R}^{d}} \left[\frac{e^{2\beta \rho \eta} \|U_{\ell,M}^{0}(\eta,x) - u(\eta,x)\|_{2}^{2}}{(V(x))^{2\beta}} \right] d\zeta \right)^{1/2} \right].$$
(73)

Combining this, the fact that $\forall t \in [0, T], x \in \mathbb{R}^d$: $\mathbb{V}(t, x) = e^{\rho(T-t)}V(x)$, and (47) ensures that for all $r \in [0, T], M, n \in \mathbb{N}$ it holds that

$$\begin{split} \left\| \left\| U_{n,M}^{0} - u \right\| \right\|_{1,r} &= \sup_{s \in [r,T]} \sup_{x \in \mathbb{R}^{d}} \frac{\left\| U_{n,M}^{0}(s,x) - u(s,x) \right\|_{2}}{(\mathbb{V}(s,x))^{\beta}} = \sup_{s \in [r,T]} \sup_{x \in \mathbb{R}^{d}} \frac{\left\| U_{n,M}^{0}(s,x) - u(s,x) \right\|_{2}}{e^{\beta \rho (T-s)} (V(x))^{\beta}} \\ &\leq \frac{2}{\sqrt{M^{n}}} + \sup_{s \in [r,T]} \sum_{\ell=0}^{n-1} \left[\frac{T^{-1/2} e^{2LT}}{\sqrt{M^{n-\ell-1}}} \left(\int_{s}^{T} \sup_{\eta \in [\zeta,T]} \sup_{x \in \mathbb{R}^{d}} \left[\frac{\left\| U_{\ell,M}^{0}(\eta,x) - u(\eta,x) \right\|_{2}}{e^{\beta \rho (T-\eta)} (V(x))^{\beta}} \right]^{2} d\zeta \right)^{1/2} \right] \\ &= \frac{2}{\sqrt{M^{n}}} + \sum_{\ell=0}^{n-1} \left[\frac{T^{-1/2} e^{2LT}}{\sqrt{M^{n-\ell-1}}} \left(\int_{r}^{T} \sup_{\eta \in [\zeta,T]} \sup_{x \in \mathbb{R}^{d}} \left[\frac{\left\| U_{\ell,M}^{0}(\eta,x) - u(\eta,x) \right\|_{2}}{(\mathbb{V}(\eta,x))^{\beta}} \right]^{2} d\zeta \right)^{1/2} \right] \\ &= \frac{2}{\sqrt{M^{n}}} + \sum_{\ell=0}^{n-1} \left[\frac{T^{-1/2} e^{2LT}}{\sqrt{M^{n-\ell-1}}} \left(\int_{r}^{T} \left\| \left\| U_{\ell,M}^{0} - u \right\| \right\|_{1,\zeta}^{2} d\zeta \right)^{1/2} \right]. \end{split}$$
(74)

This, (47), and (72) demonstrate that for all $r \in [0, T]$, $M, n \in \mathbb{N}$ it holds that

$$\left\| \left\| U_{n,M}^{0} - u \right\| \right\|_{r} \leq \frac{4}{\sqrt{M^{n}}} + \sum_{\ell=0}^{n-1} \left[\frac{49T^{-1/2}e^{2LT}}{\sqrt{M^{n-\ell-1}}} \left[\int_{r}^{T} \left\| \left\| U_{\ell,M}^{0} - u \right\| \right\|_{0,\zeta}^{2} d\zeta \right]^{1/2} \right].$$
(75)

Combining [72, Lemma 3.10] (applied for every $M, N \in \mathbb{N}, r \in [0, T]$ with $a \curvearrowleft 4, b \backsim 49T^{-1/2}e^{2LT}, c \backsim 1/\sqrt{M}, \alpha \backsim 0, \beta \curvearrowleft T, (f_n)_{n \in [0,N] \cap \mathbb{N}_0} \curvearrowleft ([0,T] \ni s \mapsto |||U_{n,M}^0 - u|||_s \in [0,\infty])_{n \in [0,N] \cap \mathbb{N}_0}$ in the notation of [72, Lemma 3.10]), the fact that $\forall k \in \mathbb{N}_0 \colon M^k/k! \leq e^M$, and (56) hence assures that for all $r \in [0,T], M, N \in \mathbb{N}$ it holds that

$$\left\| \left\| U_{N,M}^{o} - u \right\| \right\|_{0,r} \le \left[4 + 49T^{-1/2} e^{2LT} T^{1/2} \sup_{s \in [r,T]} \left\| \left\| U_{0,M}^{0} - u \right\| \right\|_{0,s} \right] \left[\sup_{k \in [0,N] \cap \mathbb{Z}} \frac{M^{-(N-k)/2}}{\sqrt{k!}} \right] \left(1 + 49T^{-1/2} e^{2LT} T^{1/2} \right)^{N-1} \le \left(4 + 49e^{2LT} 8e^{2LT} \right) e^{M/2} M^{-N/2} \left(1 + 49e^{2LT} \right)^{N-1} \le e^{M/2} M^{-N/2} \left(50e^{2LT} \right)^{N+1}.$$

$$(76)$$

The fact that $\forall M \in \mathbb{N} : U_{0,M}^0 = 0$ and (56) therefore show that for all $r \in [0,T]$, $M \in \mathbb{N}$, $N \in \mathbb{N}_0$ it holds that

$$\left\| \left\| U_{N,M}^{s,0} - u \right\| \right\|_{r} \le e^{M/2} M^{-N/2} \left(50 e^{2LT} \right)^{N+1}.$$
(77)

This establishes item (iii) and item (iv). The proof of Proposition 4.2 is thus complete. \Box

5 Computational complexity analysis for MLP approximations for backward stochastic differential equations (BSDEs)

In this section we combine the findings from Sections 2 and 4 to supply in Theorem 5.1 and Corollary 5.3 computational complexity analyses for the Monte Carlo-type approximation algorithm for BSDEs in (1)-(2) in Theorem 1.1 in Section 1 above. Corollary 5.3 specializes

Theorem 5.1 to the specific situation where the driver of the BSDEs is twice continuously differentiable with bounded derivatives and does neither depend on the time variable $t \in [0, T]$ nor on the space variable $x \in \mathbb{R}^d$ but only on the solution processes $Y^d : [0, T] \times \Omega \to \mathbb{R}, d \in \mathbb{N}$, of the BSDEs under consideration.

Our proof of Corollary 5.3 uses beside Theorem 5.1 also the elementary Lipschitz-type estimate for twice continuously differentiable functions in Lemma 5.2 below. For completeness we also include in this section a detailed proof for Lemma 5.2. Our proof of Theorem 5.1, in turn, employs Lemma 2.3 from Section 2 and Proposition 4.2 from Section 4. Theorem 1.1 in the introduction is a direct consequence of Corollary 5.3.

Theorem 5.1. Assume Setting 4.1, let $\alpha \in \mathbb{N}$, $(\theta_n)_{n \in \mathbb{N}_0} \subseteq \Theta$, let $\lfloor \cdot \rfloor_M \colon \mathbb{R} \to \mathbb{R}$, $M \in \mathbb{N}$, and $\lceil \cdot \rceil_M \colon \mathbb{R} \to \mathbb{R}$, $M \in \mathbb{N}$, satisfy for all $M \in \mathbb{N}$, $t \in [0,T]$ that $\lfloor t \rfloor_M = \max(([0,t] \setminus \{T\}) \cap \{0, \frac{T}{M}, \frac{2T}{M}, \ldots\})$ and $\lceil t \rceil_M = \min(((t, \infty) \cup \{T\}) \cap \{0, \frac{T}{M}, \frac{2T}{M}, \ldots\})$, let $\mathscr{Y}^{n,M} \colon [0,T] \times \Omega \to \mathbb{R}$, $n, M \in \mathbb{N}$, satisfy for all $n, M \in \mathbb{N}$, $t \in [0,T]$ that

$$\mathscr{Y}_{t}^{n,M} = \sum_{\ell=0}^{n-1} \left[\left[\frac{[t]_{M^{l+1}-t}}{(T/M^{l+1})} \right] U_{n-\ell,M}^{\theta_{\ell}} (\lfloor t \rfloor_{M^{l+1}}, W_{\lfloor t \rfloor_{M^{l+1}}}) + \left[\frac{t-\lfloor t \rfloor_{M^{l+1}}}{(T/M^{l+1})} \right] U_{n-\ell,M}^{\theta_{\ell}} (\lceil t \rceil_{M^{l+1}}, W_{\lceil t \rceil_{M^{l+1}}}) - \mathbb{1}_{\mathbb{N}} (\ell) \left(\left[\frac{[t]_{M^{l}-t}}{(T/M^{l})} \right] U_{n-\ell,M}^{\theta_{\ell}} (\lfloor t \rfloor_{M^{l}}, W_{\lfloor t \rfloor_{M^{l}}}) + \left[\frac{t-[t]_{M^{l}}}{(T/M^{l})} \right] U_{n-\ell,M}^{\theta_{\ell}} (\lceil t \rceil_{M^{l}}, W_{\lceil t \rceil_{M^{l}}}) \right) \right], \quad (78)$$

and let $\mathcal{C}_{n,M} \in \mathbb{N}_0$, $n, M \in \mathbb{Z}$, and $\mathfrak{C}_{n,M} \in \mathbb{N}_0$, $n, M \in \mathbb{Z}$, satisfy for all $n, M \in \mathbb{N}_0$ that

$$\mathcal{C}_{n,M} \le \alpha M^n \mathbb{1}_{\mathbb{N}}(n) + \sum_{\ell=0}^{n-1} \left[M^{n-\ell} \left(1 + \alpha + \mathcal{C}_{\ell,M} + \mathcal{C}_{\ell-1,M} \mathbb{1}_{\mathbb{N}}(\ell) \right) \right]$$
(79)

and $\mathfrak{C}_{n,M} \leq \alpha(M^n + 1) + \sum_{\ell=0}^{n-1} \left[(M^{\ell+1} + 1) \mathcal{C}_{n-\ell,M} \right]$. Then

(i) there exists an $(\mathbb{F}_t)_{t\in[0,T]}$ -predictable stochastic process $\mathbf{Y} = (Y, Z) = (Y, Z^1, Z^2, \dots, Z^d)$: $[0,T] \times \Omega \to \mathbb{R} \times \mathbb{R}^d$ with $\int_0^T \mathbb{E}[|Y_t| + ||Z_t||^2] dt < \infty$ which satisfies that for all $t \in [0,T]$ it holds \mathbb{P} -a.s. that

$$Y_t = g(W_T) + \int_t^T f(s, W_s, Y_s) \, ds - \sum_{j=1}^d \int_t^T Z_s^j \, dW_s^j, \tag{80}$$

- (ii) it holds for all $M, n \in \mathbb{N}$, $t \in [0, T]$ that $\mathscr{Y}_t^{n, M}$ is measurable,
- (iii) it holds for all $M, n \in \mathbb{N}, t \in [0, T]$ that

$$\left(\mathbb{E}\left[|\mathscr{Y}_{t}^{n,M}-Y_{t}|^{2}\right]\right)^{1/2} \leq 8ne^{M/2+4nLT+\rho T/2}M^{-n/2}50^{2n}\left|V(0)\max\left\{\mathbb{E}\left[\|\mathbf{z}^{0}\|^{4}\right],1\right\}\right|^{1/4},\quad(81)$$

and

(iv) there exists $\mathbf{n}: (0, \infty) \to \mathbb{N}$ such that for all $\varepsilon, \delta \in (0, 1]$ it holds that $\sup_{t \in [0,T]} (\mathbb{E}[|Y_t - \mathcal{Y}_t^{\mathbf{n}(\varepsilon),\mathbf{n}(\varepsilon)}|^2])^{1/2} \le \varepsilon$ and

$$\mathfrak{C}_{\mathsf{n}(\varepsilon),\mathsf{n}(\varepsilon)} \qquad (82)$$

$$\leq \alpha \left(\sup_{n \in \mathbb{N}} \left[\frac{10^{n+3} n^3 \left[8n e^{n/2 + 4nLT} 50^{2n} \right]^{2+\delta}}{n^{\delta n/2}} \right] \right) \left[e^{2\rho T} V(0) \max\{ \mathbb{E} \left[\| \mathbf{z}^0 \|^4 \right], 1\} \right]^{\frac{2+\delta}{4}} \varepsilon^{-(2+\delta)} < \infty.$$

Proof of Theorem 5.1. Throughout this proof let $c \in [1, \infty)$ satisfy $c = (\max\{\mathbb{E}[\|\mathbf{z}^0\|^4], 1\})^{1/4}$, let $\mathbb{V}: [0, T] \times \mathbb{R}^d \to [1, \infty)$ satisfy for all $t \in [0, T]$, $x \in \mathbb{R}^d$ that $\mathbb{V}(t, x) = e^{\rho(T-t)}V(x)$, for every $q \in [1, \infty)$ and every random variable $\mathfrak{X}: \Omega \to \mathbb{R}$ let $\|\mathfrak{X}\|_q \in [0, \infty]$ satisfy that $\|\mathfrak{X}\|_q = (\mathbb{E}[|\mathfrak{X}|^q])^{1/q}$, for every $r \in [0, T]$ and for every random field $H: [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ let $\|\|H\|_{j,r} \in [0, \infty], j \in \{0, 1, 2\}$, satisfy that

$$\|\|H\|\|_{0,r} = \max_{j \in \{1,2\}} \|\|H\|\|_{j,r}, \quad \|\|H\|\|_{1,r} = \sup_{\substack{x \in \mathbb{R}^d, s \in [r,T] \\ (\mathbb{V}(s,x))^\beta}} \left[\frac{\left(\mathbb{E}[|H(s,x)|^2]\right)^{1/2}}{(\mathbb{V}(s,x))^\beta} \right], \quad \text{and}$$

$$\|\|H\|\|_{2,r} = \sup_{\substack{s,t \in [r,T], x, y \in \mathbb{R}^d \\ (s,x) \neq (t,y)}} \left[\frac{T^{1/2} \left(\mathbb{E}[|H(s,x) - H(t,y)|^2]\right)^{1/2}}{\left[c|s-t|^{1/2} + \|x-y\|\right] \left(\mathbb{V}(s,x) + \mathbb{V}(t,y)\right)^{1/4}} \right]$$

$$(83)$$

and let $R: \Omega \to \mathbb{R}$ and $\mathscr{Y}: [0,T] \times \Omega \to \mathbb{R}$ satisfy for all $s \in [0,T]$ that

$$\mathscr{Y}_s = u(s, W_s)$$
 and $R = g(W_T) + \int_0^T f(t, W_t, \mathscr{Y}_t) dt.$ (84)

Note that it is well-known that (42) and (43) imply item (i) (cf., e.g., [113, Theorem 4.3.1]). Next observe that Proposition 4.2 proves that

(a) there exists a unique measurable $u: [0, T] \times \mathbb{R}^d \to \mathbb{R}$ which satisfies for all $t \in [0, T], x \in \mathbb{R}^d$ that $\mathbb{E}\left[|g(X_{t,T}^0(x))|\right] + \int_t^T \mathbb{E}\left[|(F(u))(s, X_{t,s}^0(x))|\right] ds + \sup_{r \in [0,T], y \in \mathbb{R}^d} \left(\frac{|u(r,y)|}{|V(y)|^{\beta}}\right) < \infty$ and

$$u(t,x) = \mathbb{E}\left[g(X_{t,T}^{0}(x))\right] + \int_{t}^{T} \mathbb{E}\left[(F(u))\left(s, X_{t,s}^{0}(x)\right)\right] ds,$$
(85)

- (b) it holds that $U_{n,M}^{\theta}, \theta \in \Theta, n \in \mathbb{N}_0, M \in \mathbb{N}$, are measurable, and
- (c) it holds for all $r \in [0, T]$, $M \in \mathbb{N}$, $N \in \mathbb{N}_0$ that

$$\left\| \left\| U_{N,M}^{0} - u \right\| \right\|_{0,r} \le e^{M/2} M^{-N/2} \left(50 e^{2LT} \right)^{N+1}.$$
(86)

This and the fact that W is measurable establish item (ii). Moreover, observe that (84), the triangle inequality, and (42) prove that

$$|R| \le |g(W_T)| + \int_0^T |f(t, W_t, 0)| + L|u(t, W_t)| \, dt.$$
(87)

This, the fact that $\sup_{x \in \mathbb{R}^d, t \in [0,T]} \left(\frac{|Tf(t,x,0)| + |g(x)| + |u(t,x)|}{|V(x)|^{\beta}} \right) < \infty$, the fact that $0 \le \beta \le 1/2$, the fact that $\forall x \in \mathbb{R}^d, s \in [0,T], t \in [s,T] \colon \mathbb{E}[V(x+W_{t-s})] \le e^{\rho(t-s)}V(x)$, and Jensen's inequality imply that $\mathbb{E}[|R|^2] < \infty$. This, the fact that for all $A \in \mathcal{B}(\mathbb{R})$ it holds that $R^{-1}(A) \in \mathbb{F}_T$ and the martingale representation theorem (see, e.g., [91, Theorem 4.3.4]) imply that there exists an $(\mathbb{F}_s)_{s \in [0,T]}$ -progressively measurable stochastic process $\mathcal{Z} = (\mathcal{Z}^1, \mathcal{Z}^2, \dots, \mathcal{Z}^d) \colon [0,T] \times \Omega \to \mathbb{R}^d$ which satisfies that for all $s \in [0,T]$ it holds \mathbb{P} -a.s. that $\mathbb{E}[R|\mathbb{F}_s] = \mathbb{E}[R] + \sum_{j=1}^d \int_0^s \mathcal{Z}_j^j dW_r^j$. The fact that for all $A \in \mathcal{B}(\mathbb{R})$ it holds that $R^{-1}(A) \in \mathbb{F}_T$ hence shows that for all $s \in [0,T]$ it holds \mathbb{P} -a.s. that

$$R - \mathbb{E}[R|\mathbb{F}_s] = \mathbb{E}[R|\mathbb{F}_T] - \mathbb{E}[R|\mathbb{F}_s] = \sum_{j=1}^d \int_s^T \mathscr{Z}_r^j \, dW_r^j.$$
(88)

Next note that (85) and the fact that $\forall t \in [0,T], B \in \mathcal{B}(\mathbb{R}^d) \colon \mathbb{P}(W_t \in B) = \mathbb{P}(\mathbf{z}^0 \sqrt{t} \in B)$ show that for all $s \in [0,T], z \in \mathbb{R}^d$ it holds that

$$u(s,z) = \mathbb{E}\left[g(z+W_{T-s}) + \int_{s}^{T} f(t,z+W_{t-s},u(t,z+W_{t-s}))\,dt\right].$$
(89)

The Markov property of Brownian motions, (84), the fact that for all $s \in [0,T]$, $z \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R})$ it holds that $\{\omega \in \Omega : \int_0^s f(t, z + W_t(\omega), u(t, z + W_t(\omega))) dt \in A\} \in \mathbb{F}_s$, and (88) hence show that for all $s \in [0,T]$, $x \in \mathbb{R}^d$ it holds P-a.s. that

$$\begin{aligned} \mathscr{Y}_{s} &= u(s, W_{s}) = \mathbb{E}\left[g(z + W_{T-s}) + \int_{s}^{T} f(t, z + W_{t-s}, u(t, z + W_{t-s})) dt\right]\Big|_{z=W_{s}} \\ &= \mathbb{E}\left[g(W_{T}) + \int_{s}^{T} f(t, W_{t}, u(t, W_{t})) dt\Big|\mathbb{F}_{s}\right] \\ &= g(W_{T}) + \int_{s}^{T} f(t, W_{t}, \mathscr{Y}_{t}) dt \\ &- \left(g(W_{T}) + \int_{s}^{T} f(t, W_{t}, \mathscr{Y}_{t}) dt - \mathbb{E}\left[g(W_{T}) + \int_{s}^{T} f(t, W_{t}, \mathscr{Y}_{t}) dt\Big|\mathbb{F}_{s}\right]\right) \\ &= g(W_{T}) + \int_{s}^{T} f(t, W_{t}, \mathscr{Y}_{t}) dt - (R - \mathbb{E}[R|\mathbb{F}_{s}]) \\ &= g(W_{T}) + \int_{s}^{T} f(t, W_{t}, \mathscr{Y}_{t}) dt - \sum_{j=1}^{d} \int_{s}^{T} \mathscr{X}_{r}^{j} dW_{r}^{j}. \end{aligned}$$

$$(90)$$

Combining item (i) and the fact that $[0,T] \times \Omega \ni (t,\omega) \mapsto (\mathscr{Y}_t(\omega), \mathscr{Z}_t(\omega)) \in \mathbb{R} \times \mathbb{R}^d$ is $(\mathbb{F}_t)_{t \in [0,T]}$ -progressively measurable hence implies that for all $t \in [0,T]$ it holds \mathbb{P} -a.s. that $(Y_t, Z_t) = (\mathscr{Y}_t, \mathscr{Z}_t)$. This, (84) and the fact that Y and \mathscr{Y} have continuous sample paths (see (84) and item (i)) prove that \mathbb{P} -a.s. it holds for all $t \in [0,T]$ that

$$Y_t = u(t, W_t). (91)$$

Next note that the assumption that $(\mathbf{r}^{\theta})_{\theta\in\Theta}$, $(\mathbf{z}^{\theta})_{\theta\in\Theta}$, and W are independent, the fact that $\forall M \in \mathbb{N}, \theta \in \Theta \colon U_{0,M}^{\theta} = 0$, and (41) imply that $(U_{n,M}^{\theta})_{n\in\mathbb{N}_0,M\in\mathbb{N},\theta\in\Theta}$ and W are independent. Combining (83), the fact that $\forall t \in [0,T], x \in \mathbb{R}^d \colon \mathbb{V}(t,x) = e^{\rho(T-t)}V(x)$, Hölder's inequality, the fact that $0 \leq \beta \leq 1/4$, the fact that $V \geq 1$, the triangle inequality, the fact that for all $n, M \in \mathbb{N}$ it holds that $U_{n,M}^{\theta}, \theta \in \Theta$, are identically distributed, (86), and, e.g., the disintegration-type result in [73, Lemma 2.2] hence implies that for all $s \in [0,T], t \in [s,T], x \in \mathbb{R}^d, M \in \mathbb{N}, n \in \mathbb{N}_0, \theta \in \Theta$ it holds that

$$\begin{aligned} \left\| (U_{n,M}^{\theta} - u)(t, x + W_t) \right\|_2 &= \left\| \| (U_{n,M}^{\theta} - u)(t, y) \|_2 \Big|_{y = x + W_t} \right\|_2 \\ &\leq \left\| \left\| \left\| U_{n,M}^{\theta} - u \right\| \right\|_{1,t} (\mathbb{V}(t, X_{0,t}^0(x)))^{\beta} \right\|_2 = \left\| \left\| \left\| U_{n,M}^{\theta} - u \right\| \right\|_{1,t} e^{\beta \rho (T-t)} (V(X_{0,t}^0(x)))^{\beta} \right\|_2 \\ &\leq \left\| \left\| U_{n,M}^{0} - u \right\| \right\|_{0,t} e^{\beta \rho T} (V(x))^{\beta} \leq \left\| \left\| U_{n,M}^{0} - u \right\| \right\|_{0,t} e^{\rho T/4} (V(x))^{1/4} \\ &\leq e^{M/2} M^{-n/2} \left[50 e^{2LT} \right]^{n+1} (V(x))^{1/4} e^{\rho T/4} \end{aligned}$$
(92)

and

$$\begin{split} \left\| (U_{n,M}^{\theta} - u)(s, x + W_{s}) - (U_{n,M}^{\theta} - u)(t, x + W_{t}) \right\|_{2} \\ &= \left\| \left\| (U_{n,M}^{\theta} - u)(s, a) - (U_{n,M}^{\theta} - u)(t, b) \right\|_{2} \right|_{(a,b) = (x + W_{s}, x + W_{t})} \right\|_{2} \\ &\leq \left\| \left\| \left\| U_{n,M}^{\theta} - u \right\| \right\|_{2,s} T^{-1/2} \left[c \left| s - t \right|^{1/2} + \left\| a - b \right\| \right] \left(\mathbb{V}(s, a) + \mathbb{V}(t, b) \right)^{1/4} \right|_{(a,b) = (x + W_{s}, x + W_{t})} \right\|_{2} \\ &\leq \left\| \left\| \left\| U_{n,M}^{\theta} - u \right\| \right\|_{2,s} T^{-1/2} e^{\rho T/4} \left[c \left| s - t \right|^{1/2} + \left\| a - b \right\| \right] \left(V(a) + V(b) \right)^{1/4} \right|_{(a,b) = (x + W_{s}, x + W_{t})} \right\|_{2} \\ &\leq \left\| \left\| U_{n,M}^{\theta} - u \right\| \right\|_{0,s} T^{-1/2} e^{\rho T/4} \left[c \left| s - t \right|^{1/2} + \left\| \left\| W_{s} - W_{t} \right\| \right\|_{4} \right] \left(\mathbb{E}[V(x + W_{s}) + V(x + W_{t})] \right)^{1/4} \\ &\leq 4 e^{M/2} M^{-n/2} \left(50 e^{2LT} \right)^{n+1} T^{-1/2} e^{\rho T/2} c \left| s - t \right|^{1/2} \left(V(x) \right)^{1/4}. \end{split}$$

$$\tag{93}$$

The fact that $\forall M \in \mathbb{N} : U_{0,M}^0 = 0$ therefore assures for all $s \in [0,T], t \in [s,T], x \in \mathbb{R}^d$ that

$$\|u(t, x + W_t) - u(s, x + W_s)\|_2 \le 4e^{M/2} \left(50e^{2LT}\right) T^{-1/2} e^{\rho T/2} c \left|t - s\right|^{1/2} \left(V(x)\right)^{1/4}.$$
 (94)

Combining (91), Lemma 2.3 (applied for every $n, M \in \mathbb{N}$ with $V \curvearrowleft \{\mathfrak{Z}: \Omega \to \mathbb{R}: \mathfrak{Z} \text{ is measurable}\}$, $\|\cdot\| \curvearrowleft \|\cdot\|_2$, $\alpha \curvearrowleft 1/2$, $(m_l)_{l \in \{1,2,\dots,n\}} \curvearrowleft (M^l)_{l \in \{1,2,\dots,n\}}$, $(\tau_{l,k})_{k \in \{0,1,\dots,m_l\}, l \in \{1,2,\dots,n\}} \curvearrowleft (\frac{kT}{M^l})_{k \in \{0,1,\dots,M^l\}, l \in \{1,2,\dots,n\}}$, $(Y_t^0)_{t \in [0,T]} \curvearrowleft (u(t, W_t))_{t \in [0,T]}$, $((Y_t^\ell)_{t \in [0,T]})_{\ell \in [1,n] \cap \mathbb{N}} \curvearrowleft ((U_{\ell,M}^{\theta_\ell}(t, W_t))_{t \in [0,T]})_{\ell \in [1,n] \cap \mathbb{N}}, \mathscr{Y} \curvearrowleft \mathscr{Y}^{n,M}$ in the notation of Lemma 2.3), (92), and (93) hence demonstrates that for all $n, M \in \mathbb{N}$ it holds that

$$\begin{split} \sup_{t \in [0,T]} \|\mathscr{Y}_{t}^{n,M} - Y_{t}\|_{2} &= \sup_{t \in [0,T]} \|\mathscr{Y}_{t}^{n,M} - u(t,W_{t})\|_{2} \leq \sup_{t \in [0,T]} \|U_{n,M}^{\theta_{n}}(t,W_{t}) - u(t,W_{t})\|_{2} \\ &+ 2^{-1/2} T^{1/2} M^{-n/2} \left[\sup_{t,s \in [0,T], t \neq s} \frac{\|u(t,W_{t}) - u(s,W_{s})\|_{2}}{|t-s|^{1/2}} \right] \\ &+ \sum_{\ell=1}^{n-1} \left[2^{-1/2} T^{1/2} M^{-l/2} \left[\sup_{t,s \in [0,T], t \neq s} \frac{\|(U_{n-\ell,M}^{\theta_{n-\ell}} - u)(t,W_{t}) - (U_{n-\ell,M}^{\theta_{n-\ell}} - u)(s,W_{s})\|_{2}}{|t-s|^{1/2}} \right] \right] \end{split}$$

$$\leq e^{M/2} M^{-n/2} \left[50e^{2LT} \right]^{n+1} (V(0))^{1/4} e^{\rho T/2} + 2^{-1/2} T^{1/2} M^{-n/2} 4e^{M/2} \left(50e^{2LT} \right) T^{-1/2} e^{\rho T/2} c(V(0))^{1/4} \\ + \sum_{\ell=1}^{n-1} \left[2^{-1/2} T^{1/2} M^{-\ell/2} 4e^{M/2} M^{-(n-\ell)/2} \left(50e^{2LT} \right)^{n-\ell+1} T^{-1/2} e^{\rho T/2} c(V(0))^{1/4} \right].$$

$$\tag{95}$$

The fact that $c \ge 1$ therefore proves that for all $n, M \in \mathbb{N}$ it holds that

$$\begin{split} \sup_{t\in[0,T]} \|\mathscr{Y}_{t}^{n,M} - Y_{t}\|_{2} &\leq e^{M/2} M^{-n/2} (V(0))^{1/4} e^{\rho T/2} \\ &\cdot \left[\left(50e^{2LT} \right)^{n+1} + 2^{-1/2} 4c \left(50e^{2LT} \right) + 2^{-1/2} 4c \sum_{\ell=1}^{n-1} \left(50e^{2LT} \right)^{n-\ell+1} \right] \\ &\leq ce^{M/2} M^{-n/2} (V(0))^{1/4} e^{\rho T/2} \left[\left(50e^{2LT} \right)^{n+1} + 2^{-1/2} 4 \left(50e^{2LT} \right) + 2^{-1/2} 4n \left(50e^{2LT} \right)^{n} \right] \\ &= ce^{M/2} M^{-n/2} (V(0))^{1/4} e^{\rho T/2} 50^{2n} e^{4nLT} \\ &\cdot \left[\left(50e^{2LT} \right)^{-n+1} + 2^{-1/2} 4 \left(50e^{2LT} \right)^{1-2n} + 2^{-1/2} 4n \left(50e^{2LT} \right)^{-n} \right] \\ &\leq ce^{M/2} M^{-n/2} (V(0))^{1/4} e^{\rho T/2} 50^{2n} e^{4nLT} \left[1 + 2^{-1/2} 4 \left(50e^{2LT} \right)^{-1} + 2^{-1/2} 4n \left(50e^{2LT} \right)^{-1} \right] \\ &\leq nce^{M/2} M^{-n/2} (V(0))^{1/4} e^{\rho T/2} 50^{2n} e^{4nLT} \left[1 + \frac{4}{50\sqrt{2}} + \frac{4}{50\sqrt{2}} \right] \\ &\leq 2ne^{M/2} M^{-n/2} 50^{2n} e^{4nLT} e^{\rho T/2} c (V(0))^{1/4}. \end{split}$$

This establishes item (iii). Next note that [73, Lemma 3.6] (applied with $d \curvearrowleft \alpha$, $(\mathrm{RV}_{n,M})_{n,M\in\mathbb{Z}} \curvearrowleft (\mathcal{C}_{n,M})_{n,M\in\mathbb{Z}}$ in the notation of [73, Lemma 3.6]) and (79) show that for all $n, M \in \mathbb{N}$ it holds that $\mathcal{C}_{n,M} \leq \alpha(5M)^n$ and

$$\alpha^{-1}\mathfrak{C}_{n,M} \leq M^{n} + 1 + \sum_{\ell=0}^{n-1} \left[(M^{\ell+1} + 1)\alpha^{-1}\mathcal{C}_{n-\ell,M} \right] \leq M^{n} + 1 + \sum_{\ell=0}^{n-1} \left[(M^{\ell+1} + 1)(5M)^{n-\ell} \right]$$
$$\leq M^{n} + 1 + n(5M)^{n+1} + \left[\sum_{\ell=0}^{n-1} (5M)^{n-\ell} \right] = M^{n} + 1 + n(5M)^{n+1} + \frac{(5M)^{n+1} - 5M}{5M - 1}$$
$$\leq (n+2)(5M)^{n+1}. \tag{97}$$

Hence, we obtain that for all $n \in \mathbb{N}$ it holds that $\mathfrak{C}_{n+1,n+1} \leq \alpha(n+3)(5n+5)^{n+2} \leq \alpha(10n)^{n+3}$. This and (96) demonstrate that for all $t \in [0,T]$, $\delta \in (0,\infty)$, $n \in \mathbb{N}$ it holds that

$$\mathfrak{C}_{n+1,n+1} \| \mathscr{Y}_{t}^{n,n} - Y_{t} \|_{2}^{2+\delta} \leq \left[\frac{\alpha (10n)^{n+3} \left[8ne^{n/2} 50^{2n} e^{4nLT} \right]^{2+\delta}}{n^{(2+\delta)n/2}} \right] \left[e^{\rho T/2} c(V(0))^{1/4} \right]^{2+\delta} \\
\leq \left[\frac{10^{n+3} n^{3} \left[8ne^{n/2} 50^{2n} e^{4nLT} \right]^{2+\delta}}{n^{\delta n/2}} \right] \alpha \left[e^{\rho T/2} c(V(0))^{1/4} \right]^{2+\delta} < \infty.$$
(98)

Next observe that (96) and the fact that $\limsup_{n\to\infty} \left[ne^{n/2}n^{-n/2}50^{2n}e^{4nLT} \right] = 0$ prove that

$$\limsup_{n \to \infty} \sup_{t \in [0,T]} \| \mathcal{Y}_t^{n,n} - Y_t \|_2 = 0.$$
(99)

In the next step let $n: (0, \infty) \to [0, \infty]$ satisfy for all $\varepsilon \in (0, \infty)$ that

$$\mathsf{n}(\varepsilon) = \inf\left(\left\{n \in \mathbb{N} : \sup_{t \in [0,T]} \mathbb{E}\left[\left|\mathscr{Y}_{t}^{n,n} - Y_{t}\right|^{2}\right] < \varepsilon^{2}\right\} \cup \{\infty\}\right).$$
(100)

Note that (99) and (100) imply that for all $\varepsilon \in (0,\infty)$ it holds that $\mathsf{n}(\varepsilon) \in \mathbb{N}$ and $\sup_{t \in [0,T]} \|\mathscr{Y}_t^{\mathsf{n}(\varepsilon),\mathsf{n}(\varepsilon)} - Y_t\|_2 < \varepsilon \leq \mathbb{1}_{\{1\}}(\mathsf{n}(\varepsilon))\varepsilon + \mathbb{1}_{(1,\infty)}(\mathsf{n}(\varepsilon))\sup_{t \in [0,T]} \|\mathscr{Y}_t^{\mathsf{n}(\varepsilon)-1,\mathsf{n}(\varepsilon)-1} - Y_t\|_2$. Combining (97) and (98) hence ensures that for all $\delta, \varepsilon \in (0,1]$ it holds that

$$\begin{aligned} \mathfrak{C}_{\mathsf{n}(\varepsilon),\mathsf{n}(\varepsilon)}\varepsilon^{2+\delta} &\leq \mathbb{1}_{\{1\}}(\mathsf{n}(\varepsilon))\mathfrak{C}_{\mathsf{n}(\varepsilon),\mathsf{n}(\varepsilon)}\varepsilon^{2+\delta} + \mathbb{1}_{(1,\infty)}(\mathsf{n}(\varepsilon))\left[\mathfrak{C}_{\mathsf{n}(\varepsilon),\mathsf{n}(\varepsilon)}\sup_{t\in[0,T]}\|\mathscr{Y}_{t}^{\mathsf{n}(\varepsilon)-1,\mathsf{n}(\varepsilon)-1} - Y_{t}\|_{2}^{2+\delta}\right] \\ &\leq \left(\sup_{n\in\mathbb{N}}\left[\frac{10^{n+3}n^{3}[8ne^{n/2}50^{2n}e^{4nLT}]^{2+\delta}}{n^{\delta n/2}}\right]\right)\alpha\left[e^{\rho T/2}c(V(0))^{1/4}\right]^{2+\delta} < \infty. \end{aligned}$$

$$(101)$$

This, (100), the fact that for all $\varepsilon \in (0, \infty)$ it holds that $\mathbf{n}(\varepsilon) < \infty$, and the fact that $c = (\max\{\mathbb{E}[\|\mathbf{z}^0\|^4], 1\})^{1/4}$ establish item (iv). The proof of Theorem 5.1 is thus complete.

Lemma 5.2. Let $f \in C^2(\mathbb{R}, \mathbb{R})$. Then it holds for all $v_1, v_2, w_1, w_2 \in \mathbb{R}$ that

$$|(f(v_1) - f(w_1)) - (f(v_2) - f(w_2))| \le (\sup_{x \in \mathbb{R}} |f'(x)|) |(v_1 - w_1) - (v_2 - w_2)| + \frac{1}{2} (\sup_{x \in \mathbb{R}} |f''(x)|) [|v_1 - w_1| + |v_2 - w_2|] \min\{|v_1 - v_2|, |w_1 - w_2|\}.$$
(102)

Proof of Lemma 5.2. Observe that the fundamental theorem of calculus and the triangle inequality show that for all $v_1, v_2, w_1, w_2 \in \mathbb{R}$ it holds that

$$\begin{aligned} |(f(v_{1}) - f(w_{1})) - (f(v_{2}) - f(w_{2}))| &= |(f(v_{1}) - f(v_{2})) - (f(w_{1}) - f(w_{2}))| \\ &= \left| \int_{0}^{1} f'(\lambda v_{1} + (1 - \lambda)v_{2})(v_{1} - v_{2}) - f'(\lambda w_{1} + (1 - \lambda)w_{2})(w_{1} - w_{2}) d\lambda \right| \\ &= \left| \int_{0}^{1} f'(\lambda v_{1} + (1 - \lambda)v_{2}) \left[(v_{1} - v_{2}) - (w_{1} - w_{2}) \right] d\lambda \\ &+ \int_{0}^{1} \left[f'(\lambda v_{1} + (1 - \lambda)v_{2}) - f'(\lambda w_{1} + (1 - \lambda)w_{2}) \right] (w_{1} - w_{2}) d\lambda \right| \\ &\leq (\sup_{x \in \mathbb{R}} |f'(x)|) \left| (v_{1} - w_{1}) - (v_{2} - w_{2}) \right| \\ &+ (\sup_{x \in \mathbb{R}} |f''(x)|) \left[\int_{0}^{1} (\lambda |v_{1} - w_{1}| + (1 - \lambda)|v_{2} - w_{2}|) d\lambda \right] |w_{1} - w_{2}|. \end{aligned}$$

This and the fact that $\int_0^1 \lambda \, d\lambda = \int_0^1 (1-\lambda) \, d\lambda = \frac{1}{2}$ establish (102). The proof of Lemma 5.2 is thus complete.

Corollary 5.3. Let $T, \delta \in (0, \infty)$, $\Theta = \bigcup_{n \in \mathbb{N}} \mathbb{Z}^n$, $f \in C^2(\mathbb{R}, \mathbb{R})$, let $g_d \in C^1(\mathbb{R}^d, \mathbb{R})$, $d \in \mathbb{N}$, satisfy $\sup_{d \in \mathbb{N}} \sup_{x=(x_1, x_2, \dots, x_d) \in \mathbb{R}^d} (|f(x_1)| + |f'(x_1)| + |g'(x_1)| + |g_d(x)| + \sum_{i=1}^d |\frac{\partial g_d}{\partial x_i}(x)|^2) < \infty$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0,T]})$ be a filtered probability space, let $\mathfrak{r}^{\theta} \colon \Omega \to [0,1]$, $\theta \in \Theta$, be i.i.d. random variables, assume for all $t \in (0,1)$ that $\mathbb{P}(\mathfrak{r}^0 \leq t) = t$, let $\mathbf{z}^{d,\theta} \colon \Omega \to \mathbb{R}^d$, $\theta \in \Theta$, $d \in \mathbb{N}$, be i.i.d. standard normal vectors, let $W^d = (W^{d,1}, W^{d,2}, \dots, W^{d,d}) \colon [0,T] \times \Omega \to \mathbb{R}^d$, $d \in \mathbb{N}$, be standard $(\mathbb{F}_t)_{t \in [0,T]}$ -Brownian motions, assume that $(\mathfrak{r}^{\theta})_{\theta \in \Theta}, (\mathbf{z}^{d,\theta})_{(d,\theta) \in \mathbb{N} \times \Theta}$, and $(W^d)_{d \in \mathbb{N}}$ are independent, let $U^{d,\theta}_{n,M} \colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$, $d, M, n \in \mathbb{N}_0$, $\theta \in \Theta$, satisfy for all $d, M \in \mathbb{N}$, $n \in \mathbb{N}_0, \theta \in \Theta, t \in [0,T], x \in \mathbb{R}^d$ that

$$U_{n,M}^{d,\theta}(t,x) = (T-t)f(0)\mathbb{1}_{\mathbb{N}}(n) + \frac{\mathbb{1}_{\mathbb{N}}(n)}{M^{n}} \sum_{i=1}^{M^{n}} g_{d} \left(x + [T-t]^{1/2} \mathbf{z}^{d,(\theta,0,-i)} \right) \\ + \sum_{\ell=1}^{n-1} \left[\frac{(T-t)}{M^{n-\ell}} \sum_{i=1}^{M^{n-\ell}} \left(f \circ U_{\ell,M}^{d,(\theta,\ell,i)} - f \circ U_{\ell-1,M}^{d,(\theta,-\ell,i)} \right) \left(t + (T-t) \mathbf{r}^{(\theta,\ell,i)}, x + [(T-t)\mathbf{r}^{(\theta,\ell,i)}]^{1/2} \mathbf{z}^{d,(\theta,\ell,i)} \right) \right],$$
(104)

 $\begin{array}{l} let \ \lfloor \cdot \rfloor_M \colon \mathbb{R} \to \mathbb{R}, \ M \in \mathbb{N}, \ and \ \lceil \cdot \rceil_M \colon \mathbb{R} \to \mathbb{R}, \ M \in \mathbb{N}, \ satisfy \ for \ all \ M \in \mathbb{N}, \ t \in [0,T] \ that \\ \lfloor t \rfloor_M = \max(([0,t] \setminus \{T\}) \cap \{0, \frac{T}{M}, \frac{2T}{M}, \ldots\}) \ and \ \lceil t \rceil_M = \min(((t,\infty) \cup \{T\}) \cap \{0, \frac{T}{M}, \frac{2T}{M}, \ldots\}), \ let \\ \mathscr{Y}^{d,n,M} \colon [0,T] \times \Omega \to \mathbb{R}, \ d, n, M \in \mathbb{N}, \ satisfy \ for \ all \ d, n, M \in \mathbb{N}, \ t \in [0,T] \ that \end{array}$

$$\mathscr{Y}_{t}^{d,n,M} = \sum_{\ell=0}^{n-1} \left[\left[\frac{[t]_{M^{l+1}-t}}{(T/M^{l+1})} \right] U_{n-\ell,M}^{d,\ell} (\lfloor t \rfloor_{M^{l+1}}, W_{\lfloor t \rfloor_{M^{l+1}}}^{d}) + \left[\frac{t-\lfloor t \rfloor_{M^{l+1}}}{(T/M^{l+1})} \right] U_{n-\ell,M}^{d,\ell} (\lceil t \rceil_{M^{l+1}}, W_{\lceil t \rceil_{M^{l+1}}}^{d}) - \mathbb{1}_{\mathbb{N}} (\ell) \left(\left[\frac{[t]_{M^{l}-t}}{(T/M^{l})} \right] U_{n-\ell,M}^{d,\ell} (\lfloor t \rfloor_{M^{l}}, W_{\lfloor t \rfloor_{M^{l}}}^{d}) + \left[\frac{t-[t]_{M^{l}}}{(T/M^{l})} \right] U_{n-\ell,M}^{d,\ell} (\lceil t \rceil_{M^{l}}, W_{\lceil t \rceil_{M^{l}}}^{d}) \right) \right],$$
(105)

let $\mathbf{Y}^d = (Y^d, Z^{d,1}, Z^{2,d}, \dots, Z^{d,d}) \colon [0, T] \times \Omega \to \mathbb{R}^{d+1}, d \in \mathbb{N}$, be $(\mathbb{F}_t)_{t \in [0,T]}$ -predictable stochastic processes, assume for all $d \in \mathbb{N}$ that $\int_0^T \mathbb{E}\left[|Y_s^d| + \sum_{j=1}^d |Z_s^{d,j}|^2\right] ds < \infty$, assume that for all $d \in \mathbb{N}, t \in [0, T]$ it holds \mathbb{P} -a.s. that

$$Y_t^d = g_d(W_T^d) + \int_t^T f(Y_s^d) \, ds - \sum_{j=1}^d \int_t^T Z_s^{d,j} \, dW_s^{d,j}, \tag{106}$$

and let $\mathcal{C}_{d,n,M} \in \mathbb{N}_0$, $d, n, M \in \mathbb{Z}$, and $\mathfrak{C}_{n,M} \in \mathbb{N}_0$, $d, n, M \in \mathbb{Z}$, satisfy for all $d \in \mathbb{N}$, $n, M \in \mathbb{N}_0$ that

$$\mathcal{C}_{d,n,M} \le (d+1)M^{n}\mathbb{1}_{\mathbb{N}}(n) + \sum_{\ell=0}^{n-1} \left[M^{n-\ell} \left(2 + d + \mathcal{C}_{d,\ell,M} + \mathcal{C}_{d,\ell-1,M}\mathbb{1}_{\mathbb{N}}(\ell) \right) \right],$$
(107)

and $\mathfrak{C}_{d,n,M} \leq (d+1)(M^n+1) + \sum_{\ell=0}^{n-1} \left[(M^{\ell+1}+1)\mathcal{C}_{d,n-\ell,M} \right]$. Then there exist $c \in \mathbb{R}$ and $\mathfrak{n} \colon \mathbb{N} \times (0,1] \to \mathbb{N}$ such that for all $d \in \mathbb{N}$, $\varepsilon \in (0,1]$ it holds that $\sup_{t \in [0,T]} (\mathbb{E}[|\mathscr{Y}_t^{d,\mathfrak{n}(d,\varepsilon),\mathfrak{n}(d,\varepsilon)} - Y_t^d|^2])^{1/2} \leq \varepsilon$ and $\mathfrak{C}_{d,\mathfrak{n}(d,\varepsilon),\mathfrak{n}(d,\varepsilon)} \leq cd^c \varepsilon^{-(2+\delta)}$.

Proof of Corollary 5.3. Note that Lemma 5.2 ensures that for $v_1, v_2, w_1, w_2 \in \mathbb{R}$ it holds that

$$|(f(v_1) - f(w_1)) - (f(v_2) - f(w_2))| \le (\sup_{x \in \mathbb{R}} |f'(x)|) |(v_1 - w_1) - (v_2 - w_2)| + \frac{1}{2} (\sup_{x \in \mathbb{R}} |f''(x)|) [|v_1 - w_1| + |v_2 - w_2|] \min\{|v_1 - v_2|, |w_1 - w_2|\}.$$
(108)

This and Theorem 5.1 prove that there exist $c \in \mathbb{R}$ and $\mathbf{n} \colon \mathbb{N} \times (0, 1] \to \mathbb{N}$ such that for all $d \in \mathbb{N}$, $\varepsilon \in (0, 1]$ it holds that $\sup_{t \in [0,T]} (\mathbb{E}[|\mathscr{Y}_t^{d,\mathsf{n}(d,\varepsilon)}, -Y_t^d|^2])^{1/2} \leq \varepsilon$ and $\mathfrak{C}_{d,\mathsf{n}(d,\varepsilon),\mathsf{n}(d,\varepsilon)} \leq cd^c \varepsilon^{-(2+\delta)}$. The proof of Corollary 5.3 is thus complete.

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