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Convergence Rate Analysis of Domain Decomposition Methods for Obstacle Problems

by

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1 Introduction

In this work, we shall study the constrained minimization problem

$$\min_{v \in K} F(v) \ , \tag{1}$$

where K is a nonempty closed convex set in a reflexive Banach space V in the strong topology and $F : V \mapsto \Re$ is a lower semicontinuous convex Gâteau-differentiable function. Denote $\langle \cdot, \cdot \rangle$ the duality pairing of V and its dual space V', i.e. the value of a linear functional at an element of V. We shall assume the differential of F satisfies

$$\begin{aligned} \langle F'(w) - F'(v), w - v \rangle & \geq \kappa \|w - v\|_V^2, \quad \forall w, v \in V \ , \\ \|F'(w) - F'(v)\|_{V'} & \leq \ell \|w - v\|_V, \quad \forall w, v \in V \ , \end{aligned} \tag{2}$$

for some given constants $\kappa, \ell > 0$. Under the above assumptions, minimization (1) has a unique solution u [13, p. 23].

The general theory developed for (1) will be specialized for the following concrete application in the case of domain decomposition:

$$\text{Find } u \in K, \quad \text{such that} \quad a(u, v - u) \geq l(v - u), \quad \forall v \in K, \tag{3}$$

with

$$a(v, w) = \int_\Omega \nabla v \cdot \nabla w \, dx, \quad K = \{v \in H_0^1(\Omega) | \ v(x) \geq \psi(x) \text{ a.e. in } \Omega\}. \tag{4}$$

It is well known that the above problem is equivalent to the following minimization problem

$$\min_{v \in K} F(v), \quad F(v) = \frac{1}{2} a(v, v) - l(v), \tag{5}$$

assuming that $l \in H^{-1}(\Omega)$ is a linear functional of $H_0^1(\Omega)$.

Obstacle problems arise from many important applications. Amongst many of the standard references, we refer to Baiocchi and Capelocite [3], Cottle et al. [5], Duvaut and Lions [6], Elliot and Ockendon [8], Glowinski [11], Glowinski et al. [12], Kinderlehrer and Stampaccia [22], Kornhuber [24], and Rodrigues [29]. In this work, we are concerned about the use of efficient iterative solvers for the

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obstacle problem (3). Especially, we shall concentrate on domain decomposition and multigrid methods. For general iterative methods for obstacle problems in finite dimensions, we refer, in addition to the afore mentioned references, to [28, 31, 26].

Domain decomposition methods are well known iterative methods for solving partial differential equations. Their applications to obstacle kind of problems have been studied in some recent works, see Badea and Wang [1, 2], Hoffman and Zou [18], Lu et. al. [25], Tai [32, 33, 34], Tai and Tseng [37], Tarvainen [39], Zeng and Zhou [40] etc. In the work of Tai [32, 33, 34], some general space decomposition algorithms are proposed for the minimization of convex functionals over convex constraint subsets. The algorithms can be used for domain decomposition type of techniques. Only convergence was proved in Tai [32, 33, 34], but the rate of convergence for the algorithms was not given. In the present work, we attempt to give an estimate for the rate of convergence. In Tarvainen [39], overlapping domain decomposition is used for (3) also without analysing the rate of convergence. The works of [40] and [1, 2] are intended to provide an estimate of the convergence rate. However, the estimate given in [40] is only valid for the case of two subdomains with a uniform overlapping size. In Badea and Wang [1, 2], a linear convergence rate was estimated, but the dependence of the rate of convergence on the number of subdomains and the size of overlaps was not given. Moreover, the relaxation parameters offered by [39, 1] are too pessimistic. Using a suitable coloring procedure, we shall see that a much bigger relaxation parameter may be chosen. In addition, we shall give a linear rate of convergence with an explicit estimation of the dependence of the rate of convergence on the number of subdomains and the overlapping size. When our algorithm are used for the obstacle problem (3) with an overlapping domain decomposition, the boundary condition for the subdomain problems is the same as in [1, 39, 40], but the obstacle functions for the subdomain problems are normally bigger than the obstacles used in [1, 39, 40].

The multigrid type of methods has been used for obstacle problems by Hackbusch and Mittelman [17], Hoppe et al. [21, 20, 19], Gelman and Mandel [10], Kornhuber [24], Mandel [27], Sharapov [30], etc. The convergence analysis for the multigrid method for the obstacle problem is normally divided into two steps, see [10], [21, 20, 19] and [24]. In the first step, it is shown that the constraint is identified with a fixed number of iterations. After the constraint has been identified, the obstacle problem is reduced to an unconstrained problem. Modifying the proofs for multigrid methods for general elliptic problems, it can be shown that the error reduction is independent of the mesh sizes for the unconstrained problem. For obstacle problems, it is difficult to estimate the iteration number required for the identification of the obstacle. Thus the iteration number needed to reach a given accuracy by means of multigrid methods for the obstacle problem is hard to estimate.

In this work, we try to use a different approach both in deriving the algorithms and analysing the rate of convergence. The essential idea is to use domain decomposition and multigrid methods as decomposition techniques for the decomposition of the constraint set. It is shown that the estimate for the rate of convergence is reduced to the estimation of two constants. When an overlapping domain decomposition is used, the iterative solution is monotonically increasing, This property is used to show the rate of convergence is only depending on the overlapping size, but is not on the number of subdomains and the mesh sizes. When multi-level methods are used, we can also apply our algorithms to the obstacle problem, but the iterative solution is not monotone any more. Different techniques are needed to estimate the constants and they are not included in this work.

The paper is organized in the following way: In section 2, we present the algorithm for the general constraint problem (1). The convergence rate analysis is given in section 3. It is shown that the convergence only depends on two constants C_1 and

C_2. In section 4, these constants are estimated for an overlapping domain decomposition. The implementation issues for both the overlapping domain decomposition and multi-level methods are briefly discussed in section 5.

2 Space decomposition algorithms for convex programming problems

The starting point for our algorithms is that the convex subset K can be decomposed as

$$K = \sum_{i=1}^{m} K_i. \tag{6}$$

The above decomposition means that for any $v \in K$, we can find $v_i \in K_i$, possibly not unique, such that $v = \sum_{i=1}^{m} v_i$ and , on the other hand, for any $v_i \in K_i$, we have $\sum_{i=1}^{m} v_i \in K$. If the constraint set K can be decomposed as above, we can use the following algorithms to solve the minimization problem (1).

Algorithm 1 *[A parallel subspace correction method].*

1. *Choose initial values $u_i^0 \in K_i$ and a relaxation parameter $\gamma \in (0, 1/m]$.*
2. *For $n \geq 0$, if $u_i^n \in K_i$ is defined, then find $\hat{u}_i^{n+1} \in K_i$ in parallel for $i = 1, 2, \cdots, m$ such that*

$$F\left(\sum_{j=1, j\neq i}^{m} u_j^n + \hat{u}_i^{n+1}\right) \leq F\left(\sum_{j=1, j\neq i}^{m} u_j^n + v_i\right), \quad \forall v_i \in K_i\ . \tag{7}$$

3. *Set*

$$u_i^{n+1} = u_i^n + \gamma(\hat{u}_i^{n+1} - u_i^n)\ , \tag{8}$$

and go to the next iteration.

Algorithm 2 *[A successive subspace correction method].*

1. *Choose initial values $u_i^0 \in K_i$ and a relaxation parameter γ.*
2. *For $n \geq 0$, if $u_i^n \in K_i$ is defined, find $\hat{u}_i^{n+1} \in K_i$ sequentially for $i = 1, 2, \cdots, m$ such that*

$$F\left(\sum_{j<i} u_j^{n+1} + \hat{u}_i^{n+1} + \sum_{j>i} u_j^n\right) \leq F\left(\sum_{j<i} u_j^{n+1} + v_i + \sum_{j>i} u_j^n\right), \quad \forall v_i \in K_i\ . \tag{9}$$

3. *Set*

$$u_i^{n+1} = u_i^n + \gamma(\hat{u}_i^{n+1} - u_i^n)\ , \tag{10}$$

and go to the next iteration.

For Algorithm 1, under-relaxation (i.e. $\gamma \leq 1$) must be introduced in order to guarantee the convergence. Even for the unconstrained case (i.e. $K = V$), the algorithm can diverge when $\gamma > 1$, see Remark 4.1. of [34, p.146]. For Algorithm 2, over-relaxation (i.e. $\gamma > 1$) may accelerate the convergence, but it is hard to do the analysis. In this work, the convergence of Algorithm 2 is only analysed for the case when $\gamma = 1$. An analysis for some problems with $K = V$ and $\gamma > 1$ can be found in Frommer and Renaut [9].

3 Convergence Analysis for the Algorithms

3.1 Conditions for the convergence of the algorithms

Using similar definitions as that in [38], we shall use the following notations in the proofs. u will always be used to denote the unique solution of (1). For any $n \geq 0$, we define

$$u^n = \sum_{i=1}^{m} u_i^n, \quad e_i^{n+1} = \hat{u}_i^{n+1} - u_i^n, \quad d_n = F(u^n) - F(u). \tag{11}$$

For the decomposed spaces, we assume that there exits a constant $C_1 > 0$ such that for any iteration number $n > 0$, we can find $u_i \in K_i$ that satisfy

$$\begin{aligned} u = \sum_{i=1}^{m} u_i\,, \qquad & u_i + \hat{u}_i^{n+1} - u_i^{n+1} \in K_i, \\ \text{and} \quad \left(\sum_{i=1}^{m} \|u_i - u_i^{n+1}\|_V^2 \right)^{\frac{1}{2}} & \leq C_1 \|u - u^{n+1}\|_V \,. \end{aligned} \tag{12}$$

Observe that u_i may depend on the iteration number n. In addition to the assumption of the existence of such a constant C_1, we also need to assume that there is a $C_2 > 0$ such that

$$\begin{aligned} & \sum_{i=1}^{m} \sum_{j=1}^{m} \langle F'(w_{ij} + \hat{v}_i) - F'(w_{ij}), \bar{v}_j \rangle \\ \leq \; & C_2 \Big(\sum_{i=1}^{m} \|\hat{v}_i\|_V^2 \Big)^{\frac{1}{2}} \Big(\sum_{j=1}^{m} \|\bar{v}_j\|_V^2 \Big)^{\frac{1}{2}} \,, \\ & \forall \; w_{ij} \in K, \hat{v}_i \in K_i \text{ and } \bar{v}_j \in K_j \,. \end{aligned} \tag{13}$$

It is possible that the decomposition given in (6) may differ from iteration to iteration, i.e. K_i can depend on n. In such a case, we need to assume that C_1 and C_2 are independent of n. The analysis given below remains valid because the rate of convergence of the proposed algorithms at a given iteration only depends on the decomposition at the present iteration. The error reduction at a given iteration does not depend on the decomposition of K of the earlier or later iterations. See [9] for some cases when the decomposition differs from iteration to iteration.

3.2 The convergence of the parallel subspace correction method

The convergence of Algorithm 1 is given in the following theorem.

Theorem 1 *Assuming that the space decomposition satisfies (12), (13) and that the functional F satisfies (2). Then for Algorithm 1 and d_n given by (11), we have*

$$d_{n+1} \leq \frac{1}{1 + C^*} d_n, \quad \forall n \geq 1 \,. \tag{14}$$

Here $C^* > 0$ is defined in (23) which only depends on $\gamma, \kappa, \ell, C_1$ and C_2.

Proof. Using the notations of (11) and the fact that F is differentiable and convex, it is known (see Ekeland and Temam [7]) that (7) implies

$$\langle F'(u^n + e_i^{n+1}), v_i - \hat{u}_i^{n+1} \rangle \geq 0, \quad \forall v_i \in K_i. \tag{15}$$

Under the assumption of (2), it is known that (See Tai and Epsedal [36, Lemma 3.2])

$$F(w) - F(v) \geq \langle F'(v), w - v\rangle + \frac{\kappa}{2}\|w - v\|_V^2 , \quad \forall v, w \in V . \tag{16}$$

Define

$$u^{n+\frac{i}{m}} = \sum_{j=1, j\neq i}^{m} u_j^n + \hat{u}_i^{n+1}. \tag{17}$$

From (8), we see that $u^{n+\frac{i}{m}} = u^n + e_i^{n+1}$ and

$$\begin{aligned} u^{n+1} &= \sum_{i=1}^{m} u_i^{n+1} = \sum_{i=1}^{m} u_i^n + \gamma \sum_{i=1}^{m}(\hat{u}_i^{n+1} - u_i^n) \\ &= u^n + \gamma \sum_{i=1}^{m}(\hat{u}_i^{n+1} - u_i^n) = (1 - \gamma m)u^n + \gamma \sum_{i=1}^{m} u^{n+\frac{i}{m}}. \end{aligned} \tag{18}$$

Using (15), (18), the convexity of F and (2), and applying similar techniques as in [36, p.1563], it can be proved that

$$\begin{aligned} &F(u^n) - F(u^{n+1}) \\ &\geq \quad F(u^n) - \sum_{i=1}^{m} \gamma F\left(u^n + e_i^{n+1}\right) - (1 - \gamma m)F(u^n) \qquad (19) \\ &= \quad \sum_{i=1}^{m} \gamma \Big(F(u^n) - F(u^n + e_i^{n+1}) \Big) \\ &\geq \quad -\sum_{i=1}^{m} \gamma \langle F'(u^n + e_i^{n+1}), e_i^n\rangle + \frac{\kappa}{2}\sum_{i=1}^{m} \gamma \|e_i^{n+1}\|_V^2 \\ &\geq \quad \frac{\kappa}{2}\sum_{i=1}^{m} \gamma \|e_i^{n+1}\|_V^2 . \end{aligned}$$

For simplicity, we define

$$\phi_j^n = \sum_{i=1}^{j} u_i^{n+1} + \sum_{i=j+1}^{m} u_i^n .$$

Let u_i be the functions given in assumptions (12). By assumptions (12) and (15), we see that

$$\langle F'(u^n + e_i^{n+1}), u_i^{n+1} - u_i\rangle = \langle F'(u^n + e_i^{n+1}), \hat{u}_i^{n+1} - (u_i + \hat{u}_i^{n+1} - u_i^{n+1})\rangle \leq 0. \tag{20}$$

We shall use (12), (13), (8) and (20) to estimate

$$\begin{aligned} &\left\langle F'(u^{n+1}) - F'(u), u^{n+1} - u\right\rangle \\ &\leq \quad \langle F'(u^{n+1}), u^{n+1} - u\rangle = \sum_{i=1}^{m} \Big\langle F'(u^{n+1}), u_i^{n+1} - u_i \Big\rangle \\ &\leq \quad \sum_{i=1}^{m} \Big\langle F'(u^{n+1}) - F'(u^n + e_i^{n+1}), u_i^{n+1} - u_i \Big\rangle \\ &= \quad \sum_{i=1}^{m} \Big\langle F'(u^{n+1}) - F'(u^n), u_i^{n+1} - u_i \Big\rangle \\ &- \quad \sum_{i=1}^{m} \Big\langle F'(u^n + e_i^{n+1}) - F'(u^n), u_i^{n+1} - u_i \Big\rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m}\sum_{j=1}^{m}\left\langle F'(\phi_j^n) - F'(\phi_{j-1}^n), u_i^{n+1} - u_i\right\rangle + \sum_{i=1}^{m}\left\langle F'(u^n + e_i^{n+1}) - F'(u^n), u_i^{n+1} - u_i\right\rangle \\
&\leq C_2\left(\sum_{j=1}^{m}\|u_i^{n+1} - u_i^n\|_V^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{m}\|u_i^{n+1} - u_i\|_V^2\right)^{\frac{1}{2}} + C_2\sum_{i=1}^{m}\|e_i^{n+1}\|_V\|u_i^{n+1} - u_i\|_V \\
&\leq (1+\gamma)C_2\left(\sum_{i=1}^{m}\|e_i^{n+1}\|_V^2\right)^{\frac{1}{2}} \cdot C_1\|u^{n+1} - u\|_V. \qquad (21)
\end{aligned}
$$

From (2) and (21), it is easy to see that

$$
\kappa\|u^{n+1} - u\|_V^2 \leq C_1C_2(1+\gamma)\left(\sum_{i=1}^{m}\|e_i^{n+1}\|_V^2\right)^{\frac{1}{2}}\|u^{n+1} - u\|_V,
$$

and thus

$$
\|u^{n+1} - u\|_V \leq \left(\frac{C_1C_2(1+\gamma)}{\kappa}\right)\left(\sum_{i=1}^{m}\|e_i^{n+1}\|_V^2\right)^{\frac{1}{2}}.
$$

Using the property (2) and the fact that u is the minimizer of (1), it can be proved that (see Lemma 3.2 of [36])

$$
F(u^{n+1}) - F(u) \leq \frac{\ell}{2}\|u^{n+1} - u\|_V^2 \leq \frac{\ell}{2}\left(\frac{C_1C_2(1+\gamma)}{\kappa}\right)^2\sum_{i=1}^{m}\|e_i^{n+1}\|_V^2. \qquad (22)
$$

Defining

$$
C^* = \frac{\ell}{\gamma\kappa}\left(\frac{C_1C_2(1+\gamma)}{\kappa}\right)^2, \qquad (23)
$$

we get from (19) and (22) that

$$
d_{n+1} \leq C^*(d_n - d_{n+1}),
$$

and thus

$$
d_{n+1} \leq \frac{1}{1+C^*}d_n.
$$

From the above estimate, we see that the convergence is uniformly linear with a convergence rate depending only on C^*. ∎

3.3 The convergence of the successive subspace correction method

The convergence of Algorithm 2 is similar to Algorithm 1.

Theorem 2 *Let the space decomposition satisfy (12), (13) and the functional F satisfy (2). Define*

$$
C^* = \frac{\ell}{\kappa}\left(\frac{C_1C_2}{\kappa}\right)^2.
$$

If $\gamma = 1$, we have for Algorithm 2

$$
d_{n+1} \leq \frac{C^*}{1+C^*}d_n, \quad \forall n \geq 1\ . \qquad (24)
$$

Proof. Define

$$u^{n+\frac{i}{m}} = \sum_{j<i} u_j^{n+1} + \hat{u}_i^{n+1} + \sum_{j>i} u_j^n. \tag{25}$$

We see that

$$F(u^n) - F(u^{n+1}) = \sum_{i=1}^{m} \left[F(u^{n+(i-1)/m}) - F(u^{n+i/m}) \right] . \tag{26}$$

Since $u^{n+\frac{i}{m}}$ is the minimizer of (9), it satisfies

$$\langle F'(u^{n+\frac{i}{m}}), v_i - u_i^{n+1} \rangle \geq 0, \quad \forall v_i \in K_i. \tag{27}$$

Using (16) and (27), we get that

$$F(u^{n+(i-1)/m}) - F(u^{n+i/m}) \geq \frac{\kappa}{2} \| e_i^{n+1} \|_V^2 \ . \tag{28}$$

Thus, estimates (26) and (28) together lead to

$$F(u^n) \geq F(u^{n+1}), \tag{29}$$

and

$$F(u^n) - F(u^{n+1}) \geq \frac{\kappa}{2} \sum_{i=1}^{m} \| e_i^{n+1} \|_V^2 \ . \tag{30}$$

Note that $\hat{u}_i^{n+1} = u_i^{n+1}$ when $\gamma = 1$. Similar to the proofs for (21), we use (12) and (13) to get

$$\begin{aligned}
& \langle F'(u^{n+1}) - F'(u), u^{n+1} - u \rangle \\
= & \sum_{i=1}^{m} \left\langle F'(u^{n+1}) - F'(u^{n+i/m}), u_i^{n+1} - u_i \right\rangle \\
= & \sum_{i=1}^{m} \sum_{j>i}^{m} \left\langle F'(u^{n+j/m}) - F'(u^{n+(j-1)/m}), u_i^{n+1} - u_i \right\rangle \qquad (31) \\
\leq & \ C_2 \left(\sum_{j=1}^{m} \| e_j^n \|_V^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} \| u_i^{n+1} - u_i \|_V^2 \right)^{\frac{1}{2}} \\
\leq & \ C_1 C_2 \left(\sum_{i=1}^{m} \| e_i^{n+1} \|_V^2 \right)^{\frac{1}{2}} \cdot \| u^{n+1} - u \|_V .
\end{aligned}$$

Using (2), (30) and (31), we obtain

$$d_{n+1} \leq C^*(d_n - d_{n+1}),$$

and the theorem follows. ■

4 Overlapping domain decomposition for the obstacle problem

4.1 Decomposition by overlapping subdomains

For simplicity, we shall present our decomposition for the continuous problem. The corresponding decomposition for the finite element discretized case is essentially similar to the continuous case with slightly modified analysis for the estimation of the constant C_1.

For the given domain Ω, we first divide it into nonoverlapping subdomains Ω_i and we extend each subdomains by a distance δ to get overlapping subdomains Ω_i^δ. Assume that the subdomains Ω_i^δ can be painted by n_c colors such that the subdomains of the same color will not intersect each other. Let $\Omega_i^c, i = 1, 2, \cdots, n_c$ be the union of the overlapping subdomains painted with the ith color. Denote $H_0^1(\Omega_i^c)$ the space of H^1-functions with zero traces on $\partial\Omega_i^c$ and extended by zero outside Ω_i^c. It is easy to see that

$$H_0^1(\Omega) = \sum_{i=1}^{n_c} H_0^1(\Omega_i^c).$$

First, we decompose the obstacle ψ as

$$\psi = \sum_{i=1}^{n_c} \psi_i, \quad \psi_i \in H^1(\Omega), \psi_i = 0 \text{ in } \Omega\backslash\Omega_i^c. \tag{32}$$

Due to the overlaps of the subdomains, the decomposition of ψ is not unique. The convex set K can be decomposed into a sum of

$$K_i = \{v_i |\ v_i \in H_0^1(\Omega_i^c),\ v_i \ge \psi_i \text{ a.e. in } \Omega_i^c\}, \tag{33}$$

i.e.

$$K = \sum_{i=1}^{n_c} K_i. \tag{34}$$

For finite element approximations, the subspaces $H_0^1(\Omega_i^c)$ and the subsets K_i shall be replaced by their finite element counter parts.

4.2 A technical lemma

Lemma 1 *Assume $\psi_1 \ge \psi_2$ and $g_1 \ge g_2$. Define*

$$K_i = \left\{ v \in H^1(\Omega) \Big| \quad v = g_i \ on\ \partial\Omega, \quad v \ge \psi_i\ in\ \Omega \right\}, i = 1, 2.$$

Let $u_i \in K_i$, $i = 1, 2$, be the solutions of

$$a(u_i, v - u_i) \ge l(v - u_i), \quad \forall v \in K_i.$$

Then we have

$$u_1 \ge u_2 \quad in\ \Omega.$$

Proof. Define $v = \max(0, u_2 - u_1) \in H_0^1(\Omega)$. In the region that $v > 0$, we have $u_2 > u_1 \ge \psi_1 \ge \psi_2$ which means the obstacle is not active for u_2 and it is easy to show that

$$a(u_2, v) = l(v).$$

From the definition of v, we see that $v \ge 0$ and thus $u_1 + v \in K$. Accordingly, it is true that

$$a(u_1, v) \ge l(v).$$

Subtracting the above two equations from each other we obtain

$$a(u_1 - u_2, v) \ge 0.$$

It is easy to see that

$$a(v, v) = -a(u_1 - u_2, v) \le 0,$$

which implies that $v = 0$ and thus proves the lemma. The proof for the discretized case is similar. ■

The lemma shows that if the boundary value and the obstacle of a problem is larger than the boundary value and obstacle of another problem respectively, then the corresponding solution is also larger than the solution of the other problem.

4.3 Mesh independence for the decomposition

From §4.1, we see that the number of the decomposed constraint sets is equal to the number of colors for the subdomains, i.e.

$$m = n_c.$$

For the obstacle problem (3), we have $V = H_0^1(\Omega)$. If we use $|\cdot|_1$ as the norm for V, we have $\kappa = \ell = 1$. We only need to estimate the constants C_1 and C_2 in order to show the convergence rate for the domain decomposition method.

Lemma 2 *For both Algorithms 1 and 2, assuming that*[1]

$$u^n = \sum_{i=1}^{m} u_i^n \le u, \tag{35}$$

for $n = 0$, then the inequality (35) is also true for all $n > 0$.

Proof. We shall prove the lemma by induction, i.e. assume (35) is correct for n, then we shall show that (35) is also correct for $n+1$.

Due to the decomposition (34), the solution u of problem (3) satisfies

$$\sum_{i=1}^{m} a(u, v_i - u_i) \ge \sum_{i=1}^{m} l(v_i - u_i), \quad \forall v_i \in K_i, \quad \text{and} \quad \sum_{i=1}^{m} u_i = u. \tag{36}$$

We shall first prove the lemma for Algorithm 1. For the obstacle problem (3), the functional F is given in (5) and the subproblem (7) is equivalent to

$$a(u^{n+\frac{i}{m}}, v_i - \hat{u}_i^{n+1}) \ge l(v_i - \hat{u}_i^{n+1}), \quad \forall v_i \in K_i. \tag{37}$$

Introduce

$$K_i^1 = \left\{ v + \sum_{j=1, j\ne i}^{m} u_j^n \;\middle|\; v \in K_i \right\}, i = 1, 2, \cdots, m.$$

It is easy to see that (37) is equivalent to

$$a(u^{n+\frac{i}{m}}, v_i - u^{n+\frac{i}{m}}) \ge l(v_i - u^{n+\frac{i}{m}}), \quad \forall v_i \in K_i^1. \tag{38}$$

When (35) is correct for n, we are able to find $u_j \in K_j$, $j = 1, 2, \cdots, m$ such that

$$u_j \ge u_j^n, \quad \text{and} \quad u = \sum_{j=1}^{m} u_j. \tag{39}$$

For these functions u_j, we get from (36) that u also satisfies

$$a(u, v_i - u) \ge l(v_i - u) \quad \forall v_i \in K_i^2 = \left\{ v + \sum_{j=1, j\ne i}^{m} u_j \;\middle|\; v \in K_i \right\}. \tag{40}$$

---

[1] We need $\gamma = 1$ for Algorithm 2.

Applying Lemma 1 to (38) and (40), we obtain

$$u^{n+\frac{i}{m}} \leq u.$$

When $\gamma \leq \frac{1}{m}$ and (35) is correct for n, we use (18), (35) and the above inequality to get

$$u^{n+1} \leq (1-\gamma m)u + \gamma m u = u,$$

which proves the lemma for Algorithm 1.

For Algorithm 2, we shall define $u^{n+\frac{i}{m}}$ as in (25). The minimization problem (9) is equivalent to

$$a(u^{n+\frac{i}{m}}, v_i - \hat{u}_i^{n+1}) \geq l(v_i - \hat{u}_i^{n+1}), \quad \forall v_i \in K_i,$$

which again is equivalent to

$$a(u^{n+\frac{i}{m}}, v_i - u^{n+\frac{i}{m}}) \geq l(v_i - u^{n+\frac{i}{m}})$$

$$\forall v_i \in K_i^3 = \left\{ v + \sum_{j<i} u_j^{n+1} + \sum_{j>i} u_j^n \;\middle|\; v \in K_i. \right\}. \tag{41}$$

If $\gamma = 1$ in Algorithm 2, an induction on i by using Lemma 1 to (40) and (41) will show that

$$u^{n+1} = u^{n+\frac{i}{m}}\Big|_{i=m} \leq u,$$

which proves the results of the lemma for Algorithm 2. ■

Choosing the initial values u_i^0 to be the obstacle functions ψ_i is an easy way to guarantee that (35) holds for all $n > 0$.

Let θ_i be a partition of unity with respect to the overlapping subdomains Ω_i^c, i.e. $\theta_i \in C_0^\infty(\Omega)$, $0 \leq \theta_i \leq 1$, $\theta_i = 0$ outside of Ω_i^c, and $\sum_{i=1}^m \theta_i = 1$. By defining

$$u_i = u_i^{n+1} + \theta_i(u - u^{n+1}),$$

and using the inequality (35) for $n+1$, we can easily check that $u_i \geq u_i^{n+1}$ and so

$$u_i \in K_i \quad \text{and} \quad \sum_{i=1}^m u_i = u. \tag{42}$$

Moreover,

$$u_i + \hat{u}_i^{n+1} - u_i^{n+1} = \hat{u}_i^{n+1} + \theta_i(u - u^{n+1}) \geq \hat{u}_i^{n+1},$$

and so

$$u_i + \hat{u}_i^{n+1} - u_i^{n+1} \in K_i.$$

By choosing the function θ_i in a way that $|\nabla\theta_i| \leq C/\delta$, where δ is the overlapping size, it is easy to calculate that

$$|u_i - u_i^{n+1}|_1^2 \leq C\left(\frac{1}{\delta^2}|u - u^{n+1}|_0^2 + |u - u^{n+1}|_1^2\right) \leq C\left(1 + \frac{1}{\delta^2}\right)|u - u^{n+1}|_1^2,$$

which shows that

$$C_1 \leq C\sqrt{1 + \frac{1}{\delta^2}}.$$

Using the Cauchy-Schwarz inequality, it is also easy to show that

$$C_2 \leq C n_c.$$

In case of finite element approximations, the estimation of C_1 is slightly more complicated.

From the estimates of constants C_1 and C_2, we see that the rate of convergence of Algorithm 1 is

$$\frac{d_{n+1}}{d_n} \leq \frac{1}{1+\gamma C(1+\frac{1}{\delta^2})^{-1}},$$

and the rate of convergence of Algorithm 2 is

$$\frac{d_{n+1}}{d_n} \leq \frac{1}{1+C(1+\frac{1}{\delta^2})^{-1}}.$$

In the above, the generic constant C does not depend on the mesh size or the number of subdomains.

5 Conclusion

The proposed algorithms can be implemented in two different ways. The first approach is to define $u^{n+\frac{i}{m}}$ as in (17) and (25) respectively for Algorithms 1 and 2. It is clear that the value of $u^{n+\frac{i}{m}}$ is known outside Ω_i^c. Inside Ω_i^c, $u^{n+\frac{i}{m}}$ is the solution of an obstacle problem with known Dirichlet boundary conditions. The algorithms obtained by this type of implementation are rather similar to the ones proposed in Tarvainen [39], Badea and Wang [1, 2] and Zeng and Zhou [40]. The boundary values for the subdomain problems of our algorithms are the same as the boundary values of the ones of [39, 1, 2, 40], but the obstacle functions we use for the subdomain problems are normally bigger than the obstacles used in [1, 39, 40]. We see from (38), the obstacle functions for the subdomain problems for Algorithm 1 are

$$\psi_i + \sum_{j=1, j\neq i}^{m} u_j^n \geq \psi_i + \sum_{j=1, j\neq i}^{m} \psi_j = \psi.$$

The obstacle functions for the subdomain problems for Algorithm 2 are given in (41), which are

$$\psi_i + \sum_{j<i} u_j^{n+1} + \sum_{j>i} u_j^n \geq \psi_i + \sum_{j<i} \psi_j + \sum_{j>i} \psi_j = \psi.$$

The above procedure for implementing the domain decomposition method has been used in Tai and Espedal [35, pp.723-735] for partial differential equations without constraint.

Another way of implementing the algorithms is to re-write subproblems (7) and (9) as variational inequalities for e_i^{n+1}. For Algorithm 1, the subproblem (7) is equivalent to

$$a(u^n + e_i^{n+1}, v_i - \hat{u}_i^{n+1}) \geq l(v_i - \hat{u}_i^{n+1}) \quad \forall v_i \in K_i,$$

which can be re-written as

$$\begin{aligned} a(e_i^{n+1}, v - e_i^{n+1}) &\geq l(v_i - e_i^{n+1}) - a(u^n, v_i - e_i^{n+1}), \\ \forall v_i \in K_i^A &= \{v - u_i^n \mid \quad v \in K_i\}. \end{aligned}$$

With such an implementation, we only need to compute the correction values e_i^{n+1} and then update the residuals from the correction values e_i^{n+1}. See Tai and Xu [38, §6] for a detailed algorithm for the above implementation for the case of unconstrained partial differential equations.

Subproblems (7) and (9) can also be solved by approximate solvers as in Tai and Espedal [36, §2]. Assuming that the approximate solutions guarantee that (35) is correct for all $n > 0$, the convergence can be estimated by combining the techniques given here and in [36].

An overlapping domain decomposition with a coarse mesh and multigrid methods can also be interpreted as space decomposition techniques, see Griebel and Zumbusch [14, 15, 16], Kornhuber [24, 23], Tai and Xu [38, §4 and §5] and Chan and Mathew [4]. The implementation with the two-level method and multigrid methods can be done similarly by decomposing the obstacle function ψ into a sum of functions from the subspaces (Tai and Xu [38, §4 and §5]). However, the estimation of the constants C_1 and C_2 cannot be done as for the overlapping domain decomposition method because the iterative solution is generally not monotone, and hence the inequality (35) cannot be guaranteed.

References

[1] L. Badea and J. Wang. An additive Schwarz method for variarional inequalities. Preprint, 1998.

[2] L. Badea and J. Wang. Domain decomposition methods for variational inequalities. In J. Wang et al., editor, *Iterative methods in scientific computation*, volume 4 of *Series in computational and applied mathematics*, pages 199–204. IMACS, 1998.

[3] C. Baicci and A. Capelo. *Variational and quasivariational inequalities.* Wiley, Chichester, 1984.

[4] T. F. Chan and T. P. Mathew. Domain decomposition algorithms. *Acta Numerica*, pages 61–143, 1994.

[5] R. W. Cottle, J. S. Pang, and R. E. Stone. *The linear complementary problem.* Academic Press, Boston, 1992.

[6] G. Duvaut and J. L. Lions. *Inequalities in mechanics and physics.* Springer, Berlin, 1976.

[7] I. Ekeland and R. Temam. *Convex Analysis and Variational Problems.* North–Holland, Amsterdam, 1976.

[8] C. M. Elliot and J. R. Ockendon. *Weak and variational methods for moving boundary problems*, volume 53 of *Research notes in mathematics.* Pitman, London, 1982.

[9] A. Frommer and R. A. Renaut. A unified approach to parallel space decomposition methods. Technical Report Preprint BUGHW-SC 98/6, University of Wuppertal, Germany, October, 1998.

[10] E. Gelman and J. Mandel. On multilevel iterative methods for optimization problems. *Math. Programming*, 48:1–17, 1990.

[11] R. Glowinski. *Numerical methods for nonlinear variational problems.* Springer-Verlag, 1984.

[12] R. Glowinski, J. L. Lions, and R. Trémoléres. *Numerical analysis of variational inequalities.* North-Holland, 1984.

[13] R. Glowinski and P. Le Tallec. *Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics.* SIAM, Philadelphia, 1989.

[14] M. Griebel. Domain oriented multilevel methods. *Domain decomposition methods in scientific and engineering computing (Proc. of the 7th international conference on domain decomposition, Penn. State University, 1993), D. Keyes and J. Xu eds, American Mathematical Society*, pages 223–229, 1993.

[15] M. Griebel. Multilevel algorithms considered as iterative methods on semidefinite systems. *SIAM J. SCi. Comput.*, 15:547–565, 1994.

[16] M. Griebel and G. Zumbusch. Hash-storage techniques for adaptive multilevel solvers and their domain decomposition parallization. In J. Mandel, G. Farhat, and X. Cai, editors, *Domain decomposition 10, 10th international conference on domain decomposition methods*, volume 218 of *Contemporary Matehmatics*, pages 286–293. AMS, 1998.

[17] W. Hackbusch and H. D. Mittelmann. On multilevel methods for variational inequalities. *Numer. Math.*, 42:65–76, 1983.

[18] K. H. Hoffmand and J. Zou. Parallel algorithms of schwarz variant for variational inequalities. *Numer. Funct. Anal. Optim.*, 13:449–462, 1992.

[19] R. Hoppe. Two-sided approximations for unilateral variational inequalities by multi-grid methods. *Optimization*, 18:867–881, 1987.

[20] R. H. Hoppe. Multigrid solutions to the elastic plastic torsion problem in multiply connected domains. *Int. J. Numer. Methods Engin.*, 26:631–646, 1988.

[21] R. H. W. Hoppe. Two-sided approximations for unilateral variational inequalities by multigrid methods. *Optimization*, 16:867—881, 1987.

[22] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications.* Academic Press, New York, 1994.

[23] R. Kornhuber. Monotone multigrid methods for elliptic variational inequalities i. *Numer. Math*, 69:167–184, 1994.

[24] R. Kornhuber. *Adaptive monotone multigrid methods for nonlinear variational problems.* Teubner, Stuttgart, 1997.

[25] T. Lu, T. M. Shih, and C. B. Liem. Parallel algorithm for variational inequalities based on domain decomposition. *Systems Sci. Math. Sci.*, 4:341–348, 1991.

[26] Z.-Q Luo and P. Tseng. Error bounds and convergence analysis of feasible descent methods: a general approach. *Annals of Operations Research*, 46:157–178, 1993.

[27] J. Mandel. Algebric study of a multigrid method for some free boundary problems. *Comptes Rendus Academic of Science Paris, Series I*, 298:77–95, 1984.

[28] J. Pang. Necessary and sufficient conditions for the convergence of iterative methods for the linear complementary problem. *J. Optim. theory Appl.*, 42:1–17, 1984.

[29] R. Rodrigues. *Obstacle problems in mathematical physics.* North-Holland, Amsterdam, 1987.

[30] I. Sharapov. Multilevel subspace correction for large scale optimization problems. Technical Report cam-97-31, UCLA, computational and applied mathematics, 1997.

[31] M. V. Solodov and P. Tseng. Modified projection type methods for monotone variational inequalities. *SIAM J. Control Optim.*, 34:1814–1830, 1996.

[32] X.-C. Tai. Parallel function decomposition and space decomposition methods with applications to optimisation, splitting and domain decomposition. *Preprint No. 231–1992, Institut für Mathematik, TechnischeUniversität Graz*, 1992. See also http://www.mi.uib.no/˜tai.

[33] X.-C. Tai. Parallel function and space decomposition methods. In P. Neittaanmäki, editor, *Finite element methods, fifty years of the Courant element, Lecture notes in pure and applied mathematics*, volume 164, pages 421–432, New York, 1994. Marcel Dekker. See also http://www.mi.uib.no/˜tai.

[34] X.-C. Tai. Parallel function decomposition and space decomposition methods: Part II. Space decomposition. *Beijing Mathematics*, 1, part 2:135–152, 1995. See also http://www.mi.uib.no/˜tai.

[35] X.-C. Tai and M. Espedal. Applications of a space decomposition method to linear and nonlinear elliptic problems. *Numer. Meth. for Part. Diff. Equat.*, 14:717–737, 1998.

[36] X.-C. Tai and M. Espedal. Rate of convergence of some space decomposition method for linear and nonlinear elliptic problems. *SIAM J. Numer. Anal.*, 35:1558–1570, 1998.

[37] X.-C. Tai and P. Tseng. Convergence rate analysis of an asynchronous space decomposition method for convex minimization. Technical Report 120, Department of Mathematics, University of Bergen, August, 1998.

[38] X-C. Tai and J.-C. Xu. Global convergence of subspace correction methods for convex optimization problems. Technical Report No.-114, Department of Mathematics, University of Bergen, Norway, March, 1998. See also http://www.mi.uib.no/˜tai.

[39] P. Tarvainen. Block relaxation methods for algebraic obstacle problems with m-matrices: Theory and applications. *ThD Thesis, Jyvaskyla University, Finland*, 1994.

[40] J. Zeng and S. Zhou. On monotone and geometric convergence of schwarz methods for two sided obstacle problems. *SIAM J. Numer. Anal.*, 35:600–616, 1998.