# Comparison of Sobol' sequences in financial applications

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**Abstract.** Sobol' sequences are widely used for quasi-Monte Carlo methods that arise in financial applications. Sobol' sequences have parameter values called direction numbers, which are freely chosen by the user, so there are several implementations of Sobol' sequence generators. The aim of this paper is to provide a comparative study of (noncommercial) high-dimensional Sobol' sequences by calculating financial models. Additionally, we implement the Niederreiter sequence (in base 2) with a slight modification, that is, we reorder the rows of the generating matrices, and analyze and compare it with the Sobol' sequences.

Keywords. Quasi-Monte Carlo method; Sobol' sequence; Computational finance.

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# 1 Introduction

Monte Carlo (MC) methods are an important numerical tool for pricing many financial derivatives and calculating the Greeks. Generally speaking, these values can be expressed as mathematical expectations, and the expectations reduce to integrals over the *s*-dimensional unit cube  $(0, 1)^s$  after a suitable change of variables, that is,  $\int_{(0,1)^s} f(\mathbf{x}) d\mathbf{x}$  for a function  $f : (0, 1)^s \to \mathbb{R}$  and  $\mathbf{x} := (x_1, \ldots, x_s)$ . However, it is often difficult to evaluate the exact value analytically and the dimension *s* is over hundreds or thousands, so we use MC integration:

$$\int_{(0,1)^s} f(\mathbf{x}) d\mathbf{x} \approx \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n), \tag{1.1}$$

where  $\{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\} \subset (0, 1)^s$  is a point set of independent random samples from the uniform distribution on  $(0, 1)^s$ . MC has a probabilistic error of  $O(N^{-1/2})$ , which does not depend on the dimension s but is significantly slow. To improve the rate of convergence, we apply quasi-Monte Carlo (QMC) methods using

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*low-discrepancy point sets* or *sequences* that are more uniformly distributed than random points (see [7, 22] for the precise definition). Around the middle of the 1990s, a series of studies reported that QMC attains a higher rate of convergence than MC for certain types of high-dimensional numerical integration in finance [1, 4, 15, 23, 25]. Because of this, Sobol' sequences have been widely used since then.

Sobol' sequences are a class of low-discrepancy sequences originally proposed by Sobol' [27] in 1967 and have parameters called *direction numbers*, which are freely chosen by the user. Thus, there are several implementations of Sobol' sequences with distinct parameter values [2, 13, 14, 18, 28]. Some of them have been optimized with the aim of applying them to finance. A comparison of Sobol' sequences for high-dimensional problems in finance was presented in [28], but we want to know further numerical examples, including randomization and effective dimension reduction techniques. According to [14], Joe and Kuo conducted some preliminary calculations for financial models and found that their new Sobol' sequence [14] provided better results in some cases and, at worst, was comparable with the old sequence [13]; however, specific numerical examples were not included in their paper.

The aim of this paper is to provide a comprehensive comparative study of (noncommercial) high-dimensional Sobol' sequences [13, 14, 18] in financial applications. Niederreiter [21, 22] proposed another class of low-discrepancy sequences, called *Niederreiter sequences*. Recently, Faure and Lemieux [8] described the relationships between Sobol' and Niederreiter sequences in detail. Additionally, Faure and Lemieux [9] reported that the Niederreiter sequence (in base 2) with a slight modification, i.e., reordering the rows of the generating matrices, demonstrated high performance in some applications. Motivated by their report, we also analyze the modified Niederreiter sequence and compare it with Sobol' sequences.

In the theory of "analysis of variance" (ANOVA) decomposition [4, 10, 11, 34, 35], it is known that the integrand  $f(\mathbf{x})$  for certain high-dimensional problems in finance is dominated by the first few variables (*low effective dimension in the truncation sense*) or is well approximated by a sum of functions of at most one or two variables (*low effective dimension in the superposition sense*), i.e.,

$$f(\mathbf{x}) = \underbrace{f_0}_{\text{constant}} + \underbrace{\sum_{i=1}^{s} f_i(x_i)}_{\text{order-1 terms}} + \underbrace{\sum_{1 \le i < j \le s} f_{i,j}(x_i, x_j)}_{\text{order-2 terms}} + (\text{small higher-order terms}).(1.2)$$

It is believed that these are reasons why QMC succeeds in high-dimensional numerical integration even if the nominal dimension *s* is over hundreds or thousands. The Sobol' sequence provided by Joe and Kuo [14] was optimized so as to have good two-dimensional (2D) projections for the assumption (1.2). As we shall see later, if the latter condition (1.2) is satisfied but the former condition is not satisfied, that is, if  $f(\mathbf{x})$  has low effective dimension in the superposition sense but high effective dimension in the truncation sense, then such optimization seems to be effective.

The remainder of this paper is organized as follows: In Section 2, we review digital nets and sequences, the *t*-value, which is a criterion of uniformity, and Sobol' and Niederreiter sequences. In Sections 3 and 4, we present our main results. In Section 3, we calculate the frequency of *t*-values of Sobol' and Niederreiter sequences for 2D projections in high dimensions and show that the new Sobol' sequence provided by Joe and Kuo [14] and the modified Niederreiter sequence avoid the existence of extremely large *t*-values. In Section 4, we compare Sobol' and Niederreiter sequences for numerical integration problems, e.g., Asian, digital, and basket options, with or without effective dimension reduction. In Section 5, we conclude this paper.

# 2 Preliminaries

#### 2.1 Digital nets and digital sequences

Following [5, 7, 22], we recall a digital method to construct QMC point sets P and (infinite) sequences S. Sobol' sequences are included in these classes. Let  $\mathbb{F}_2 := \{0, 1\}$  be the two-element field. We perform addition and multiplication over  $\mathbb{F}_2$  (or modulo 2).

**Definition 2.1** (Digital nets). Let  $s \ge 1$  and  $m \ge 1$  be integers. Let  $C_1, ..., C_s \in \mathbb{F}_2^{m \times m}$  be  $m \times m$  matrices over  $\mathbb{F}_2$ . For each  $n = 0, ..., 2^m - 1$ , let  $n = \sum_{l=0}^{m-1} n_l 2^l$  with  $n_l \in \mathbb{F}_2$  be the expansion in base 2. For each  $1 \le i \le s$ , set  $(x_{n,i,0}, \ldots, x_{n,i,m-1})^\top := C_i(n_0, \ldots, n_{m-1})^\top$ , where  $\top$  is the transpose, and  $x_{n,i} := \sum_{l=0}^{m-1} x_{n,i,l} 2^{-l-1}$ . Then, the point set  $P = \{\mathbf{x}_n := (x_{n,1}, \ldots, x_{n,s}) \mid n = 0, \ldots, 2^m - 1\}$  is called a *digital net* over  $\mathbb{F}_2$  and  $C_1, \ldots, C_s$  are called the *generating matrices* of the digital net P.

The concept of digital nets can be extended to (infinite) sequences  $S = {\mathbf{x}_0, \mathbf{x}_1, \ldots} \subset [0, 1)^s$  for  $\infty \times \infty$  generating matrices  $C_1, \ldots, C_s \in \mathbb{F}_2^{\infty \times \infty}$  and infinite expansions  $n = \sum_{l=0}^{\infty} n_l 2^l$  and  $x_{n,i} := \sum_{l=0}^{\infty} x_{n,i,l} 2^{-l-1}$  that contain only a finite number of nonzero terms. The resulting sequence S is called a *digital sequence* and the matrices  $C_1, \ldots, C_s$  are called the *generating matrices* of the digital sequence.

# 2.2 (t, m, s)-nets

As a quality parameter of uniformity for a point set P, we recall the definition of the *t*-value. See [5,7,22] for details.

**Definition 2.2.** ((t, m, s)-nets). Let  $s \ge 1$ , and t be an integer with  $0 \le t \le m$ . A point set  $P = {\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{2^m-1}}$  consisting of  $2^m$  points in  $[0, 1)^s$  is called a (t, m, s)-net (in base 2) if every subinterval  $J = \prod_{i=1}^s [a_i/2^{d_i}, (a_i + 1)/2^{d_i}) \in [0, 1)^s$  with integers  $d_i \ge 0$  and  $0 \le a_i < 2^{d_i}$  for  $1 \le i \le s$  and of volume  $2^{t-m}$ contains exactly  $2^t$  points of P.

**Definition 2.3** (*t*-value for a (t, m, s)-net). The minimum *t* that satisfies the above property is called the *t*-value for a (t, m, s)-net.

A point set P is well distributed if the t-value is small. The integration error is bounded by  $O(2^t(\log N)^{s-1}/N)$  for  $N = 2^m$  when f is smooth. The factor  $(\log N)^{s-1}$  is not negligible if s is large, but QMC works well for high-dimensional numerical integration in finance possibly because  $f(\mathbf{x})$  has low effective dimension. In the case of digital nets, the t-value can be easily calculated by some algorithms [6, 26].

## 2.3 Sobol' and Niederreiter sequences

Sobol' [27] proposed a construction method for generating matrices  $C_1, \ldots, C_s \in \mathbb{F}_2^{\infty \times \infty}$  that have a good structure of (t, m, s)-nets. His sequences are now called *Sobol' sequences* and are included in a subclass of *generalized Niederreiter sequences* [31, 32]. Recently, Faure and Lemieux [8] described the relationships between them in detail. From this viewpoint, Sobol' sequences are formulated as follows:

- (i) Let p<sub>1</sub>(x) = x ∈ F<sub>2</sub>[x] and p<sub>i</sub>(x) ∈ F<sub>2</sub>[x], 2 ≤ i ≤ s, be the (i − 1)th primitive polynomials in a list of primitive polynomials that are sorted in non-decreasing order of degree, i.e., p<sub>2</sub>(x) = x + 1, p<sub>3</sub>(x) = x<sup>2</sup> + x + 1, and so on. Let e<sub>i</sub> := deg(p<sub>i</sub>).
- (ii) For each  $1 \leq i \leq s$ , set polynomials  $g_{i,0}(x), \ldots, g_{i,e_i-1}(x) \in \mathbb{F}_2[x]$  such that

$$\deg g_{i,k}(x) = e_i - 1 - k \tag{2.1}$$

for  $0 \le k \le e_i - 1$ , in advance. These polynomials are the parameters that can be freely chosen by the user and correspond one-to one to the so-called *direction numbers* (see Remark 2.4 for details).

(iii) For u = 1, 2, ..., consider the formal power series expansion

$$\frac{g_{i,k}(x)}{p_i(x)^u} = \sum_{v=1}^{\infty} a^{(i)}(u,k,v) x^{-v} \in \mathbb{F}_2((x^{-1})).$$
(2.2)

(iv) Define  $C_i = (c_{j,v}^{(i)})_{j \ge 1, v \ge 1} \in \mathbb{F}_2^{\infty \times \infty}$  as  $c_{j,v}^{(i)} = a^{(i)}(Q+1,k,v) \in \mathbb{F}_2$  for  $1 \le i \le s, j \ge 1, v \ge 1$ , where

$$j - 1 = Qe_i + k, \tag{2.3}$$

with integers Q = Q(i, j) and k = k(i, j) satisfying  $0 \le k \le e_i - 1$ . Note that each row of  $C_i$  corresponds to each formal power series expansion in (2.2). Note that the conditions (2.1) and (2.3) correspond to the reordering of rows of  $C_i$  so as to obtain non-singular upper triangular (NUT) matrices.

The first  $2^m$  points P can be viewed as a digital net generated by the upper-left  $m \times m$  submatrices of  $C_1, \ldots, C_s \in \mathbb{F}_2^{\infty \times \infty}$ . We can prove that P is a (t, m, s)-net with the following properties (see [7, 22]):

- Each one-dimensional (1D) projection is a (0, m, 1)-net, which means that 1D projections have already been optimized, that is, each *t*-value is 0.
- The t-value is ≤ ∑<sup>s</sup><sub>i=1</sub>(e<sub>i</sub> − 1) for any m, which means that the initial dimensions have already been optimized.
- For any low-dimensional projection, the *t*-value is ≤ ∑(e<sub>i</sub> − 1), where ∑ is taken over the corresponding projections.

The condition (2.1) can be described as

$$g_{i,k}(x) = x^{e_i - 1 - k} + (\text{lower terms}) \in \mathbb{F}_2[x]$$
(2.4)

for  $0 \le k \le e_i - 1$ , and a good selection of lower terms makes us obtain *t*-values smaller than those of the above upper bounds.

**Remark 2.4.** Sobol' [27] originally proposed a column-by-column construction for generating matrices  $C_i$  using recurrences of columns based on primitive polynomials  $p_i(x)$  for each  $1 \le i \le s$ . In this construction, the upper-left  $e_i \times e_i$ submatrices of generating matrices  $C_i$  are initial values, and were originally called the *direction numbers*, which exactly correspond one-to-one to the polynomials  $g_{i,0}(x), \ldots, g_{i,e_i-1}(x)$  with (2.1). See [8] for details. **Remark 2.5.** Niederreiter [21,22] proposed another construction method for generating matrices  $C_1, \ldots, C_s \in \mathbb{F}_b^{\infty \times \infty}$  for low-discrepancy sequences, where *b* is a prime power and  $\mathbb{F}_b$  is a finite field with *b* elements. These sequences are called *Niederreiter* sequences. In the standard implementation of [3] in base b = 2, the main differences from Sobol' sequences are that  $p_i(x) \in \mathbb{F}_2[x], 1 \le i \le s$ , are taken to be *irreducible* polynomials (sorted in non-decreasing order of degree) instead of primitive polynomials, and  $g_{i,0}(x), \ldots, g_{i,e_i-1}(x)$  are taken to be

$$g_{i,k}(x) = x^k \in \mathbb{F}_2[x]$$

for  $e_i = \deg(p_i)$  and  $0 \le k \le e_i - 1$ , instead of polynomials with the condition (2.1). Note that there are no freely chosen parameters. Note that the resulting generating matrices  $C_1, \ldots, C_s \in \mathbb{F}_2^{\infty \times \infty}$  do not have the NUT properties (see Figs. 1 and 2 in [8]), so we suffer from the *leading-zero phenomenon*, that is, there are too many points close to the origin at the beginning of the sequences. Additionally, note that we obtain the NUT generating matrices after reordering the rows of  $C_i$ . According to Theorem 4.3 of [8], such NUT generating matrices are obtained by the original column-by-column construction, which implies that the primitivity of  $p_i(x)$  is not necessary.

# **3** Analysis of Sobol' and Niederreiter sequences with NUT generating matrices

In this section, we compare high-dimensional Sobol' and Niederreiter sequences in terms of the *t*-values. In 1976, Sobol' and Levitan [29] provided direction numbers in terms of Property A and Property A' [30], which are the criteria for the equidistribution property of the 1 and 2 most significant bits, respectively. In 1988, Bratley and Fox [2] provided a FORTRAN implementation of the Sobol' sequence using this set of direction numbers up to dimension 40. In our tests, we investigate the following (non-commercial) high-dimensional Sobol' and Niederreiter sequences released after 2000:

- (a) In 2003, Joe and Kuo [13] provided a Sobol' sequence generator up to dimension 1111. The direction numbers for 1 ≤ s ≤ 40 are the same as those of Bratley and Fox [2]. The direction numbers for 40 < s ≤ 1111 are selected so as to satisfy Property A. We refer to this generator as Sobol' (JoeKuo03).</p>
- (b) Lemieux et al. [18] provided a Sobol' sequence generator up to dimension 360. The direction numbers for  $1 \le s \le 40$  are the same as those of Bratley and Fox [2]. The direction numbers for  $40 < s \le 360$  are optimized in terms of the resolution criterion for eight successive dimensions, see [17,

Chapter 3.5.2]. We use the 2004 version and refer to this generator as Sobol' (Lemieux).

- (c) In 2008, Joe and Kuo [14] indicated that the 2003 version of the Sobol' sequence generator has a bad structure (i.e., extremely large *t*-values) for some 2D projections and searched the new direction numbers up to 21201. Their approach was to choose the direction numbers so that (i) Property A holds for  $1 \le s \le 1111$ ; and (ii) the t-values of 2D projections of the point sets are minimized by proposing the search criterion  $D^{(6)}$ . Consequently, extremely large *t*-values are avoided. We refer to this new generator as Sobol' (JoeKuo08).
- (d) Recently, Faure and Lemieux [8] discussed the Niederreiter sequence (in base 2) with NUT generating matrices after reordering the rows of the generating matrices. This sequence can be viewed as the Sobol' sequence based on irreducible polynomials p<sub>1</sub>(x) = x, p<sub>2</sub>(x), ..., p<sub>s</sub>(x) ∈ F<sub>2</sub>[x] with nondecreasing order of degree and given by g<sub>i,k</sub>(x) = x<sup>e<sub>i</sub>-1-k</sup> for e<sub>i</sub> = deg(p<sub>i</sub>) and 0 ≤ k ≤ e<sub>i</sub> 1, so as to satisfy the condition (2.1). Additionally, Faure and Lemieux [9] reported that the Niederreiter sequence with NUT generating matrices already demonstrated high performance in some high-dimensional numerical integrations without optimizing the lower terms of g<sub>i,k</sub>(x) in (2.4). To confirm their new findings, we implement this generator and refer to it as Niederreiter (NUT). The irreducible polynomials and direction numbers are available at https://github.com/sharase/niederreiter-nut.

In addition, Kucherenko et al. [28] released commercial software for Sobol' sequences up to dimension 65536 with Property A for all dimensions and Property A' for successive dimensions, but we exclude it from our tests because it requires a commercial license.

To compare the sequences (a)–(d), we assume that the integrand  $f(\mathbf{x})$  satisfies the condition (1.2). The *t*-values of 1D projections are all 0, so we calculate the *t*-values of 2D projections. Let  $m \ge 1$  and

denote the *t*-value of the digital net that corresponds to the (i, s)-projection (i.e., the 2D projection of dimensions *i* and *s* with  $1 \le i \le s - 1$ ) of the first  $2^m$  points. Tables 1 and 2 show the frequency of all the values of t(i, s; m)  $(1 \le i \le s - 1)$  for  $2 \le s \le 360$  and  $2 \le s \le 1024$ , respectively. Sobol' (Lemieux) is up to dimension 360, and hence it is excluded in Table 2. From the tables, there exist extremely large *t*-values for Sobol' (Lemieux) and (JoeKuo03), but Niederreiter (NUT) tends to avoid such large *t*-values, as well as Sobol' (JoeKu08). Conversely, the occurrence of small *t*-values (e.g., 1 or 2) for Niederreiter (NUT) is more frequent than

for Sobol' sequences. This implies that Niederreiter (NUT) has high uniformity for 2D projections in high dimensions without optimizing direction numbers.

# 4 Comparison in financial applications

We compare Niederreiter (NUT) and Sobol' sequences in Section 3 from the viewpoint of financial applications. In the QMC setting, we apply *randomizations* using *linear scrambling* and *digital shift* to point sets (see [12, 16, 17] for details). This technique preserves the *t*-values of (t, m, s)-nets and avoids the problem that the first point is always the origin. We apply the randomizations M times, make Mpoint sets  $\tilde{P}_l = {\{\tilde{\mathbf{x}}_n^{(l)}\}} \subset (0, 1)^s$  (l = 1, ..., M), and compute M independent estimates of (1.1):

$$Q_l := \frac{1}{N} \sum_{n=0}^{N-1} f(\tilde{\mathbf{x}}_n^{(l)}).$$

Further, we compute the mean and the standard error of  $Q_1, \ldots, Q_M$ , i.e.,

$$\bar{Q} := \frac{1}{M} \sum_{l=1}^{M} Q_l$$
, stderr $(\bar{Q}) := \sqrt{\frac{1}{M(M-1)} \sum_{l=1}^{M} (Q_l - \bar{Q})^2}$ .

Throughout this paper, we set M = 100 as the number of randomizations.

#### 4.1 Asian option

Assume that under the risk-neutral measure the asset price  $S_t$  follows the Black–Scholes model (i.e., geometric Brownian motion):

$$dS_t = rS_t dt + \sigma S_t dB_t, \tag{4.1}$$

where r is the risk-free interest rate,  $\sigma$  is the volatility,  $B_t$  is a standard Brownian motion. The problem of pricing an Asian call option on the discrete arithmetic average is formulated as follows: the payoff function is given by  $\max(0, \frac{1}{s}\sum_{i=1}^{s} S_{t_i} - K)$ , where K is the strike price at maturity T, and a time interval [0, T] is discretized at equally spaced times  $t_i = i\Delta t$  for  $i = 1, \ldots, s$ , where  $\Delta t = T/s$ . Then, the value of the option at time 0 is given by

$$E\left[e^{-rT}\max(0, \frac{1}{s}\sum_{i=1}^{s}S_{t_i} - K)\right].$$
(4.2)

The analytical solution to (4.1) is given by  $S_t = S_0 \exp((r - \sigma^2/2)t + \sigma B_t)$ , so it is sufficient to simulate sample paths of Brownian motion. The standard construction of Brownian motion is to generate  $B_{t_i}$  sequentially in time: given  $B_0 = 0$ ,

$$B_{t_i} = B_{t_i-1} + \sqrt{\Delta t} Z_i, \quad i = 1, \dots, s,$$
 (4.3)

where  $Z_1, \ldots, Z_s \sim N(0, 1)$  are i.i.d. standard normally distributed random variables. The standard construction (4.3) can be written as

$$(B_{t_1}, \dots, B_{t_s})^{\top} = A(Z_1, \dots, Z_s)^{\top}, \quad A = \sqrt{\Delta t} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$
 (4.4)

where A is an  $s \times s$  lower triangular matrix. Thus, the expectation (4.2) can be written as

$$\int_{(0,1)^s} e^{-rT} \max\left(0, \frac{1}{s} \sum_{i=1}^s S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)t_i + \sigma w_i\right] - K\right) d\mathbf{x}$$

where  $\Phi : (0, 1) \to \mathbb{R}$  denotes the cumulative distribution function of the standard normal distribution and  $(w_1, \ldots, w_s)^\top := A(\Phi^{-1}(x_1), \ldots, \Phi^{-1}(x_s))^\top$  for  $\mathbf{x} = (x_1, \ldots, x_s) \in (0, 1)^s$ . We use the following parameters:  $T = 1, r = 0.1, \sigma = 0.2, S_0 = 100, K = 100$ , which were used in [34].

First, we consider the case of dimension s = 360. Figure 1 shows a summary of the standard error stderr( $\bar{Q}$ ) in  $\log_2$  scale for  $m = 1, \ldots, 20$ . In our experiments, we applied QMC methods based on Sobol' and Niederreiter sequences (a)–(d) and crude MC methods using random number sequences from Mersenne Twister [19]. For this, we observed that Niederreiter (NUT) and Sobol' (JoeKuo08) are more effective than the others, particularly for  $m = 10, \ldots, 18$ , which are often used in practice. Additionally, our result seems to agree with the frequency of *t*-values for 2D projections in Table 1. Thus, it is inferred that the Asian option using the standard construction (4.3) has low effective dimension in the superposition sense but high effective dimension in the truncation sense.

Further, we consider the higher dimensional case s = 1024. Figure 2 shows a summary of stderr( $\bar{Q}$ ) in log<sub>2</sub> scale. In the standard construction (4.3), Niederreiter (NUT) and Sobol' (JoeKuo08) also provide better results than Sobol' (JoeKuo03). Further, we recall *dimension reduction techniques*, such as the principal component analysis (PCA) construction [1] for generating Brownian motion  $B_t$ , which enhance the efficiency of QMC methods. Here, the sampled Brownian motion  $(B_{t_1}, \ldots, B_{t_s})^{\top}$  is normally distributed with mean **0** and covariance matrix  $C = (\min(t_i, t_j))_{i,j=1}^s$ , i.e.,  $(B_{t_1}, \ldots, B_{t_s})^{\top} \sim N(\mathbf{0}, C)$ . Generally, we obtain the equivalent paths of Brownian motion:

$$(B_{t_1},\ldots,B_{t_s})^{\top} = A(Z_1,\ldots,Z_s)^{\top}, \quad (Z_1,\ldots,Z_s)^{\top} \sim N(\mathbf{0},I_s),$$

provided we apply the change of variables  $\mathbf{x} = A\mathbf{z}$  with  $AA^{\top} = C$ . The matrix A in the standard construction (4.4) is the Cholesky matrix of C, i.e.,  $AA^{\top} = C$ . Conversely, the PCA construction is a method used to choose  $A = [\sqrt{\lambda_1}\mathbf{v}_1, \dots, \sqrt{\lambda_s}\mathbf{v}_s]$ , where  $\lambda_1 \geq \dots \geq \lambda_s$  are the eigenvalues and  $\mathbf{v}_1, \dots, \mathbf{v}_s$  are the corresponding unit-length eigenvectors of C. In our experiment, the Niederreiter (NUT) and Sobol' sequences with PCA outperform those with the standard construction, but those with PCA have exactly the same convergence rates. Our result implies that PCA transforms the integrand so as to have low effective dimension in the truncation sense, that is, the important variables are concentrated in the first few dimensions (e.g.,  $\leq 2$  or 3). However, for Sobol' sequences, the lower terms of  $g_{i,k}(x)$  in (2.4) have almost no choice and are fixed in these first dimensions because the degree  $e_i$  in (2.4) is sufficiently small. Thus, it seems to be difficult to expect further improvement for dimension reduction techniques as a result of changing the direction numbers for Sobol' sequences.

We also tested the Brownian bridge (BB) construction [20] as another dimension reduction technique and observed that there is no difference among Niederreiter (NUT) and Sobol' sequences for the convergence rates, which are better than the standard construction but worse than PCA, so we omitted the results.

# 4.2 Digital option

Assume that the asset price  $S_t$  follows the Black–Scholes model (4.1). Papageorgiou [24] considered the following *digital option*:

$$E\left[\frac{1}{s}\sum_{i=1}^{s}(S_{t_i}-S_{t_{i-1}})_+^0S_{t_i}\right],$$
(4.5)

where  $(x)^0_+$  is equal to 1 if x > 0 and is 0 otherwise,  $x \in \mathbb{R}$ . He indicated that effective dimension reduction techniques perform worse than the standard construction (4.3). Wang and Tan [36] and Wang [33] found that if the paths are generated by the standard construction, then the discontinuities of the payoff function of the sum of the indicator functions are aligned with the coordinate axes, so good performance is expected, but BB and PCA do not have this type of discontinuity. Thus, the standard construction is a good choice in this QMC setting. Figure 3 shows a summary of stderr( $\overline{Q}$ ) in log<sub>2</sub> scale. We used the parameters  $s = 128, T = 1, r = 0.045, \sigma = 0.3, S_0 = 100$  from [24]. Indeed, PCA is worse than the standard construction. Niederreiter (NUT) and Sobol' (JoeKuo08) with good 2D projections are useful for such a problem. Note that this example is very simple and the value (4.5) can be calculated analytically.

### 4.3 Basket option

Following [34, 35], under the risk-neutral measure, we consider a European-style basket call option on the arithmetic average over s assets  $S_t^{(1)}, \ldots, S_t^{(s)}$ , and assume that each asset satisfies

$$dS_t^{(i)} = rS_t^{(i)}dt + \sigma^{(i)}S_t^{(i)}dB_t^{(i)} \quad (i = 1, \dots, s),$$
(4.6)

for a mean return parameter r and volatility parameters  $\sigma^{(i)}$ . Assume  $B_t^{(1)}, \ldots, B_t^{(s)}$  are correlated Brownian motions with correlations  $\rho_{ij}$ , and the terminal pay off at T is given by  $\max(0, \frac{1}{s} \sum_{i=1}^{s} S_T^{(i)} - K)$ . For this, we compute the price of the basket option:

$$E\left[e^{-rT}\max\left(0,\frac{1}{s}\sum_{i=1}^{s}S_{T}^{(i)}-K\right)\right].$$
(4.7)

Note that s is the number of assets, not the number of discretization steps. The solutions to (4.6) are given by  $S_t^{(i)} = S_0^{(i)} \exp((r - (\sigma^{(i)})^2/2)t + \sigma^{(i)}B_t^{(i)})$ . Here, the random vector  $(B_T^{(1)}, \ldots, B_T^{(s)})^\top$  is normally distributed with mean **0** and covariance matrix  $C = (\rho_{ij}T)_{i,j=1}^s$ . Let  $(Z_1, \ldots, Z_s)^\top \sim N(\mathbf{0}, I_s)$ . The standard construction for generating Brownian motion is  $(B_T^{(1)}, \ldots, B_T^{(s)})^\top = A(Z_1, \ldots, Z_s)^\top$ , where A is the Cholesky matrix of C. By contrast, the PCA chooses  $A = [\sqrt{\lambda_1}\mathbf{v}_1, \cdots, \sqrt{\lambda_s}\mathbf{v}_s]$ where  $\lambda_1 \ge \cdots \ge \lambda_s$  are the eigenvalues and  $\mathbf{v}_1, \ldots, \mathbf{v}_s$  are the corresponding unit-length eigenvectors of C. Expectation (4.7) is expressed as

$$\int_{(0,1)^s} e^{-rT} \max\left(0, \frac{1}{s} \sum_{i=1}^s S_0^{(i)} \exp\left[\left(r - \frac{(\sigma^{(i)})^2}{2}\right)T + \sigma^{(i)} w_i\right] - K\right) d\mathbf{x},$$

where  $(w_1, \ldots, w_s)^{\top} := A(\Phi^{-1}(x_1), \ldots, \Phi^{-1}(x_s))^{\top}$ . We set the parameters  $s = 128, T = 1, r = 0.1, \sigma^{(i)} = 0.2, \rho_{ij} = 0.3 (i \neq j), S_0^{(i)} = 100, K = 100$ , which are taken from [34, 35], and conduct experiments on the standard and PCA constructions. Figure 4 shows a summary of stderr( $\bar{Q}$ ) in log<sub>2</sub> scale. Unlike the previous examples, the Niederreiter (NUT) and Sobol' sequences using the standard (Cholesky) construction have exactly the same convergence rates. According to [34, Table 3 and 6] and [35, Table 3 and Table 5], it is inferred that the value

of basket options using the standard (Cholesky) construction is determined by depending on the first few variables or depending on a high proportion to order-1 terms  $\sum_{i=1}^{s} f_i(x_i)$  in (1.2), compared with those of Asian options.

#### 4.4 Asian option under the Heston model

As a more complicated model, under the risk-neutral measure, we consider the pricing of an Asian call option (4.2) with maturity T and strike K written on an asset whose price process  $S_t$  satisfies the Heston stochastic volatility model:

$$dS_t = rS_t dt + \sigma_t S_t \left[ \rho dB_t^{(1)} + \sqrt{1 - \rho^2} dB_t^{(2)} \right],$$
  
$$d\sigma_t^2 = \kappa \left[ \theta - \sigma_t^2 \right] dt + \xi \sigma_t dB_t^{(1)},$$

where  $\sigma_t^2$  is the volatility process,  $B_t^{(1)}$  and  $B_t^{(2)}$  are two independent standard Brownian motions, r is the risk-free interest rate,  $\kappa$  is the *speed of mean reversion*,  $\theta > 0$  is the *long-run mean variance*,  $\xi$  is the *volatility of the volatility*,  $\rho$  is the correlation between the Brownian motions driving  $S_t$  and  $\sigma_t^2$ . The volatility process  $\sigma_t^2$  follows a CIR process, which is always positive under the assumption  $2\kappa\theta > \xi^2$ . We use the Euler–Maruyama scheme with s steps to discretize both  $S_t$  and  $\sigma_t^2$  as in [17, Fig. 7.3 in Chapter 7.2.1]. Let  $\Delta t = T/s$ . Then, we need 2s-dimensional points to simulate both  $S_{t_i}$  and  $\sigma_{t_i}^2$  for  $t_i = i\Delta t$  (i = 1, ..., s).

Figure 5 gives results for an Asian option under the Heston model with s = 512. We use the parameters T = 0.5, r = 0,  $\kappa = 2$ ,  $\theta = 0.01$ ,  $\xi = 0.1$ ,  $\rho = 0.5$ ,  $S_0 = 100$ ,  $\sigma_0 = 0.1$ , K = 100, which are from [17, Chapter 7.3]. Note that Niederreiter (NUT) and Sobol' (JoeKuo08) give better results than Sobol' (JoeKu003).

# 5 Concluding remarks

Sobol' sequences have been used successfully in high-dimensional numerical integration in financial applications. There are several implementations of Sobol' sequence generators with distinct direction numbers, so it is natural to assess which of them is better. Hence, we tested Sobol' sequences for calculating financial models and observed that the Sobol' sequence with good 2D projections [14] outperforms the previous Sobol' sequences [13, 18], particularly in the case in which the integrands have low effective dimension in the superposition sense but high effective dimension in the truncation sense. Additionally, we implemented the Niederreiter sequence with NUT generating matrices, suggested by Faure and Lemieux [8, 9]. Surprisingly, the modified Niederreiter sequence has already had good 2D projections, and had high performance for calculating some financial models without optimizing direction numbers.



Figure 1. Comparison of Niederreiter (NUT) and Sobol' sequences for the pricing of an Asian option with s = 360.



Figure 2. Comparison of Niederreiter (NUT) and Sobol' sequences for the pricing of an Asian option using the standard and PCA constructions for s = 1024.



Figure 3. Comparison of Niederreiter (NUT) and Sobol' sequences for the pricing of a digital option with s = 128.



Figure 4. Comparison of Niederreiter (NUT) and Sobol' sequences for the pricing of a basket option with number of assets s = 128.



Figure 5. Comparison of Niederreiter (NUT) and Sobol' sequences for the pricing of an Asian option under the Heston model for s = 512.

Finally, we mention the possibility of the further improvement of Sobol' sequences. In fact, we attempted the further improvement of Sobol' type digital nets by optimizing direction numbers in terms of the framework of generalized Niederreiter sequences based on irreducible polynomials. For example, we attempted to search direction numbers so as to have even better *t*-values for 2D projections or better three-dimensional projections in addition to good 2D projections. The aim appears to be theoretically achieved, but we could not observe a clear difference from the Niederreiter sequence with the NUT generating matrices for calculating financial models. Perhaps, there might be a limit to the further improvement of Sobol' sequences by optimizing direction numbers from a practical perspective. Therefore, further improvement of Sobol' sequences is left for future work.

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Table 1. Frequency of t(i, s; m) for  $2 \le s \le 360$ .

		Number	of occurrer	nces of the t	t-value						
m		0	1	2	3	4	5	6	7	8	9
10	Niederreiter (NUT)	163	11321	23097	16270	7947	3495	1472	576	231	48
	Sobol' (JoeKuo08)	214	8201	20243	18004	10275	4819	1924	777	163	
	Sobol' (Lemieux)	204	8210	20040	18000	10092	4865	1991	863	233	122
	Sobol' (JoeKuo03)	204	8208	20285	17854	9886	4997	1961	835	250	140
12	Niederreiter (NUT)	71	7679	22265	17458	9527	4353	1954	864	332	79
	Sobol' (JoeKuo08)	62	4752	17648	19105	12303	6334	2848	1127	389	52
	Sobol' (Lemieux)	49	4757	17452	18973	12091	6408	2842	1270	498	191
	Sobol' (JoeKuo03)	56	4774	17342	19027	12307	6249	2814	1276	467	215
14	Niederreiter (NUT)	21	5119	19870	19164	11111	5373	2375	1020	405	134
	Sobol' (JoeKuo08)	14	2857	14942	19442	14020	7581	3516	1551	557	140
	Sobol' (Lemieux)	15	2913	14912	19155	13589	7531	3581	1690	705	329
	Sobol' (JoeKuo03)	16	2864	14967	19131	13669	7414	3634	1698	696	329
16	Niederreiter (NUT)	6	3044	16906	20478	13001	6584	2903	1115	434	130
	Sobol' (JoeKuo08)	5	1771	12568	19566	14939	8566	4252	1893	750	285
	Sobol' (Lemieux)	4	1815	12586	19696	14735	8257	4077	1917	846	408
	Sobol' (JoeKuo03)	6	1745	12503	19418	15039	8354	4097	1902	890	390
18	Niederreiter (NUT)	8	1804	14312	20990	14831	7440	3267	1343	455	144
	Sobol' (JoeKuo08)	3	1119	10985	19412	15999	9175	4595	2087	830	335
	Sobol' (Lemieux)	1	1168	10897	19421	15963	9064	4538	2028	854	380
20	Sobol (JoeKuo03)	1	1075	10866	19422	15870	9236	4540	2003	909	397
20	Niederreiter (NUT)	3	1183	12437	20942	15387	8394	3806	1594	030	189
	Sobol' (JoeKuo08)	1	787	9470	19570	16932	9897	4/36	1992	831	320
	Sobol (Lemieux)	1	762	9615	19188	10888	9/98	4884	2052	808	325
	50001 (JOEKU005)	1	/04	9326	19188	17006	9834	4790	2117	817	331
		Number	of occurren	ces of the t	-value (cor	itinued)	15	16	17	19	10
10	Niederreiter (NUT)	10	11	12	15	14	15	10	17	10	19
10	Sobol' (JoeKuo(8)										
	Sobol' (Joerdoos)										
	Sobol' (Lenneux)										
12	Niederreiter (NUT)	38									
12	Sobol' (JoeKuo08)	50									
	Sobol' (Lemieux)	56	33								
	Sobol' (JoeKuo03)	56	37								
14	Niederreiter (NUT)	28									
	Sobol' (JoeKuo08)										
	Sobol' (Lemieux)	124	50	17	9						
	Sobol' (JoeKuo03)	135	50	8	9						
16	Niederreiter (NUT)	19									
	Sobol' (JoeKuo08)	25									
	Sobol' (Lemieux)	150	76	28	19	3	3				
	Sobol' (JoeKuo03)	161	71	32	11		1				
18		24	2								
	Niederreiter (NUT)	24	2								
	Niederreiter (NUT) Sobol' (JoeKuo08)	80	2								
	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (Lemieux)	80 167	82	31	17	4	3		2		
	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (Lemieux) Sobol' (JoeKuo03)	80 167 174	82 73	31 32	17 14	4 7	3		2		
20	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (Lemieux) Sobol' (JoeKuo03) Niederreiter (NUT)	80 167 174 41	82 73 8	31 32	17 14	4 7	3 1		2		
20	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (Lemieux) Sobol' (JoeKuo03) Niederreiter (NUT) Sobol' (JoeKuo08)	24 80 167 174 41 82	82 73 8 2	31 32	17 14	4 7	3 1		2		
20	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (Lemieux) Sobol' (JoeKuo03) Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (Lemieux)	24 80 167 174 41 82 140	82 73 8 2 55	31 32 24	17 14 12	4 7 4	3 1 2	1	2		

Table 2. Frequency of t(i, s; m) for  $2 \le s \le 1024$ .

		Number	of occurrer	nces of the $t$	-value						
m		0	1	2	3	4	5	6	7	8	9
10	Niederreiter (NUT)	1217	91368	187247	131306	64096	28622	12135	5079	1881	825
	Sobol' (JoeKuo08)	1713	66135	163425	146133	81378	39763	15828	6864	1920	617
	Sobol' (JoeKuo03)	1761	65788	163011	146129	80810	40093	16015	7134	2025	1010
12	Niederreiter (NUT)	358	61934	178807	141504	77363	35943	16214	7030	2920	1188
	Sobol' (JoeKuo08)	464	37931	140291	154369	99570	51840	23418	10403	3884	1380
	Sobol' (JoeKuo03)	420	37691	139255	154868	99220	51931	23360	10514	4002	1756
14	Niederreiter (NUT)	131	41284	161893	153405	87958	43456	20196	9017	3904	1672
	Sobol' (JoeKuo08)	113	21774	116341	156401	113622	62704	30350	13787	5688	2326
	Sobol' (JoeKuo03)	109	21745	115939	155545	112483	63286	30287	14127	5909	2706
16	Niederreiter (NUT)	51	25934	142119	159612	101194	51623	24259	10993	4912	2035
	Sobol' (JoeKuo08)	29	12562	94650	153576	123743	73026	36970	17348	7471	3128
	Sobol' (JoeKuo03)	37	12580	94525	152964	123493	72503	36228	17522	7846	3583
18	Niederreiter (NUT)	20	15374	118340	164947	114523	60295	28790	12922	5415	2118
	Sobol' (JoeKuo08)	14	7362	77577	148682	130582	81034	42592	20643	9316	3984
	Sobol' (JoeKuo03)	8	7464	77049	148627	131116	80131	42178	20536	9189	4254
20	Niederreiter (NUT)	9	8790	95347	163784	127680	71672	33322	14166	5747	2254
	Sobol' (JoeKuo08)	4	4609	64037	144019	137413	87005	46421	22518	10407	4664
	Sobol' (JoeKuo03)	3	4524	64039	143480	137250	87005	46332	22273	10359	4831
		NT 1	c	6.4 /	1 (	· 1)					
		Number	of occurren	ces of the $t$ -	value (con	inued)					
m		Number 10	or occurren 11	12	Value (con 13	inued) 14	15	16	17	18	19
m 10	Niederreiter (NUT)	Number 10	or occurren	12	13	111ued) 14	15	16	17	18	19
m 10	Niederreiter (NUT) Sobol' (JoeKuo08)	Number 10	11	12	13	14	15	16	17	18	19
m 10	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (JoeKuo03)	10	or occurren	12	13	14	15	16	17	18	19
m 10 12	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (JoeKuo03) Niederreiter (NUT)	10 432	83	12	13	14	15	16	17	18	19
m 10 12	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (JoeKuo03) Niederreiter (NUT) Sobol' (JoeKuo08)	432 226	83	12	13	14	15	16	17	18	19
m 10 12	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (JoeKuo03) Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (JoeKuo03)	432 226 519	83 240	12	13	14	15	16	17	18	19
m 10 12 14	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (JoeKuo03) Niederreiter (NUT) Sobol' (JoeKuo03) Niederreiter (NUT)	432 226 519 675	83 240 139	46	13	14	15	16	17	18	19
m 10 12 14	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (JoeKuo03) Niederreiter (NUT) Sobol' (JoeKu008) Sobol' (JoeKu003) Niederreiter (NUT) Sobol' (JoeKu008)	432 226 519 675 622	83 240 139 48	46	13	14	15	16	17	18	19
m 10 12 14	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (JoeKuo03) Niederreiter (NUT) Sobol' (JoeKu008) Sobol' (JoeKu008) Sobol' (JoeKu008) Sobol' (JoeKu008)	Number 10 432 226 519 675 622 1020	83 240 139 48 436	46 12	58	14	15	16	17	18	19
m 10 12 14	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (JoeKuo03) Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (JoeKuo03) Niederreiter (NUT) Sobol' (JoeKuo03) Niederreiter (NUT)	432 226 519 675 622 1020 793	83 240 139 48 436 220	46 12 46 126 31	58	14	15	16	17	18	19
m 10 12 14 16	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (JoeKuo03) Niederreiter (NUT) Sobol' (JoeKuo03) Niederreiter (NUT) Sobol' (JoeKu003) Sobol' (JoeKu003) Niederreiter (NUT) Sobol' (JoeKu003)	Aumber 10 432 226 519 675 622 1020 793 1044	83 240 139 48 436 220 226	46 12 46 126 31 3	58	14	15	16	17	18	19
m 10 12 14 16	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (JoeKuo03) Niederreiter (NUT) Sobol' (JoeKu008) Sobol' (JoeKu008) Sobol' (JoeKu008) Sobol' (JoeKu008) Sobol' (JoeKu008) Sobol' (JoeKu008) Sobol' (JoeKu003)	432 226 519 675 622 1020 793 1044 1439	83 240 139 48 436 220 226 659	46 12 46 126 31 3 240	58 118	26	15	16	17	18	19
m 10 12 14 16 18	Niederreiter (NUT) Sobol' (JoeKuo03) Niederreiter (NUT) Sobol' (JoeKuo03) Niederreiter (NUT) Sobol' (JoeKu003) Niederreiter (NUT) Sobol' (JoeKu003) Niederreiter (NUT) Sobol' (JoeKu003) Niederreiter (NUT)	Aumeer           10           432           226           519           675           622           1000           793           1044           1439           767	11 83 240 139 436 220 226 659 234	46 12 46 126 31 3 240 31	58 118	<u>14</u> <u>26</u>	15	16	17	18	19
m 10 12 14 16 18	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (JoeKu003) Niederreiter (NUT) Sobol' (JoeKu008) Sobol' (JoeKu008) Sobol' (JoeKu008) Sobol' (JoeKu003) Niederreiter (NUT) Sobol' (JoeKu008) Sobol' (JoeKu008) Sobol' (JoeKu008)	Number 10 432 226 519 675 622 1020 793 1044 1439 767 1497	11 83 240 139 48 436 220 659 234 446	46 126 31 3 240 31 47	58 118	<u>14</u>	15	16	17	18	19
m           10           12           14           16           18	Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (JoeKuo03) Niederreiter (NUT) Sobol' (JoeKuo08) Sobol' (JoeKu003) Niederreiter (NUT) Sobol' (JoeKu003) Sobol' (JoeKu003) Sobol' (JoeKu003) Niederreiter (NUT) Sobol' (JoeKu008) Sobol' (JoeKu003)	Number 10 432 226 519 675 622 1020 793 1044 1439 767 1497 1834	83 240 139 48 436 220 226 659 234 446 805	46 12 46 126 31 3 240 31 31 47 331	58 118 153	26 61	15 13 32	16	17	18	19
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