Convergence of Langevin-Simulated Annealing algorithms with multiplicative noise II: Total Variation

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Abstract

We study the convergence of Langevin-Simulated Annealing type algorithms with multiplicative noise, i.e. for $V:\mathbb{R}^d\to\mathbb{R}$ a potential function to minimize, we consider the stochastic differential equation $dY_t = -\sigma\sigma^\top\nabla V(Y_t)dt + a(t)\sigma(Y_t)dW_t + a(t)^2\Upsilon(Y_t)dt$, where (W_t) is a Brownian motion, where $\sigma:\mathbb{R}^d\to\mathcal{M}_d(\mathbb{R})$ is an adaptive (multiplicative) noise, where $a:\mathbb{R}^+\to\mathbb{R}^+$ is a function decreasing to 0 and where Υ is a correction term. Allowing σ to depend on the position brings faster convergence in comparison with the classical Langevin equation $dY_t = -\nabla V(Y_t)dt + \sigma dW_t$. In a previous paper we established the convergence in L^1 -Wasserstein distance of Y_t and of its associated Euler scheme \bar{Y}_t to argmin(V) with the classical schedule $a(t) = A\log^{-1/2}(t)$. In the present paper we prove the convergence in total variation distance. The total variation case appears more demanding to deal with and requires regularization lemmas.

Keywords– Stochastic Optimization, Langevin Equation, Simulated Annealing, Neural Networks *MSC Classification*– 62L20, 65C30, 60H35

1 Introduction

Langevin-based algorithms are used to solve optimization problems in high dimension and have gained much interest in relation with Machine Learning. The Langevin equation is a Stochastic Differential Equation (SDE) which consists in a gradient descent with noise. More precisely, let $V: \mathbb{R}^d \to \mathbb{R}^+$ be a coercive potential function, then the associated Langevin equation reads

$$dX_t = -\nabla V(X_t)dt + \sigma dW_t, \ t \ge 0,$$

where (W_t) is a d-dimensional Brownian motion and where $\sigma > 0$. Under standard assumptions, the invariant measure of this SDE is the Gibbs measure ν_{σ^2} of density proportional to $e^{-2V(x)/\sigma^2}$ and for small enough σ , this measure concentrates around $\operatorname{argmin}(V)$ [Dal17] [Bra21]. Adding a small noise to the gradient descent allows to explore the space and to escape from traps such as local minima or saddle points appearing in non-convex optimization problems [Laz92] [DPG⁺14]. Such methods have been recently brought up to light again with Stochastic Gradient Langevin Dynamics (SGLD) algorithms [WT11] [LCCC16], especially for the deep learning and the calibration of large artificial neural networks.

The Langevin-simulated annealing SDE is the Langevin equation where the noise parameter is slowly decreasing to 0, namely

$$dX_t = -\nabla V(X_t)dt + a(t)\sigma dW_t, \ t \ge 0, \tag{1.1}$$

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where $a: \mathbb{R}^+ \to \mathbb{R}^+$ is non-increasing and converges to 0. The idea is that the "instantaneous" invariant measure $\nu_{a(t)\sigma}$ which is the Gibbs measure of density $\propto \exp(-2V(x)/(a(t)^2\sigma^2))$ converges itself to $\operatorname{argmin}(V)$. Although the additive case i.e. where σ is constant has been extensively studied, little attention has been paid to the multiplicative case i.e. where $\sigma: \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ depends on X_t .

The objective of the present paper is to study the convergence in total variation of the Langevin-Simulated annealing SDE, i.e. (1.1) with non-constant σ . Following [PP20, Proposition 2.5], we need to add a correction term in the drift, giving

$$dY_t = -(\sigma \sigma^\top \nabla V)(Y_t)dt + a(t)\sigma(Y_t)dW_t + \left(a^2(t) \left[\sum_{j=1}^d \partial_j(\sigma \sigma^*)(Y_t)_{ij}\right]_{1 \le i \le d}\right)dt, \tag{1.2}$$

$$a(t) = \frac{A}{\sqrt{\log(t+e)}},\tag{1.3}$$

so that $\nu_{a(t)}$ is still the the "instantaneous" invariant measure. We also study the convergence of its Euler-Maruyama scheme \bar{Y}_t with decreasing steps and with noisy gradient estimates coming from stochastic gradient algorithms. We assume in particular the convex uniformity of the potential V outside a compact set (but we do not assume that the potential is convex) and the ellipticity and the boundedness of σ .

We studied this SDE and proved the convergence in L^1 -Wasserstein distance of Y and \bar{Y} to ν^* which is the limit measure of ν_a as $a \to 0$, in a previous paper [BP21], which the present paper is a companion paper of. More precisely, we proved that $W_1(Y_t, \nu^*)$ is of order a(t) as $t \to \infty$ and that $W_1(Y_t, \nu_{(a(t))})$ is of order $t^{-\alpha}$ for every $\alpha \in (0, 1)$. For more details, we refer to the introduction of [BP21]. In particular, for applications to optimization problems arising in Stochastic Optimization and in Machine Learning and for choices of $\sigma : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ used by practitioners, we refer to [BP21, Section 3].

The proof for the total variation distance case relies on the same strategy developed in [BP21]. We first introduce the process X where the coefficient (a(t)) is "by plateaux" i.e. non-increasing and piecewise constant on time intervals $[T_n, T_{n+1}]$. Then we give bounds on $d_{\text{TV}}(X_t, Y_t)$ using a domino strategy [BP21, (1.2)]. However the main difference with the L^1 -Wasserstein distance concerns the total variation distance between X and Y in small time as in general, it is more difficult to give bounds in small time for the total variation distance between two processes with close coefficients. Indeed, considering the functional characterization and comparing it with the L^1 -Wasserstein distance, if x and $y \in \mathbb{R}^d$ are close to each other and if $f : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz-continuous, then we can bound |f(x) - f(y)| by $[f]_{\text{Lip}}|x - y|$; however if f is measurable bounded, then we cannot directly bound |f(x) - f(y)| in terms of |x - y|. Instead, the common strategy of proof in the literature is to use Malliavin calculus in order to perform an integration by parts and to use bounds on the derivatives of the density. In this context, [PP20] relies on a highly technical Malliavin approach inducing a "regularization from the past" (see [PP20, Theorem 3.7 and Appendix C]).

We give bounds in small time relying on the recent paper [BPP21] and we adapt some of the proofs to the non-homogeneous Markovian setting. These bounds rely on estimates of the density of the solutions to SDE's and their derivatives [Fri64]. The strategy of proof is the following: we first reduce to the null drift case using a Girsanov change of measure. Then we introduce an artificial regularization in order to perform a Malliavin-type integration by parts and we use Aronson's bounds on the density and its derivatives; we need to pay attention to the dependency in the parameter a, controlling the ellipticity of the SDE and which converges to 0, of the constants that appear in the Aronson bounds. Moreover, we rely on [DMR18] to give bounds on the total variation between two Gaussian laws.

Contrary to the L^1 -Wasserstein distance, we do not prove the convergence as $t \to \infty$ of Y_t and \bar{Y}_t to ν^* since in most of the cases, ν^* is supported by a finite number of points and then if Y_t has a

density then $d_{\text{TV}}(Y_t, \nu^*) = 2$. Instead, we prove the convergence in total variation of Y_t and \bar{Y}_t to their "instantaneous invariant measure" $\nu_{a(t)}$ which itself converges to ν^* (in law, for the L^1 -Wasserstein distance etc, see for example [Hwa80, Theorem 2.1] and [BP21, Lemma 4.6]) and we give bounds on $d_{\text{TV}}(Y_t, \nu_{a(t)})$ and on $d_{\text{TV}}(\bar{Y}_t, \nu_{a(t)})$ as $t \to \infty$.

The paper is organized as follows. In Section 2 we give the setting and assumptions of the problem we consider and state our main results of convergence with convergence rates. This setting is the same as in [BP21]. In Section 3 we establish bounds in small time for $d_{TV}(X_t, Y_t)$ and for $d_{TV}(X_t, \bar{Y}_t)$, in inspired from [BPP21]. In Section 4, we prove the convergence of the plateaux SDE X using exponential contraction properties. Using this convergence, the convergences of $d_{TV}(Y_t, \nu_{a(t)})$ and $d_{TV}(\bar{Y}_t, \nu_{a(t)})$ are proved in Section 5 and 6 respectively.

Notations

We endow the space \mathbb{R}^d with the canonical Euclidean norm denoted by $|\cdot|$ and we denote by $\langle\cdot,\cdot\rangle$ the associated canonical inner product. For $x\in\mathbb{R}^d$ and for R>0, we denote $\mathbf{B}(x,R)=\{y\in\mathbb{R}^d:|y-x|\leq R\}$.

For $M \in (\mathbb{R}^d)^{\otimes k}$, we denote by ||M|| its operator norm, i.e. $||M|| = \sup_{u \in \mathbb{R}^{d \times k}, |u|=1} M \cdot u$. If $M : \mathbb{R}^d \to (\mathbb{R}^d)^{\otimes k}$, we denote $||M||_{\infty} = \sup_{x \in \mathbb{R}^d} ||M(x)||$. We say that M is \mathcal{C}_b^r for some $r \in \mathbb{N} \cup \{0\}$ if M is bounded and has bounded derivatives up to the order r.

For $k \in \mathbb{N}$ and if $f : \mathbb{R}^d \to \mathbb{R}$ is C^k , we denote by $\nabla^k f : \mathbb{R}^d \to (\mathbb{R}^d)^{\otimes k}$ its differential of order k. If f is Lipschitz-continuous, we denote by $[f]_{\text{Lip}}$ its Lipschitz constant.

We denote the total variation distance between two distributions π_1 and π_2 on \mathbb{R}^d :

$$d_{\text{TV}}(\pi_1, \pi_2) = 2 \sup_{A \in \mathcal{B}(\mathbb{R}^d)} |\pi_1(A) - \pi_2(A)|.$$

Without ambiguity, if Z_1 and Z_2 are two \mathbb{R}^d -valued random vectors, we also write $d_{\text{TV}}(Z_1, Z_2)$ to denote the total variation distance between the law of Z_1 and the law of Z_2 . We have as well

$$d_{\text{TV}}(\pi_1, \pi_2) = \sup \left\{ \int_{\mathbb{R}^d} f d\pi_1 - \int_{\mathbb{R}^d} f d\pi_1, \ f : \mathbb{R}^d \to [-1, 1] \text{ measurable} \right\}.$$

Moreover, we recall that if π_1 and π_2 admit densities with respect to some measure reference λ , then

$$d_{\text{TV}}(\pi_1, \pi_2) = \int_{\mathbb{R}^d} \left| \frac{d\pi_1}{d\lambda} - \frac{d\pi_2}{d\lambda} \right| d\lambda.$$

We denote the L^p -Wasserstein distance between two distributions π_1 and π_2 on \mathbb{R}^d :

$$W_p(\pi_1, \pi_2) = \inf \left\{ \left(\int_{\mathbb{R}^d} |x - y|^p \pi(dx, dy) \right)^{1/p} : \pi \in \mathcal{P}(\pi_1, \pi_2) \right\},$$

where $\mathcal{P}(\pi_1, \pi_2)$ stands for the set of probability distributions on $(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{B}or(\mathbb{R}^d)^{\otimes 2})$ with respective marginal laws π_1 and π_2 . For p = 1, let us recall the Kantorovich-Rubinstein representation of the Wasserstein distance of order 1 [Vil09, Equation (6.3)]:

$$W_1(\pi_1, \pi_2) = \sup \left\{ \int_{\mathbb{R}^d} f(x)(\pi_1 - \pi_2)(dx) : f : \mathbb{R}^d \to \mathbb{R}, [f]_{\text{Lip}} = 1 \right\}.$$

For $x \in \mathbb{R}^d$, we denote by δ_x the Dirac mass at x.

In this paper, we use the notation C and c to denote real positive constants, which may change from line to line.

2 Assumptions and main results

2.1 Assumptions

Let us briefly recall the setting adopted in [BP21]. Let $V : \mathbb{R}^d \to (0, +\infty)$ be a \mathcal{C}^2 potential function such that V is coercive and

$$(x \mapsto |x|^2 e^{-2V(x)/A^2}) \in L^1(\mathbb{R}^d) \text{ for some } A > 0.$$
 (2.1)

Then V admits a minimum on \mathbb{R}^d . Moreover, let us assume that

$$V^{\star} := \min_{\mathbb{R}^d} V > 0, \quad \operatorname{argmin}(V) = \{x_1^{\star}, \dots, x_{m^{\star}}^{\star}\}, \quad \forall \ i = 1, \dots, m^{\star}, \ \nabla^2 V(x_i^{\star}) > 0, \qquad (2.2, \mathcal{H}_{V1})$$

i.e. $\min_{\mathbb{R}^d} V$ is attained at a finite number m^* of points and at each point the Hessian matrix is positive definite. We then define for $a \in (0, A]$ the Gibbs measure ν_a of density:

$$\nu_a(dx) = \mathcal{Z}_a e^{-2(V(x) - V^*)/a^2} dx, \quad \mathcal{Z}_a = \left(\int_{\mathbb{R}^d} e^{-2(V(x) - V^*)/a^2} dx \right)^{-1}$$
 (2.3)

Following [Hwa80, Theorem 2.1], the measure ν_a converges weakly to ν^* as $a \to 0$, where ν^* is the weighted sum of Dirac measures:

$$\nu^* = \left(\sum_{j=1}^{m^*} \left(\det \nabla^2 V(x_j^*)\right)^{-1/2}\right)^{-1} \sum_{i=1}^{m^*} \left(\det \nabla^2 V(x_i^*)\right)^{-1/2} \delta_{x_i^*}.$$
 (2.4)

Following [BP21, Lemma 4.6], ν_a also converges to ν^* as $a \to 0$ for the L^1 -Wasserstein distance.

We consider the following Langevin SDE in \mathbb{R}^d :

$$Y_0^{x_0} = x_0 \in \mathbb{R}^d, \quad dY_t^{x_0} = b_{a(t)}(Y_t^{x_0})dt + a(t)\sigma(Y_t^{x_0})dW_t, \tag{2.5}$$

where, for $a \geq 0$, the drift b_a is given by

$$b_a(x) = -(\sigma \sigma^{\top} \nabla V)(x) + a^2 \left[\sum_{j=1}^d \partial_j (\sigma \sigma^{\top})_{ij}(x) \right]_{1 \le i \le d} =: -(\sigma \sigma^{\top} \nabla V)(x) + a^2 \Upsilon(x), \qquad (2.6)$$

where W is a standard \mathbb{R}^d -valued Brownian motion defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where $\sigma : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ is \mathcal{C}^2 and

$$a(t) = \frac{A}{\sqrt{\log(t+e)}}\tag{2.7}$$

where A is defined in (2.1) and with $\log(e) = 1$. This equation corresponds to a gradient descent on the potential V with preconditioning σ and multiplicative noise; the second term in the drift (2.6) is a correction term (see [PP20, Proposition 2.5]) which is zero for constant σ .

We make the following assumptions on the potential V:

$$|\nabla V|^2 \le CV$$
 and $\sup_{x \in \mathbb{R}^d} ||\nabla^2 V(x)|| < +\infty,$ (2.8, \mathcal{H}_{V2})

which implies in particular that V has at most a quadratic growth. Let us also assume that

 σ is bounded and Lipschitz-continuous, $\nabla^2 \sigma$ is bounded, $\nabla(\sigma \sigma^\top) \nabla V$ is bounded, (2.9, \mathcal{H}_{σ})

and that σ is uniformly elliptic, i.e.

$$\exists \underline{\sigma}_0 > 0, \ \forall x \in \mathbb{R}^d, \ (\sigma \sigma^\top)(x) \ge \underline{\sigma}_0^2 I_d. \tag{2.10}$$

Assumptions (2.8, \mathcal{H}_{V2}) and (2.9, \mathcal{H}_{σ}) imply that Υ is also bounded and Lipschitz-continuous and that b_a is Lipschitz-continuous uniformly in $a \in [0, A]$. Let the minimal constant $[b]_{\text{Lip}}$ be such that:

$$\forall a \in [0, A], \ b_a \text{ is } [b]_{\text{Lip}}\text{-Lipschitz continuous.}$$
 (2.11)

We make the non-uniform dissipative (or convexity) assumption outside of a compact set: there exists $\alpha_0 > 0$ and $R_0 > 0$ such that

$$\forall x, y \in \mathbf{B}(0, R_0)^c, \left\langle \left(\sigma \sigma^\top \nabla V \right)(x) - \left(\sigma \sigma^\top \nabla V \right)(y), \ x - y \right\rangle \ge \alpha_0 |x - y|^2. \tag{2.12, } \mathcal{H}_{cf}(x) = 0$$

Taking $y \in \mathbf{B}(0, R_0)^c$ fixed, letting $|x| \to \infty$ and using the boundedness of σ , (2.12, \mathcal{H}_{cf}) implies that $|\nabla V|$ is coercive. Using (2.8, \mathcal{H}_{V2}) and the boundedness of σ , there exists C > 0 (depending on A) such that:

$$\forall a \in [0, A], \ 1 + |b_a(x)| \le CV^{1/2}(x).$$

Let $(\gamma_n)_{n\geq 1}$ be a non-increasing sequence of varying positive steps. We define $\Gamma_n := \gamma_1 + \cdots + \gamma_n$ and for $t\geq 0$:

$$N(t) := \min\{k \ge 0 : \ \Gamma_{k+1} > t\} = \max\{k \ge 0 : \ \Gamma_k \le t\}.$$
(2.13)

We make the classical assumptions on the step sequence, namely

$$\gamma_n \downarrow 0, \quad \sum_{n>1} \gamma_n = +\infty \quad \text{and} \quad \sum_{n>1} \gamma_n^2 < +\infty$$
 (2.14, $\mathcal{H}_{\gamma 1}$)

and we also assume that

$$\varpi := \limsup_{n \to \infty} \frac{\gamma_n - \gamma_{n+1}}{\gamma_{n+1}^2} < \infty. \tag{2.15, } \mathcal{H}_{\gamma 2})$$

For example, if $\gamma_n = \gamma_1/n^{\eta}$ with $\eta \in (1/2, 1)$ then $\varpi = 0$; if $\gamma_n = \gamma_1/n$ then $\varpi = \gamma_1$.

In stochastic gradient algorithms, the true gradient is measured with a zero-mean noise ζ , which law only depends on the current position. That is, let us consider a family of random fields $(\zeta_n(x))_{x \in \mathbb{R}^d, n \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$, $(\omega, x) \in \Omega \times \mathbb{R}^d \mapsto \zeta_n(x, \omega)$ is measurable and for all $x \in \mathbb{R}^d$, the law of $\zeta_n(x)$ only depends on x and $(\zeta_n(x))_{n \in \mathbb{N}}$ is an i.i.d. sequence independent of W. We make the following assumptions:

$$\forall x \in \mathbb{R}^d, \ \forall p \ge 1, \ \mathbb{E}[\zeta_1(x)] = 0 \quad \text{and} \quad \mathbb{E}[|\zeta_1(x)|^p] \le C_p V^{p/2}(x). \tag{2.16}$$

We then consider the Euler-Maruyama scheme with decreasing steps associated to (Y_t) :

$$\bar{Y}_{0}^{x_{0}} = x_{0}, \quad \bar{Y}_{\Gamma_{n+1}}^{x_{0}} = \bar{Y}_{\Gamma_{n}} + \gamma_{n+1} \left(b_{a(\Gamma_{n})} (\bar{Y}_{\Gamma_{n}}^{x_{0}}) + \zeta_{n+1} (\bar{Y}_{\Gamma_{n}}^{x_{0}}) \right) + a(\Gamma_{n}) \sigma(\bar{Y}_{\Gamma_{n}}^{x_{0}}) (W_{\Gamma_{n+1}} - W_{\Gamma_{n}}), \quad (2.17)$$

We extend \bar{Y}^{x_0} on \mathbb{R}^+ by considering its genuine continuous interpolation:

$$\forall t \in [\Gamma_n, \Gamma_{n+1}), \ \bar{Y}_t^{x_0} = \bar{Y}_{\Gamma_n}^{x_0} + (t - \Gamma_n) \left(b_{a(\Gamma_n)} (\bar{Y}_{\Gamma_n}^{x_0}) + \zeta_{n+1} (\bar{Y}_{\Gamma_n}^{x_0}) \right) + a(\Gamma_n) \sigma(\bar{Y}_{\Gamma_n}^{x_0}) (W_t - W_{\Gamma_n}). \ (2.18)$$

2.2 Main results

Theorem 2.1. (a) Let Y be defined in (2.5). Assume (2.2, \mathcal{H}_{V1}), (2.8, \mathcal{H}_{V2}), (2.9, \mathcal{H}_{σ}), (2.10) and (2.12, \mathcal{H}_{cf}). Then, for every $\alpha \in (0,1)$, if A is large enough, then for every $x_0 \in \mathbb{R}^d$ and for every t > 0:

$$d_{\text{TV}}\left(Y_t^{x_0}, \nu_{a(t)}\right) \le Ce^{C\sqrt{\log(t)}(1+|x_0|^2)}t^{-\alpha}.$$
(2.19)

(b) Let \bar{Y} be defined in (2.17). Assume (2.2, \mathcal{H}_{V1}), (2.8, \mathcal{H}_{V2}), (2.9, \mathcal{H}_{σ}), (2.10) and (2.12, \mathcal{H}_{cf}). Assume furthermore that $\sigma \in \mathcal{C}_b^{2r}$. Assume furthermore (2.14, $\mathcal{H}_{\gamma 1}$) and (2.15, $\mathcal{H}_{\gamma 2}$), that V is \mathcal{C}^3 with $\|\nabla^3 V\| \leq CV^{1/2}$ and that σ is \mathcal{C}^3 with $\|\nabla^3(\sigma\sigma^\top)\| \leq CV^{1/2}$. Then, for every $\alpha \in (0,1)$, if A is large enough, then for every $x_0 \in \mathbb{R}^d$ and for every t > 0:

$$d_{\text{TV}}\left(\bar{Y}_{t}^{x_{0}}, \nu_{a(t)}\right) \leq C\left(\log^{1/2}(t) \max\left[V^{2}(x_{0}), 1 + |x_{0}|\right] t^{-\alpha} + e^{C\sqrt{\log(t)}(1 + |x_{0}|^{2})} t^{C/A^{2}} \gamma_{N(Ct)}^{r/(2r+1)}\right). \tag{2.20}$$

Remark 2.2. Depending on the step sequence (γ_n) , we can compare the two terms arising in the right-hand side of (2.20). For example, if $\gamma_n = \gamma_1 n^{-\eta}$ for some $\eta \in (1/2, 1]$, then

- If $\eta = 1$, then $\gamma_{N(Ct)} \approx e^{-Ct}$ and the first term is the dominating term.
- If $\eta \in (1/2,1)$ then $\gamma_{N(Ct)} \simeq (Ct)^{-\eta/(1-\eta)}$.

2.3 Extensions and interpolations of the processes

Let us define the following processes that will be used as auxiliary tools in the proofs.

• We define (X_t) as the solution the following SDE where the coefficients piecewisely depend on the time; X is then said to be "by plateaux":

$$X_0^{x_0} = x_0, \quad dX_t^{x_0} = b_{a_{k+1}}(X_t^{x_0})dt + a_{k+1}\sigma(X_t^{x_0})dW_t, \quad t \in [T_k, T_{k+1}], \tag{2.21}$$

where b_a is defined in (2.6) and the time schedule (T_n) is defined by

$$T_n := C_{(T)} n^{1+\beta},$$
 (2.22)

where $C_{(T)} > 0$, $\beta > 0$ and $a_n := a(T_n)$. More generally, we define $(X_t^{x,n})$ as the solution of

$$X_0^{x,n} = x, \quad dX_t^{x,n} = b_{a_{k+1}}(X_t^{x,n})dt + a_{k+1}\sigma(X_t^{x,n})dW_t, \quad t \in [T_k - T_n, T_{k+1} - T_n], \ k \ge n, \quad (2.23)$$

i.e. $(X_t^{x,n})$ has the conditional law of $(X_{T_n+t})_{t\geq 0}$ given $X_{T_n}=x$. We have $X_t^x=X_t^{x,0}$. The Markov transition kernel associated to $X^{\cdot,n}$ denoted $P_t^{X,n}$ reads on Borel functions $f:\mathbb{R}^d\to\mathbb{R}^+$, $P_t^{X,n}f(x)=\mathbb{E}[f(X_t^{x,n})]$.

• Considering now the original SDE (2.5), we also define for every $x \in \mathbb{R}^d$ and every fixed $u \geq 0$:

$$Y_{0,u}^{x} = x, \quad dY_{t,u}^{x} = b_{a(t+u)}(Y_{t,u}^{x})dt + a(t+u)\sigma(Y_{t,u}^{x})dW_{t}, \tag{2.24}$$

so that $Y^x = Y^x_{.,0}$. We define the Markov transition kernel associated to Y between the times t and t+u by $P^Y_{t,u}$ such that for all Borel functions $f: \mathbb{R}^d \to \mathbb{R}^+$, $P^Y_{t,u}f(x) = \mathbb{E}[f(Y^x_{t,u})]$.

• Considering finally (2.17) and (2.18), we define for every $n \ge 0$, $(\bar{Y}_{t,\Gamma_n}^x)_{t\ge 0}$, first at times $\Gamma_k - \Gamma_n$, $k \ge n$, by

$$\bar{Y}_{0,\Gamma_{n}}^{x} = x, \quad \bar{Y}_{\Gamma_{k+1}-\Gamma_{n},\Gamma_{n}}^{x} = \bar{Y}_{\Gamma_{k}-\Gamma_{n},\Gamma_{n}}^{x} + \gamma_{k+1} \left(b_{a(\Gamma_{k})} (\bar{Y}_{\Gamma_{k}-\Gamma_{n},\Gamma_{n}}^{x}) + \zeta_{k+1} (\bar{Y}_{\Gamma_{k}-\Gamma_{n},\Gamma_{n}}^{x}) \right) \\
+ a(\Gamma_{k}) \sigma(\bar{Y}_{\Gamma_{k}-\Gamma_{n},\Gamma_{n}}^{x}) (W_{\Gamma_{k+1}} - W_{\Gamma_{k}}), \tag{2.25}$$

then at every time t by the genuine interpolation on the intervals $([\Gamma_k - \Gamma_n, \Gamma_{k+1} - \Gamma_n))_{k \geq n}$ as before. In particular $\bar{Y}^x = \bar{Y}^x_{,0}$. Still more generally, we define $\bar{Y}^x_{t,u}$ where $u \in (\Gamma_n, \Gamma_{n+1})$ as

$$\bar{Y}_{0,u}^x = x, \quad \bar{Y}_{t,u}^x = \begin{cases} x + t(b_a(x) + \zeta_{n+1}(x)) + a^2(u)\sigma(x)(W_t - W_{\Gamma_u}) & \text{if } t \in [u, \Gamma_{n+1}] \\ = \bar{Y}_{\Gamma_{n+1}-u, \Gamma_{n+1}}^{\bar{Y}_x} & \text{if } t > \Gamma_{n+1}. \end{cases}$$

For $n, k \geq 0$, for $u \in [\Gamma_k, \Gamma_{k+1})$ and $\gamma \in [0, \Gamma_{k+1} - u]$, let $P_{\gamma, u}^{\bar{Y}}$ be the Markov transition kernel associated to $\bar{Y}_{\cdot, u}$ between the times 0 and γ i.e. for all Borel functions $f : \mathbb{R}^d \to \mathbb{R}^+$, $P_{\gamma, u}^{\bar{Y}} f(x) = \mathbb{E}[f(\bar{Y}_{\gamma, u}^x)]$.

3 Bounds in total variation for small t

In this section we give bounds for the total variation distance between the processes X, Y and \overline{Y} . Although such bounds are straightforward for L^p -distances, they are more difficult to establish for d_{TV} . To this end we adopt a strategy similar to [BPP21].

For $x \in \mathbb{R}^d$ and for $a \in \mathbb{R}^+$ we define the "cut" drift $\tilde{b}_a^x : \mathbb{R}^d \to \mathbb{R}^d$ which is the drift b_a which is null outside a compact set centred on x. More precisely, we choose R > 0 and we consider a \mathcal{C}^{∞} decreasing function $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi = 1$ on $[0, R^2]$ and $\psi = 0$ on $[(R+1)^2, \infty)$ and we define $\tilde{b}_a^x(y) := b_a(y)\psi(|y-x|^2)$, so that $|\tilde{b}_a^x|$ is bounded by C(1+|x|) since b_a is Lipschitz-continuous. For $\sigma: \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$, we denote the martingale:

$$M(\sigma)_0^x = x, \quad dM(\sigma)_t^x = \sigma(M(\sigma)_t^x)dW_t$$
 (3.1)

with its associated one-step Euler-Maruyama scheme:

$$\bar{M}(\sigma)_t^x = x + \sigma(x)W_t. \tag{3.2}$$

Lemma 3.1. Let Z be solution of the following SDE:

$$dZ_t^x = u(t)\sigma_Z(Z_t^x)dW_t,$$

where $u: \mathbb{R}^+ \to (0,\infty)$ is \mathcal{C}^1 and bounded. Then $(Z_t) \sim (M(\sigma)_{F^{(-1)}(t)})$, where $F: \mathbb{R}^+ \to \mathbb{R}^+$ is solution of the differential equation

$$F(0) = 0, \quad F'(t) = \frac{1}{u^2(F(t))}$$

and where $F^{(-1)}$ denotes the (continuous) inverse function of F.

Proof. First, F is well defined and is strictly increasing with $F(t) \to \infty$ as $t \to \infty$ since u is bounded, so that $F^{(-1)}: \mathbb{R}^+ \to \mathbb{R}^+$ is well defined as well. We have

$$d\left(Z_{F(t)}^x\right) = u(F(t))\sigma_Z(Z_{F(t)}^x)d\left(W_{F(t)}\right) = F'(t)^{1/2}u(F(t))\sigma_Z(Z_{F(t)}^x)d\widetilde{W}_t = \sigma_Z(Z_{F(t)}^x)d\widetilde{W}_t.$$

where \widetilde{W} is the Brownian motion defined by $\widetilde{W}_t = \int_0^t (F'(s))^{-1/2} dW_{F(s)}$.

Total variation bound in small time for the Euler-Maruyama scheme

Proposition 3.2. Assume that $\sigma \in C_b^{2r}$. There exists C > 0 such that for every $n, k \geq 0$, for every $u \in [\Gamma_k, \Gamma_{k+1})$ and every t > 0 such that $u \in [T_n, T_{n+1}], t \leq \Gamma_{k+1} - u$ and $u + t \in [T_n, T_{n+1}],$

$$d_{\text{TV}}(X_t^{x,n}, \bar{Y}_{t,u}^x) \le Ce^{Ca_{n+1}^{-1}(1+|x|^2)} t^{r/(2r+1)} + Ca_n^{-2}(a_n - a_{n+1}). \tag{3.3}$$

Proof. We apply a strategy of proof similar to [BPP21, Theorem 2.2]. However we need to pay attention to the dependency of the constants in the bounds in (a_n) . Let us write

$$d_{\text{TV}}(X_t^{x,n}, \bar{Y}_{t,u}^x) \le d_{\text{TV}}(X_t^{x,n}, \tilde{X}_t^{x,n}) + d_{\text{TV}}(\tilde{X}_t^{x,n}, Z_t^{x,n}) + d_{\text{TV}}(Z_t^{x,n}, \bar{Z}_t^{x,n}) + d_{\text{TV}}(\bar{Z}_t^{x,n}, \bar{X}_t^{x,n}) + d_{\text{TV}}(\bar{X}_t^{x,n}, \bar{Y}_{t,u}^x),$$
(3.4)

where

$$\begin{split} \widetilde{X}_{0}^{x,n} &= x, \quad d\widetilde{X}_{t}^{x,n} = \widetilde{b}_{a_{n+1}}^{x}(\widetilde{X}_{t}^{x,n})dt + a_{n+1}\sigma(\widetilde{X}_{t}^{x,n})dW_{t}, \\ Z_{0}^{x,n} &= x, \quad dZ_{t}^{x,n} = a_{n+1}\sigma(Z_{t}^{x,n})dW_{t}, \\ \bar{Z}_{0}^{x,n} &= x, \quad \bar{Z}_{t}^{x,n} = x + a_{n+1}\sigma(x)W_{t}. \end{split}$$

• Using [BPP21, Lemma 3.2], we have

$$d_{\text{TV}}(X_t^{x,n}, \widetilde{X}_t^{x,n}) \le C(1 + |x|^2)t,$$

where the constant C does not depend on n.

• We use [QZ04, Theorem 2.4] and we rework the bound from [BPP21, Lemma 3.5] to make explicit the dependency in a_n . Reworking [BPP21, Lemma 3.4], we have for $q \ge 1$:

$$\mathbb{E}\left[\sup_{s\in[0,t]}|U_s^{x,n}|^{2q}\right] \leq Ce^{C_q a_{n+1}^{-1}(1+|x|^2)},$$

$$U_0^{x,n} = 1, \quad dU_s^{x,n} = a_{n+1}^{-1} U_s^{x,n} \left\langle \sigma^{-1}(Z_s^{x,n}) \tilde{b}_{a_{n+1}}^x(Z_s^{x,n}), dW_s \right\rangle.$$
(3.5)

Moreover, following Lemma 3.1 we have $(Z_t^{x,n}) \sim (M(\sigma)_{F^{(-1)}(t)}^x)$ where the process $(M(\sigma)_t)$ does not depend on n and where $F^{(-1)}(t) = a_{n+1}^2 t$. Thus following [Fri64, Chapter 9, Theorem 7] (also see [BPP21, Theorem 3.1] for the application to SDE's) and since $\sigma \in \mathcal{C}_b^2$ we have

$$|\nabla_x p_{M(\sigma)}(t, x, y)| \le \frac{C}{t^{(d+1)/2}} e^{-c|y-x|^2/t}$$
 (3.6)

and then

$$|\nabla_x p_Z(t,x,y)| = |\nabla_x p_{M(\sigma)}(a_{n+1}^2t,x,y)| \leq \frac{Ca_{n+1}^{-(d+1)}}{t^{(d+1)/2}}e^{-ca_{n+1}^{-2}|y-x|^2/t} \leq \frac{Ca_{n+1}^{-(d+1)}}{t^{(d+1)/2}}e^{-c|y-x|^2/t}.$$

Then using [BPP21, Lemma 3.5] with the adapted bound on $U_s^{x,n}$ (3.5) along with [QZ04, Theorem 2.4], we obtain

$$d_{\text{TV}}(\widetilde{X}_t^{x,n}, Z_t^{x,n}) \le Ce^{Ca_{n+1}^{-1}(1+|x|^2)}t^{1/2}.$$
(3.7)

The same way, we obtain

$$d_{\text{TV}}(\bar{Z}_t^{x,n}, \bar{X}_t^{x,n}) \le Ce^{Ca_{n+1}^{-1}(1+|x|^2)}t^{1/2}.$$

• Following Lemma 3.1, we have $(Z_t^{x,n}) \sim (M(\sigma)_{a_{n+1}^2}^x)$ and $(\bar{Z}_t^{x,n}) \sim (\bar{M}(\sigma)_{a_{n+1}^2}^x)$, where both processes $M(\sigma)$ and $\bar{M}(\sigma)$ do not depend on n. We then use [BPP21, Theorem 2.2] to get

$$\mathrm{d}_{\mathrm{TV}}(M(\sigma)_t^x, \bar{M}(\sigma)_t^x) \le C e^{C|x|^2} t^{r/(2r+1)}$$

which implies

$$d_{\text{TV}}(Z_t^{x,n}, \bar{Z}_t^{x,n}) \le Ce^{C|x|^2} a_{n+1}^{2r/(2r+1)} t^{r/(2r+1)} \le Ce^{C|x|^2} t^{r/(2r+1)}.$$

• Let us now investigate $d_{\text{TV}}(\bar{X}_t^{x,n}, \bar{Y}_{t,u}^x)$. Conditionally to $\zeta(x)$, both random vectors are Gaussian vectors with

$$\bar{X}_t^{x,n} \sim \mathcal{N}\left(x + tb_{a_{n+1}}(x), a_{n+1}^2 t\sigma\sigma^\top(x)\right) \quad \text{and} \quad \bar{Y}_{t,u}^x \sim \mathcal{N}\left(x + tb_{a(u)}(x) + t\zeta(x), a^2(u)t\sigma\sigma^\top(x)\right).$$

Then, conditionally to $\zeta(x)$ we have

$$d_{\text{TV}}(\bar{X}_{t}^{x,n}, \bar{Y}_{t,u}^{x}) \leq d_{\text{TV}}\left(\mathcal{N}\left(x + tb_{a_{n+1}}(x), a_{n+1}^{2}t\sigma\sigma^{\top}(x)\right), \mathcal{N}\left(x + tb_{a(u)}(x) + t\zeta(x), a_{n+1}^{2}t\sigma\sigma^{\top}(x)\right)\right) + d_{\text{TV}}\left(\mathcal{N}\left(x + tb_{a(u)}(x) + t\zeta(x), a_{n+1}^{2}t\sigma\sigma^{\top}(x)\right), \mathcal{N}\left(x + tb_{a(u)}(x) + t\zeta(x), a^{2}(u)t\sigma\sigma^{\top}(x)\right)\right) =: D_{1} + D_{2}.$$

We then refer to [DMR18] which gives bounds on the total variation between two Gaussian laws, first in the case d > 1. Using [DMR18, Theorem 1.1] with $\lambda_1 = \cdots = \lambda_d = (a(u)^2 - a_{n+1}^2)/a_{n+1}^2$, we have

$$D_2 \le C\left(\frac{a^2(u) - a_{n+1}^2}{a_{n+1}^2}\right) \le Ca_n^{-1}(a_n - a_{n+1}).$$

Using [DMR18, Theorem 1.2], since the ρ_i 's are bounded independently of n and since for every $y \in \mathbb{R}^d$, $y^{\top} \sigma \sigma^{\top}(x) y \geq \underline{\sigma}_0^2 |y|^2$, we have

$$D_1 \le C\sqrt{t}a_{n+1}^{-1}(1+|\zeta(x)|^{1/2}).$$

Now, integrating over the law of $\zeta(x)$ and using that $\mathbb{E}|\zeta(x)| \leq CV(x)$, we obtain

$$d_{\text{TV}}(\bar{X}_t^{x,n}, \bar{Y}_{t,u}^x) \le Ca_n^{-1}(a_n - a_{n+1}) + C\sqrt{t}(1 + V^{1/2}(x)).$$

In the case d = 1, we use [DMR18, Theorem 1.3] and obtain the same bounds.

• Conclusion: Considering (3.4), we get

$$\begin{split} \mathrm{d_{TV}}(X_t^{x,n},\bar{Y}_{t,u}^x) &\leq C(1+|x|^2)t + Ce^{Ca_{n+1}^{-1}(1+|x|^2)}t^{1/2} + Ce^{C|x|^2}t^{r/(2r+1)} \\ &\quad + Ca_n^{-1}(a_n-a_{n+1}) + C\sqrt{t}(1+V^{1/2}(x)) \\ &\leq Ce^{Ca_{n+1}^{-1}(1+|x|^2)}t^{r/(2r+1)} + Ca_n^{-1}(a_n-a_{n+1}). \end{split}$$

3.2 Total variation bound in small time for the continuous SDE

Proposition 3.3. Assume that $\sigma \in \mathcal{C}_b^{2r}$ and let $\bar{\gamma} > 0$. There exists C > 0 such that for all $\varepsilon > 0$, $n \geq 0$, $u, t \geq 0$ such that $u \in [T_n, T_{n+1}]$, $u + t \in [T_n, T_{n+1}]$ and $t \leq \bar{\gamma}$,

$$d_{\text{TV}}(X_t^{x,n}, Y_{t,u}^x) \le Ce^{Ca_{n+1}^{-1}(1+|x|^2)}t^{1/2} + Ca_{n+1}^{-(d+r)}(a(u) - a_{n+1})^{2r/(2r+1)}. \tag{3.8}$$

Proof. We have

$$d_{\text{TV}}(X_t^{x,n}, Y_{t,u}^x) \le d_{\text{TV}}(X_t^{x,n}, \widetilde{X}_t^{x,n}) + d_{\text{TV}}(\widetilde{X}_t^{x,n}, Z_t^{x,n}) + d_{\text{TV}}(Z_t^{x,n}, \widetilde{Z}_{t,u}^x) + d_{\text{TV}}(\widetilde{Z}_{t,u}^x, \widetilde{Y}_{t,u}^x) + d_{\text{TV}}(\widetilde{Y}_{t,u}^x, Y_{t,u}^x)$$
(3.9)

where

$$d\widetilde{X}_{t}^{x,n} = \widetilde{b}_{a_{n+1}}^{x}(\widetilde{X}_{t}^{x,n})dt + a_{n+1}\sigma(\widetilde{X}_{t}^{x,n})dW_{t},$$

$$dZ_t^{x,n} = a_{n+1}\sigma(Z_t^{x,n})dW_t,$$

$$d\widetilde{Z}_{t,u}^x = a(u+t)\sigma(\widetilde{Z}_{t,u}^x)dW_t,$$

$$d\widetilde{Y}_{t,u}^x = \widetilde{b}_{a(u+t)}^x(\widetilde{Y}_{t,u}^x)dt + a(u+t)\sigma(\widetilde{Y}_{t,u}^x)dW_t.$$

Using [BPP21, Lemma 3.2], we have

$$d_{\text{TV}}(X_t^{x,n}, \widetilde{X}_t^{x,n}) + d_{\text{TV}}(\widetilde{Y}_{t,u}^x, Y_{t,u}^x) \le C(1 + |x|^2)t.$$

Using (3.7) again, we have

$$d_{\text{TV}}(\widetilde{X}_t^{x,n}, Z_t^{x,n}) \le Ce^{Ca_{n+1}^{-1}(1+|x|^2)}t^{1/2}.$$

Moreover, using [QZ04, Theorem 2.4] (with an immediate adaptation to the non-homogeneous case) and establishing the same bounds as in [BPP21, Lemma 3.5], we also have

$$d_{\text{TV}}(\widetilde{Z}_{t,u}^x, \widetilde{Y}_{t,u}^x) \le Ce^{Ca_{n+1}^{-1}(1+|x|^2)}t^{1/2}.$$
(3.10)

We now turn to $d_{\text{TV}}(Z_t^{x,n}, \widetilde{Z}_{t,u}^x)$. Using Lemma 3.1 as in (3.6) we have

$$|\nabla_y^{2r} p_Z(t,x,y)| = |\nabla_y^{2r} p_{M(\sigma)}(a_{n+1}^2t,x,y)| \le C \frac{a_{n+1}^{-(d+r)}}{t^{(d+r)/2}} e^{-c|x-y|^2/t}.$$

To bound $p_{\widetilde{Z}}$ we use the change of time F satisfying $F'(t) = a^{-2}(u + F(t))$ so that

$$a_n^{-2}t \le F(t) \le a_{n+1}^{-2}t$$
 and $a_{n+1}^2t \le F^{(-1)}(t) \le a_n^2t$

and then

$$|\nabla_y^{2r} p_{\widetilde{z}}(t,x,y)| = |\nabla_y^{2r} p_{M(\sigma)}(F^{(-1)}(t),x,y)| \le C \frac{a_{n+1}^{-(d+r)}}{t^{(d+r)/2}} e^{-c|x-y|^2/t}.$$

We prove as in [BP21, Lemma 6.2] that

$$||Z_t^{x,n} - \widetilde{Z}_{t,u}^x||_1 \le C(a(u) - a_{n+1})t^{1/2}$$

and then using [BPP21, Proposition 2.3] we get for every $\varepsilon > 0$:

$$d_{\text{TV}}(Z_t^{x,n}, \widetilde{Z}_{t,u}^x) \le C a_{n+1}^{-(d+r)} \varepsilon^r t^{-r} + C \varepsilon^{-1/2} (a(u) - a_{n+1}) t^{1/2}.$$

Choosing $\varepsilon = (a(u) - a_{n+1})^{2/(2r+1)}t$ yields

$$d_{\text{TV}}(Z_t^{x,n}, \widetilde{Z}_{t,u}^x) \le C a_{n+1}^{-(d+r)} (a(u) - a_{n+1})^{2r/(2r+1)}.$$

• Conclusion: considering (3.9), we get

$$d_{\text{TV}}(X_t^{x,n}, Y_{t,u}^x) \le C(1 + |x|^2)t + Ce^{Ca_{n+1}^{-1}(1+|x|^2)}t^{1/2} + Ca_{n+1}^{-(d+r)}(a(u) - a_{n+1})^{2r/(2r+1)}.$$

Remark 3.4. As in [BPP21, Theorem 2.3], we could improve the dependency in |x| in (3.3) and (3.8), at the expanse of further assumptions on V. However it would require to track the dependency in the ellipticity (in a_n) in the bounds proved in [MPZ21], which rely on Malliavin calculus. We believe that it would considerably increase the length and the technicality of the present article, while bringing no significant improvement to our final results.

4 Convergence of the plateau SDE X_t in total variation

In this section, we prove the convergence of the plateau SDE (X_t) defined in (2.21).

4.1 Exponential contraction in total variation

We first show that the property of exponential contraction that holds for the L^1 -Wasserstein distance under the setting described in Section 2.1 (see [BP21, Theorem 4.2]) also holds for the total variation distance.

Theorem 4.1. Let X be the solution to

$$X_0^x = x, \quad dX_t^x = b_a(X_t^x)dt + a\sigma(X_t^x)dW_t, \tag{4.1}$$

with $a \in (0, A]$ and where b_a is defined in (2.6), so that ν_a defined in (2.3) is the unique invariant distribution of X ([PP20, Proposition 2.5]). Let $t_0 \in (0, 1]$. Under the assumption (2.12, \mathcal{H}_{cf}),

(a) For every $x, y \in \mathbb{R}^d$ and for every $t \geq t_0$ we have

$$d_{\text{TV}}(X_t^x, X_t^y) \le Ca^{-1}e^{C_1/a^2}e^{-\rho_a t}|x - y|, \quad \rho_a := e^{-C_2/a^2}. \tag{4.2}$$

(b) For every $x \in \mathbb{R}^d$ and for every $t \geq t_0$ we have

$$d_{\text{TV}}(X_t^x, \nu_a) \le Ca^{-1}e^{C_1/a^2}e^{-\rho_a t}\nu_a(|x - \cdot|). \tag{4.3}$$

Proof. (a) Following [BP21, Theorem 4.2], we have

$$\forall x, y \in \mathbb{R}^d, \ \mathcal{W}_1(X_t^x, X_t^y) \le Ce^{C_1/a^2} |x - y|e^{-\rho_a t}.$$

Let $t \geq t_0$ and let $f: \mathbb{R}^d \to \mathbb{R}$ a Borel bounded function. Then

$$\mathbb{E}[f(X_t^x)] - \mathbb{E}[f(X_t^y)] = \mathbb{E}[P_{t_0}^X f(X_{t-t_0}^x)] - \mathbb{E}[P_{t_0}^X f(X_{t-t_0}^x)],$$

where P^X denotes the kernel associated to X. But using [PP20, Proposition 3.1] we have for every z_1 and $z_2 \in \mathbb{R}^d$,

$$P_{t_0}^X f(z_2) - P_{t_0}^X f(z_1) = \langle \nabla P_{t_0}^X f(\xi), z_2 - z_1 \rangle = \frac{1}{t_0} \mathbb{E} \left[f(X_t^{\xi}) \left\langle \int_0^{t_0} (a^{-1} \sigma^{-1} (X_s^{\xi}) Y_s^{\xi})^{\top} dW_s, z_2 - z_1 \right\rangle \right],$$

where $\xi \in (z_1, z_2)$ and $(Y_s^{\xi})_{s \geq 0}$ denotes the tangent process of (X_s^{ξ}) , i.e.

$$Y_0^{\xi} = I_d, \quad dY_s^{\xi} = \nabla b_a(X_s^{\xi}) Y_s^{\xi} ds + a \nabla \sigma(X_s^{\xi}) Y_s^{\xi} \otimes dW_s. \tag{4.4}$$

Since $\nabla \sigma$ and ∇b_a are bounded (uniformly in a), we have

$$\sup_{\xi \in \mathbb{R}^d, s \in [0, t_0]} \mathbb{E} \|Y_s^{\xi}\|_2^2 < +\infty,$$

where the bound does not depend on a. So that

$$P_{t_0}^X f(z_2) - P_{t_0}^X f(z_1) \le C||f||_{\infty}|z_2 - z_2|a^{-1} \sup_{\xi \in \mathbb{R}^d, s \in [0, t_0]} \mathbb{E}||Y_s^{\xi}||_2,$$

and then $[P_{t_0}^X f]_{\text{Lip}} \leq Ca^{-1} ||f||_{\infty}$. Then we obtain

$$d_{\text{TV}}(X_t^x, X_t^y) \le Ca^{-1}\mathcal{W}_1(X_{t-t_0}^x, X_{t-t_0}^y) \le Ca^{-1}e^{C_1/a^2}e^{-\rho_a t}|x-y|.$$

(b) As ν_a is the invariant distribution of the diffusion (4.1) we have

$$d_{\text{TV}}(X_t^x, \nu_a) \le \int_{\mathbb{R}^d} d_{\text{TV}}(X_t^x, X_t^y) \, \nu_a(dy) \le C e^{C_1/a^2} e^{-\rho_a t} \int_{\mathbb{R}^d} |x - y| \nu_a(dy)$$

$$\le C e^{C_1/a^2} e^{-\rho_a t} \nu_a(|x - \cdot|).$$

4.2 Convergence of the plateau SDE

Let (T_n) be the time schedule defined in (2.22) and by a slight abuse of notation we define

$$a_n := a(T_n) = \frac{A}{\sqrt{\log(T_n + e)}}$$
 and $\rho_n := \rho_{a_n} = e^{-C_2/a_n^2}$. (4.5)

We recall that [BP21, Lemma 4.3]

$$0 \le a_n - a_{n+1} \asymp (n \log^{3/2}(n))^{-1}. \tag{4.6}$$

Proposition 4.2. Let ν_a , $a \in (0, A]$, be the Gibbs measure defined in (2.3). Assume that V is coercive, that $(x \mapsto |x|^2 e^{-2V(x)/A^2}) \in L^1(\mathbb{R}^d)$ and (2.2, \mathcal{H}_{V1}). Then for $n \geq 2$,

$$d_{\text{TV}}(\nu_{a_n}, \nu_{a_{n+1}}) \le \frac{C}{n \log(n)}.$$
(4.7)

Moreover, for every $s, t \in [a_{n+1}, a_n]$, we have

$$d_{\text{TV}}(\nu_s, \nu_t) \le \frac{C}{n \log(n)}.$$
(4.8)

The proof is given in the Appendix A.1.

We now prove the convergence of the SDE "by plateaux" for the total variation distance.

Theorem 4.3. Let X be the process defined in (2.21) and (2.23). Let t_0 be defined as in Theorem 4.1. If $A > \max(\sqrt{(1+\beta^{-1})C_2}, \sqrt{(1+\beta)C_1})$ where C_1 and C_2 are defined in (4.2), then for all $x_0 \in \mathbb{R}^d$ and for all $C'_{(T)} < C_{(T)}$, for all large enough $n \ge n(C'_{(T)})$, on the time schedule (T_n) we have

$$d_{\text{TV}}(X_{T_n}^{x_0}, \nu_{a_n}) \le Ca_n^{-1} n^{-1 + (\beta + 1)C_1/A^2} \exp\left(-(C_{(T)}')^{1 - C_2/A^2} (\beta + 1) n^{\beta - (\beta + 1)C_2/A^2}\right) (1 + |x_0|) \tag{4.9}$$

and for every $t \in \mathbb{R}^+ \setminus (\bigcup_{n \geq 1} [T_n, T_n + t_0])$ we have

$$d_{\text{TV}}(X_t^{x_0}, \nu_{a(t)}) \le \frac{C(1+|x_0|)}{t^{(1+\beta)^{-1}-C_1/A^2} \log(t+e)}.$$
(4.10)

Proof. For fixed $x \in \mathbb{R}^d$ and using Theorem 4.1, we have for every bounded Borel function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\mathbb{E}[f(X_{T_{n+1}-T_n}^{x,n})] - \mathbb{E}[f(Z_{a_{n+1}})] \le Ca_{n+1}^{-1}e^{C_1/a_{n+1}^2}e^{-\rho_{a_{n+1}}(T_{n+1}-T_n)}||f||_{\infty}\mathbb{E}|x - Z_{a_{n+1}}|,$$

where $Z_{a_{n+1}} \sim \nu_{a_{n+1}}$. Now integrating x with respect to the law of $X_{T_n}^{x_0}$ yields

$$d_{\text{TV}}(X_{T_{n+1}}^{x_0}, \nu_{a_{n+1}}) \leq C a_{n+1}^{-1} e^{C_1/a_{n+1}^2} e^{-\rho_{a_{n+1}}(T_{n+1}-T_n)} \left(\mathcal{W}_1(X_{T_n}^{x_0}, \nu_{a_n}) + \mathcal{W}_1(\nu_{a_{n+1}}, \nu_{a_n}) \right)$$

$$\leq C \frac{a_{n+1}^{-1} \mu_{n+1}}{n \log^{3/2}(n)} (1 + |x_0|),$$

$$\mu_n := e^{C_1/a_{n+1}^2} e^{-\rho_{a_{n+1}}(T_{n+1}-T_n)}$$

$$(4.11)$$

where we used [BP21, Theorem 5.1] and [BP21, Proposition 4.4]. We use the bound on μ_n given by [BP21, (5.5)]. Then to bound $d_{\text{TV}}(X_t^{x_0}, \nu_{a_{n+1}})$ for any $t \in (T_n + t_0, T_{n+1})$, we apply Theorem (4.1) on the time interval $[T_n, t]$ which length is not smaller than t_0 and we conclude as in the proof of [BP21, Theorem 5.1].

Remark 4.4. The condition that t does not belong in any interval $[T_n, T_n + t_0]$ is a technical condition which is specific to our strategy of proof. However this condition is not a problem for the convergence of Y_t and \bar{Y}_t since for these two processes, the time schedule (T_n) is only a tool for the proof.

5 Convergence of Y_t in total variation

We now consider (Y_t) as defined in (2.5) with extended definition (2.24).

5.1 Preliminary lemmas

Lemma 5.1. Let $\lambda \in \mathbb{R}^+$. There exists C > 0 such that for every $n \geq 0$, $u \geq 0$ and every $x \in \mathbb{R}^d$:

$$\sup_{t\geq 0} \mathbb{E}\left[e^{\lambda|X_t^{x,n}|^2}\right] \leq Ce^{\lambda|x|^2} \quad and \quad \sup_{t\geq 0} \mathbb{E}\left[e^{\lambda|Y_{t,u}^{x}|^2}\right] \leq Ce^{\lambda|x|^2}. \tag{5.1}$$

Sketch of proof. By Itō's Lemma, we have for $k \geq n$ and for $t \in [T_k - T_n, T_{k+1} - T_n)$:

$$\begin{split} d\left(e^{\lambda|X_t^{x,n}|^2}\right) &= \lambda e^{\lambda|X_t^{x,n}|^2} \left(2\langle X_t^{x,n}, dX_t^{x,n}\rangle + d\langle X^{x,n}\rangle_t\right) + 2\lambda^2|X_t^{x,n}|^2 e^{\lambda|X_t^{x,n}|^2} d\langle X^{x,n}\rangle_t \\ &= \lambda e^{\lambda|X_t^{x,n}|^2} \Big(-2\langle \sigma\sigma^\top \nabla V(X_t^{x,n}), X_t^{x,n}\rangle dt + 2a_{n+1}^2\langle X_t^{x,n}, \Upsilon(X_t^{x,n})\rangle dt \\ &\quad + 2a_{n+1}\langle X_t^{x,n}, \sigma(X_t^{x,n}) dW_t\rangle + a_{n+1}^2 \operatorname{Tr}(\sigma\sigma^\top(X_t)) dt\Big) \\ &\quad + 2\lambda^2 e^{\lambda|X_t^{x,n}|^2} a_{n+1}^2(X_t^{x,n})^\top \sigma\sigma^\top(X_t^{x,n}) X_t^{x,n} dt \end{split}$$

the "dominating" term is $-\langle \sigma \sigma^{\top} \nabla V(X_t^{x,n}), X_t^{x,n} \rangle dt$ which makes $\mathbb{E}[e^{\lambda |X_t^{x,n}|^2}]$ decrease. Using assumption (2.12, \mathcal{H}_{cf}), we have for $|X_t^{x,n}|$ large enough,

$$-\langle \sigma \sigma^{\top} \nabla V(X_t^{x,n}), X_t^{x,n} \rangle \le -C \underline{\sigma}_0^2 \alpha_0 |X_t^{x,n}|^2.$$

Moreover, using the facts that Υ and σ are bounded, that $a_n \to 0$, that $|\nabla V| \leq CV^{1/2}$ and that $\sigma \sigma^{\top} \geq \sigma_0^2 I_d$, for large enough $|X_t^{x,n}|$ and large enough n, the coefficient in dt in the last equation is negative. We deal with the cases where $|X_t^{x,n}|$ is not large enough or where n is not large enough the same way as in the proof of [BP21, Lemma 6.1] and [BP21, Lemma 7.1], where more details can be found.

The proof is the same for Y, replacing a_{k+1} by a(u+t).

Proposition 5.2. Let T, $\bar{\gamma} > 0$. There exists C > 0 such that for every Borel bounded function $f: \mathbb{R}^d \to \mathbb{R}$ and every $t \in (0,T]$, for all $n \geq 0$, for all $\gamma < \bar{\gamma}$ such that $u \in [T_n, T_{n+1}]$ and $u + t + \gamma \in [T_n, T_{n+1}]$,

$$\left| \mathbb{E} \left[P_t^{X,n} f(Y_{\gamma,u}^x) \right] - \mathbb{E} \left[P_t^{X,n} f(X_{\gamma}^{x,n}) \right] \right| \le C a_{n+1}^{-2} (a_n - a_{n+1}) \|f\|_{\infty} \gamma t^{-1} V(x).$$
 (5.2)

Proof. We apply [BP21, Proposition 6.4] to $g_t := P_t^{X,n} f$ with t > 0. Following [PP20, Proposition 3.2(b)], we have

$$\Phi_{g_t}(x) \le C \|f\|_{\infty} a_{n+1}^{-2} t^{-1} \max \left(V^{1/2}(x), \left\| \sup_{\xi \in (X_{\gamma}^{x,n}, Y_{\gamma,u}^x)} V^{1/2}(\xi) \right\|_2, V^{1/2}(x) \left\| \sup_{\xi \in (x, X_{\gamma}^{x,n})} V^{1/2}(\xi) \right\|_2 \right).$$

We conclude as in the proof of [BP21, Proposition 6.5].

5.2 Proof of Theorem 2.1(a)

More precisely, we prove that for all $\beta > 0$, if

$$A > \max\left(\sqrt{(\beta+1)(2C_1+C_2)}, \sqrt{(1+\beta^{-1})C_2}\right),$$
 (5.3)

then

$$d_{\text{TV}}\left(Y_t^{x_0}, \nu_{a(t)}\right) \le \frac{Ce^{C\sqrt{\log(t)}(1+|x_0|^2)}}{t^{(1+\beta)^{-1}-(2C_1+C_2)/A^2}}.$$
(5.4)

Proof. We follow the proof of [BP21, Theorem 2.1(b)] in [BP21, Section 7.3] based on a domino strategy with respect to some decreasing step sequence (γ_n) , even though Y is not an Euler-Maruyama scheme. In this case, the step sequence (γ_n) is only a tool for the proof. This way we can choose freely the sequence (γ_n) in this section. We use Theorem 4.1 in place of [BP21, Theorem 4.2] and Proposition 5.2 in place of [BP21, Proposition 7.4]. For $f: \mathbb{R}^d \to \mathbb{R}$ bounded measurable and for $x \in \mathbb{R}^d$ we write

$$\begin{split} & \left| \mathbb{E} f(X_{T_{n+1}-T_n}^{x,n}) - \mathbb{E} f(Y_{T_{n+1}-T_n,T_n}^{x}) \right| \leq \left| (P_{\gamma^{\text{init}},T_n}^{Y} - P_{\gamma^{\text{init}}}^{X,n}) \circ P_{T_{n+1}-\Gamma_{N(T_n)+1}}^{X,n} f(x) \right| \\ & + \sum_{k=N(T_n)+2}^{N(T_{n+1}-T)} \left| P_{\gamma^{\text{init}},T_n}^{Y} \circ P_{\gamma_{N(T_n)+2},\Gamma_{N(T_n)+1}}^{Y} \circ \cdots \circ P_{\gamma_{k-1},\Gamma_{k-2}}^{Y} \circ (P_{\gamma_k,\Gamma_{k-1}}^{Y} - P_{\gamma_k}^{X,n}) \circ P_{T_{n+1}-\Gamma_k}^{X,n} f(x) \right| \\ & + \sum_{k=N(T_{n+1})-1}^{N(T_{n+1})-1} \left| P_{\gamma^{\text{init}},T_n}^{Y} \circ P_{\gamma_{N(T_n)+2},\Gamma_{N(T_n)+1}}^{Y} \circ \cdots \circ P_{\gamma_{k-1},\Gamma_{k-2}}^{Y} \circ (P_{\gamma_k,\Gamma_{k-1}}^{Y} - P_{\gamma_k}^{X,n}) \circ P_{T_{n+1}-\Gamma_k}^{X,n} f(x) \right| \\ & + \left| P_{\gamma^{\text{init}},T_n}^{Y} \circ P_{\gamma_{N(T_n)+2},\Gamma_{N(T_n)+1}}^{Y} \circ \cdots \circ P_{\gamma_{N(T_{n+1})-1},\Gamma_{N(T_{n+1})-2}}^{Y} \circ (P_{\gamma^{\text{end}}+\gamma_{N(T_{n+1})},\Gamma_{N(T_{n+1})-1}}^{Y} - P_{\gamma^{\text{end}}+\gamma_{N(T_{n+1})}}^{X,n}) f(x) \right| \\ & =: (c^{\text{init}}) + (a) + (b) + (c^{\text{end}}), \end{split}$$

where

$$\gamma^{\text{init}} := \Gamma_{N(T_n)+1} - T_n \le \gamma_{N(T_n)+1}$$
 and $\gamma^{\text{end}} := T_{n+1} - \Gamma_{N(T_{n+1})} \le \gamma_{N(T_{n+1})+1}$.

Then we have

$$(a) \leq C a_{n+1}^{-3} e^{C_1 a_{n+1}^{-2}} e^{-\rho_{n+1} T_{n+1}} \|f\|_{\infty} V(x) (a_n - a_{n+1}) \sum_{k=N(T_n)+2}^{N(T_{n+1} - T)} \gamma_k e^{\rho_{n+1} \Gamma_k}$$

$$\leq C a_{n+1}^{-3} e^{C_1 a_{n+1}^{-2}} \|f\|_{\infty} (a_n - a_{n+1}) V(x) \rho_{n+1}^{-1}.$$

We obtain likewise

$$(c^{\text{init}}) \le Ca_{n+1}^{-3}e^{-\rho_{n+1}(T_{n+1}-T_n)} ||f||_{\infty} (a_n - a_{n+1})\gamma_{N(T_n)} V(x).$$

Applying Proposition 5.2 yields

$$(b) \le C a_{n+1}^{-2} (a_n - a_{n+1}) \|f\|_{\infty} V(x) \sum_{k=N(T_{n+1} - T)+1}^{N(T_{n+1})-1} \frac{\gamma_k}{T_{n+1} - \Gamma_k}$$

$$\le C a_{n+1}^{-2} (a_n - a_{n+1}) \|f\|_{\infty} V(x) \log(1/\gamma_{N(T_{n+1})}).$$

Applying Proposition 3.3 with r = 1 along with Lemma 5.1 yields

$$(c^{\text{end}}) \le C \|f\|_{\infty} \left(e^{Ca_{n+1}^{-1}(1+|x|^2)} \gamma_{N(T_n)}^{1/2} + a_{n+1}^{-(d+1)} \left(a(T_{n+1} - \gamma_{N(T_{n+1})}) - a_{n+1} \right)^{2/3} \right).$$

But we have

$$a(T_{n+1} - \gamma_{N(T_{n+1})}) - a_{n+1} = a(T_{n+1} - \gamma_{N(T_{n+1})}) - a(T_{n+1}) \le C \frac{da}{dt}(T_{n+1}) \cdot \gamma_{N(T_{n+1})} \le \frac{C\gamma_{N(T_{n+1})}}{T_{n+1}}.$$

We now choose $\gamma_n = \gamma_1 n^{-2/3}$ so that $\gamma_{N(T_n)} \approx n^{-2}$ and then

$$(c^{\text{end}}) \le Ce^{Ca_{n+1}^{-1}(1+|x|^2)}n^{-1}.$$

This way we obtain for every $x \in \mathbb{R}^d$:

$$|\mathbb{E}f(X_{T_{n+1}-T_n}^{x,n}) - \mathbb{E}f(Y_{T_{n+1}-T_n,T_n}^{x})| \le C||f||_{\infty} \underbrace{a_{n+1}^{-3}e^{C_1a_{n+1}^{-2}}(a_n - a_{n+1})V(x)\rho_{n+1}^{-1}}_{=:v_{n+1}} e^{Ca_{n+1}^{-1}(1+|x|^2)}. \quad (5.5)$$

We integrate this inequality with respect to the laws of $X_{T_n}^{x_0}$ and $\bar{Y}_{T_n}^{x_0}$ and obtain, temporarily setting $x_n := X_{T_n}^{x_0}$ and $y_n := Y_{T_n}^{x_0}$ and using [BP21, Lemma 6.1] and Lemma 5.1,

$$d_{\text{TV}}(X_{T_{n+1}}^{x_0}, Y_{T_{n+1}}^{x_0}) \leq d_{\text{TV}}(X_{T_{n+1}-T_n}^{x_n, n}, X_{T_{n+1}-T_n}^{y_n, n}) + d_{\text{TV}}(X_{T_{n+1}-T_n}^{y_n, n}, Y_{T_{n+1}-T_n}^{\bar{y}_n})$$

$$\leq \underbrace{Ca_{n+1}^{-1}e^{C_1a_{n+1}^{-2}}e^{-\rho_{n+1}(T_{n+1}-T_n)}}_{:=\mu'_{n+1}=a_{n+1}^{-1}\mu_{n+1}} d_{\text{TV}}(X_{T_n}^{x_0}, Y_{T_n}^{x_0}) + \underbrace{Cv_{n+1}e^{Ca_{n+1}^{-1}(1+|x_0|^2)}}_{:=w_{n+1}},$$

where μ_n is defined in (4.11). Iterating this inequality yields

$$d_{\text{TV}}(X_{T_{n+1}}^{x_0}, Y_{T_{n+1}}^{x_0}) \le C(w_{n+1} + \mu'_{n+1}w_n + \dots + \mu'_{n+1} \dots + \mu'_2w_1) \le Cw_{n+1},$$

where we used, since A satisfies (5.3), that $\mu'_n = O(e^{-Cn^{\eta}})$ for some $\eta > 0$ (see [BP21, (5.5)]) and that w_n is bounded as it converges to 0. Moreover using Theorem 4.3 we have

$$d_{\text{TV}}(Y_{T_n}^{x_0}, \nu_{a_n}) \le d_{\text{TV}}(X_{T_n}^{x_0}, Y_{T_n}^{x_0}) + d_{\text{TV}}(X_{T_n}^{x_0}, \nu_{a_n}) \le \frac{Ce^{C\sqrt{\log(n)}(1+|x_0|^2)}}{n^{1-(\beta+1)(C_1+C_2)/A^2}}.$$
(5.6)

Finally, let us bound $d_{\text{TV}}(X_t^{x_0}, Y_t^{x_0})$ for any $t \in [T_n, T_{n+1}]$. If $t \in [T_n + t_0, T_{n+1}]$ then we can apply Theorem 4.1 and we proceed as in the end of [BP21, Section 6.3]. If $t \in [T_n, T_n + t_0]$, then we consider another shifted time schedule $\bar{T}_n := C_{(T)} n^{1+\beta} + 2t_0$ such that

$$\bigcup_{i=0}^{\infty} [T_n, T_n + t_0] \cap \bigcup_{i=0}^{\infty} [\bar{T}_n, \bar{T}_n + t_0] = \varnothing.$$

Making use of the new time schedule we obtain as before a bound on $d_{\text{TV}}(Y_t^{x_0}, \nu_{a(t)})$ for every $t \notin \bigcup_{i=0}^{\infty} [\bar{T}_n, \bar{T}_n + t_0]$. Since the time schedules (T_n) and (\bar{T}_n) are only tools for the proof of convergence of Y_t , we then obtain a bound on $d_{\text{TV}}(Y_t, \nu_{a(t)})$ for every $t \in \mathbb{R}^+$.

6 Convergence of the Euler-Maruyama scheme in total variation

We now consider (\bar{Y}_n) as in (2.17) with extended definition (2.25).

6.1 Preliminary lemmas

Lemma 6.1. Let $\lambda \in \mathbb{R}^+$. There exists a constant C > 0 such that for every $k \geq 0$, for every $u \in [\Gamma_k, \Gamma_{k+1})$ and for every $x \in \mathbb{R}^d$:

$$\sup_{n>k+1} \mathbb{E}\left[e^{\lambda|Y_{\Gamma_n-u,u}^x|^2}\right] \le Ce^{\lambda|x|^2}.$$
(6.1)

Proof. The prove is the same as for Lemma 5.1. For the adaptation to discrete time, we refer to the proof of [BP21, Lemma 7.1]. \Box

Proposition 6.2. Let T > 0. There exists C > 0 such that for every Lipschitz continuous function f and every $t \in (0,T]$, for all $n \geq 0$, for all γ such that $\Gamma_k \in [T_n,T_{n+1}]$, $\gamma \leq \gamma_{k+1}$ and $\Gamma_k + t + \gamma \in [T_n,T_{n+1}]$,

$$\left| \mathbb{E} \left[P_t f(\bar{Y}_{\gamma,\Gamma_k}^x) \right] - \mathbb{E} \left[P_t f(X_{\gamma}^{x,n}) \right] \right| \\
\leq C \|f\|_{\infty} V^2(x) \left(a_{n+1}^{-2} t^{-1} \left(\gamma^2 + (a(\Gamma_k) - a_{n+1}) \gamma \right) + a_{n+1}^{-3} t^{-3/2} \left(\gamma^2 + \gamma^{3/2} (a(\Gamma_k) - a_{n+1}) \right) \right). \tag{6.2}$$

Proof. The proof is the same as the proof of Proposition 5.2, using [BP21, Proposition 7.3]. We also remark that we can directly improve the bound in $(a_n - a_{n+1})$ into $(a(\Gamma_k) - a_{n+1})$.

6.2 Proof of Theorem 2.1(b)

More precisely, we prove that for all $\beta > 0$, if $\sigma \in \mathcal{C}_b^{2r}$ and if

$$A > \max\left(\sqrt{(\beta+1)(2C_1+C_2)}, \sqrt{(1+\beta^{-1})C_2}\right)$$
(6.3)

and if A is large enough so that

$$n^{(\beta+1)C_1/A^2} \gamma_{N(T_n)}^{r/(2r+1)} \underset{n \to \infty}{\longrightarrow} 0, \tag{6.4}$$

then

$$d_{\text{TV}}(\bar{Y}_t^{x_0}, \nu_{a(t)}) \le C \left(\frac{\log^{1/2}(t) \max \left[V^2(x_0), 1 + |x_0| \right]}{t^{(\beta+1)^{-1} - (2C_1 + C_2)/A^2}} + e^{C\sqrt{\log(t)}(1 + |x_0|^2)} t^{C_1/A^2} \gamma_{Ct}^{r/(2r+1)} \right). \tag{6.5}$$

Proof. We still follow the proof of [BP21, Theorem 2.1(b)] in [BP21, Section 7.3] based on a domino strategy, using Theorem 4.1 in place of [BP21, Theorem 4.2] and Proposition 6.2 in place of [BP21, Proposition 7.4]. Let $n \geq 0$, for $f: \mathbb{R}^d \to \mathbb{R}$ bounded measurable, we split $|\mathbb{E}f(X_{T_{n+1}-T_n}^{x,n}) - \mathbb{E}f(\bar{Y}_{T_{n+1}-T_n,T_n}^x)|$ into four terms (c^{init}) , (a), (b), (c^{end}) .

Using Theorem 4.1, [BP21, Lemma 7.1] and Proposition 6.2 we get as in [BP21, Section 7.3]:

$$(a) \le C a_{n+1}^{-4} e^{C_1 a_{n+1}^{-2}} \|f\|_{\infty} (a_n - a_{n+1}) V^2(x) \rho_{n+1}^{-1}.$$

$$(c^{\text{init}}) \le C a_{n+1}^{-4} e^{C_1 a_{n+1}^{-2}} e^{-\rho_n (T_{n+1} - T_n)} \|f\|_{\infty} (a_n - a_{n+1}) \gamma_{N(T_n) + 1} V^2(x).$$

Using Proposition 6.2 and [BP21, Lemma 7.1], we obtain

$$(b) \leq Ca_{n+1}^{-3} \left(\gamma_{N(T_{n+1}-T)} + \sqrt{\gamma_{N(T_{n+1}-T)}} (a_n - a_{n+1}) \right) \|f\|_{\infty} V^2(x) \sum_{k=N(T_{n+1}-T)+1}^{N(T_{n+1})-1} \frac{\gamma_k}{(T_{n+1} - \Gamma_k)^{3/2}} + Ca_{n+1}^{-2} \left(\sum_{k=N(T_{n+1}-T)+1}^{N(T_{n+1})-1} \frac{\gamma_{N(T_{n+1}-T)} \gamma_k}{T_{n+1} - \Gamma_k} + \sum_{k=N(T_{n+1}-T)+1}^{N(T_{n+1})-1} \frac{\gamma_k (a(\Gamma_k) - a_{n+1})}{T_{n+1} - \Gamma_k} \right) \|f\|_{\infty} V^2(x).$$

But we remark that

$$a(\Gamma_k) - a_{n+1} = a(\Gamma_k) - a(T_{n+1}) \le C \frac{da}{dt} (T_{n+1}) \cdot (\Gamma_k - T_{n+1}) \le \frac{C(\Gamma_k - T_{n+1})}{T_{n+1} \log^{3/2} (T_{n+1})}$$

and then

$$(b) \leq Ca_{n+1}^{-3} \left(\gamma_{N(T_{n+1}-T)} + \sqrt{\gamma_{N(T_{n+1}-T)}} (a_n - a_{n+1}) \right) \|f\|_{\infty} V^2(x) \int_{T_{n+1}-T}^{T_{n+1}-\gamma_{N(T_{n+1})}} \frac{du}{(T_{n+1}-u)^{3/2}}$$

$$+ Ca_{n+1}^{-2} \left(\gamma_{N(T_{n+1}-T)} \int_{T_{n+1}-T}^{T_{n+1}-\gamma_{N(T_{n+1})}} \frac{du}{T_{n+1}-u} + \frac{1}{T_{n+1}} \int_{T_{n+1}-T}^{T_{n+1}-\gamma_{N(T_{n+1})}} du \right) \|f\|_{\infty} V^2(x)$$

$$\leq Ca_{n+1}^{-3} \left(\gamma_{N(T_{n+1}-T)} + \sqrt{\gamma_{N(T_{n+1}-T)}} (a_n - a_{n+1}) \right) \|f\|_{\infty} V^2(x) \gamma_{N(T_{n+1})}^{-1/2}$$

$$+ Ca_{n+1}^{-2} \left(\gamma_{N(T_{n+1})} |\log(\gamma_{N(T_{n+1})})| + T_{n+1}^{-1} \right) \|f\|_{\infty} V^2(x)$$

$$\leq Ca_{n+1}^{-3} \left(\gamma_{N(T_{n+1})}^{1/2} + (a_n - a_{n+1}) \right) \|f\|_{\infty} V^2(x).$$

Applying Proposition 3.2 along with Lemma 6.1 yields

$$(c^{\text{end}}) \le C \|f\|_{\infty} \left(e^{Ca_{n+1}^{-1}(1+|x|^2)} \gamma_{N(T_{n+1})}^{r/(2r+1)} + a_n^{-2}(a_n - a_{n+1}) \right).$$

We finally obtain for every $x \in \mathbb{R}^d$:

$$|\mathbb{E}f(X_{T_{n+1}-T_n}^{x,n}) - \mathbb{E}f(\bar{Y}_{T_{n+1}-T_n,T_n}^x)| \leq C||f||_{\infty} \left(a_{n+1}^{-4}e^{C_1a_{n+1}^{-2}}(a_n - a_{n+1})V^2(x)\rho_{n+1}^{-1} + e^{Ca_{n+1}^{-1}(1+|x|^2)}\gamma_{N(T_{n+1})}^{r/(2r+1)}\right).$$

The same way as in Section 5.2 we get

$$\begin{aligned} &\mathrm{d_{TV}}(\bar{Y}_{T_{n+1}}^{x_0},\nu_{a_{n+1}}) \leq C\left(a_{n+1}^{-4}e^{C_1a_{n+1}^{-2}}(a_n-a_{n+1})\max\left[V^2(x_0),1+|x_0|\right]\rho_{n+1}^{-1}+e^{Ca_{n+1}^{-1}(1+|x_0|^2)}\gamma_{N(T_{n+1})}^{r/(2r+1)}\right)\\ &\mathrm{and, for } \ t\in[T_n,T_{n+1}], \end{aligned}$$

$$\mathrm{d}_{\mathrm{TV}}(\bar{Y}_t^{x_0},\nu_{a(t)}) \leq Ce^{C_1a_{n+1}^{-2}} \left(a_{n+1}^{-4}e^{C_1a_{n+1}^{-2}}(a_n - a_{n+1}) \max \left[V^2(x_0), 1 + |x_0| \right] \rho_{n+1}^{-1} + e^{Ca_{n+1}^{-1}(1 + |x_0|^2)} \gamma_{N(T_{n+1})}^{r/(2r+1)} \right).$$

A Appendix

A.1 Proof of Proposition 4.2

Proof. We use the characterization of the total variation distance as the L^1 -distance between the densities, which reads

$$\begin{split} \mathrm{d}_{\mathrm{TV}}(\nu_{a_{n}},\nu_{a_{n+1}}) &= \int_{\mathbb{R}^{d}} \left| \mathcal{Z}_{a_{n}} e^{-2(V(x)-V^{\star})/a_{n}^{2}} - \mathcal{Z}_{a_{n+1}} e^{-2(V(x)-V^{\star})/a_{n+1}^{2}} \right| dx \\ &\leq \mathcal{Z}_{a_{n+1}} \int_{\mathbb{R}^{d}} \left| e^{-2(V(x)-V^{\star})/a_{n}^{2}} - e^{-2(V(x)-V^{\star})/a_{n+1}^{2}} \right| dx + \left| \mathcal{Z}_{a_{n}} - \mathcal{Z}_{a_{n+1}} \right| \int_{\mathbb{R}^{d}} e^{-2(V(x)-V^{\star})/a_{n}^{2}} dx \\ &= \mathcal{Z}_{a_{n+1}} a_{n+1}^{d} \int_{\mathbb{R}^{d}} \left| e^{-2(V(a_{n+1}x)-V^{\star})/a_{n}^{2}} - e^{-2(V(a_{n+1}x)-V^{\star})/a_{n+1}^{2}} \right| dx \\ &+ \left| 1 - \frac{\mathcal{Z}_{a_{n}}}{\mathcal{Z}_{a_{n+1}}} \right| \mathcal{Z}_{a_{n+1}} a_{n}^{d} \int_{\mathbb{R}^{d}} e^{-2(V(a_{n}x)-V^{\star})/a_{n}^{2}} dx. \end{split}$$

Using [BP21, (B.3)] and [BP21, (B.5)], the first term is bounded by

$$C\frac{a_n - a_{n+1}}{a_n} \int_{\mathbb{R}^d} e^{-2(V(a_{n+1}y) - V^*)/a_n^2} \frac{V(a_{n+1}y) - V^*}{a_n^2} dx \le C\frac{a_n - a_{n+1}}{a_n},$$

because the integral converges by dominated convergence as for the proof of [BP21, (B.3)]. Using [BP21, (B.3)] and [BP21, (B.4)], the second term is bounded by $C(n \log(n))^{-1}$.

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