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Research Article

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Some Open Questions in the Numerical Analysis of Singularly Perturbed Differential Equations

Abstract: Several open questions in the numerical analysis of singularly perturbed differential equations are discussed. These include whether certain convergence results in various norms are optimal, when superclose-ness is obtained in finite element solutions, the validity of defect correction in finite difference approximations, and desirable adaptive mesh refinement results that remain to be proved or disproved.

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1 Introduction

In the last decade, a large number of papers dealing with the numerical analysis of singularly perturbed differential equations have appeared in the research literature. A search of the MathSciNet database for papers published in the years 2005–2014 with MSC Primary Classification 65 (viz., Numerical Analysis) and the phrase " singular* perturb* " [in MathSciNet asterisks are wildcards] yields 879 published works. An overview of this body of work is given in the monograph [81], in Linß's book [60] on layer-adapted meshes, and in Roos's survey article [73].

Clearly there is a very healthy level of research activity in this area. But regrettably, the "new" results in many recent papers are merely minor extensions and/or syntheses of older results. (Or worse, they are results that were already known!) Perhaps this is an indication that our area of numerical analysis has reached a mature stage in its development?

Despite this remarkable amount of activity, some old and fairly important research questions in the numerical analysis of singularly perturbed differential equations remain unanswered. In this article we shall describe some of these open problems – confining our attention to questions that we regard as interesting. Of course, our selection is inevitably personal and reflects our own main research interests. We hope that our exposition will stimulate further *worthwhile* research on the numerical analysis of singularly perturbed differential equations.

1.1 The Classes of Problems Considered

Our discussions in this paper centre on two classes of problems, which we now describe.

Let ε be a small positive parameter. In our differential equations, which are all second-order, this parameter will be the diffusion coefficient and ε is a singular perturbation parameter. If instead one had $\varepsilon = 1$,

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then the differential equation would be diffusion-dominated and would be amenable to classical analysis and to the standard numerical methods that one finds in typical undergraduate textbooks. The interesting and challenging case for numerical analysts is when ε is close to zero.

Our main focus is the *convection-diffusion* problem

$$Lu := -\varepsilon\Delta u - \mathbf{b}\cdot\nabla u + cu = f \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{1.1a}$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{1.1b}$$

where $\mathbf{b} = (b_1, b_2, \dots, b_n)$ and c are smooth functions ($\mathbf{b}$ models convection while c models reaction) and $f \in L_2(\Omega)$. Here Ω is some bounded domain with boundary $\partial\Omega$, and $n \geq 1$. Some additional hypotheses will be needed to ensure that (1.1) has a unique solution in some suitable normed space. For example, in finite element analysis one typically assumes that

$$c + \frac{1}{2}\operatorname{div}\mathbf{b} \geq \alpha_0 > 0 \quad \text{on } \Omega \tag{1.2}$$

for some constant α_0; then the problem (1.1), (1.2) has a unique solution $u \in H_0^1(\Omega)$. Moreover, if Ω is convex then $u \in H^2(\Omega)$.

The problem (1.1) can also be called a *convection-reaction-diffusion* problem, reflecting the presence of the reaction term cu. Both names for (1.1) place diffusion last to emphasise that the influence of the highest-order diffusion term is weakened by its small coefficient.

The second main class of singularly perturbed differential equation in our sphere of interest is the (linear) *reaction-diffusion* problem

$$Lu := -\varepsilon\Delta u + cu = f \quad \text{in } \Omega, \tag{1.3a}$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{1.3b}$$

where c is a smooth function and $f \in L_2(\Omega)$. The domain Ω is as above, and the condition (1.2) simplifies to $c \geq \alpha_0 > 0$ on Ω. Results for problems in this class frequently can be extended to the semilinear reaction-diffusion equation $-\varepsilon\Delta u(x) + f(x,u) = 0$ where one assumes that $f_u(x,u) \geq \alpha_0$ on $\mathbb{R}\times\mathbb{R}$, possibly with additional hypotheses on f.

The reaction-diffusion problem (1.3) is more easily solved (and analysed) than the convection-diffusion problem (1.1), but it does present some challenges to the numerical analyst, as we shall see.

Notation. Throughout the paper C denotes a generic positive constant that is independent of ε and of the mesh diameter in any numerical method. Standard notation is used for the Lebesgue spaces $L_p(\Omega)$ and the Sobolev spaces $H^k(\Omega)$, with their respective associated norms $\|\cdot\|_{L_p}$ and $\|\cdot\|_k$, and the space $H_0^1(\Omega)$ comprising those functions in $H^1(\Omega)$ whose traces vanish on $\partial\Omega$. We also use the Sobolev seminorms $|\cdot|_j$ where $\|\cdot\|_k \equiv [\sum_{j=0}^k |\cdot|_j^2]^{1/2}$. Thus $|\cdot|_0 = \|\cdot\|_0 = \|\cdot\|_{L_2}$. The $L_2(\Omega)$ inner product is denoted by $(\cdot,\cdot)$.

We include here the first two of our open questions – even though they change the boundary conditions in (1.1) and do not involve numerical analysis – because they are quite basic.

Question 1.1. Suppose that one has a homogeneous Neumann outflow condition in (1.1), i.e., along that part of $\partial\Omega$ where $\mathbf{b}$ points out of Ω, the boundary condition $u = 0$ is replaced by $\partial u/\partial n = 0$ where n denotes the unit normal to $\partial\Omega$. Suppose also that the characteristic boundary (where $\mathbf{b}$ is tangent to $\partial\Omega$) is a set of $(n-1)$-dimensional measure zero. Define as in [81, Section III.1] the reduced solution u_0 of (1.1) by $-\mathbf{b}\cdot\nabla u_0 + cu_0 = f$ on Ω and $u_0 = 0$ on the inflow boundary of Ω (that part of $\partial\Omega$ where $\mathbf{b}$ points into Ω). Can one show that $\|u - u_0\|_1 = \mathcal{O}(\varepsilon^{1/2})$?

A related problem, where $n = 2$, $\mathbf{b} = (1,0)$, c is a positive constant and $\Omega = (0,1)^2$, is analysed in [68].

Question 1.2. In the previous question, suppose that the Neumann condition is replaced by the condition $-\mathbf{b}\cdot\nabla u + cu = f$ on the outflow boundary (cf. [7]). What bound can one then prove for $\|u - u_0\|_1$?

1.2 Decompositions of the Solution

In the simplest convection-diffusion problem where $n = 1$ in (1.1) and $\mathbf{b} = b_1 > \beta > 0$ on $[0, 1]$ for some constant β, the solution u will have an exponential boundary layer at $x = 0$. That is, for $x \in [0, 1]$ one has the sharp bounds

$$|u^{(i)}(x)| \le C\big[1 + \varepsilon^{-i}\exp(-\beta x/\varepsilon)\big] \quad \text{for } i = 0, 1, \dots, q, \tag{1.4}$$

where q depends on the regularity of the data of the problem. In [55] it is shown that (1.4) is equivalent to the decomposition $u = S + E$ where the smooth component S and the layer component E satisfy

$$|S^{(i)}(x)| \le C, \quad |E^{(i)}(x)| \le C\varepsilon^{-i}\exp(-\beta x/\varepsilon) \quad \text{for } i = 0, 1, \dots, q,$$
$$LS = f \quad \text{and} \quad LE = 0.$$

We call this an *S-decomposition* because it is originally due to Shishkin [86, 88, 89].

When $n = 2$ in (1.1), sufficient conditions for the existence of an S-decomposition of the convection-diffusion solution u are known only for small values of q and provided that the locations and nature of the layers in u are known. For problems where u has only exponential layers and for some problems with characteristic layers, see [42, 43, 63, 69].

To analyse finite difference or (linear and bilinear) finite element methods for a two-dimensional convection-diffusion problem posed on the unit square (i.e., $\Omega = (0, 1)^2$) with exponential layers, one assumes typically that $\mathbf{b} > (\beta_1, \beta_2) > (0, 0)$ for some constants β_i and

$$u = S + E_1 + E_2 + E_{12} \tag{1.5a}$$

with

$$\left|\frac{\partial^{i+j}S}{\partial x^i \partial y^j}(x, y)\right| \le C, \tag{1.5b}$$

$$\left|\frac{\partial^{i+j}E_1}{\partial x^i \partial y^j}(x, y)\right| \le C\varepsilon^{-i}e^{-\beta_1 x/\varepsilon}, \quad \left|\frac{\partial^{i+j}E_2}{\partial x^i \partial y^j}(x, y)\right| \le C\varepsilon^{-j}e^{-\beta_2 y/\varepsilon}, \tag{1.5c}$$

$$\left|\frac{\partial^{i+j}E_{12}}{\partial x^i \partial y^j}(x, y)\right| \le C\varepsilon^{-(i+j)}e^{-(\beta_1 x+\beta_2 y)/\varepsilon} \tag{1.5d}$$

for all $(x, y) \in \bar{\Omega}$ and $0 \le i + j \le k$, where $k = 2$ or 3. Note that increasing k also increases the number of compatibility conditions required of the data of (1.1) at the corners of the domain Ω, as described in the papers cited above, but sometimes [47, Remark 3.3] it is possible to use fewer corner compatibility conditions than in those papers.

To avoid the (possibly excessive) corner compatibility assumptions needed for the validity of (1.5) with $k = 3$, for FEMs one can try to use the pointwise information of (1.5) for $0 \le i + j \le 2$ only, combined with weaker L_2 information for certain third-order derivatives. We do not discuss this approach here; see [81] for more details.

Remark 1.3. The remaining sections of this paper, which are largely independent of each other, will be presented in the order that is most convenient for our exposition. This order has nothing to do with their relative importance.

2 Stabilised FEM L_2 Errors: Is $h^{1/2}$ Always Missing?

Choose a finite element space $V_h \subset H_0^1(\Omega)$, where h is the mesh diameter. Then the standard Galerkin finite element method for solving (1.1) is: Find $u_h \in V_h$ such that

$$a_G(u_h, v_h) := \varepsilon(\nabla u_h, \nabla v_h) - (\mathbf{b} \cdot \nabla u_h - cu_h, v_h) = (f, v_h) \quad \text{for all } v_h \in V_h. \tag{2.1}$$

Define the ε-weighted $H^1(\Omega)$ norm by

$$\|v\|_\varepsilon := \varepsilon^{1/2}|v|_1 + \|v\|_0; \tag{2.2}$$

this norm is natural for the analysis of finite element methods for (1.1). One can prove easily that the Galerkin solution u_h satisfies the stability inequality

$$\|u_h\|_\varepsilon \le C\|f\|_0 \tag{2.3}$$

on quite general triangulations of diameter h. This bound is sharp, but the norm $\|\cdot\|_\varepsilon$ is so weak that in practice the computed solution u_h typically exhibits large oscillations when V_h comprises piecewise linear or bilinear elements, and is therefore unsatisfactory.

This instability of the standard Galerkin method has lead to the development of several types of stabilised Galerkin methods when solving (1.1). Most stabilised methods are modifications of the standard Galerkin method: Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) := a_G(u_h, v_h) + a_{\mathrm{st}}(u_h, v_h) = f_h(v_h) \quad \text{for all } v_h \in V_h, \tag{2.4}$$

where $a_{\mathrm{st}}(\cdot,\cdot)$ represents a stabilisation term and f_h is some modification of (f, v_h). If the discrete bilinear form $a_h(\cdot,\cdot)$ is V_h-elliptic (or satisfies an inf-sup condition) that is uniform in ε with respect to some norm $|||\cdot|||$ that is stronger than $\|\cdot\|_\varepsilon$, then typically one has

$$|||u_h||| \le C\|f\|^* \tag{2.5}$$

for some norm $\|\cdot\|^*$. One constructs the stabilised method to yield a norm $|||\cdot|||$ that is so strong that large oscillations in u_h are excluded by (2.5); see for example [84].

As well as stability in our computed solutions, of course we also want accuracy. This is where our open question arises. Take for example the best-known example of a stabilised FEM: the streamline diffusion finite element method (SDFEM), where in (2.4) one takes

$$a_{\mathrm{st}}(u_h, v_h) = \sum_K \delta_K(-\varepsilon\Delta u_h - \mathbf{b}\cdot\nabla u_h + cu_h, -\mathbf{b}\cdot\nabla v_h)_K$$

with δ_K a user-chosen parameter that is constant on each finite element K in the decomposition of Ω and $(\cdot,\cdot)_K$ the $L_2(K)$ inner product. On shape-regular meshes, typically $\delta_k = \mathcal{O}(h_K)$. Then, define the streamline diffusion norm $|||\cdot|||_{\mathrm{SD}}$ by

$$|||v|||_{\mathrm{SD}} := \|v\|_\varepsilon + \Big(\sum_K \delta_K\|\mathbf{b}\cdot\nabla v\|_{0,K}^2\Big)^{1/2},$$

where $\|w\|_{0,K}^2 := (w, w)_K$. The usual analysis of the SDFEM – see for example [81, Section III.3.2.1] – leads (under reasonable constraints on the formulation of the method) to the error bound

$$|||u - u_h|||_{\mathrm{SD}} \le C(\varepsilon^{1/2} + h^{1/2})h^k|u|_{k+1} \tag{2.6}$$

when V_h contains all piecewise polynomials of degree k and $u \in H^{k+1}(\Omega)$. Here the presence of boundary layers implies that the factor $|u|_{k+1}$ is typically $\mathcal{O}(\varepsilon^{-k+1/2})$, which is very large when ε is near zero, so the bound (2.6) is not by itself evidence of accuracy, but using cut-off functions one can usually obtain analogous bounds for stabilised methods on subdomains of Ω that exclude layers in u. Thus, we can expect that away from the layers, the SDFEM solution u_h satisfies

$$|||u - u_h|||' \le C(\varepsilon^{1/2} + h^{1/2})h^k, \tag{2.7}$$

where the notation $|||\cdot|||'$ means that $|||\cdot|||$ is restricted to some subdomain Ω' that does not intersect any layer of u.

So far so good; but usually $\varepsilon < h$ and as we are working with a finite element method, one typically has $\|\cdot\|_{L_2(\Omega')} \le C|||\cdot|||'$ for some constant C. Thus, we can infer that

$$\|u - u_h\|_{L_2(\Omega')} \le Ch^{k+1/2}. \tag{2.8}$$

But the optimal bound for the L_2 error, when some standard interpolant $i_h u$ is used, is

$$\|u - i_h u\|_{L_2(\Omega')} \le Ch^{k+1}. \tag{2.9}$$

Why is (2.8) order $h^{1/2}$ less than optimal?

One might surmise that (2.8) is due to a lack of sharpness in the analysis, but numerical experiments in [102] for piecewise linears (so $k = 1$) show that for specially-chosen shape-regular meshes one can indeed have $\|u - u_h\|_{L_2(\Omega')} = \mathcal{O}(h^{3/2})$. Furthermore, a micro-analysis of the behaviour of the computed solution in [102] proves rigorously that precisely this order of convergence is attained. This settles the issue for the case $k = 1$ and piecewise linears in the SDFEM, but the case $k > 1$ remains open and will form part of our Question 2.2 a little later.

We remark in passing that if one imposes a lot of structure on the mesh one can then prove an optimal L_2 result; for example, on a so-called three-directional mesh the inequality $\|u - u_h\|_0 \le Ch^2|u|_5$ holds true [81, Theorem 3.36].

When one goes on to consider the analyses of other stabilised FEMs on fairly general meshes for (1.1), one encounters a similar phenomenon: for each stabilised FEM the main error bound that is derived in some appropriate stronger norm implies a bound similar to (2.8), i.e., an error that is order $h^{1/2}$ less than optimal in L_2. The stabilised FEMs in this observation include methods based on local projection stabilisation, various discontinuous Galerkin FEMs, methods based on continuous interior penalty stabilisation, and the Galerkin least squares method; see [14, 18, 44, 45, 81] .

Question 2.1. Zhou [102] has shown that for the SDFEM with piecewise linears, the known L_2 estimate of $\mathcal{O}(h^{3/2})$ is best possible. Can one devise examples showing for some value of $k \ge 1$ that other known suboptimal L_2 estimates are best possible? Special cases include (i) the SDFEM with piecewise bilinears where $\mathcal{O}(h^{3/2})$ is known (ii) piecewise linears or bilinears for any of the other FEMs listed above, where $\mathcal{O}(h^{3/2})$ is known.

Question 2.2. For general shape-regular meshes of diameter h and a finite element space V_h that includes all polynomials of degree $k \ge 1$, can one construct a finite element method whose solution $u_h \in V_h$ has the optimal L_2 error property

$$\|u - u_h\|_0 \le Ch^{k+1}\|u\|_m \quad \text{for some } m? \tag{2.10}$$

At present no such method is known for any value of k.

3 The Il'in–Allen–Southwell Scheme in 2D

This famous finite difference method (which is closely related to the Scharfetter–Gummel scheme), derived independently by Allen and Southwell [5] and Il'in [40] for the one-dimensional analogue of (1.1), was shown by Il'in to be first-order convergent in the discrete maximum norm, uniformly in ε. This result is discussed, for instance, in [81, Section I.2.1]. But what is known about the extension of this scheme to the case $n = 2$ in (1.1)?

Suppose that $n = 2$ and $\Omega = (0, 1)^2$ in (1.1). Consider the equidistant mesh $\{(x_i, y_j) : i, j = 0, \dots, N\}$ where $N \in \mathbb{N}$, $x_i = i/N$ and $y_j = j/N$. Set $h = 1/N$. Write g_{ij} for $g(x_i, y_j)$, where g can be b_1, b_2, c or f. Denote the finite difference solution at (x_i, y_j) by u_{ij}^N for $i, j = 0, \dots, N$. Then, at each node (x_i, y_j) with $0 < i, j < N$, the I-A-S scheme is

$$\begin{aligned}
&-\varepsilon\frac{(b_1)_{ij}h}{2\varepsilon}\coth\Big(\frac{(b_1)_{ij}h}{2\varepsilon}\Big)\frac{u_{i+1,j}^N - 2u_{ij}^N - u_{i-1,j}^N}{h^2} + (b_1)_{ij}\frac{u_{i+1,j}^N - u_{i-1,j}^N}{h}\\
&\quad-\varepsilon\frac{(b_2)_{ij}h}{2\varepsilon}\coth\Big(\frac{(b_2)_{ij}h}{2\varepsilon}\Big)\frac{u_{i,j+1}^N - 2u_{ij}^N - u_{i,j-1}^N}{h^2} + (b_2)_{ij}\frac{u_{i,j+1}^N - u_{i,j-1}^N}{h} + c_{ij}u_{ij}^N = f_{ij}.
\end{aligned} \tag{3.1}$$

One also imposes the boundary condition from (1.1): $u_{ij}^N = 0$ if either i or j equals 0 or N. Here we have chosen to write the scheme in a form that portrays it as an "artificial viscosity" scheme; alternatively, it could be expressed in an upwinded form – cf. [81, p. 52] or see [79].

It is straightforward to verify that the matrix associated with (3.1) is an M-matrix, so the scheme satisfies a discrete maximum principle. Numerical results in [37] show that the I-A-S scheme is often first-order convergent, uniformly in ε, but may be less accurate when the solution u is less smooth because of incompatibility of the data of (1.1) at the corners of the domain.

Assuming that $b_1 > 0$, $b_2 > 0$ and that u lies in the Hölder space $C^{k,\lambda}(\bar{\Omega})$, Emel'janov [21] proves that

$$\max_{i,j} |u(x_i, y_j) - u_{ij}^N| \le Ch^{2/(4+\lambda)}. \tag{3.2}$$

Note the strangeness of this bound: as λ increases – so the solution u becomes smoother – the rate of convergence guaranteed by (3.2) decreases! Furthermore, the order of convergence in (3.2) is at best $\mathcal{O}(h^{1/2})$, which is inferior to the first-order convergence attained by the I-A-S scheme in 1D problems.

In fact Emel'janov's result is derived for both the 2D and 3D versions of the I-A-S scheme.

A discussion of Emel'janov's argument is given by Roos and Schopf [79]. They then improve his result: assuming that (1.5) is valid for $k = 3$ and that $b_1 = b_1(x)$, $b_2 = b_2(y)$, bounds for the discrete Green's function associated with the I-A-S scheme are used to show that the scheme has the anisotropic stability property

$$\max_{ij} |u_{ij}^N| \le C\Big(\max_{ij} |(f_0)_{ij}| + h\sum_{i=0}^{N} \max_j |(f_1)_{ij}| + h\sum_{j=0}^{N} \max_i |(f_2)_{ij}|\Big) \tag{3.3}$$

for any decomposition $f = f_0 + f_1 + f_2$. This leads to the discrete maximum norm first-order convergence result

$$\max_{i,j} |u(x_i, y_j) - u_{ij}^N| \le Ch. \tag{3.4}$$

Numerical results in [79] show that this rate of convergence can deteriorate if the solution u is less smooth than was assumed in the analysis. Further numerical experiments carried out by the first author together with M. Schopfshow that, for u satisfying (1.5) for $k = 3$, apparently one still obtains first-order convergence in the discrete maximum norm when $b_1 = b_1(x, y)$, $b_2 = b_2(x, y)$.

Question 3.1. (i) Can one prove (3.4) when $b_1 = b_1(x, y)$, $b_2 = b_2(x, y)$ and u satisfies (1.5) for $k = 3$?
(ii) Can one extend the result (3.4) to the 3D case? Note that the determination of the regularity of the solution u and its decomposition then becomes more complicated; see [90].

Note that in the case $b = (b_1, 0, \dots, 0)$, $c = 0$ and for any n, Emel'janov [22] proves that for the I-A-S scheme the bound (3.4) is valid in the subdomain $[0, 1] \times [d, 1-d] \times \cdots \times [d, 1-d]$ for any fixed $d \in (0, 1)$, i.e., outside the characteristic boundary layers one obtains first-order convergence that is uniform in ε.

4 Error Estimates on Layer-Adapted Meshes

The derivative bound (1.4) has inspired the construction of various layer-adapted meshes for one-dimensional convection-diffusion problems with $b > \beta > 0$ on the interval $[0, 1]$. We now describe the main ideas in these constructions.

The inequality (1.4) tells us essentially that u has a layer of the form $\exp(-\beta x/\varepsilon)$. This layer is located at $x = 0$. As early as 1969, Bakhvalov [9] proposed a special mesh with mesh points x_i near $x = 0$ defined by the inverse function of this boundary layer; outside the layer an equidistant mesh is used.

We describe a simpler version of the Bakhvalov mesh that is known as a *B-type mesh*. Set $x_i = \varphi(i/N)$ for $i = 0, 1, \dots, N$, where

$$\varphi(\xi) = \begin{cases} \chi(\xi) := -\dfrac{\sigma\varepsilon}{\beta} \ln \dfrac{q-\xi}{q} & \text{for } \xi \in [0, \tau], \\ \chi(\tau) + \dfrac{\xi-\tau}{1-\tau}(1-\chi(\tau)) & \text{for } \xi \in [\tau, 1]. \end{cases}$$

Here τ is a transition point between the fine and coarse submeshes, the parameter $q \in (0, 1)$ determines how many mesh points are used to resolve the layer, and $\sigma > 0$ controls the spacing within the layer region.

Originally Bakhvalov chose τ to ensure that the mesh generating function φ lay in $C^1[0, 1]$ with $\varphi(1) = 1$, but this gives a nonlinear scalar equation for τ that must be solved numerically; instead, one can simplify the mesh construction by defining explicitly

$$\tau = \frac{\gamma\varepsilon}{\beta}|\ln \varepsilon| \text{ for some user-chosen positive constant } \gamma, \text{ so } e^{-\beta\tau/\varepsilon} = \varepsilon^{\gamma}.$$

For both these choices of τ, the layer function $\exp(-\beta x/\varepsilon)$ is small when $x \geq \tau$. But from the point of view of numerical analysis, the choice of transition point τ should reflect the smallness of the layer term component of the discretisation error instead of the smallness of $\exp(-\beta x/\varepsilon)$. Assume the formal order of the numerical method to be σ. Then imposing the condition

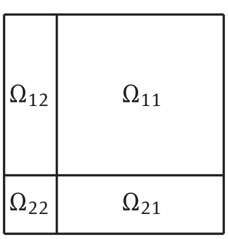
$$\exp\Big(-\frac{\beta\tau}{\varepsilon}\Big) = N^{-\sigma}$$

yields the choice $\tau = (\sigma\varepsilon/\beta)\ln N$ for the transition point. We call a mesh an *S-type mesh* if it is generated by

$$\varphi(\xi) = \begin{cases} \dfrac{\sigma\varepsilon}{\beta}\hat{\varphi}(\xi) \quad \text{with} \quad \hat{\varphi}(0) = 0,\ \hat{\varphi}(1/2) = \ln N & \text{for } \xi \in [0, 1/2], \\ 1 - 2\Big(1 - \dfrac{\sigma\varepsilon}{\beta}\ln N\Big)(1 - \xi) & \text{for } \xi \in [1/2, 1], \end{cases} \tag{4.1}$$

where $\hat{\varphi}$ is some monotonic function. For the particular choice $\hat{\varphi}(\xi) = 2(\ln N)\xi$, the mesh generated is piecewise equidistant; this *S-mesh* was introduced by Shishkin in 1988 [87].

More thorough discussions of layer-adapted meshes can be found in [56, 57, 60]. In 2D, when $\Omega = (0, 1)^2$ and $b_1 > 0,\ b_2 > 0$, then only exponential layers along the sides $x = 0$ and $y = 0$ of Ω are present, and one takes a tensor product of the one-dimensional S-meshes or B-meshes to get the analogous rectangular mesh on the unit square; see [60, 81] and Figure 1, where the mesh in the x direction is fine on $\Omega_{12} \cup \Omega_{22}$ and coarse on $\Omega_{11} \cup \Omega_{21}$, with analogous statements for the mesh in the y direction.

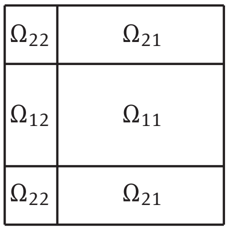


$$\Omega_{11} := [\tau_x, 1] \times [\tau_y, 1]$$
$$\Omega_{12} := [0, \tau_x] \times [\tau_y, 1]$$
$$\Omega_{21} := [\tau_x, 1] \times [0, \tau_y]$$
$$\Omega_{22} := [0, \tau_x] \times [0, \tau_y]$$

Figure 1. Mesh subregions of Ω when u has exponential layers.

If instead one has $b_1 > \beta_1 > 0$ and $b_2 \equiv 0$, then as well as an exponential boundary layer along $x = 0$, there are parabolic boundary layers along $y = 0$ and $y = 1$ and they have width $\mathcal{O}(\varepsilon^{1/2}|\ln \varepsilon|)$. Consequently, a different choice of transition point is made in the y-direction: $\tau_y = \mathcal{O}(\varepsilon^{1/2}\ln N)$, while $\tau_x = \mathcal{O}(\varepsilon \ln N)$ remains unchanged. See [60, 81] and Figure 2 for more details.

The analysis of stable finite difference schemes for 1D convection-diffusion problems in [57, 60] shows that typically the maximum nodal error (i.e., the error measured in the discrete L_∞ norm) of a particular method on an S-mesh is $\mathcal{O}(N^{-1}\ln N)^\sigma$ for some constant $\sigma > 0$, and on a B-type mesh (and on S-type meshes with certain optimality properties of the function $\hat{\varphi}$) the error for the same scheme is $\mathcal{O}(N^{-\sigma})$. In 2D one can prove that the maximum nodal error for the well-known simple upwind scheme is $\mathcal{O}(N^{-1}\ln N)$ on the S-mesh and $\mathcal{O}(N^{-1})$ on the particular S-type mesh for which $\hat{\varphi}(\xi) = -\ln(1 - 2\xi(N-1)/N)$ in (4.1); see [60, Theorem 9.1]. But for B-type meshes no similar result is known, which motivates our next question.



$$\Omega_{11} := [\lambda_x, 1] \times [\lambda_y, 1 - \lambda_y]$$
$$\Omega_{12} := [0, \lambda_x] \times [\lambda_y, 1 - \lambda_y]$$
$$\Omega_{21} := [\lambda_x, 1] \times ([0, \lambda_y] \cup [1 - \lambda_y, 1])$$
$$\Omega_{22} := [0, \lambda_x] \times ([0, \lambda_y] \cup [1 - \lambda_y, 1])$$

Figure 2. Mesh subregions of Ω with u has exponential and parabolic layers.

Question 4.1. For an upwind finite difference method applied on a B-type mesh to the convection-diffusion problem (1.1) with $n = 2$, can one prove the discrete maximum norm convergence result

$$\max_{i,j} |u(x_i, y_j) - u_{ij}^N| \le CN^{-1}$$

under reasonable hypotheses on the data (e.g., if $\Omega = (0, 1)^2$, assume that (1.5) is valid for $k = 3$)? Here N mesh intervals are used in each coordinate direction, and u_{ij}^N denotes the computed solution at the point (x_i, y_j).

For finite element methods applied to convection-diffusion problems, the situation is as follows: while satisfactory analyses (energy norm interpolation error and convergence result) on S-meshes are well established even in two dimensions and for polynomials of higher degree for the Galerkin and SDFEM methods (see [60, 93]), on B-type meshes one can derive optimal interpolation error bounds but no optimal convergence results are known – with the exception of [72], where a special quasi-interpolant is used on a B-type mesh for a two-point boundary value problem, but this technique cannot be extended to two dimensions.

Question 4.2. For the convection-diffusion problem (1.1) with $n \ge 2$, under reasonable hypotheses on the data (e.g., in the case $n = 2$ and $\Omega = (0, 1)^2$, assume that (1.5) is valid for $k = 3$), can one prove an optimal convergence result of the form

$$\|u - u_h\|_\varepsilon \le CN^{-k},$$

where u_h is the solution computed by some FEM using piecewise polynomials of some degree k on a B-type mesh?

The bound (5.2) below comes close to attaining the target set by Question 4.2 when $k = 1$.

In classical elliptic problems, it is more difficult to derive error bounds in the L_∞ norm than in the energy norm. Of course one would expect the same to be true in singularly perturbed problems. In fact, singularly perturbed problems may present even greater obstacles to L_∞ analysis than classical problems, as illustrated by the erratic behaviour of computed solutions described and analysed in Kopteva [50]. In this paper one takes a two-dimensional reaction-diffusion problem (1.3) posed on the unit square with a pure layer solution $u(x) = \exp(x/\varepsilon)$ and solves this problem by a standard Galerkin method on a tensor product S-type or B-type mesh; the order of convergence of the L_∞ error depends on how exactly one bisects the mesh rectangles into triangles! This extraordinary result shows that the next question in our list cannot be easy.

Question 4.3. For the convection-diffusion problem (1.1) or the reaction-diffusion problem (1.3) with $n \ge 2$, under reasonable hypotheses on the data, can one prove a convergence result for $\|u - u_h\|_{L_\infty}$ where u_h is the solution computed by some FEM on an S-type or B-type mesh?

5 Superclose Error Estimates

Much of the terminology used in this section comes from Section 4, while the norm $\|\cdot\|_\varepsilon$ and the SDFEM are defined in Section 2.

When the Galerkin finite element method with linear or bilinear elements on Shishkin meshes with N mesh intervals in each coordinate direction is applied to convection-diffusion problems on $(0,1)^2$ with exponential layers, then [15, 91] the computed solution u^N satisfies

$$\|u - u^N\|_\varepsilon \le C N^{-1} \ln N. \tag{5.1}$$

This bound is also valid on S-type meshes [75]. It is more difficult to analyse the same method on Bakhvalov-type meshes; so far, the only result [80] is

$$\|u - u^N\|_\varepsilon \le C N^{-1} Q(\varepsilon, N), \tag{5.2}$$

with $Q \le \sqrt{\ln 10}$ for $N \ge 10$ and $\varepsilon \ge 10^{-100}$. Durán and Lombardi [16] considered a similar problem using a novel graded mesh for which they proved $\|u - u^N\|_\varepsilon \le C N^{-1}|\ln \varepsilon|^2$. For S-type meshes applied to problems with characteristic/parabolic boundary layers, see Franz and Linß [29].

Linß and Stynes [64] were the first to observe numerically that for both the standard Galerkin FEM and the SDFEM applied to this problem using a Shishkin mesh, the nodal L_∞ convergence rates for linear and bilinear elements on the layer regions $\Omega \setminus \Omega_{11}$ of Figure 1 differ significantly: the rates for bilinears (almost second order) are twice the rates for linears!

This phenomenon can be explained via the superconvergent property of *supercloseness*: if one can define in the finite element space an interpolant or projection u^I of u such that $\|u^I - u^N\|$ converges at a faster rate than $\|u - u^N\|$ as $N \to \infty$ (here $\|\cdot\|$ is any norm), we then say that the finite element method has the superclose property in that norm. See [81, p. 395] for a comparison with other forms of superconvergence.

For the Galerkin FEM with bilinear elements on a Shishkin mesh and u^I the standard Lagrange nodal interpolant of u from the finite element space, one has [54, 101] by comparison with (5.1) the supercloseness result

$$\|u^I - u^N\|_\varepsilon \le C(N^{-1} \ln N)^2. \tag{5.3}$$

In contrast, linear elements do not enjoy this property.

The almost-optimal estimate $\|u - u^N\|_{L_2} \le C(N^{-1} \ln N)^2$ follows easily from this supercloseness bound. Furthermore, supercloseness enables a simple postprocessing of the computed solution u^N that yields a solution Pu^N for which $\|u - Pu^N\|_\varepsilon \ll \|u - u^N\|_\varepsilon$ (see [92], where postprocessing is discussed for the SDFEM). The so-called Lin identities for bilinears (see for instance [34]) are often used to prove supercloseness; alternatively, one can follow the simpler approach of Zlamal that is exploited in [17].

In [92] Stynes and Tobiska analysed the SDFEM for bilinears on an S-mesh. Assuming that $\varepsilon \le CN^{-1}$, the SD-parameter δ_K is specified on each element K by

$$\delta_K = \begin{cases} N^{-1} & \text{if } K \subset \Omega_{11}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

Here a detailed analysis shows that for $K \subset \Omega \setminus \Omega_{11}$ one should choose $\delta_K \le C\varepsilon N^{-2}$. As this value is so much smaller than the natural diffusion parameter ε, one can set $\delta_K = 0$. Then, [92, Theorem 4.5] shows that

$$\|u^I - u^N\|_{\mathrm{SD}} \le C[\varepsilon N^{-3/2} + (N^{-1} \ln N)^2], \tag{5.5}$$

which implies trivially that

$$\|u^I - u^N\|_\varepsilon \le C[\varepsilon N^{-3/2} + (N^{-1} \ln N)^2]. \tag{5.6}$$

Recalling (5.1), we see that the bound (5.6) is a supercloseness result.

When $b_1 > 0$ and $b_2 \equiv 0$ so u has characteristic boundary layers along $y = 0$ and $y = 1$, it is more difficult to tune the SD parameter δ_K. When $K \subset \Omega_{21}$ in Figure 2, the general recommendation often found in the literature – that the SDFEM parameter should be proportional to the length of the element in the streamline direction – gives $\delta_K = \mathcal{O}(N^{-1})$, but this choice is in fact inappropriate [48]. For bilinears it is shown in [30] that one should choose $\delta_{21} \le C\varepsilon^{-1/4} N^{-2}$.

For the rest of Section 5, consider a convection-diffusion problem posed on $(0,1)^2$ whose solution u has only exponential boundary layers as in (1.5).

All the discussion so far in Section 5 pertains to linear and bilinear elements. When piecewise polynomial higher-order finite elements Q_p with $p > 1$ are used in the SDFEM, one can use results from [93] to prove, analogously to (5.1), that

$$\|u - u^N\|_\varepsilon \le C(N^{-1}\ln N)^p \tag{5.7}$$

on Shishkin meshes, provided that (1.5) is satisfied for sufficiently large k. Furthermore, it is shown in [93, Theorem 15] that one has the supercloseness property

$$\|\pi u - u^N\|_\varepsilon \le \|\pi u - u^N\|_{\mathrm{SD}} \le CN^{-(p+1/2)}, \tag{5.8}$$

where instead of the Lagrange nodal interpolant u^I, a vertices-edge-cell interpolant πu is used. But note that (5.8) gains only almost $\mathcal{O}(N^{-1/2})$ over (5.7), although for $p = 1$ the supercloseness result (5.5) gained almost $\mathcal{O}(N^{-1})$; this inconsistency is explained in [93, p. 1801].

Somewhat similar results are obtained (in a norm appropriate to the method) for variants of the discontinuous Galerkin method applied to this 2D problem with exponential layers in the papers [82] by Roos and Zarin and [98, 99, 103, 104] by Zhang et al.

In [25], numerical experiments by Franz using Q_p elements in the Galerkin and streamline diffusion finite element methods yielded some surprising results. He investigated three different interpolation operators on each mesh element: Lagrange interpolation at uniformly distributed points (denoted by J^N_{eq}), Lagrange interpolation at the Gauss–Lobatto points (J^N_{GL}) and vertex-edge-cell interpolation (π). (An identity relating J^N_{GL} and π is derived in [26].) The numerical results showed for the Galerkin solution u^N_{Gal} that

$$\|K^N u - u^N_{\mathrm{Gal}}\|_\varepsilon \le CN^{-2} \qquad \text{for } p = 2 \text{ and } K^N = J^N_{\mathrm{eq}}, J^N_{\mathrm{GL}}, \pi, \tag{5.9a}$$

$$\|J^N_{\mathrm{eq}} u - u^N_{\mathrm{Gal}}\|_\varepsilon \le CN^{-p} \qquad \text{for } p \ge 3 \tag{5.9b}$$

and

$$\|J^N_{\mathrm{GL}} u - u^N_{\mathrm{Gal}}\|_\varepsilon + \|\pi u - u^N_{\mathrm{Gal}}\|_\varepsilon \le CN^{-(p+1)} \qquad \text{for } p \ge 3. \tag{5.9c}$$

For the SDFEM, it was seen that

$$N^{-1}\|J^N_{\mathrm{eq}} u - u^N_{\mathrm{SD}}\|_\varepsilon + \|J^N_{\mathrm{GL}} u - u^N_{\mathrm{Gal}}\|_\varepsilon + \|\pi u - u^N_{\mathrm{Gal}}\|_\varepsilon \le CN^{-(p+1)} \quad \text{for } p \ge 2. \tag{5.10}$$

So far no theoretical proof of these observations is known. In [31] Franz and Roos prove that for odd p one has the result

$$\|\pi u - u^N_{\mathrm{Gal}}\|_\varepsilon \le C\left[(N^{-1}\ln N)^{p+1} + N^{-(p+1/4)}\right] \tag{5.11}$$

but this is weaker than (5.10).

Question 5.1. For the convection-diffusion problem (1.1) with $n \ge 2$, under reasonable hypotheses on the data (e.g., in the case $n = 2$ and $\Omega = (0, 1)^2$, assume that (1.5) is valid for $k = 3$), can one prove any of the superclosenesss bounds in (5.9) or (5.10)?

6 Defect Correction on Layer-Adapted Meshes

Techniques for convergence acceleration that are cheap to implement but yield enhanced orders of convergence in computed solutions are evidently desirable. Two well-known approaches in this area are *Richardson extrapolation* and *defect correction*; the former is based on the use of different meshes while the latter uses different discretisations. In this section we shall consider only defect correction on layer-adapted meshes.

We begin with a general description of defect correction. Let the given boundary value problem be $Lu = f$ in Ω with $u = 0$ on $\partial\Omega$. Consider two different discretisations on the same mesh (where both discretisations satisfy the boundary condition):

$$L^1_h u^1_h = f^1_h \quad \text{and} \quad L^2_h u^2_h = f^2_h,$$

where L_h^1 is of lower order and stable while L_h^2 is of higher order but unstable, and f_h^1, f_h^2 are some discretisations of f. Defect correction attempts to exploit the good features of each scheme (stability of L_h^1 and accuracy of L_h^2) in the following way: First solve the stable low-order problem

$$L_h^1 u_h^1 = f_h^1, \quad u_h^1|_{\partial\Omega} = 0.$$

Then compute the defect correction solution u_h^{dc} by

$$L_h^1(u_h^{\mathrm{dc}} - u_h^1) = f_h^2 - L_h^2 u_h^1, \quad u_h^{\mathrm{dc}}|_{\partial\Omega} = 0;$$

here we modify u_h^1 using the "defect" $L_h^2 u_h^1 - f_h^2$, i.e., the amount by which u_h^1 fails to be a solution of the higher-order method. Note that in this method, discrete systems of equations are solved using only the stable operator L_h^1.

One can describe defect correction in a variational setting. Let the given problem be

$$a(u, v) = (f, v) \quad \text{for all } v \in V,$$

where $a(\cdot\,,\cdot)$ is a bilinear form. Using two bilinear forms $a^1(\cdot\,,\cdot)$, $a^2(\cdot\,,\cdot)$ we first solve

$$a^1(u_h^1, v_h) = (f_h^1, v) \quad \text{for all } v_h \in V_h$$

and then correct u_h^1:

$$a^1(u_h^{\mathrm{dc}} - u_h^1, v_h) = (f_h^2, v) - a^2(u_h^1, v_h) \quad \text{for all } v_h \in V_h.$$

Standard discretisations of singularly perturbed equations often have stability problems, and these can be remedied at the cost of using a lower-order method; see for example the discussion of central differencing (second-order but unstable) and simple upwinding (stable but only first-order) applied to 1D convection-diffusion problems in [81, Section I.2.1]. Thus, it is natural to apply defect correction as first proposed in 1982 by Hemker [38]. In later papers [8, 24], Layton et al. present certain error estimates on standard (i.e., non-layer-adapted) meshes for finite difference methods in 1D and finite element methods in higher space dimensions; they obtain – roughly speaking – good estimates for defect correction in subdomains away from the layer regions.

It is natural to consider a defect correction method that combines the simple upwind operator L^1 and the central difference operator L^2 on Shishkin meshes. Indeed, in [32] it was shown that for a 1D convection-diffusion problem with an exponential boundary layer, on a class of meshes that includes the Shishkin mesh $\{x_i : i = 0, 1, \dots, N\}$, the defect correction solution u_N^{dc} satisfies

$$\max_i \big|(u - u_N^{\mathrm{dc}})(x_i)\big| \le C(N^{-1} \ln N)^2. \tag{6.1}$$

The proof of (6.1) decomposes the consistency error as

$$L^1(u - u_N^{\mathrm{dc}}) = (L^1 - L^2)(u - u_N^1) + (L^2 u - f_h^2). \tag{6.2}$$

Here the first component on the right-hand side is the relative consistency error which is difficult to analyse. It is known [81, Lemma I.2.91] that in 1D the simple upwind operator L^1 is $(W_{-1,\infty}, L_\infty)$ stable, viz.,

$$\|v_N\|_{\infty,d} \le C\|L^1 v_N\|_{-1,\infty,d}. \tag{6.3}$$

(These norms are discrete analogues of the standard Sobolev norms in L_∞ and $W_{-1,\infty}$.) One can use (6.3) to estimate $\|u - u_N^{\mathrm{dc}}\|_{\infty,d}$ from (6.2) by first showing that

$$\big\|(L^1 - L^2)(u - u_N^1)\big\|_{-1,\infty,d} \le C\big(\max|e_{j+1} - e_j| + N^{-1}\max|e_j|\big), \tag{6.4}$$

where e_j is the error in the upwind solution at the mesh point x_j. To estimate $e_{j+1} - e_j$, one can use an expansion of Linß [60, Section 4.2.3] for the error in simple upwinding (see also the full discussion of defect correction in [60, Section 4.3.3]). This expansion also facilitates the analysis of Richardson extrapolation in 1D.

What about 2D? In [64], Linß and Stynes compare various numerical methods for (1.1) when $\Omega = (0,1)^2$ and only exponential boundary layers appear in the solution. For the above defect correction method (upwinding and central differencing on Shishkin meshes), they observe almost second-order convergence in the norm $\|\cdot\|_{\infty,d}$ and conclude that this "appears to be the most efficient of the finite difference methods considered". But for this method no rigorous proof of almost second-order uniform convergence on a Shishkin mesh (or a more general layer-adapted mesh) is known.

Question 6.1. Consider the convection-diffusion problem (1.1) with $\Omega = (0,1)^2$, under reasonable hypotheses on the data (e.g., assume that (1.5) is valid for $k = 3$). Solve this problem numerically using a defect correction method based on simple upwinding and central differencing on a Shishkin mesh (or other layer-adapted mesh). Can one prove almost second-order convergence in the discrete maximum norm?

What are the difficulties in addressing Question 6.1? In 2D no stability result for simple upwinding that is analogous to (6.3) is known, and consequently no analogue of (6.4) is available. Kopteva [47] provides an error expansion for the upwind scheme on a Shishkin mesh in 2D:

$$u_{ij} - u(x_i, y_j) = H\Phi(x_i, y_j) + \frac{h}{\varepsilon}\Psi(x_i, y_j) + R_{ij} \tag{6.5}$$

which in 1D is a special case of the error expansion of Linß for general meshes. Here Φ and ψ are explicitly known and the remainder R_{ij} is second order:

$$|R_{ij}| \le C N^{-2} F_{ij}(1, \Omega_0; (\ln N)^2),$$

where close to the layers F_{ij} is $O(\ln N)^2$ and otherwise it is $O(1)$.

For finite elements on Shishkin meshes or more general layer-adapted meshes it is unclear how best to use defect correction. Could one combine viscosity stabilisation [60, Section 9.2.3] and the Galerkin finite element method? Should one combine SDFEM with a higher-order method?

Question 6.2. Consider the convection-diffusion problem (1.1) with $\Omega = (0,1)^2$. Devise, implement and analyse a defect correction finite element method.

7 Adaptive Generation of Uniform Convergence

Adaptive finite element methods compute an approximate solution to a given problem, then refine the mesh (h-method) or change locally the polynomial degree (p-method) based on some *a posteriori error estimator* η. This estimator should be locally computable from the computed numerical solution u_h and the given data of the problem. Ideally η should be equivalent to the numerical error in some norm:

$$d_\ell \eta \le \|u - u_h\| \le d_u \eta \quad \text{for some constants } d_\ell,\ d_u. \tag{7.1}$$

(Alternatively, in the DWR method [10] one tries to control some functional instead of a norm.) For a singularly perturbed problem, if the constants d_ℓ, d_u are independent of ε, we then say that the estimator is *robust* with respect to $\|\cdot\|$; if the constant d_u is independent of ε but d_l depends weakly on ε, we say the estimator is *semi-robust* with respect to $\|\cdot\|$.

In this section we examine estimators for the convection-diffusion and reaction-diffusion problems (1.1) and (1.3). First we discuss error estimators for energy and similar norms. Consider piecewise linear elements and assume for simplicity that b, c, f are piecewise linear (otherwise, additional data error terms will appear). By a careful study of the dependence on ε of the constants in the standard residual error estimators, Verfürth [94] discussed the residual estimator η_T defined by

$$\eta_T^2 := \alpha_T^2 \|r_T\|_{0,T}^2 + \sum_{E(T)} \varepsilon^{-1/2} \alpha_E \|r_E\|_{0,E}^2.$$

Here $r_T := (f + \varepsilon\Delta u_h - b\nabla u_h - cu_h)|_T$ and $r_E := [n_e \cdot \nabla u_h]_E$ are the standard element and edge residuals. The weights α_S for $S = T, E$ are defined by

$$\alpha_S = \min\{h_S \varepsilon^{-1/2}, \alpha_0^{-1/2}\},$$

where h_T and h_E are the element and edge diameters and α_0 is the constant of (1.2). For reaction-diffusion problems, Verfürth proved robustness of this estimator for $\|\cdot\|_\varepsilon$. For convection-diffusion problems, unfortunately the estimator is only semi-robust for this norm.

Nevertheless, for convection-diffusion problems the residual estimator η_T is robust with respect to the so-called *dual norm* $|||\cdot|||$ defined by

$$|||v||| := \|v\|_\varepsilon + \sup_\varphi \frac{(b \cdot \nabla v, \varphi)}{\|\varphi\|_\varepsilon}.$$

This was first observed by Sangalli [83] for the residual-free bubble FEM and later analysed for the Galerkin and streamline diffusion FEMs in [95]; see also [1–4, 6, 20]. Unfortunately, the norm $|||\cdot|||$ is not computable so these results are of limited practical value.

Robust a posteriori estimators for Lebesgue norms are developed in [35, 36] and for the SDFEM norm in [41].

Alternative a posteriori error estimators that are discussed in many papers are based on *flux reconstruction* in the space $H(\mathrm{div}, \Omega)$. Consider first, for simplicity, the reaction-diffusion problem (1.3) as in [11]. The derivation of the error estimator starts from

$$a(u - u_h, v) = (f - cu_h - \nabla \cdot \sigma_h, v) - (\varepsilon\nabla u_h + \sigma_h, \nabla v). \tag{7.2}$$

Now $\sigma_h \in H(\mathrm{div}, \Omega)$ is chosen to approximate the numerical flux $-\varepsilon\nabla u_h$. Additionally, σ_h is required to satisfy

$$(\nabla \cdot \sigma_h + cu_h, 1)_K = (f, 1)_K \quad \text{for all mesh elements } K. \tag{7.3}$$

(For details of the computation of the recovered flux σ_h, see [12].) The combination of the first term of the right-hand side of (7.2) with (7.3) yields the residual part of the new estimator (where the weights m_K are specified in [95])

$$\eta_{K,\mathrm{res}} := m_K \|f - cu_h - \nabla \cdot \sigma_h\|_K .$$

The second term of (7.2) yields after some manipulation (a direct application of the Cauchy-Schwarz inequality yields a non-robust estimator) a more complicated diffusive flux estimator η_{DF}. The resulting full estimator of [11] is robust and equivalent to Verfürth's residual estimator for reaction-diffusion problems.

For convection-diffusion problems the derivation of estimators using flux reconstruction works similarly. Consider a convection-diffusion problem of the form

$$-\varepsilon\Delta u + \nabla \cdot (bu) + cu = f \quad \text{in } \Omega = (0, 1)^2, \tag{7.4a}$$
$$u = 0 \quad \text{on } \partial\Omega. \tag{7.4b}$$

Now σ_h has to approximate $-\varepsilon\nabla u_h + bu_h$ and should again satisfy (7.3).

In [23] the authors extend the approach to an interior penalty discontinuous Galerkin method, introducing additional nonconformity, convection and upwinding estimators. They prove robustness in some augmented norm similar as the dual norm above. See also [19, 97].

Some attempts have been made to derive pointwise a posteriori error estimates for singularly perturbed problems. Using the Green's function G of the continuous problem, the usual starting point is the representation

$$(u - u_h)(x) = a(u - u_h, G) = (f, G) - a(u_h, G) \tag{7.5}$$

or (in the strong form using distributions)

$$(u - u_h)(x) = \int L(u - u_h)G. \tag{7.6}$$

Linß [61] studies higher-order FEMs in 1D and derives estimates from (7.5) using L_1-norm information about G; his estimator contains discrete derivatives of the numerical solution u_h.

Kopteva [49] considered a reaction-diffusion problem in 2D and obtained an a posteriori error bound for a finite difference method using (7.6). For convection-diffusion in 2D, detailed estimates of the Green's function G were proved in [27, 28]; these were used successfully in [13] to derive a posteriori error estimates for finite elements in 2D on isotropic meshes for reaction-diffusion problems.

Assuming that L_∞ error estimators in space are available, in [51] the authors construct error estimators in space and time for time-dependent problems with various time discretisations.

A good a posteriori error estimator tells us *where* the mesh should be refined, but typically we do not have additional information about the directional behaviour of the error that will guide *how* the mesh should be refined. Some work in this direction appears in [39, 71], but currently in the singularly perturbed case there seems to be no satisfactory theory of metric-based algorithms for mesh generation.

A posteriori error estimators are usually analysed with the underlying assumptions that the mesh is shape-regular and locally uniform, but this excludes the long thin mesh elements that in practice are needed to deal with layers in solutions. For anisotropic meshes, Kunert [53] developed an a posteriori theory of energy norm estimates which relies on a measure of alignment between mesh element and layer; this has been used for example in [33], which examines hp-DG methods for convection-diffusion problems on anisotropic meshes, and [100], which introduces a modified alignment measure for nonconforming elements when solving reaction-diffusion problems.

Recently, Kopteva [46] derived residual-type a posteriori error estimates in the maximum norm when linear finite elements on anisotropic triangulations are used to solve singularly perturbed semilinear reaction-diffusion equations posed in polygonal domains; significantly, the error constants in her estimates are independent of the diameters and aspect ratios of mesh elements and of the singular perturbation parameter ε; perhaps surprisingly, no alignment measure appears in these estimates.

A survey of anisotropic refinement methods in FEMs is given in [85]. Verfürth [96] presents in detail the theory of a posteriori error estimation.

Question 7.1. Can one extend the results of [46] to convection-diffusion problems?

Question 7.2. For convection-diffusion or reaction-diffusion problems, using some a posteriori error estimator combined with some strategy for mesh refinement (or for changing the local polynomial degree in FEMs), can one prove convergence of the computed solution in some norm, independently of the singular perturbation parameter ε? (The only rigorous published result of this type is the adaptive mesh algorithm in [52] for upwind finite differences in 1D for which the authors prove first-order nodal convergence, uniformly in ε, starting from an arbitrary mesh; while many other papers describe their methods as "adaptive" and prove some convergence result, their analysis frequently makes the very strong assumption that the mesh is specially suited to the unknown solution u without any rigorous justification that their algorithm will produce such a mesh.)

8 Strongly Coupled Singularly Perturbed Systems

In 2009, Linß and Stynes [65] surveyed methods for the numerical solution of singularly perturbed systems. Here we describe some recent results that are not in [65] and present some open questions.

First we sketch the situation for systems of reaction-diffusion equations of the form

$$-E^2u'' + Au = f \quad \text{in } (0,1), \qquad u(0) = u(1) = 0,$$

where E and $A = (a_{ij})$ are $\ell \times \ell$ matrices and u is an $\ell \times 1$ column vector. The matrix E is diagonal, defined as $E = \operatorname{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_\ell)$ with $0 < \varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_\ell \le 1$. Assume that A has positive diagonal entries and that the matrix $\Gamma = (\gamma_{ij})$ defined by

$$\gamma_{ii} = 1, \quad \gamma_{ij} = -\left\|\frac{a_{ij}}{a_{ii}}\right\|_\infty \quad \text{for } i \ne j$$

is invertible with $\Gamma^{-1} \geq 0$ (i.e., this inequality holds for each entry in Γ^{-1}). Then [62, 65] one can decompose u into a smooth component and boundary layers. Other authors assume that A is an M-matrix or that $(Ax, x) \geq 0$ where $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^\ell$. A connection between this positive definite property and $\Gamma^{-1} \geq 0$ is described in [65, Theorem 2.2]. In [67] a full asymptotic expansion is derived for positive definite A in the case $\ell = 2$, including information on analytic regularity. For systems of reaction-diffusion equations posed in two dimensions there are some results concerning the layer structure [65, Section 3].

Systems of convection-diffusion equations are more delicate to handle. Consider first *weakly coupled* systems (i.e., coupled only through their reaction terms) of the form

$$Lu := -Eu'' - \operatorname{diag}(b)u' + Au = f, \quad u(0) = u(1) = 0,$$

where $u = (u_1, \dots, u_\ell)^T$ and the matrix E is as above. Assume that $|b_i| \geq \beta_i > 0$ for all i and $\tilde{\Gamma}^{-1} \geq 0$, where $\tilde{\Gamma} = (\tilde{\gamma}_{ij})$ with

$$\tilde{\gamma}_{ii} = 1, \qquad \tilde{\gamma}_{ij} = -\min\left(\left\|\frac{a_{ij}}{a_{ii}}\right\|_\infty, \left\|\frac{a_{ij}}{b_i}\right\|_\infty\right) \quad \text{for } i \neq j.$$

Then [58] for $\nu = 0, 1$ one has

$$|u_k^{(\nu)}(x)| \leq C \begin{cases} 1 + \varepsilon_k^{-\nu} e^{-\beta_k(1-x)/\varepsilon} & \text{if } b_k < 0, \\ 1 + \varepsilon_k^{-\nu} e^{-\beta_k x/\varepsilon} & \text{if } b_k > 0. \end{cases}$$

Thus, when only first order-derivatives are considered, there is no strong interaction between the different u_i (incidentally, this is simpler than the reaction-diffusion case).

But consider now a set of two equations with $\varepsilon_1 = \varepsilon_2$, $b_1 > 0$ and $b_2 < 0$. Then the layer at $x = 1$ in u_1 generates a weak layer at $x = 1$ in u_2, and the situation at $x = 0$ is analogous. Under certain conditions [77], one can prove the following solution decomposition for $\nu \leq 2$ (α is some positive parameter):

$$\begin{aligned} u_1 &= S_1 + E_{10} + E_{11}, \\ u_2 &= S_2 + E_{20} + E_{21} \end{aligned}$$

with

$$\begin{aligned} &\|S_1^{(\nu)}\|_0 \leq C, && \|S_2^{(\nu)}\|_0 \leq C, \\ &|E_{10}^{(\nu)}(x)| \leq C\epsilon^{1-\nu} e^{-\alpha\frac{x}{\epsilon}}, && |E_{11}^{(\nu)}(x)| \leq C\epsilon^{-\nu} e^{-\alpha\frac{1-x}{\epsilon}}, \\ &|E_{20}^{(\nu)}(x)| \leq C\epsilon^{-\nu} e^{-\alpha\frac{x}{\epsilon}}, && |E_{21}^{(\nu)}(x)| \leq C\epsilon^{1-\nu} e^{-\alpha\frac{1-x}{\epsilon}}. \end{aligned}$$

For instance, this tells us that u_1 has a strong layer at $x = 1$ and also a weak layer at $x = 0$.

The general case of ℓ weakly coupled convection-diffusion equations with $b_i \geq \beta_i > 0$ for each i is discussed in detail in [78], and an error estimate for the Galerkin FEM with piecewise linear finite elements on a layer-adapted mesh is derived in the weighted energy norm $\|\cdot\|_\varepsilon$.

For *strongly coupled* systems of convection-diffusion equations (i.e., coupled through their convection terms) a full interaction between the layers of the various u_i takes place. Consider the system of two equations

$$Lu := -\varepsilon u'' - Bu' + Au = f, \quad u(0) = u(1) = 0 \tag{8.1}$$

assuming as in [76] that

(V_1) $B = B(x)$ is symmetric,

(V_2) $A + B'/2$ is positive semidefinite,

(V_3) the eigenvalues λ_1, λ_2 of $B(x)$ satisfy $\min\{|\lambda_1|, |\lambda_2|\} > \alpha > 0$ for all x.

If both eigenvalues of B are positive, then u_1 and u_2 have overlapping layers at $x = 0$ and the reduced solution of (8.1) is the solutions of an initial-value problem [70]. But if λ_1 and λ_2 have different signs, then u_1 and u_2 each have strong layers at $x = 0$ and $x = 1$; we have full layer interaction.

Let $\tilde\lambda_1$ be the positive eigenvalue of $B(0)$ with $[\alpha_1, \beta_1]^T$ an associated eigenvector. Similarly let $\tilde\lambda_2$ be the negative eigenvalue of $B(1)$ with eigenvector $[\alpha_2, \beta_2]^T$. Then [76] one can define an asymptotic approximation of u by

$$u^0_{\mathrm{as}}(x) := w_0(x) + w_h(x)\tilde c_0 + d_1 \begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix} \exp(-\tilde\lambda_1 x/\varepsilon) + d_2 \begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix} \exp(\tilde\lambda_2(1-x)/\varepsilon), \tag{8.2}$$

where $w_0 + w_h\tilde c_0$, with an unknown vector $\tilde c_0$, is the general solution of the reduced equation $-Bw' + Aw = f$. One can compute $\tilde c_0$ and the constants d_1, d_2 from the boundary conditions of (8.1). Neglecting the layer terms in (8.2), we define the "reduced solution" of (8.1) to be

$$u^0 = w_0 + w_h c_0 \quad \text{with } c_0 = \lim_{\varepsilon\to 0} \tilde c_0.$$

Note that – surprisingly – in general u^0 does not satisfy any of the boundary conditions in (8.1).

Applying this analysis to the example

$$-\varepsilon u'' - \begin{bmatrix}3 & 4\\ 4 & -3\end{bmatrix} u' = \begin{bmatrix}1\\ 2\end{bmatrix}, \qquad u(0) = u(1) = 0,$$

yields the reduced solution $u_0 = (u^0_1, u^0_2)$ where

$$u^0_1 = 11/25x - 8/25, \quad u^0_2 = -2/25x - 4/25.$$

It was not observed in [76] that the correct boundary conditions for the reduced problem can be explicitly described as

$$(u^0_1 - 2u^0_2)(0) = 0, \quad (2u^0_1 + u^0_2)(1) = 0$$

and can be derived, for instance, from the decomposition $B = B^+ + B^-$, where B^+ is positive semidefinite and B^- negative semidefinite. In our example, we have

$$(Bv, v) = 3v_1^2 + 8v_1v_2 - 3v_2^2 = (2v_1 + v_2)^2 - (v_1 - 2v_2)^2.$$

Question 8.1. Can one analyse the solution structure of the strongly coupled convection-diffusion system (8.1) when one or more of the assumptions (V_1), (V_2), (V_3) are violated? Can the results for (8.1) be extended to the general case of $\ell \ge 2$ equations?

When strongly coupled systems of the form

$$-Eu'' - Bu' + Au = f, \quad u(0) = u(1) = 0 \tag{8.3}$$

have different small parameters, i.e., $E = \operatorname{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_\ell)$ with $0 < \varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_\ell \le 1$, the situation is even more complicated. Some a priori estimates can be found in [59], and some information about the layer structure of u is derived in [66, 74] under restrictive conditions; many results – even the boundary conditions for the reduced problem – depend on the relative scalings $\varepsilon_i/\varepsilon_j$ of the singular perturbation parameters.

Question 8.2. For the strongly coupled system (8.3) of convection-diffusion equations on the interval $[0, 1]$ with different small parameters, can one identify the location and structure of the layers in the solution u without placing overly restrictive hypotheses on the problem? Can one then prove convergence, uniformly in the ε_i, for some numerical method for (8.3)?

There are almost no published results for systems of strongly coupled convection-diffusion equations posed in two (or more) dimensions. Consider the system

$$-\varepsilon\Delta u + A_1\frac{\partial u}{\partial x_1} + A_2\frac{\partial u}{\partial x_2} + \rho\, u = f \quad \text{in } \Omega, \tag{8.4a}$$

$$u = 0 \quad \text{on } \Gamma = \partial\Omega, \tag{8.4b}$$

assuming that the matrices $A_1, A_2 \in C^1(\Omega)$ are symmetric and that the unit outer normal $\nu = (\nu_1, \nu_2)$ exists almost everywhere on the boundary $\partial\Omega$. Then only in simple cases do we have some information about location and structures of layers. For instance, if A_1 and A_2 are simultaneously diagonalisable then the system (8.4) can be decoupled.

Question 8.3. Can one analyse the structure of the solution u of the strongly coupled system (8.4) posed in some 2D domain Ω without restricting the problem to some extremely special situation? Can one then prove convergence, uniformly in ε, for some numerical method for (8.4)?

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References

[1] B. Achchab, A. Agouzal, M. El Fatini and A. Souissi,Robust hierarchical a posteriori error estimators for stabilized convection-diffusion problems, *Numer. Methods Partial Differential Equations* **28** (2012), no. 5, 1717–1728.

[2] B. Achchab, A. Benjouad, M. El Fatini, A. Souissi and G. Warnecke, Robust a posteriori error estimates for subgrid stabilization of non-stationary convection dominated diffusive transport, *Appl. Math. Comput.* **218** (2012), no. 9, 5276–5291.

[3] B. Achchab, M. El Fatini, A. Ern and A. Souissi, A posteriori error estimates for subgrid viscosity stabilized approximations of convection-diffusion equations, *Appl. Math. Lett.* **22** (2009), no. 9, 1418–1424.

[4] B. Achchab, M. El Fatini and A. Souissi, A posteriori error analysis for multiscale stabilization of convection-diffusion-reaction problems: unsteady state, *Int. J. Math. Stat.* **4** (2009), no. S09, 3–11.

[5] D. N. de G. Allen and R. V. Southwell, Relaxation methods applied to determine the motion, in two dimensions, of a viscous fluid past a fixed cylinder, *Quart. J. Mech. Appl. Math.* **8** (1955), 129–145.

[6] T. Apel, S. Nicaise and D. Sirch, A posteriori error estimation of residual type for anisotropic diffusion-convection-reaction problems, *J. Comput. Appl. Math.* **235** (2011), no. 8, 2805–2820.

[7] O. Axelsson and G. F. Carey, On the numerical solution of two-point singularly perturbed boundary value problems, *Comput. Methods Appl. Mech. Engrg.* **50** (1985), no. 3, 217–229.

[8] O. Axelsson and W. Layton, Defect correction methods for convection dominated convection-diffusion problems, *Math. Modelling and Num. Anal.* **24** (1990), 423–455.

[9] N. Bakhvalov, Towards optimization of methods for solving boundary value problems in the presence of boundary layers, *Zh. Vychisl. Mat. i Mat. Fiz.* **9** (1969), 841–859.

[10] W. Bangerth and R. Rannacher, *Adaptive Finite Element Methods for Differential Equations*, Birkhäuser, Zürich, 2003.

[11] I. Cheddadi, R. Fučík, M. I. Prieto and M. Vohralík, Guaranteed and robust a posteriori error estimates for singularly perturbed reaction-diffusion problems, *M2AN Math. Model. Numer. Anal.* **43** (2009), no. 5, 867–888.

[12] B. Cockburn, B. Dong, J. Guzmán, M. Restelli and R. Sacco, A hybridizable discontinuous Galerkin method for steady-state convection-diffusion-reaction problems, *SIAM J. Sci. Comput.* **31** (2009), no. 5, 3827–3846.

[13] A. Demlow and N. Kopteva, Maximum-norm a posteriori error estimates for singularly perturbed reaction-diffusion problems, submitted.

[14] D. A. Di Pietro and A. Ern, *Mathematical Aspects of Discontinuous Galerkin Methods*, Math. Appl. 69, Springer, Berlin, 2012.

[15] M. Dobrowolski and H.-G. Roos, A priori estimates for the solution of convection-diffusion problems and interpolation on Shishkin meshes, *Z. Anal. Anwend.* **16** (1997), no. 4, 1001–1012.

[16] R. G. Durán and A. L. Lombardi, Finite element approximation of convection diffusion problems using graded meshes, *Appl. Numer. Math.* **56** (2006), no. 10-11, 1314–1325.

[17] R. G. Durán, A. L. Lombardi and M. I. Prieto, Superconvergence for finite element approximation of a convection-diffusion equation using graded meshes, *IMA J. Numer. Anal.* **32** (2012), no. 2, 511–533.

[18] H. Egger and J. Schöberl, A hybrid mixed discontinuous Galerkin finite-element method for convection-diffusion problems, *IMA J. Numer. Anal.* **30** (2010), no. 4, 1206–1234.

[19] M. Eigel and T. Samrowski, Functional a posteriori error estimation for stationary reaction-convection-diffusion problems, *Comput. Meth. Appl. Math.* **14** (2014), no. 2, 135–150.

[20] L. El Alaoui, A. Ern and E. Burman, A priori and a posteriori analysis of non-conforming finite elements with face penalty for advection-diffusion equations, *IMA J. Numer. Anal.* **27** (2007), no. 1, 151–171.

[21] K. V. Emel'janov, A difference scheme for a three-dimensional elliptic equation with a small parameter multiplying the highest derivative, in: *Boundary Value Problems for Equations of Mathematical Physics*, USSR Academy of Sciences, Sverdlovsk (1973), 30–42.

[22] K. V. Emel'janov, A difference scheme for the equation $-\varepsilon\Delta u + au_{x_1} = f$, *Trudy Inst. Mat. i Mekh. Ural. Nauchn. Tsentr Akad. Nauk SSSR* (1976), no. 60, 19–37.

[23] A. Ern, A. F. Stephansen and M. Vohralík, Guaranteed and robust discontinuous Galerkin a posteriori error estimates for convection-diffusion-reaction problems, *J. Comput. Appl. Math.* **234** (2010), no. 1, 114–130.

[24] V. Ervin and W. Layton, An analysis of a defect-correction method for a model convection-diffusion equation, *SIAM J. Num. Anal.* **26** (1989), 169–179.

[25] S. Franz, Convergence phenomena of Q_p-elements for convection-diffusion problems, *Numer. Methods Partial Differential Equations* **29** (2013), no. 1, 280–296.

[26] S. Franz, Superconvergence using pointwise interpolation in convection-diffusion problems, *Appl. Numer. Math.* **76** (2014), 132–144.

[27] S. Franz and N. Kopteva, Green's function estimates for a singularly perturbed convection-diffusion problem in three dimensions, *Int. J. Numer. Anal. Model. Ser. B* **2** (2011), no. 2-3, 124–141.

[28] S. Franz and N. Kopteva, Green's function estimates for a singularly perturbed convection-diffusion problem, *J. Differential Equations* **252** (2012), no. 2, 1521–1545.

[29] S. Franz and T. Linß, Superconvergence analysis of the Galerkin FEM for a singularly perturbed convection-diffusion problem with characteristic layers, *Numer. Methods Partial Differential Equations* **24** (2008), no. 1, 144–164.

[30] S. Franz, T. Linß and H.-G. Roos, Superconvergence analysis of the SDFEM for elliptic problems with characteristic layers, *Appl. Numer. Math.* **58** (2008), no. 12, 1818–1829.

[31] S. Franz and H.-G. Roos, Superconvergence for higher order elements in convection-diffusion problems, *Numer. Math. Theory Methods Appl.* **7** (2014), 356–373.

[32] A. Fröhner, T. Linß and H.-G. Roos, Defect correction on Shishkin-type meshes, *Numer. Algorithms* **26** (2001), 281–299.

[33] S. Giani, D. Schötzau and L. Zhu, An a-posteriori error estimate for hp-adaptive DG methods for convection-diffusion problems on anisotropically refined meshes, *Comput. Math. Appl.* **67** (2014), no. 4, 869–887.

[34] C. Grossmann, H.-G. Roos and M. Stynes, *Numerical Treatment of Partial Differential Equations*, Universitext, Springer, Berlin, 2007.

[35] G. Hauke, M. H. Doweidar, D. Fuster, A. Gómez and J. Sayas, Application of variational a-posteriori multiscale error estimation to higher-order elements, *Comput. Mech.* **38** (2006), no. 4–5, 356–389.

[36] G. Hauke, D. Fuster and M. H. Doweidar, Variational multiscale a-posteriori error estimation for multi-dimensional transport problems, *Comput. Methods Appl. Mech. Engrg.* **197** (2008), no. 33–40, 2701–2718.

[37] A. F. Hegarty, E. O'Riordan and M. Stynes, A comparison of uniformly convergent difference schemes for two-dimensional convection-diffusion problems, *J. Comput. Phys.* **105** (1993), no. 1, 24–32.

[38] P. W. Hemker, Mixed defect correction iteration for the accurate solution of the convection diffusion equation, in: *Multigrid Methods* (Cologne 1981), Lecture Notes in Math. 960, Springer, Berlin (1982), 485–501.

[39] W. Huang, L. Kamenski and J. Lang, A new anisotropic mesh adaptation method based upon hierarchical a posteriori error estimates, *J. Comput. Phys.* **229** (2010), no. 6, 2179–2198.

[40] A. M. Il'in, A difference scheme for a differential equation with a small parameter multiplying the highest derivative, *Mat. Zametki* **6** (1969), 237–248.

[41] V. John and J. Novo, A robust SUPG norm a posteriori error estimator for stationary convection-diffusion equations, *Comput. Methods Appl. Mech. Engrg.* **255** (2013), 289–305.

[42] R. B. Kellogg and M. Stynes, Corner singularities and boundary layers in a simple convection-diffusion problem, *J. Differential Equations* **213** (2005), no. 1, 81–120.

[43] R. B. Kellogg and M. Stynes, Sharpened bounds for corner singularities and boundary layers in a simple convection-diffusion problem, *Appl. Math. Lett.* **20** (2007), no. 5, 539–544.

[44] P. Knobloch, A generalization of the local projection stabilization for convection-diffusion-reaction equations, *SIAM J. Numer. Anal.* **48** (2010), no. 2, 659–680.

[45] P. Knobloch and L. Tobiska, On the stability of finite-element discretizations of convection-diffusion-reaction equations, *IMA J. Numer. Anal.* **31** (2011), no. 1, 147–164.

[46] N. Kopteva, Maximum-norm a posteriori error estimates for singularly perturbed reaction-diffusion problems on anisotropic meshes, submitted.

[47] N. Kopteva, Error expansion for an upwind scheme applied to a two-dimensional convection-diffusion problem, *SIAM J. Num. Anal.* **41** (2003), 1851–1869.

[48] N. Kopteva, How accurate is the streamline-diffusion FEM inside characteristic (boundary and interior) layers?, *Comput. Methods Appl. Mech. Engrg.* **193** (2004), no. 45–47, 4875–4889.

[49] N. Kopteva, Maximum norm a posteriori error estimate for a 2D singularly perturbed semilinear reaction-diffusion problem, *SIAM J. Numer. Anal.* **46** (2008), no. 3, 1602–1618.

[50] N. Kopteva, Linear finite elements may be only first-order pointwise accurate on anisotropic triangulations, *Math. Comp.* **83** (2014), no. 289, 2061–2070.

[51] N. Kopteva and T. Linß, Maximum norm a posteriori error estimation for a time-dependent reaction-diffusion problem, *Comput. Methods Appl. Math.* **12** (2012), no. 2, 189–205.

[52] N. Kopteva and M. Stynes, A robust adaptive method for a quasi-linear one-dimensional convection-diffusion problem, *SIAM J. Numer. Anal.* **39** (2001), no. 4, 1446–1467.

[53] G. Kunert, Toward anisotropic mesh construction and error estimation in the finite element method, *Numer. Methods Partial Differential Equations* **18** (2002), no. 5, 625–648.

[54] T. Linß, Uniform superconvergence of a Galerkin finite element method on Shishkin-type meshes, *Numer. Methods Partial Differential Equations* **16** (2000), no. 5, 426–440.
[55] T. Linß, The necessity of Shishkin decompositions, *Appl. Math. Lett.* **14** (2001), no. 7, 891–896.
[56] T. Linß, Layer-adapted meshes for convection-diffusion problems, *Comput. Methods Appl. Mech. Engrg.* **192** (2003), no. 9–10, 1061–1105.
[57] T. Linß, *Layer adapted meshes for convection-diffusion problems*, Habilitation, TU Dresden, 2006.
[58] T. Linß, Analysis of an upwind finite-difference scheme for a system of coupled singularly perturbed convection-diffusion equations, *Computing* **79** (2007), no. 1, 23–32.
[59] T. Linß, Analysis of a system of singularly perturbed convection-diffusion equations with strong coupling, *SIAM J. Numer. Anal.* **47** (2009), no. 3, 1847–1862.
[60] T. Linß, *Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems*, Lecture Notes in Math. 1985, Springer, Berlin, 2010.
[61] T. Linß, A posteriori error estimation for higher order fem applied to one-dimensional reaction-diffusion problems, *Applications of Math.* **59** (2014), 241–256.
[62] T. Linß and N. Madden, Layer-adapted meshes for a linear system of coupled singularly perturbed reaction-diffusion problems, *IMA J. Numer. Anal.* **29** (2009), no. 1, 109–125.
[63] T. Linß and M. Stynes, Asymptotic analysis and Shishkin-type decomposition for an elliptic convection-diffusion problem, *J. Math. Anal. Appl.* **261** (2001), no. 2, 604–632.
[64] T. Linß and M. Stynes, Numerical methods on Shishkin meshes for linear convection-diffusion problems, *Comput. Methods Appl. Mech. Engrg.* **190** (2001), 3527–3542.
[65] T. Linß and M. Stynes, Numerical solution of systems of singularly perturbed differential equations, *Comput. Methods Appl. Math.* **9** (2009), no. 2, 165–191.
[66] J. M. Melenk, L. Oberbroeckling, B. Pichler and C. Xenophontos,Remarks on asymptotic expansions for singularly perturbed second order ODEs with multiple scales. Lecture presented on the Workshop “Numerical analysis of singularly perturbed problems”, Dresden, 16–18 November 2011.
[67] J. M. Melenk, C. Xenophontos and L. Oberbroeckling, Analytic regularity for a singularly perturbed system of reaction-diffusion equations with multiple scales, *Adv. Comput. Math.* **39** (2013), no. 2, 367–394.
[68] A. Naughton, R. B. Kellogg and M. Stynes, Regularity and derivative bounds for a convection-diffusion problem with a Neumann outflow condition, *J. Differential Equations* **247** (2009), no. 9, 2495–2516.
[69] E. O’Riordan and G. I. Shishkin, A technique to prove parameter-uniform convergence for a singularly perturbed convection-diffusion equation, *J. Comput. Appl. Math.* **206** (2007), no. 1, 136–145.
[70] E. O’Riordan and M. Stynes, Numerical analysis of a strongly coupled system of two singularly perturbed convection-diffusion problems, *Adv. Comput. Math.* **30** (2009), no. 2, 101–121.
[71] T. Richter, A posteriori error estimation and anisotropy detection with the dual-weighted residual method, *Internat. J. Numer. Methods Fluids* **62** (2010), no. 1, 90–118.
[72] H.-G. Roos, Error estimates for linear finite elements on Bakhvalov-type meshes, *Appl. Math.* **51** (2006), no. 1, 63–72.
[73] H.-G. Roos, Robust numerical methods for singularly perturbed differential equations: A survey 2008–2012, *Int. Scholarly Res. Not.* (2012), Article ID 379547.
[74] H.-G. Roos, Special features of strongly coupled systems of convection-diffusion equations with two small parameters, *Appl. Math. Lett.* **25** (2012), no. 8, 1127–1130.
[75] H.-G. Roos and T. Linß, Sufficient conditions for uniform convergence on layer-adapted grids, *Computing* **63** (1999), no. 1, 27–45.
[76] H.-G. Roos and C. Reibiger, Analysis of a strongly coupled system of two convection-diffusion equations with full layer interaction, *ZAMM Z. Angew. Math. Mech.* **91** (2011), no. 7, 537–543.
[77] H.-G. Roos and C. Reibiger, Numerical analysis of a system of singularly perturbed convection-diffusion equations related to optimal control, *Numer. Math. Theory Methods Appl.* **4** (2011), no. 4, 562–575.
[78] H.-G. Roos and M. Schopf, Analysis and finite element approximation for a system of singularly perturbed convection-diffusion equations with multiple scales, submitted.
[79] H.-G. Roos and M. Schopf, An optimal a priori error estimate for the Il’in scheme in 2D, *BIT*, to appear.
[80] H.-G. Roos and M. Schopf, Analysis of finite element methods on Bakhvalov-type meshes for linear convection-diffusion problems in 2D, *Appl. Math.* **57** (2012), no. 2, 97–108.
[81] H.-G. Roos, M. Stynes and L. Tobiska, *Robust Numerical Methods for Singularly Perturbed Differential Equations*, 2nd ed., Springer Ser. Comput. Math. 24, Springer, Berlin, 2008.
[82] H.-G. Roos and H. Zarin, A supercloseness result for the discontinuous Galerkin stabilization of convection-diffusion problems on Shishkin meshes, *Numer. Methods Partial Differential Equations* **23** (2007), no. 6, 1560–1576.
[83] G. Sangalli, A robust a posteriori estimator for the residual-free bubbles method applied to advection-diffusion problems, *Numer. Math.* **89** (2001), no. 2, 379–399.
[84] F. Schieweck, On the role of boundary conditions for CIP stabilization of higher order finite elements, *Electron. Trans. Numer. Anal.* **32** (2008), 1–16.

[85] R. Schneider, A review of anisotropic refinement methods for triangular meshes in FEM, in: *Advanced Finite Element Methods and Applications*, Lecture Notes in Appl. Comput. Mech. 66, Springer, Berlin (2013), 133–152.

[86] G. I. Shishkin, Difference scheme on a nonuniform grid for a differential equation with small parameter multiplying the highest derivative, *Zh. Vychisl. Mat. i Mat. Fiz.* **23** (1983), no. 3, 609–619; translation in *U.S.S.R. Comput. Math. and Math. Phys.* **23**, no. 5, 59–66.

[87] G. I. Shishkin, A difference scheme for a singularly perturbed equation of parabolic type with a discontinuous initial condition, *Dokl. Akad. Nauk SSSR* **300** (1988), no. 5, 1066–1070; translation in *Soviet Math. Dokl.* **37** (1988), no. 3, 792–796.

[88] G. I. Shishkin, Approximation of solutions of singularly perturbed boundary value problems with a parabolic boundary layer, *Zh. Vychisl. Mat. i Mat. Fiz.* **29** (1989), no. 7, 963–977; translation in *U.S.S.R. Comput. Math. and Math. Phys.* **29** (1989), no. 4, 1–10.

[89] G. I. Shishkin, A difference scheme for a singularly perturbed equation of parabolic type with discontinuous coefficients and concentrated factors, *Zh. Vychisl. Mat. i Mat. Fiz.* **29** (1989), no. 9, 1277–1290; translation in *U.S.S.R. Comput. Math. and Math. Phys.* **29** (1989), no. 5, 9–19.

[90] G. I. Shishkin and L. P. Shishkina, *Difference Methods for Singular Perturbation Problems*, Chapman & Hall/CRC Monogr. Surv. in Pure Appl. Math. 140, CRC Press, Boca Raton, 2009.

[91] M. Stynes and E. O'Riordan, A uniformly convergent Galerkin method on a Shishkin mesh for a convection-diffusion problem, *J. Math. Anal. Appl.* **214** (1997), no. 1, 36–54.

[92] M. Stynes and L. Tobiska, The SDFEM for a convection-diffusion problem with a boundary layer: Optimal error analysis and enhancement of accuracy, *SIAM J. Numer. Anal.* **41** (2003), no. 5, 1620–1642.

[93] M. Stynes and L. Tobiska, Using rectangular Q_p elements in the SDFEM for a convection-diffusion problem with a boundary layer, *Appl. Numer. Math.* **58** (2008), no. 12, 1789–1802.

[94] R. Verfürth, A posteriori error estimators for convection-diffusion equations, *Numer. Math.* **80** (1998), no. 4, 641–663.

[95] R. Verfürth, Robust a posteriori error estimates for stationary convection-diffusion equations, *SIAM J. Numer. Anal.* **43** (2005), no. 4, 1766–1782.

[96] R. Verfürth, *A posteriori Error Estimation Techniques for Finite Element Methods*, Numer. Math. Sci. Comput., Oxford University Press, Oxford, 2013.

[97] M. Vohralík, *A Posteriori Error Estimates for Efficiency and Error Control in Numerical Simulations*, Lecture Notes, Université Pierre et Marie Curie, Paris, 2012.

[98] Z. Xie and Z. Zhang, Superconvergence of DG method for one-dimensional singularly perturbed problems, *J. Comput. Math.* **25** (2007), no. 2, 185–200.

[99] Z. Xie and Z. Zhang, Uniform superconvergence analysis of the discontinuous Galerkin method for a singularly perturbed problem in 1-D, *Math. Comp.* **79** (2010), no. 269, 35–45.

[100] B. Zhang, S. Chen and J. Zhao, A posteriori error estimation based on conservative flux reconstruction for nonconforming finite element approximations to a singularly perturbed reaction-diffusion problem on anisotropic meshes, *Appl. Math. Comput.* **232** (2014), 1062–1075.

[101] Z. Zhang, Finite element superconvergence on Shishkin mesh for 2-D convection-diffusion problems, *Math. Comp.* **72** (2003), no. 243, 1147–1177.

[102] G. Zhou, How accurate is the streamline diffusion finite element method?, *Math. Comp.* **66** (1997), no. 217, 31–44.

[103] H. Zhu and Z. Zhang, Convergence analysis of the LDG method applied to singularly perturbed problems, *Numer. Methods Partial Differential Equations* **29** (2013), no. 2, 396–421.

[104] H. Zhu and Z. Zhang, Uniform convergence of the LDG method for a singularly perturbed problem with the exponential boundary layer, *Math. Comp.* **83** (2014), 635–663.