A GENERAL SUPERAPPROXIMATION RESULT

SUSANNE C. BRENNER

ABSTRACT. A general superapproximation result is derived in this paper which is useful for the local/interior error analysis of finite element methods.

Superapproximation results are useful tools for the local/interior error analysis of finite element methods [10]. Our goal is to derive a general superapproximation result that includes many of the results in the literature as special cases.

We will adopt the usual notation of function spaces and norms that can be found for example in [1, 6, 5]. The set-up is as follows.

• ω is a C^{∞} function on \mathbb{R}^N such that, for a positive number d,

$$|\omega|_{W^j_{\infty}(\mathbb{R}^N)} \le C_{\dagger} d^{-j} \qquad j = 0, 1, 2, \dots$$

$$\tag{1}$$

• $(K, \mathcal{P}, \mathcal{N})$ is a finite element space (à la Ciarlet), where K is a connected compact subset of \mathbb{R}^N , \mathcal{P} is a finite dimensional vector space of polynomials, \mathcal{N} is a set of nodal variables (degrees of freedom), and the diameter h_K of K satisfies

$$h_K \le d. \tag{2}$$

We will allow a slight abuse of notation to write the Sobolev spaces $H^{\ell}(\operatorname{int}(K))$ (resp., $W^{\ell}_{\infty}(\operatorname{int}(K))$) as $H^{\ell}(K)$ (resp., $W^{\ell}_{\infty}(K)$).

• ω_K is the mean of ω over K so that

$$|\omega_{\kappa}| \le C_{\dagger},\tag{3}$$

$$|\omega - \omega_{\kappa}|_{L_{\infty}(K)} \le C_{\dagger} d^{-1} h_{\kappa}.$$
(4)

• The linear operator $\Pi_{K} : C^{\infty}(K) \to \mathcal{P}$ is the nodal interpolation operator that satisfies

$$\zeta - \Pi_K \zeta = 0 \qquad \forall \zeta \in \mathcal{P},\tag{5}$$

and there exists a positive integer k (depending on the maximum order of differentiation among the nodal variables in \mathcal{N}) such that

$$|\zeta - \prod_{K} \zeta|_{W^{\ell}_{\infty}(K)} \le C_{\flat} h^{k+1-\ell}_{K} |\zeta|_{W^{k+1}_{\infty}(K)} \qquad \forall \zeta \in C^{\infty}(K), \ 0 \le \ell \le k+1.$$
(6)

• The following inverse estimates are satisfied:

$$|\chi|_{W_s^q(K)} \le C_{\sharp} h_K^{p-q} |\chi|_{W_s^p(K)} \qquad \forall \, \chi \in \mathcal{P}, \ s = 2, \infty, \text{ and } 0 \le p < q \le k+1,$$
(7)

$$|\chi|_{W^p_{\infty}(K)} \le C_{\diamond} h_K^{-N/2} |\chi|_{H^p(K)} \qquad \forall \, \chi \in \mathcal{P} \text{ and } 0 \le p \le k.$$
(8)

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SUSANNE C. BRENNER

Inverse estimates and interpolation error estimates for finite elements can be found in [6, 5]. Note that, by the Bramble-Hilbert lemma [3], the interpolation error estimates in (6) are valid provided the space \mathcal{P} of shape functions contains all the polynomials of total degree $\leq k$.

The following theorem is the main result of this paper.

Theorem 1. Given $0 \le m \le k$, $0 \le \ell \le m + 1$ and $n \ge 1$, we have, under assumptions (1)-(8),

$$|\omega^n \chi - \Pi_K(\omega^n \chi)|_{H^\ell(K)} \le Ch_K^{m+1-\ell} d^{-(m+1)} \sum_{j=0}^m d^j |\chi|_{H^j(K)} \qquad \forall \chi \in \mathcal{P},$$
(9)

and, for $n \ge m+1$,

$$|\omega^n \chi - \Pi_K(\omega^n \chi)|_{H^\ell(K)} \le Ch_K^{m+1-\ell} d^{-(m+1)} \sum_{j=0}^m d^j |\omega^j \chi|_{H^j(K)} \qquad \forall \, \chi \in \mathcal{P}, \tag{10}$$

where the positive constant C in (9) and (10) depends only on N, n, k, C_{\dagger} , C_{\flat} , C_{\sharp} and C_{\diamond} .

Remark 1. For an arbitrary smooth function χ , the interpolation error estimates in (9) and (10) will involve the (m + 1)-order Sobolev norm $|\chi|_{H^{m+1}(K)}$. The fact that the right-hand sides of (9) and (10) only involve Sobolev norms up to order m can be exploited to show that the energy norm of a discrete harmonic (or biharmonic) function on a subdomain can be bounded by a lower order norm on a larger subdomain, which is the discrete version of a Caccioppoli estimate. Such estimates play a key role in the local/interior error analysis of finite element methods (cf. [10, Section 9.1] and [7, Section 3]).

Remark 2. The key observation is that, by the binomial theorem,

$$\omega^{n}\chi - \Pi_{K}(\omega^{n}\chi) = \sum_{q=1}^{n} \binom{n}{q} \left[(\omega - \omega_{K})^{q} \omega_{K}^{n-q} \chi - \Pi_{K} \left((\omega - \omega_{K})^{q} \omega_{K}^{n-q} \chi \right) \right] \qquad \forall \chi \in \mathcal{P}, \quad (11)$$

where the summation is over $q \ge 1$ because $\omega_{\kappa}^n \chi - \prod_{\kappa} (\omega_{\kappa}^n \chi) = 0$ by (5). This is why only derivatives up to order *m* appear on the right-hand side of (9) and (10) (cf. Lemma 1 below).

Example 1. For n = 1, $m = \ell = 1$ or $m = \ell = 0$, it follows from (9) that

$$\begin{aligned} |\omega\chi - \Pi_{K}(\omega\chi)|_{H^{1}(K)} &\leq Ch_{K}d^{-2}(\|\chi\|_{L_{2}(K)} + d|\chi|_{H^{1}(K)}), \\ \|\omega\chi - \Pi_{K}(\omega\chi)\|_{L_{2}(K)} &\leq Ch_{K}d^{-1}\|\chi\|_{L_{2}(K)}, \end{aligned}$$

which cover the results in [9, Section 2], [4, Section 6] and [2, Section 3].

Example 2. For n = 2 and $m = \ell = 1$, it follows from (10) that

$$|\omega^2 \chi - \Pi_K(\omega^2 \chi)|_{H^1(K)} \le Ch_K d^{-2} \big(\|\chi\|_{L_2(K)} + d|\omega\chi|_{H^1(K)} \big),$$

which is the result in [7, Section 2].

Example 3. For n = 4, m = 2 and $1 \le \ell \le 3$, it follows from (10) that

 $|\omega^4 - \Pi_K(\omega^4\chi)|_{H^\ell(K)} \le Ch_K^{3-\ell} d^{-3} (||\chi||_{L_2(K)} + d|\omega\chi|_{H^1(K)} + d^2|\omega^2\chi|_{H^2(K)}),$ which is the result in [8, Section 3.3].

2

We will use the notation $A \leq B$ to represent the inequality $A \leq (\text{constant})B$, where the hidden generic positive constant only depends on N, n, k, C_{\dagger} , C_{\flat} , C_{\sharp} and C_{\diamond} .

The proof of Theorem 1 requires two lemmas that are based on the product rule (Leibniz formula) and the binomial theorem. The first lemma allows the reduction of the order of the Sobolev seminorm.

Lemma 1. For $1 \le q \le n$, $0 \le p \le k+1$ and $s = 2, \infty$, we have

$$d^{p}|(\omega-\omega_{\kappa})^{q}\chi|_{W^{p}_{s}(K)} \lesssim \sum_{j=0}^{(p-q)\vee 0} d^{j}|\chi|_{W^{j}_{s}(K)} \qquad \forall \chi \in \mathcal{P},$$

$$(12)$$

where $a \lor b = \max(a, b)$.

Proof. Note that (1), (2), (4) and the product rule imply

$$|(\omega - \omega_{\kappa})^{q}|_{W^{r}_{\infty}(K)} \lesssim \begin{cases} d^{-r} & \text{if } r > q\\ d^{-q}h_{\kappa}^{q-r} & \text{if } r \le q \end{cases}.$$

$$(13)$$

Let $\chi \in \mathcal{P}$ be arbitrary. It follows from (13) and the product rule again that

$$d^{p}|(\omega - \omega_{\kappa})^{q}\chi|_{W^{p}_{s}(K)} \lesssim d^{p}\sum_{r=0}^{p}|(\omega - \omega_{\kappa})^{q}|_{W^{r}_{\infty}(K)}|\chi|_{W^{p-r}_{s}(K)}$$
$$\lesssim \sum_{r=0}^{p\wedge q} d^{p-q}h^{q-r}_{\kappa}|\chi|_{W^{p-r}_{s}(K)} + \sum_{r=(p\wedge q)+1}^{p} d^{p-r}|\chi|_{W^{p-r}_{s}(K)},$$
(14)

where $p \wedge q = \min(p, q)$ and we have adopted the convention that a sum is void if the lower index is strictly greater than the upper index. Note that the second sum on the right-hand side of (14) is void if and only if $p \leq q$.

In the case where $p \leq q$, the first sum on the right-hand side of (14) satisfies

$$\sum_{r=0}^{p \wedge q} d^{p-q} h_{K}^{q-r} |\chi|_{W_{s}^{p-r}(K)} = \sum_{r=0}^{p} (h_{K}/d)^{q-p} h_{K}^{p-r} |\chi|_{W_{s}^{p-r}(K)} \lesssim \|\chi\|_{L_{s}(K)}$$

because of (2) and (7). On the other hand, if p > q, then we have

$$\sum_{r=0}^{p\wedge q} d^{p-q} h_{\kappa}^{q-r} |\chi|_{W_{s}^{p-r}(K)} = \sum_{r=0}^{q} d^{p-q} h_{\kappa}^{q-r} |\chi|_{W_{s}^{p-r}(K)} \lesssim d^{p-q} |\chi|_{W_{s}^{p-q}(K)}$$

because of (7).

It follows that

$$\sum_{r=0}^{p \wedge q} d^{p-q} h_K^{q-r} |\chi|_{W_s^{p-r}(K)} \lesssim d^{p-(p \wedge q)} |\chi|_{W_s^{p-(p \wedge q)}(K)}$$

which together with (14) implies

$$d^p |(\omega - \omega_K)^q \chi|_{W^p_s(K)} \lesssim \sum_{r=p \wedge q}^p d^{p-r} |\chi|_{W^{p-r}_s(K)}.$$

The estimate (12) is then obtained by the change of index j = p - r.

The second lemma allows the switching from $|\omega_K^j \chi|_{H^j(K)}$ to $|\omega^j \chi|_{H^j(K)}$.

Lemma 2. For $0 \le j \le m$, we have

$$d^{j}|\omega_{K}^{j}\chi|_{H^{j}(K)} \lesssim \sum_{p=0}^{j} d^{p}|\omega^{p}\chi|_{H^{p}(K)} \qquad \forall \chi \in \mathcal{P}.$$
(15)

Proof. According to the binomial theorem, we have

$$\omega_{\kappa}^{j} = \omega^{j} - \sum_{\ell=1}^{j} {j \choose \ell} (\omega - \omega_{\kappa})^{\ell} \omega_{\kappa}^{j-\ell}.$$
(16)

It follows from (3), Lemma 1 (with s = 2) and (16) that

$$\begin{aligned} d^{j}|\omega_{K}^{j}\chi|_{H^{j}(K)} &\lesssim d^{j}|\omega^{j}\chi|_{H^{j}(K)} + \sum_{\ell=1}^{j}|\omega_{K}|^{j-\ell}d^{j}|(\omega-\omega_{K})^{\ell}\chi|_{H^{j}(K)} \\ &\lesssim d^{j}|\omega^{j}\chi|_{H^{j}(K)} + \sum_{\ell=1}^{j}\sum_{p=0}^{j-\ell}d^{p}|\omega_{K}^{j-\ell}\chi|_{H^{p}(K)} \\ &\lesssim d^{j}|\omega^{j}\chi|_{H^{j}(K)} + \sum_{\ell=1}^{j}\sum_{p=0}^{j-\ell}d^{p}|\omega_{K}^{p}\chi|_{H^{p}(K)} \\ &= d^{j}|\omega^{j}\chi|_{H^{j}(K)} + \sum_{p=0}^{j-1}(j-p)d^{p}|\omega_{K}^{p}\chi|_{H^{p}(K)} \\ &\lesssim d^{j}|\omega^{j}\chi|_{H^{j}(K)} + \sum_{p=0}^{j-1}d^{p}|\omega_{K}^{p}\chi|_{H^{p}(K)}, \end{aligned}$$

which implies (15) through a recursive argument.

We are now ready to prove Theorem 1. From (6) and Lemma 1 (with $s = \infty$), we find, for $n \ge q \ge 1$,

$$\begin{aligned} |(\omega - \omega_{\kappa})^{q} \omega_{\kappa}^{n-q} \chi - \Pi_{\kappa} \big((\omega - \omega_{\kappa})^{q} \omega_{\kappa}^{n-q} \chi \big) |_{H^{\ell}(K)} \\ &\lesssim (h_{\kappa}^{N/2} |\omega_{\kappa}^{n-q}|) |(\omega - \omega_{\kappa})^{q} \chi - \Pi_{\kappa} \big((\omega - \omega_{\kappa})^{q} \chi \big) |_{W_{\infty}^{\ell}(K)} \\ &\lesssim (h_{\kappa}^{N/2} |\omega_{\kappa}^{n-q}|) h_{\kappa}^{k+1-\ell} |(\omega - \omega_{\kappa})^{q} \chi |_{W_{\infty}^{k+1}(K)} \\ &\lesssim (h_{\kappa}^{N/2} |\omega_{\kappa}^{n-q}|) \Big(h_{\kappa}^{k+1-\ell} d^{-(k+1)} \sum_{j=0}^{(k+1-q)\vee 0} d^{j} |\chi|_{W_{\infty}^{j}(K)} \Big). \end{aligned}$$
(17)

The next step is to reduce the summation on the right-hand side of (17) from the range $0 \le j \le [(k+1-q) \lor 0]$ to the range $0 \le j \le [(m+1-q) \lor 0]$.

In the case where $k + 1 - q \leq 0$, we also have $m + 1 - q \leq 0$ (since $m \leq k$ is one of the assumptions in Theorem 1) and hence, in view of (2),

$$h_{K}^{k+1-\ell} d^{-(k+1)} \sum_{j=0}^{(k+1-q)\vee 0} d^{j} |\chi|_{W_{\infty}^{j}(K)} = (h_{K}/d)^{k-m} h_{K}^{m+1-\ell} d^{-(m+1)} ||\chi||_{L_{\infty}(K)}$$
$$\leq h_{K}^{m+1-\ell} d^{-(m+1)} \sum_{j=0}^{(m+1-q)\vee 0} d^{j} |\chi|_{W_{\infty}^{j}(K)}.$$
(18)

Suppose k + 1 - q > 0 and let j be an index within the range of the summation on the right-hand side of (17), i.e.,

$$0 \le j \le k+1-q. \tag{19}$$

If j is strictly greater than m + 1 - q, we have, by (2), (7) and (19),

$$h_{K}^{k+1-\ell} d^{-(k+1)} \left(d^{j} |\chi|_{W_{\infty}^{j}(K)} \right) \lesssim h_{K}^{k+1-\ell} d^{-(k+1)} \left(d^{j} h_{K}^{m+1-q-j} |\chi|_{W_{\infty}^{m+1-q}(K)} \right) = (h_{K}/d)^{k+1-q-j} h_{K}^{m+1-\ell} d^{-(m+1)} \left(d^{m+1-q} |\chi|_{W_{\infty}^{m+1-q}(K)} \right) \leq h_{K}^{m+1-\ell} d^{-(m+1)} \left(d^{m+1-q} |\chi|_{W_{\infty}^{m+1-q}(K)} \right).$$
(20)

If j is less than or equal to m + 1 - q, we have

$$h_{K}^{k+1-\ell} d^{-(k+1)} \left(d^{j} |\chi|_{W_{\infty}^{j}(K)} \right) = (h_{K}/d)^{k-m} h_{K}^{m+1-\ell} d^{-(m+1)} \left(d^{j} |\chi|_{W_{\infty}^{j}(K)} \right)$$
$$\leq h_{K}^{m+1-\ell} d^{-(m+1)} \left(d^{j} |\chi|_{W_{\infty}^{j}(K)} \right), \tag{21}$$

again because $m \leq k$ is one of the assumptions in Theorem 1.

Since all the possibilities are covered by (18), (20) and (21), we conclude from (8) and (17) that

$$|(\omega - \omega_{\kappa})^{q} \omega_{\kappa}^{n-q} \chi - \Pi_{\kappa} ((\omega - \omega_{\kappa})^{q} \omega_{\kappa}^{n-q} \chi)|_{H^{\ell}(K)} \lesssim (h_{\kappa}^{N/2} |\omega_{\kappa}^{n-q}|) \left(h_{\kappa}^{m+1-\ell} d^{-(m+1)} \sum_{j=0}^{(m+1-q)\vee 0} d^{j} |\chi|_{W_{\infty}^{j}(K)} \right) \lesssim h_{\kappa}^{m+1-\ell} d^{-(m+1)} \sum_{j=0}^{(m+1-q)\vee 0} d^{j} |\omega_{\kappa}^{n-q} \chi|_{H^{j}(K)}.$$
(22)

Note that $n-q \ge 0$ and hence, in view of (3), we can simply drop ω_K^{n-q} from the right-hand side of (22) to arrive at

$$|(\omega - \omega_{\kappa})^{q} \omega_{\kappa}^{n-q} \chi - \prod_{\kappa} \left((\omega - \omega_{\kappa})^{q} \omega_{\kappa}^{n-q} \chi \right)|_{H^{\ell}(K)} \lesssim h_{\kappa}^{m+1-\ell} d^{-(m+1)} \sum_{j=0}^{(m+1-q)\vee 0} d^{j} |\chi|_{H^{j}(K)}, \quad (23)$$

and the estimate (9) follows immediately from (11) and (23).

In the case where $n \ge m+1$, we have

$$n-q \ge \left[(m+1-q) \lor 0 \right]$$

and hence

$n-q-j \ge 0$

for any index j in the range of the summation on the right-hand side of (22). Therefore we can replace ω_K^{n-q} by ω_K^j inside the summation to obtain, through Lemma 2,

$$|(\omega - \omega_{\kappa})^{q} \omega_{\kappa}^{n-q} \chi - \Pi_{\kappa} ((\omega - \omega_{\kappa})^{q} \omega_{\kappa}^{n-q} \chi)|_{H^{\ell}(K)} \lesssim h_{\kappa}^{m+1-\ell} d^{-(m+1)} \sum_{j=0}^{(m+1-q)\vee 0} d^{j} |\omega_{\kappa}^{j} \chi|_{H^{j}(K)} \lesssim h_{\kappa}^{m+1-\ell} d^{-(m+1)} \sum_{j=0}^{(m+1-q)\vee 0} d^{j} |\omega^{j} \chi|_{H^{j}(K)}.$$
(24)

The estimate (10) follows immediately from (11) and (24).

References

- [1] R.A. Adams and J.J.F. Fournier. Sobolev Spaces (Second Edition). Academic Press, Amsterdam, 2003.
- [2] D.N. Arnold and X.B. Liu. Local error estimates for finite element discretizations of the Stokes equations. RAIRO Modél. Math. Anal. Numér., 29:367–389, 1995.
- [3] J.H. Bramble and S.R. Hilbert. Estimation of linear functionals on Sobolev spaces with applications to Fourier transforms and spline interpolation. SIAM J. Numer. Anal., 7:113–124, 1970.
- [4] J.H. Bramble, J.A. Nitsche, and A.H. Schatz. Maximum-norm interior estimates for Ritz-Galerkin methods. *Math. Comp.*, 29:677–688, 1975.
- [5] S.C. Brenner and L.R. Scott. The Mathematical Theory of Finite Element Methods (Third Edition). Springer-Verlag, New York, 2008.
- [6] P.G. Ciarlet. The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam, 1978.
- [7] A. Demlow, J. Guzmán, and A.H. Schatz. Local energy estimates for the finite element method on sharply varying grids. *Math. Comp.*, 80:1–9, 2011.
- [8] D. Leykekhman. Pointwise error estimates for C^0 interior penalty approximation of biharmonic problems. *preprint*, 2019.

(https://www2.math.uconn.edu/~leykekhman/publications.html)

- [9] J.A. Nitsche and A.H. Schatz. Interior estimates for Ritz-Galerkin methods. Math. Comp., 28:937—958, 1974.
- [10] L.B. Wahlbin. Local Behavior in Finite Element Methods. In P.G. Ciarlet and J.L. Lions, editors, Handbook of Numerical Analysis, II, pages 353–522. North-Holland, Amsterdam, 1991.

SUSANNE C. BRENNER, DEPARTMENT OF MATHEMATICS AND CENTER FOR COMPUTATION AND TECH-NOLOGY, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70803, USA

E-mail address: brenner@math.lsu.edu

6