# On the finite element approximation of fourth order singularly perturbed eigenvalue problems 

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#### Abstract

We consider fourth order singularly perturbed eigenvalue problems in one-dimension and the approximation of their solution by the $h$ version of the Finite Element Method (FEM). In particular, we use piecewise Hermite polynomials of degree $p \geq 3$ defined on an exponentially graded mesh. We show that the method converges uniformly, with respect to the singular perturbation parameter, at the optimal rate when the error in the eigenvalues is measured in absolute value and the error in the eigenvectors is measured in the energy norm. We also illustrate our theoretical findings through numerical computations for the case $p=3$.


Keywords: fourth order singularly perturbed eigenvalue problem; boundary layers; finite element method; exponentially graded mesh; uniform convergence

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[^0]
## 1 Introduction

Singularly perturbed boundary value problems, and their numerical solution, is a much studied topic in the last few decades (see the books [12, [13], [19] and the references therein). It is well known that a main difficulty in the approximation to the solution of these problems is the presence of boundary layers in the solution. In order for the approximate solution to be considered reliable, it must account for these layers. In the context of the Finite Element Method (FEM), the robust approximation of boundary layers requires either the use of the $h$ version on non-uniform, layer-adapted meshes (such as the Shishkin [21] or Bakhvalov [2] mesh), or the use of the high order $p$ and $h p$ versions on specially designed (variable) meshes [20]. One other layer-adapted mesh that has appeared in the literature is the exponentially graded mesh (eXp) [25]. The finite element analysis on this mesh appears in [5] for one-dimensional reaction-diffusion and convection-diffusion problems, in [27] for a twodimensional convection-diffusion problem posed in a square and in [4] for two-dimensional reaction-diffusion problems posed in smooth domains. All the aforementioned works concern second order singularly perturbed problems. Only recently have fourth order singularly perturbed problems truly attracted the attention of the numerical analysis research community (see, e.g., [7, 8, 9, 15, 26] for some recent results and [16, 18, 23] for some earlier results). In [26] the finite element analysis for a one-dimensional fourth order problem was carried out on the eXp mesh, in the context of the $h$ version with piecewise polynomials of degree $p \geq 3$. The purpose of this article is to extend the results of [26] to one-dimensional fourth order singularly perturbed eigenvalue problems. To our knowledge, numerical analysis results for such problems are scarce in the literature. The only relevant ones we could find are the following: [6] in which a hybrid scheme based on asymptotic expansions is employed in order to solve the thin hanging rod problem and [17] where the author presents a finite element discretization of problem (1)-(2) (see ahead), using a Shishkin mesh and polynomials of degree $p=3$. We will present a finite element discretization using the eXp mesh and polynomials of degree $p \geq 3$, proving robust, optimal convergence in both the eigenvalues and the eigenvectors, assuming they are simple. The error in the eigenvalues is shown to decrease at the (expected) double rate, and the error in the eigenvectors, measured in the energy norm, decreases at the optimal rate; both do so independently of the singular perturbation parameter $\varepsilon$.

The rest of the paper is organized as follows: in Section 2 we present the model problem and its regularity. The discretization using the exponentially graded mesh is presented in Section 3 and in Section 4 we present our main results of parameter robust convergence in the eigenvalues and the eigenvectors. Section 5 shows the results of some numerical computations that illustrate the theoretical findings and in Section 6 we give our conclusions.

With $I \subset \mathbb{R}$ a bounded open interval with boundary $\partial I$ and measure $|I|$, we will denote by $C^{k}(I)$ the space of continuous functions on $I$ with continuous derivatives up to order $k$. We will use the usual Sobolev spaces $H^{k}(I)=W^{k, 2}(I)$ of functions on $I$ with $0,1,2, \ldots, k$ generalized derivatives in $L^{2}(I)$, equipped with the norm and seminorm $\|\cdot\|_{k, I}$ and $|\cdot|_{k, I}$,
respectively. We will also use the space

$$
H_{0}^{k}(I)=\left\{u \in H^{k}(I):\left.u^{(i)}\right|_{\partial I}=0, i=0, \ldots, k-1\right\} .
$$

The norm of the space $L^{\infty}(I)$ of essentially bounded functions is denoted by $\|\cdot\|_{\infty, I}$. Finally, the notation " $a \lesssim b$ " means " $a \leq C b$ " with $C$ being a generic positive constant, independent of any discretization or singular perturbation parameters and possibly having different values in each occurrence - dependence on various other constants will be indicated.

## 2 The model problem and its regularity

We consider the following eigenvalue problem: Find $0 \neq u(x) \in C^{4}(I), \lambda \in \mathbb{C}$ such that

$$
\begin{equation*}
\varepsilon^{2} u^{(4)}(x)-\left(a(x) u^{\prime}(x)\right)^{\prime}+b(x) u(x)=\lambda u(x) \text { in } I=(0,1) \tag{1}
\end{equation*}
$$

along with the boundary conditions

$$
\begin{equation*}
u(0)=u^{\prime}(0)=u^{\prime}(1)=u(1)=0 \tag{2}
\end{equation*}
$$

The parameter $0<\varepsilon \leq 1$ is given, as are the functions $a, b>0$, which are assumed to be sufficiently smooth on the closed interval $\bar{I}=[0,1]$. Moreover, we assume that $\exists a_{0} \in \mathbb{R}$ such that

$$
a(x) \geq a_{0}>0, b(x) \geq 0 \forall x \in \bar{I}
$$

It is easy to see that the problem (1), (2) is self-adjoint. As a result, the behavior of the eigenvalues is simplified, as $\varepsilon \rightarrow 0$, as follows: for all positive eigenvalues $\lambda_{k}(\varepsilon)$ there holds $\lim _{\varepsilon \rightarrow 0} \lambda_{k}(\varepsilon)=\lambda_{k}(0)$. The values $\lambda_{k}(0)$ are the eigenvalues of the reduced problem and if they are real then so are the $\lambda_{k}(\varepsilon)$. Moreover, $\lambda_{k}(\varepsilon)$ can be expanded in a power series in $\varepsilon$ (see [14] for details).

The variational formulation of (1), (2) reads: Find $0 \neq u_{k} \in H_{0}^{2}(I), \lambda_{k} \in \mathbb{C}$ such that

$$
\begin{equation*}
\mathcal{B}_{\varepsilon}\left(u_{k}, v\right)=\lambda_{k}\left\langle u_{k}, v\right\rangle_{I} \quad \forall v \in H_{0}^{2}(I), \tag{3}
\end{equation*}
$$

where, with $\langle\cdot, \cdot\rangle_{I}$ the usual $L^{2}(I)$ inner product,

$$
\begin{equation*}
\mathcal{B}_{\varepsilon}(u, v)=\varepsilon^{2}\left\langle u^{\prime \prime}, v^{\prime \prime}\right\rangle_{I}+\left\langle a u^{\prime}, v^{\prime}\right\rangle_{I}+\langle b u, v\rangle_{I} . \tag{4}
\end{equation*}
$$

It follows that the bilinear form $\mathcal{B}_{\varepsilon}(\cdot, \cdot)$ given by (4) is coercive with respect to the energy norm

$$
\|u\|_{E, I}^{2}:=\varepsilon^{2}|u|_{2, I}^{2}+\|u\|_{1, I}^{2}, u \in H_{0}^{2}(I)
$$

i.e., there exists $\gamma \in \mathbb{R}^{+}$, independent of $\varepsilon$, such that

$$
\mathcal{B}_{\varepsilon}(u, u) \geq \gamma\|u\|_{E, I}^{2} \quad \forall u \in H_{0}^{2}(I) .
$$

The eigenfunctions $u_{k}$ are sufficiently smooth in $I$ and their first derivative features boundary layers at the endpoints. This is described in the following result.

Theorem 1. Let $u \equiv u_{k} \in H_{0}^{2}(I)$ satisfy (3). Then

$$
u=u_{S}+u_{B L}^{L}+u_{B L}^{R}
$$

and for $j=0,1,2, \ldots$

$$
\left|u_{S}^{(j)}(x)\right| \lesssim C_{j}(k),\left|\left(u_{B L}^{L}\right)^{(j)}(x)\right| \lesssim \bar{C}_{j}(k) \varepsilon^{1-j} e^{-\beta x / \varepsilon},\left|\left(u_{B L}^{R}\right)^{(j)}(x)\right| \lesssim \hat{C}_{j}(k) \varepsilon^{1-j} e^{-\beta(1-x) / \varepsilon},
$$

where $C_{j}, \bar{C}_{j}, \hat{C}_{j}, \beta$ are positive constants independent of $\varepsilon$.

Proof. In [14] we find the following decomposition for the eigenfunctions:

$$
u(x)=G_{0}(x, \varepsilon)+\varepsilon G_{1}(x, \varepsilon) \exp \left(-\frac{1}{\varepsilon} \int_{0}^{x} a^{1 / 2}(s) d s\right)+\varepsilon G_{2}(x, \varepsilon) \exp \left(-\frac{1}{\varepsilon} \int_{x}^{1} a^{1 / 2}(s) d s\right)
$$

where $G_{i}, i=0,1,2$ have asymptotic power series expansions with respect to $\varepsilon$ (we omitted the dependence on $\lambda$.) The decomposition and desired bounds follow from the above expression.

Remark 2. The dependence of the constants in the previous theorem, on $j$ and $k$ is not explicitly known. Thus, if $C_{j}(k) \rightarrow \infty$ as $j \rightarrow \infty$ and/or $k \rightarrow \infty$, our results deteriorate. Moreover, as our numerical results suggest, the computation of higher modes becomes more difficult as $k$ is increased. This is in line with classical results for non singularly perturbed eigenvalue problems, see, e.g. [3].

## 3 Discretization by an exponentially graded $h$-FEM

The discrete version of (3) reads: Find $u_{k}^{h} \in V_{h} \subset H_{0}^{2}(I), \lambda_{k}^{h} \in \mathbb{C}$ such that

$$
\begin{equation*}
\mathcal{B}_{\varepsilon}\left(u_{k}^{h}, v\right)=\lambda_{k}^{h}\left\langle u_{k}^{h}, v\right\rangle_{I} \quad \forall v \in V_{h} \subset H_{0}^{2}(I), \tag{5}
\end{equation*}
$$

with the finite dimensional subspace $V_{h}$ defined as follows: let

$$
\Delta=\left\{0=x_{0}<x_{1}<\ldots<x_{N}=1\right\}
$$

be an arbitrary partition of $I$ and set

$$
I_{j}=\left(x_{j-1}, x_{j}\right), \quad h_{j}=x_{j}-x_{j-1}, \quad j=1, \ldots, N
$$

With $\mathbb{P}_{p}(\alpha, \beta)$ the space of polynomials of degree less than or equal to $p \geq 2 N+1$ on the interval $(\alpha, \beta)$, we define the subspace $V_{h} \subset H_{0}^{2}(I)$ as

$$
\begin{equation*}
V_{h}=\left\{u \in H_{0}^{2}(I):\left.u\right|_{I_{j}} \in \mathbb{P}_{p}\left(I_{j}\right), j=1, \ldots, N\right\} . \tag{6}
\end{equation*}
$$

We note that the space $V_{h}$ consists of the classical (piecewise) Hermite polynomials (see, e.g., [1]), hence we quote the following relevant results.

Definition 3. [1] Let $\left\{x_{i}\right\}_{i=0}^{N}$ be an arbitrary partition of the interval [a,b] and suppose that for a sufficiently smooth function $f(x), x \in[a, b]$, the values

$$
f\left(x_{i}\right)=y_{i} \in \mathbb{R}, f^{\prime}\left(x_{i}\right)=y_{i}^{\prime} \in \mathbb{R}, i=0,1, \ldots, N
$$

are given. Then there exists a unique polynomial $f^{I} \in \mathbb{P}_{2 N+1}(a, b)$, called the Hermite interpolant of $f$, given by

$$
f^{I}(x)=\sum_{i=0}^{N}\left(y_{i} H_{0, i}(x)+y_{i}^{\prime} H_{1, i}(x)\right),
$$

where, with $L_{i}(x)$ the Lagrange polynomial of degree $N$ associated with node $x_{i}$,

$$
H_{0, i}(x)=\left[1-2\left(x-x_{i}\right) \frac{d L_{i}}{d x}\left(x_{i}\right)\right] L_{i}^{2}(x), H_{1, i}(x)=\left(x-x_{i}\right) L_{i}^{2}(x) .
$$

Theorem 4. [1, Thm 1.12] Let $v \in C^{2 n+2}([a, b])$ and let $\Delta=\left\{x_{i}\right\}_{i=0}^{N}$ be a mesh on $[a, b]$ with maximum meshsize $h$ and with $N$ a multiple of $n$. If $v^{I}$ is the piecewise Hermite interpolant of $v$ from Definition 3, having degree at most $2 n+1$ on each subinterval $\left[x_{i-1}, x_{i}\right], i=1, \ldots, N$, then

$$
\left\|v^{(\ell)}-\left(v^{I}\right)^{(\ell)}\right\|_{\infty, I} \lesssim h^{2 n+2-\ell}\left\|v^{(2 n+2)}\right\|_{\infty, I}, \ell=0,1, \ldots, 2 n+1 .
$$

We mention in passing that the classical theory of eigenvalue problems (see, e.g., [22]) gives, in the case when $\varepsilon$ is fixed and piecewise cubic polynomials are used on a uniform mesh with meshsize $h$,

$$
\lambda_{k} \leq \lambda_{k}^{h} \leq \lambda_{k}+C(\varepsilon) \lambda_{k}^{2} h^{4},
$$

with $h \leq h_{0}(\varepsilon)$ for some $h_{0}$. Numerical experiments, however, indicate that this estimate does not hold uniformly with respect to $\varepsilon$. This is due to the boundary layer components that are present in the (first derivative of the) eigenfunctions and in view of Theorem 1 , the 'challenge' lies in approximating the one-dimensional boundary layer function

$$
\begin{equation*}
e^{-\beta x / \varepsilon}, \beta \in \mathbb{R}^{+}, x \in[0,1], \varepsilon \in(0,1] . \tag{7}
\end{equation*}
$$

As mentioned before, there are several layer adapted meshes in the literature, perhaps the most widely known being the Shishkin or S-type meshes. In this article we choose to use the exponentially graded (eXp) mesh from [25] - therein the mesh appears for the first time in the literature. (See also [10] for a connection between the eXp mesh and S-type meshes.) To define the mesh, let the mesh points be chosen as follows: with $N>4$ a multiple of 4, we split the interval $[0,1]$ into

$$
\left[0, x_{N / 4-1}\right],\left[x_{N / 4-1}, x_{3 N / 4+1}\right],\left[x_{3 N / 4+1}, 1\right]
$$

and on $\left[x_{N / 4-1}, x_{3 N / 4+1}\right]$ we choose an equidistant mesh with $N / 4+1$ elements. For the other two subintervals the mesh will be exponentially graded with $N / 4-1$ elements. In particular,
the mesh is given by a continuous, monotonically increasing, piecewise continuously differentiable, generating function $\phi$ with $\phi(0)=0$. Then, the nodal points in our mesh are given by

$$
x_{j}=\left\{\begin{array}{cc}
\frac{\varepsilon}{\beta}(p+1) \phi\left(\frac{j}{N}\right) & , \quad j=0,1, \ldots, N / 4-1  \tag{8}\\
x_{N / 4-1}+\left(\frac{x_{3 N / 4}-x_{N / 4-1}}{N / 2+2}\right)\left(j-\frac{N}{4}+1\right) & , \quad j=N / 4, \ldots, 3 N / 4 \\
1-\frac{\varepsilon}{\beta}(p+1) \phi\left(\frac{N-j}{N}\right) & , \quad j=3 N / 4+1, \ldots, N
\end{array}\right.
$$

with

$$
\begin{equation*}
\phi(t)=-\ln \left[1-4 C_{p, \varepsilon} t\right], t \in[0,1 / 4-1 / N] \tag{9}
\end{equation*}
$$

where

$$
\begin{equation*}
C_{p, \varepsilon}=1-\exp \left(-\frac{\beta}{(p+1) \varepsilon}\right) \in \mathbb{R}^{+} \tag{10}
\end{equation*}
$$

An example of this mesh is shown in Figure 1.


Figure 1: Example of the exponential mesh.
We also define the function $\psi$ by $\phi=-\ln \psi$, which gives $\psi(t)=1-2 C_{p, \varepsilon} t$ as well as $\psi^{\prime}(t)=-2 C_{p, \varepsilon} \in \mathbb{R}^{-}$. The meshwidth $h_{j}$ in the intervals $\left[0, x_{N / 4-1}\right],\left[x_{3 N / 4+1}, 1\right]$ satisfies [5],

$$
\begin{equation*}
h_{j} \leq \frac{\varepsilon}{\beta}(p+1) N^{-1} \max _{I_{j}} \phi^{\prime} \leq \frac{\varepsilon}{\beta}(p+1) e^{\frac{x_{j}}{(p+1) \varepsilon}}, j=1, \ldots, N / 4-1,3 N / 4+1, \ldots, N . \tag{11}
\end{equation*}
$$

Moreover, under the assumption $\frac{\varepsilon}{\beta}(p+1) \ln (N-4)<1$, which means that $\varepsilon$ is small and we are in the singularly perturbed case, it was shown in [5] that

$$
\begin{equation*}
e^{-\beta x_{N / 4-1} / \varepsilon}+e^{-\left(1-\beta x_{3 N / 4+1}\right) / \varepsilon} \lesssim N^{-(p+1)} \tag{12}
\end{equation*}
$$

The interpolation result below (Lemma 5) was established in [26] under the the (stronger, but common) assumption

$$
\begin{equation*}
\varepsilon<N^{-1} . \tag{13}
\end{equation*}
$$

(This is needed in order to be able to approximate the smooth part of the solution at the correct rate.) Note that under this assumption, one has $h_{j} \lesssim N^{-1}$ for all $I_{j} \subset I$ and the problem is singularly perturbed.

## 4 Error estimates

We begin by noting that in our setting, Theorem 4 gives

$$
\begin{equation*}
\left\|v^{(k)}-\left(v^{I}\right)^{(k)}\right\|_{\infty, I_{j}} \lesssim h_{j}^{p+1-k}\left\|v^{(p+1)}\right\|_{\infty, I}, k=0,1, \ldots, p, j=1, \ldots, N . \tag{14}
\end{equation*}
$$

Using the above and the definition of the exponential mesh the following lemma was established in [26].

Lemma 5. Let $u_{B L}$ be given by (7) and let $u_{B L}^{I} \in V_{h}$ be its interpolant as in Theorem 4 based on the mesh $\Delta=\left\{x_{j}\right\}_{j=1}^{N}$ with nodes (8) obtained with the mesh generating function $\phi$ given by (9). Then

$$
\begin{equation*}
\left\|\left(u_{B L}-u_{B L}^{I}\right)^{(\ell)}\right\|_{\infty, I} \lesssim \varepsilon^{1-k} N^{-(p+1-\ell)}, \ell=0,1, \ldots, p \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|u_{B L}-u_{B L}^{I}\right|_{2, I} \lesssim \varepsilon^{-1 / 2} N^{-p+1} . \tag{16}
\end{equation*}
$$

The above lemma allows us to prove the following
Lemma 6. Let $u$ be the solution of (3) and let $u^{I} \in V_{h}$ be its interpolant as in Theorem 4 based on the mesh $\Delta=\left\{x_{j}\right\}_{j=1}^{N}$ with nodes (8) obtained with the mesh generating function $\phi$ given by (9). Then

$$
\left\|\left(u-u^{I}\right)^{(\ell)}\right\|_{\infty, I} \lesssim \varepsilon^{1-\ell} N^{-(p+1-\ell)}, \ell=0,1, \ldots, p
$$

and

$$
\left|u-u^{I}\right|_{2, I} \lesssim \varepsilon^{-1 / 2} N^{-p+1}
$$

hence

$$
\left\|u-u^{I}\right\|_{E, I} \lesssim N^{-p+1}
$$

Proof. We use the decomposition of Theorem 1, $u=u_{S}+u_{B L}^{L}+u_{B L}^{R}$, and denote the interpolant by $u^{I}=u_{S}^{I}+\left(u_{B L}^{L}\right)^{I}+\left(u_{B L}^{R}\right)^{I}$, with the obvious notation. Then,

$$
\left\|\left(u-u^{I}\right)^{(\ell)}\right\|_{\infty, I} \lesssim\left\|\left(u_{S}-u_{S}^{I}\right)^{(\ell)}\right\|_{\infty, I}+\left\|\left(u_{B L}^{L}-\left(u_{B L}^{L}\right)^{I}\right)^{(\ell)}\right\|_{\infty, I}+\left\|\left(u_{B L}^{R}-\left(u_{B L}^{R}\right)^{I}\right)^{(\ell)}\right\|_{\infty, I}
$$

with the last two terms being handled by Lemma 5 and the first one by standard techniques and (13). The other estimates are shown in a similar fashion.

Returning to the eigenvalue problem, set

$$
E_{k}=\operatorname{span}\left\{u_{k}\right\}, E_{k}^{h}=\operatorname{span}\left\{u_{k}^{h}\right\}, E^{h}=\oplus_{i=1}^{k} E_{i}^{h} .
$$

Then, the discrete min-max condition says (see [3, eq. (7.6)])

$$
\begin{equation*}
\lambda_{k}^{h}=\min _{E^{h} \in V_{h}^{(k)}} \max _{v \in E^{h}} \frac{\mathcal{B}_{\varepsilon}(v, v)}{\langle v, v\rangle_{I}}, \tag{17}
\end{equation*}
$$

where $V_{h}^{(k)}$ denotes the set of all subspaces of $V_{h}$ with dimension $k$. We choose

$$
\begin{equation*}
E^{h}=\Pi_{h} V^{(k)} \tag{18}
\end{equation*}
$$

in (17), where

$$
V^{(k)}=\oplus_{i=1}^{k} E_{i}
$$

and $\Pi_{h}: V \rightarrow V_{h}$ is the Ritz projection, defined by

$$
\begin{equation*}
\mathcal{B}_{\varepsilon}\left(u-\Pi_{h} u, v\right)=0 \forall v \in V_{h} \tag{19}
\end{equation*}
$$

We may do so since, for $h$ sufficiently small, the bound

$$
\begin{equation*}
\left\|\Pi_{h} v\right\|_{E, I} \geq\|v\|_{E, I}-\left\|v-\Pi_{h} v\right\|_{E, I} \forall v \in V \tag{20}
\end{equation*}
$$

ensures that the dimension of $E^{h}$ is equal to $k$. In particular, if we take $h$ such that

$$
\left\|v-\Pi_{h} v\right\|_{E, I} \leq \frac{1}{2}\|v\|_{E, I} \forall v \in V^{(k)}
$$

then $\Pi_{h}$ is injective from $V^{(k)}$ to $E^{h}$. (The smallness of $h$ depends on $k$ ). See [3] for more details.

As in [23], we have

$$
\begin{aligned}
\left\|u-\Pi_{h} u\right\|_{E, I}^{2} & =\mathcal{B}_{\varepsilon}\left(u-\Pi_{h} u, u-\Pi_{h} u\right)=\mathcal{B}_{\varepsilon}\left(u-\Pi_{h} u, u-\Pi_{h} u-v\right) \\
& =\mathcal{B}_{\varepsilon}\left(u-\Pi_{h} u, u-\widetilde{v}\right) \\
& \lesssim\left\|u-\Pi_{h} u\right\|_{E, I}\|u-\widetilde{v}\|_{E, I}
\end{aligned}
$$

with $\widetilde{v}=\Pi_{h} u-v \in V_{h}$ arbitrary. Hence, with $u^{I}$ the $p^{t h}$ degree interpolant of $u$ on the exponential mesh, we have by Lemma 6

$$
\begin{equation*}
\left\|u-\Pi_{h} u\right\|_{E, I} \lesssim\|u-\widetilde{v}\|_{E, I} \lesssim\left\|u-u^{I}\right\|_{E, I} \lesssim N^{-p+1} \lesssim h^{p-1} . \tag{21}
\end{equation*}
$$

The above will be utilized in establishing the following result for the approximation of the eigenvalues.

Theorem 7. Let $\lambda_{k}$, $u_{k}$ be the solution of (3) and $\lambda_{k}^{h}, u_{k}^{h}$ the solution of (5) on the eXp mesh. Assuming $\left\langle u_{k}, u_{k}\right\rangle_{I}=1=\left\langle u_{k}^{h}, u_{k}^{h}\right\rangle_{I}$ as well as $\left\langle u_{k}, u_{k}^{h}\right\rangle_{I}>0$, we have for all $h \leq h_{0}$, with $h_{0}$ independent of $\varepsilon$, the bound

$$
\lambda_{k} \leq \lambda_{k}^{h} \lesssim \bar{C}(k) \lambda_{k}\left(1+h^{2 p-2}\right)
$$

with $\bar{C}(k)$ independent of $\varepsilon$.

Proof. The proof follows [3, Sec. 2.8] and [22, Ch. 6]. Let $k$ be fixed. Using (18) in (17) gives

$$
\lambda_{k}^{h} \leq \max _{w \in E^{h}} \frac{\mathcal{B}_{\varepsilon}(w, w)}{\langle w, w\rangle_{I}}=\max _{v \in V^{(k)}} \frac{\mathcal{B}_{\varepsilon}\left(\Pi_{h} v, \Pi_{h} v\right)}{\left\langle\Pi_{h} v, \Pi_{h} v\right\rangle_{I}} .
$$

Note that

$$
\mathcal{B}_{\varepsilon}\left(\Pi_{h} v, \Pi_{h} v\right)=\mathcal{B}_{\varepsilon}(v, v)+2 \mathcal{B}_{\varepsilon}\left(\Pi_{h} v, \Pi_{h} v-v\right)-\mathcal{B}_{\varepsilon}\left(\Pi_{h} v-v, \Pi_{h} v-v\right)
$$

with the last term positive and the second to last zero. Thus,

$$
\mathcal{B}_{\varepsilon}\left(\Pi_{h} v, \Pi_{h} v\right) \leq \mathcal{B}_{\varepsilon}(v, v)
$$

Writing

$$
v=\sum_{i=1}^{k} c_{i} u_{i}, c_{i} \in \mathbb{R}
$$

we have

$$
\begin{aligned}
\mathcal{B}_{\varepsilon}\left(\Pi_{h} v, \Pi_{h} v\right) & \leq \mathcal{B}_{\varepsilon}\left(\sum_{i=1}^{k} c_{i} u_{i}, \sum_{j=1}^{k} c_{j} u_{j}\right)=\sum_{i=1}^{k} c_{i}^{2} \mathcal{B}_{\varepsilon}\left(u_{i}, u_{i}\right)=\sum_{i=1}^{k} c_{i}^{2} \lambda_{i}\left\langle u_{i}, u_{i}\right\rangle_{I} \\
& \leq \sum_{i=1}^{k} c_{i}^{2} \lambda_{i} \leq C(k) \lambda_{k}
\end{aligned}
$$

and thus,

$$
\lambda_{k}^{h} \leq C(k) \lambda_{k} \max _{v \in V^{(k)}} \frac{1}{\left\|\Pi_{h} v\right\|_{0, I}^{2}}
$$

Note that

$$
\|v\|_{0, I}^{2}=\langle v, v\rangle_{I}=\left\langle\sum_{i=1}^{k} c_{i} u_{i}, \sum_{j=1}^{k} c_{j} u_{j}\right\rangle_{I}=\sum_{i=1}^{k} c_{i}^{2}\left\langle u_{i}, u_{i}\right\rangle_{I}=\sum_{i=1}^{k} c_{i}^{2}=C(k)
$$

Moreover,

$$
\left\|v-\Pi_{h} v\right\|_{0, I}^{2}=\left\langle v-\Pi_{h} v, v-\Pi_{h} v\right\rangle_{I}=\|v\|_{0, I}^{2}-2\left\langle v, \Pi_{h} v\right\rangle_{I}+\left\|\Pi_{h} v\right\|_{0, I}^{2}
$$

hence,

$$
\left\|\Pi_{h} v\right\|_{0, I}^{2}=\left\|v-\Pi_{h} v\right\|_{0, I}^{2}-C(k)+2\left\langle v, \Pi_{h} v\right\rangle_{I}
$$

The term $\left\|v-\Pi_{h} v\right\|_{0, I}^{2}$ may be handled by Lemma 6 . For the term $\left\langle v, \Pi_{h} v\right\rangle_{I}$, we have

$$
\begin{aligned}
\left|\left\langle v, \Pi_{h} v\right\rangle_{I}\right| & =\left|\sum_{i=1}^{k} c_{i}\left\langle u_{i}, \Pi_{h} v\right\rangle_{I}\right| \leq \sum_{i=1}^{k}\left|c_{i}\right|\left|\left\langle u_{i}, \Pi_{h} v\right\rangle_{I}\right| \leq \sum_{i=1}^{k}\left|c_{i}\right|\left|\lambda_{i}^{-1} \mathcal{B}_{\varepsilon}\left(u_{i}, \Pi_{h} v\right)\right| \\
& \leq \sum_{i=1}^{k}\left|c_{i}\right|\left|\lambda_{i}^{-1}\right|\left|\mathcal{B}_{\varepsilon}\left(u_{i}-\Pi_{h} u_{i}, v-\Pi_{h} v\right)\right| \leq \sum_{i=1}^{k}\left|c_{i}\right|\left|\lambda_{i}^{-1}\right|\left\|u_{i}-\Pi_{h} u_{i}\right\|_{E, I}\left\|v-\Pi_{h} v\right\|_{E, I} \\
& \leq \sum_{i=1}^{k}\left|\frac{c_{i}}{\lambda_{i}}\right| h^{2 p-2} \lesssim\left[\sum_{i=1}^{k} \frac{c_{i}^{2}}{\lambda_{i}^{2}}\right]^{1 / 2} h^{2 p-2}=\tilde{C}(k) h^{2 p-2},
\end{aligned}
$$

where Galerkin orthogonality and the coercivity of the bilinear form were used. Since

$$
\left\|\Pi_{h} v\right\|_{0, I}^{2} \geq \max _{v \in V^{(k)}}\left|2\left\langle v, \Pi_{h} v\right\rangle_{I}+\left\|v-\Pi_{h} v\right\|_{0, I}^{2}\right|-C(k)
$$

we obtain

$$
\left\|\Pi_{h} v\right\|_{0, I}^{2} \gtrsim \hat{C}(k)\left(h^{2 p-2}-1\right),
$$

with $\hat{C}(k)=\min \{1, C(k), \tilde{C}(k)\}$. This gives

$$
\lambda_{k}^{h} \leq C(k) \lambda_{k} \frac{1}{\hat{C}(k)\left(h^{2 p-2}-1\right)} \lesssim \bar{C}(k) \lambda_{k}\left(1+2 h^{2 p-2}\right),
$$

as desired.

For the approximation of the eigenfunctions, we have the following result, under the assumption that all eigenvalues are distinct.

Theorem 8. Let $\lambda_{k}$, $u_{k}$ be the solution of (3) and $\lambda_{k}^{h}, u_{k}^{h}$ the solution of (5) on the eXp mesh. Assume that $\left\langle u_{k}, u_{k}\right\rangle_{I}=1=\left\langle u_{k}^{h}, u_{k}^{h}\right\rangle_{I},\left\langle u_{k}, u_{k}^{h}\right\rangle_{I}>0$ and that all eigenvalues are distinct. Then,

$$
\left\|u_{k}-u_{k}^{h}\right\|_{E, I} \lesssim C(k) h^{p-1}
$$

with $C(k) \in \mathbb{R}$ independent of $\varepsilon, u$ and $p$.

Proof. We again follow [3] (see also [22]), and introduce the following quantity:

$$
\rho_{k}^{h}=\max _{k \neq j} \frac{\left|\lambda_{k}\right|}{\left|\lambda_{k}-\lambda_{j}^{h}\right|} .
$$

We also consider the $L^{2}$ projection of $\Pi_{h} u_{k}$ onto $\operatorname{span}\left\{u_{k}^{h}\right\}$,

$$
\begin{equation*}
w_{k}^{h}=\left\langle\Pi_{h} u_{k}, u_{k}^{h}\right\rangle_{I} u_{k}^{h} \tag{22}
\end{equation*}
$$

which we use as follows:

$$
\begin{equation*}
\left\|u_{k}-u_{k}^{h}\right\|_{0, I} \leq\left\|u_{k}-\Pi_{h} u_{k}\right\|_{0, I}+\left\|\Pi_{h} u_{k}-w_{k}^{h}\right\|_{0, I}+\left\|w_{k}^{h}-u_{k}^{h}\right\|_{0, I} \tag{23}
\end{equation*}
$$

The first term in (23) is estimated using Lemma 6. To deal with the second term, note that

$$
\Pi_{h} u_{k}-w_{k}^{h}=\sum_{j \neq k}\left\langle\Pi_{h} u_{k}, u_{j}^{h}\right\rangle_{I} u_{j}^{h}
$$

which gives

$$
\begin{equation*}
\left\|\Pi_{h} u_{k}-w_{k}^{h}\right\|_{0, I}^{2}=\sum_{j \neq k}\left\langle\Pi_{h} u_{k}, u_{j}^{h}\right\rangle_{I}^{2} \tag{24}
\end{equation*}
$$

We have

$$
\left\langle\Pi_{h} u_{k}, u_{j}^{h}\right\rangle_{I}=\frac{1}{\lambda_{j}^{h}} \mathcal{B}_{\varepsilon}\left(\Pi_{h} u_{k}, u_{j}^{h}\right)=\frac{1}{\lambda_{j}^{h}} \mathcal{B}_{\varepsilon}\left(u_{k}, u_{j}^{h}\right)=\frac{\lambda_{k}}{\lambda_{j}^{h}}\left\langle u_{k}, u_{j}^{h}\right\rangle_{I}
$$

hence

$$
\lambda_{j}^{h}\left\langle\Pi_{h} u_{k}, u_{j}^{h}\right\rangle_{I}=\lambda_{k}\left\langle u_{k}, u_{j}^{h}\right\rangle_{I}
$$

We subtract $\lambda_{k}\left\langle\Pi_{h} u_{k}, u_{j}^{h}\right\rangle_{I}$ from both sides above and we get

$$
\left(\lambda_{j}^{h}-\lambda_{k}\right)\left\langle\Pi_{h} u_{k}, u_{j}^{h}\right\rangle_{I}=\lambda_{k}\left\langle u_{k}-\Pi_{h} u_{k}, u_{j}^{h}\right\rangle_{I}
$$

which in turn gives

$$
\left|\left\langle\Pi_{h} u_{k}, u_{j}^{h}\right\rangle_{I}\right| \leq \rho_{k}^{h}\left|\left\langle u_{k}-\Pi_{h} u_{k}, u_{j}^{h}\right\rangle_{I}\right| .
$$

From (24) we have

$$
\begin{equation*}
\left\|\Pi_{h} u_{k}-w_{k}^{h}\right\|_{0, I}^{2} \leq\left(\rho_{k}^{h}\right)^{2} \sum_{j \neq k}\left\langle u_{k}-\Pi_{h} u_{k}, u_{j}^{h}\right\rangle_{I}^{2} \leq\left(\rho_{k}^{h}\right)^{2}\left\|u_{k}-\Pi_{h} u_{k}\right\|_{0, I}^{2} \tag{25}
\end{equation*}
$$

To deal with the last term in (23), we point out that if we establish

$$
\begin{equation*}
\left\|u_{k}^{h}-w_{k}^{h}\right\|_{0, I} \leq\left\|u_{k}-w_{k}^{h}\right\|_{0, I} \tag{26}
\end{equation*}
$$

then

$$
\begin{equation*}
\left\|u_{k}^{h}-w_{k}^{h}\right\|_{0, I} \leq\left\|u_{k}-\Pi_{h} u_{k}\right\|_{0, I}+\left\|\Pi_{h} u_{k}-w_{k}^{h}\right\|_{0, I} \tag{27}
\end{equation*}
$$

with both terms on the right hand side above having been estimated. From (22) we have

$$
u_{k}^{h}-w_{k}^{h}=u_{k}^{h}\left(1-\left\langle\Pi_{h} u_{k}, u_{k}^{h}\right\rangle_{I}\right)
$$

Also

$$
\left\|u_{k}\right\|_{0, I}=\left\|u_{k}^{h}-w_{k}^{h}\right\|_{0, I} \leq\left\|w_{k}^{h}\right\|_{0, I} \leq\left\|u_{k}\right\|_{0, I}+\left\|u_{k}^{h}-w_{k}^{h}\right\|_{0, I}
$$

and since $u_{k}, u_{k}^{h}$ are normalized, we have

$$
1-\left\|u_{k}-w_{k}^{h}\right\|_{0, I} \leq\left|\left\langle\Pi_{h} u_{k}, u_{k}^{h}\right\rangle_{I}\right| \leq 1+\left\|u_{k}-w_{k}^{h}\right\|_{0, I}
$$

from which we see that

$$
\left|\left|\left\langle\Pi_{h} u_{k}, u_{k}^{h}\right\rangle_{I}\right|-1\right| \leq\left\|u_{k}-w_{k}^{h}\right\|_{0, I}
$$

By choosing

$$
\left\langle\Pi_{h} u_{k}, u_{k}^{h}\right\rangle_{I} \geq 0
$$

we conclude that (26) is satisfied. Utilizing (23), (25) and (27) we conclude that there is an appropriate choice of the sign of $u_{k}^{h}$ such that

$$
\left\|u_{k}-u_{k}^{h}\right\|_{0, I} \leq 2\left(1+\rho_{k}^{h}\right)\left\|u_{k}-\Pi_{h} u_{k}\right\|_{0, I} \lesssim C(k) h^{p+1}
$$

To get the energy norm estimate we proceed as follows:

$$
\begin{aligned}
\left\|u_{k}-u_{k}^{h}\right\|_{E, I}^{2} & \lesssim \mathcal{B}_{\varepsilon}\left(u_{k}-u_{k}^{h}, u_{k}-u_{k}^{h}\right)=\mathcal{B}_{\varepsilon}\left(u_{k}, u_{k}\right)-2 \mathcal{B}_{\varepsilon}\left(u_{k}, u_{k}^{h}\right)+\mathcal{B}_{\varepsilon}\left(u_{k}^{h}, u_{k}^{h}\right) \\
& =\lambda_{k}-2 \lambda_{k}\left\langle u_{k}, u_{k}^{h}\right\rangle_{I}+\lambda_{k}^{h}=\lambda_{k}\left[1-2\left\langle u_{k}, u_{k}^{h}\right\rangle_{I}\right]-\lambda_{k}+\lambda_{k}^{h} \\
& =\lambda_{k}\left\|u_{k}-u_{k}^{h}\right\|_{0, I}^{2}+\lambda_{k}^{h}-\lambda_{k} \\
& \lesssim C(k) h^{2(p-1)} .
\end{aligned}
$$

This completes the proof.

## 5 Numerical results

In this section we present the results of numerical computations for the approximation of (11) by cubic Hermite polynomials (i.e. $p=3$ ) in the case when the data is chosen as $a(x)=e^{x}, b(x)=x$. Since no exact solution is available, we use a reference solution for the calculation of the errors computed with higher accuracy. First we would like to verify the result of Theorem 7, so in Figure 2 we show the estimated percentage relative error in the first two (smallest) eigenvalues, $100 \times\left|\lambda_{i}-\lambda_{i}^{h}\right| /\left|\lambda_{i}\right|, i=1,2$ versus the number of degrees of freedom $D O F$ (i.e. the dimension of the subspace) in a log-log scale. We used $p=3$ and the resulting lines have slope $-4(=-2 p+2)$, just as Theorem 7 predicts.


Figure 2: Estimated convergence in $\lambda_{1}$ (left) and $\lambda_{2}$ (right).
We also show in Table 1 the computations for the first 5 eigenvalues, for $\varepsilon=10^{-6}$ (the same behavior was noticed for other values of $\varepsilon$ ). We see that for larger eigenvalues the convergence takes longer to set in, as was also observed for non-singularly perturbed eigenvalue problems (see, e.g. [3]). In Figure 3 we illustrate this phenomenon, by comparing the convergence between $\lambda_{1}$ and $\lambda_{5}$, for $\varepsilon=10^{-3}, 10^{-6}$. As can be seen, while $\varepsilon \rightarrow 0$ does not affect the behavior (after all, the method is proven to be robust), there is a clear difference between the case $\lambda_{1}$ and the case $\lambda_{5}$, which suggests that the constants $C(k)$ in Theorem 7, grow with $k$.

| $\lambda_{i}^{N} \backslash D O F$ | 2 | 8 | 14 | 20 | 26 | 32 | 38 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\lambda_{1}^{N}$ | 22.1093 | 16.6812 | 16.6803 | 16.6801 | 16.6801 | 16.6801 | 16.6801 |
| $\lambda_{2}^{N}$ | 94.9592 | 64.6500 | 64.5403 | 64.5203 | 64.5148 | 64.5130 | 64.5122 |
| $\lambda_{3}^{N}$ | - | 145.7632 | 144.7402 | 144.3536 | 144.2593 | 144.2278 | 144.2149 |
| $\lambda_{4}^{N}$ | - | 264.6963 | 258.3972 | 257.0769 | 256.2126 | 255.9574 | 255.8615 |
| $\lambda_{5}^{N}$ | - | 423.2341 | 410.7243 | 402.9403 | 401.7117 | 400.1930 | 399.6647 |

Table 1: Approximate eigenvalues for $\varepsilon=10^{-6}$.


Figure 3: Convergence comparison for $\lambda_{1}$ and $\lambda_{5}$.
We now turn our attention to the eigenfunctions: Figure 4 shows the first two approximate eigenfunctions $u_{1}^{h}, u_{2}^{h}$ and their derivatives. The computations shown were performed for $\varepsilon=10^{-3}$ and $p=3$ with $N=32$ nodal points. We see the boundary layers being present in the derivatives and how the proposed method is able to capture them.


Figure 4: Approximate eigenvectors (left) and their derivatives (right) for $\varepsilon=10^{-3}$.
In terms of convergence, we compute the percentage relative error in the energy norm

$$
\text { Error }=100 \times \frac{\left\|u_{i}-u_{i}^{h}\right\|_{E, I}}{\left\|u_{i}\right\|_{E, I}}
$$

and plot it versus the number of $D O F$, in a $\log -\log$ scale. We do so for $i=1$ and show the result in Figure 5. The slope is approximately $-2(=p-1)$, which verifies the prediction of Theorem 8 .


Figure 5: Energy norm convergence for the fist eigenfunction.

We also consider the error in the first eigenfunction and its derivative measured in a 'discrete maximum norm', defined as

$$
\text { error }=100 \times \frac{\max _{x_{\ell} \in[0,1]}| | u_{1}\left(x_{\ell}\right)\left|-\left|u_{1}^{h}\left(x_{\ell}\right)\right|\right|}{\left|u_{1}\left(x_{\ell}\right)\right|} .
$$

The points $x_{\ell} \in[0,1]$ are chosen so that we have equal number of points in the layer regions and outside - we used 1000 point in each. This is not covered by our theory, so it may be seen as an extension of our results. Figure 6 shows the convergence rate which seems to be robust and of order $O\left(h^{p}\right)$ for the eigenvector and $O\left(h^{p-1}\right)$ for its derivative.


Figure 6: Discrete maximum norm convergence for the fist eigenfunction (left) and its first derivative (right).

## 6 Conclusions

We considered a singularly perturbed fourth order eigenvalue problem and the numerical approximation of its solution using the $h$-version FEM with Hermite polynomials of degree $p \geq 3$ defined on an exponentially graded mesh. We established optimal, uniform (in $\varepsilon$ ) convergence for both the eigenvalues and the eigenfunctions, when the error was measured in absolute value and in the energy norm, respectively. We should point out that a smallness assumption on $h$ is necessary to establish our results and this is seen in our numerical experiments, especially for higher modes. While the analysis was performed in one-dimension, the results are extendable to higher dimensions, since the boundary layer effect is one-dimensional (in the direction normal to the boundary). Unfortunately, constructing $C^{1}$ elements in twodimensions is difficult - even for simple domains. Some progress has been made [24], but we believe that a mixed formulation is a viable alternative choice. This is the focus of our current research efforts.

## References

[1] M. B. Allen III and E. L. Isaacson, Numerical Analysis for Applied Science, Wiley \& Sons, 1998.
[2] N. S. Bakhvalov, Towards optimization of methods for solving boundary value problems in the presence of boundary layers (in Russian), Zh. Vychisl. Mat. Mat. Fiz. 9 (1969) 841-859.
[3] D. Boffi, Finite element approximation of eigenvalue problems, Acta Numerica 19 (2010) 1-120.
[4] P. Constantinou, S. Franz, L. Ludwig and C. Xenophontos, Finite element approximation of reaction-diffusion problems using an exponentially graded mesh, Comp. Math. Appl., 76 (2018) 2523-2534.
[5] P. Constantinou and C. Xenophontos, Finite element analysis of an exponentially graded mesh for singularly perturbed problems, Comp. Meth. Appl. Math. 15 (2015) 135-143.
[6] Y. Farjoun and D. G. Schaeffer, The hanging thin rod: A singularly perturbed eigenvalue problem, arXiv:1008.1912v1, 2010.
[7] P. Constantinou, C. Varnava and C. Xenophontos, An hp finite element method for fourth order singularly perturbed problems, Num. Alg. 73 (2016) 567-590.
[8] S. Franz and H.-G. Roos, Robust error estimation in energy and balanced norms for singularly perturbed fourth order problems, Comp. Math. Appl., 72 (2016), pp. 233-247.
[9] S. Franz and H.-G. Roos, Error Estimates in Balanced Norms of Finite Element Methods for Higher Order Reaction-Diffusion problems, Int. J. Numer. Anal. Mod., 17 (2020), pp. 532-542.
[10] S. Franz and C. Xenophontos, On a connection between layer-adapted exponentially graded and S-type meshes, Comp. Meth. Appl. Math. 18 (2017) 199-203.
[11] T. Linß, Layer-adapted meshes for reaction-convection-diffusion problems, Lecture Notes in Mathematics 1985, Springer-Verlag, 2010.
[12] J. J. H. Miller, E. O'Riordan and G. I. Shishkin, Fitted Numerical Methods Singular Perturbation Problems, World Scientific, 1996.
[13] K. W. Morton, Numerical Solution of Convection-Diffusion Problems, Volume 12 of Applied Mathematics and Mathematical Computation, Chapman \& Hall, 1996.
[14] J. Moser, Singular perturbation of eigenvalue problems for linear differential equations of even order, Comm. Pure Appl. Math., 8 (1955), pp. 251-278.
[15] P. Panaseti, A. Zouvani, N. Madden and C. Xenophontos, A $C^{1}$-conforming hp finite element method for fourth order singularly perturbed boundary value problems, Appl. Num. Math., 104 (2016) 81-97.
[16] H.-G. Roos, A uniformly convergent discretization method for a singularly perturbed boundary value problem of the fourth order, Review of Research, Faculty of Science, Mathematics Series, Univ. Novi Sad 19, (1989) 51-64.
[17] H.-G. Roos, A uniformly convergent scheme for a singularly perturbed eigenvalue problem, Proceedings of the International Conference on Boundary and Interior LayersComputational and Asymptotic Methods, BAIL 2004, ONERA, Toulouse, 5th-9th July 2004.
[18] H.-G. Roos, M. Stynes, A uniformly convergent discretization method for a fourth order singular perturbation problem, Bonn. Math. Schr. 228 (1991) 30-40.
[19] H.-G. Roos, M. Stynes, and L. Tobiska, Robust numerical methods for singularly perturbed differential equations. Convection-diffusion-reaction and flow problems. Volume 24 of Springer Series in Computational Mathematics, Springer-Verlag, 2008.
[20] C. Schwab and M. Suri, The p and hp versions of the finite element method for problems with boundary layers, Math. Comp. 65 (1996) 1403-1429.
[21] G. I. Shishkin, Grid approximation of singularly perturbed boundary value problems with a regular boundary layer, Sov. J. Numer. Anal. Math. Model. 4 (1989) 397-417.
[22] G. Strang, and G. Fix, An analysis of the finite element method, Prentice Hall, 1973.
[23] G. Sun, M. Stynes, Finite-element methods for singularly perturbed high order elliptic two point boundary value problems I: reaction-diffusion-type problems, IMA J. Numer. Anal., 15 (2005) 117-139.
[24] Y. Wu, Y. Xing and B. Liu, Hierarchical p-version $C^{1}$ finite elements on quadrilateral and triangular domains with curved boundaries and their applications to Kirchhoff plate, Int. J. Numer. Meth. Eng., 119 (2019) 177-207.
[25] C. Xenophontos, The hp finite element method for singularly perturbed problems, Ph.D. Dissertation, University of Maryland, Baltimore Co, 1996.
[26] C. Xenophontos, A parameter robust finite element method for fourth order singularly perturbed problems, Comp. Meth. Appl. Math. 17 (2017) 337-350.
[27] C. Xenophontos, S. Franz and L. Ludwig, Finite element approximation of convectiondiffusion problems using an exponentially graded mesh, Computers and Mathematics with Applications, 72 (2016) 1532-1540.


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