Cylinders, multi-cylinders and the induced action of $\mathrm{Aut}(F_n)$

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Abstract

A cylinder C^1_u is the set of infinite words with fixed prefix u. A double-cylinder $C^2_{[1,u]}$ is "the same" for bi-infinite words. We show that for every word u and any automorphism φ of the free group F the image $\varphi(C^1_u)$ is a finite union of cylinders. The analogous statement is true for double cylinders. We give (a) an algorithm, and (b) a precise formula which allows one to determine this finite union of cylinders.

1 Introduction

This paper goes back to a remark of a rather well known member of the "Outer space" community, who some years ago during a talk in Bonn explained that rational currents are dense in the space of currents, but that, other than using this fact and a bit of approximation, she didn't know how to compute the image of a current under the induced action of an automorphisms φ of a finitely generated free group F.

By definition, a current μ is a measure on the *double boundary* $\partial^2 F$, i.e. the space $\partial F \times \partial F$ minus the diagonal. The image measure $\varphi_*(\mu)$, of course, is simply the measure μ evaluated on the preimages of subsets of $\partial^2 F$ under the homeomorphisms induced by φ. The problem, it turns out, is that even for the simplest sets in $\partial^2 F$, the so called *double cylinders* $C^2_{[u,v]}$ (see Definition 5.4), given by two distinct elements $u, v \in F$ and the choice of a basis A of F, it is not at all evident how to describe $\varphi(C^2_{[u,v]})$ (or $\varphi^{-1}(C^2_{[u,v]})$). For example, using the results of this paper, it is easy to give examples of double cylinders with $\varphi(C^2_{[u,v]}) \neq C^2_{[\varphi(u),\varphi(v)]}$.

Indeed, we prove here (see §5):

Theorem 1.1. *Let φ be an automorphism of the free group F with finite basis A. For any $u, v \in F$ with $u \neq v$ there exist finite sets $U, V \subset F$ such that:*

$$\varphi(C^2_{[u,v]}) = \dot{\bigcup_{\substack{u_i \in U \\ v_j \in V}}} C^2_{[u_i,v_j]}$$

The sets U and V can be algorithmically derived from $u, v \in F$ and from the elements of $\varphi(A)$ and of $\varphi^{-1}(A)$, all expressed as reduced words in $A \cup A^{-1}$.

To simplify the arguments, one considers first one-sided cylinders $C^1_w \subset \partial F$: they too depend on the chosen basis A of F, since one has to pass from the element $w \in F$ to the corresponding element of $F(A)$, by which we denote the set of reduced words in $A \cup A^{-1}$. One thus obtains C^1_w as the set of all elements of ∂F that are represented by one-sided infinite reduced words in $A \cup A^{-1}$ which have w as prefix. We also need to consider multi-cylinders $C^1_U = \bigcup_{u \in U} C^1_u$ for finite subsets $U \subset F$. In §4 below we show:

Theorem 1.2. *Let φ be an automorphism of the free group F with finite basis A.*

(a) For any $u \in F(A)$ there exists a finite set $U \subset F(A)$ such that:

$$\varphi(C^1_u) = C^1_U$$

(b) A set U as in statement (a) can be algorithmically derived from $u \in F(A)$ and from the words in the finite subsets $\varphi(A)$ and $\varphi^{-1}(A)$ of $F(A)$. Indeed, the equality in (a) is true for

$$U = \{\varphi(u')|_{S(\varphi)^2} \mid u' \in u|^k\},$$

with $k = S(\varphi)^4 + S(\varphi)^3 + S(\varphi)^2$, where $S(\varphi)$ is the maximal length of any $\varphi(a_i)$ or $\varphi^{-1}(a_i)$ among all $a_i \in A$, see §2.

Here for any reduced word $w \in F(A)$ and any integer $l \geq 0$ we denote by $w|_l$ the word obtained from w by erasing the last l letters, and by $w|^l$ the set of reduced words obtained from w by adding l letters from $A \cup A^{-1}$ at the end of w.

The set U from the above Theorem 1.2 is not uniquely determined by u, A and φ: The set U exhibited in part (b) is only one of infinitely many finite subsets $U' \subset F(A)$ which all satisfy the equality $\varphi(C^1_u) = C^1_{U'}$ from part (a).

This non-uniqueness can be easily understood by considering the following two typical examples, given by the pairs $U_1 = \{ab, aba^{-1}\}, U_2 = \{ab\}$ and by $U_3 = \{aba^{-1}, aba, abb\}, U_4 = \{ab\}$, which satisfy $C^1_{U_1} = C^1_{U_2}$ and $C^1_{U_3} = C^1_{U_4}$. The resulting ambiguity is resolved by the following proposition, which is proved below in §3:

Proposition 1.3. *For every multi-cylinder C^1_U, determined by a finite set $U \subset F(A)$, there is a unique finite subset $U_{\min} \subset F(A)$ of minimal cardinality which determines the same multi-cylinder:*

$$C^1_{U_{\min}} = C^1_U$$

The set $U_{\min}$ can be derived algorithmically from U by a finite sequence of elementary operations (of two types, illustrated by the two examples presented in the previous paragraph), each of which strictly decreases the cardinality.

This enables us to define a map φ^*_A on elements (and on finite subsets of $F(A)$) by associating to $u \in F(A)$ the minimal set $U_{\min}$ for the multi-cylinder $\varphi(C_u)$:

the set $U_{\min}$ can be derived algorithmically from any finite set $U \in F(A)$ as in Theorem 1.2, with $C_U^1 = \varphi(C_u)$.

We can thus reformulate and specify the main case of Theorem 1.1 slightly, by stating (see §5):

Proposition 1.4. *Let $u, v \in F(A)$ be such that none is prefix of the other. Then one has*

$$\varphi(C^2_{[u,v]}) = \dot{\bigcup_{\substack{u_i \in \varphi_A^*(u) \\ v_j \in \varphi_A^*(v)}}} C^2_{[u_i, v_j]}$$

The extra hypothesis in the last proposition is necessary since double cylinders behave properly under the action of F on the indices (see Lemma 5.7), while for a single cylinder C_u^1 one has $wC_u^1 = C_{wu}^1$ only if u is not a prefix of w^{-1}. For a general formula see Remark 5.10.

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2 Notation, set-up and basic facts

Throughout this paper we denote by F a finitely generated non-abelian free group, and by φ an automorphism of F. We choose a basis A of F once and for all, which allows us to identify F with the set $F(A)$ of finite reduced words in the elements of A and their inverses. We denote by $\partial F(A)$ the set of infinite reduced words:

$$\partial F(A) = \{x_1 x_2 x_3 \cdots \mid x_i \in A \cup A^{-1},\ x_i \neq x_{i+1}^{-1}\}$$

The set $\partial F(A)$ is in a canonical bijective correspondence with the *end completion* ∂F of F. The latter also coincides with the *Gromov boundary* of F. The set ∂F (and thus $\partial F(A)$) carries a topology; indeed it is homeomorphic to a Cantor set. Every automorphism φ of F induces canonically a homeomorphism of ∂F, which for simplicity we denote also by φ. For background and details about these classical facts see [2].

The *word length* of an element $w \in F(A)$ with respect to A will be denoted by $|w|_A$ or simply by $|w|$. We write $v \leq w$, if v is a prefix (= initial subword) of w, and we write $v < w$ if in addition one has $|v| < |w|$. This puts a partial ordering on F (which heavily depends on A). The longest prefix common to elements w_1 and w_2 of $F(A) \cup \partial F(A)$ is denoted $w_1 \wedge w_2$. One has $|w_1^{-1} \wedge w_2| = 0$ if and only if the product $w_1 w_2$ is reduced; in this case we denote $w_1 w_2$ by $w_1 \cdot w_2$.

The *size* of an automorphism $\varphi \in \mathrm{Aut}(F)$ (with respect to A) is defined by

$$S(\varphi) := S_A(\varphi) := \max_{a \in A \cup A^{-1}} \{|\varphi(a)|, |\varphi^{-1}(a)|\}$$

We obtain directly from this definition:

Lemma 2.1. *For any $w \in F(A)$ and any $\varphi \in \mathrm{Aut}(F)$ one has:*

$$\frac{|w|}{S(\varphi)} \ \leq\ |\varphi(w)| \ \leq\ |w| \cdot S(\varphi)$$

□

The following is a classical result of D. Cooper, see [3].

Proposition 2.2. *Let φ be an automorphism of the finitely generated free group F, and let A be a basis of F. Then there exists a constant $C \geq 0$ such that for any elements $u, v \in F$ one has:*

$$0 \leq |\varphi(u)|_A + |\varphi(v)|_A - |\varphi(uv)|_A \leq C$$

The smallest such constant C will be denoted by $C(\varphi)$.

In the literature the above proposition is sometimes referred to as "bounded cancellation lemma". It follows directly from this proposition that the analogous statement, i.e. the upper bound on the possible cancellation, remains true if u^{-1} or v (or both) are replaced by elements from ∂F, i.e. by infinite words.

Remark 2.3. In [3] it has been shown that for any $\varphi \in \mathrm{Aut}(F)$ the constant $C(\varphi)$ is always bounded above by $S(\varphi)^2$.

Definition 2.4. Let $w = a_1 \cdots a_r \in F(A)$. For any integer $k \geq 0$ we define:

(1) $w|_k = a_1 \cdots a_{r-k} \quad$ (if $k \leq r$), and

(2) $w|^k = \{v \mid w < v \text{ and } |v| = |w| + k\}$

From this definition we obtain directly, for any $u \in F(A)$ and any integers $m, n \geq 0$ with $k = m + n$, that $u|^k = \bigcup\limits_{v \in u|^m} v|^n$.

3 Cylinders and multi-cylinders

It is crucial in this section that one distinguishes between elements of the free group F, with basis $A = \{a_1, \ldots, a_n\}$, and reduced words in the a_i and a_i^{-1} which are used to represent these elements. We denote the set of reduced words by $F(A)$.

Similarly, we denote by $\partial F(A)$ the set of infinite reduced words $X = x_1 x_2 \ldots$ in $A \cup A^{-1}$ which are used to represent the elements of the Gromov boundary ∂F.

We will denote in this section by $\mathbb{U}$ the set of all finite subsets of $F(A)$.

Definition 3.1. For any $u \in F(A)$ we define $C_u^1 = \{X \in \partial F(A) \mid u < X\}$. The set C_u^1 is called the *cylinder* defined by u (and by A).

Remark 3.2. Let $u, v \in F(A)$. Then from the definition of C_u^1 one derives directly:

(1) If $C_u^1 = C_v^1$ then $u = v$.

(2) If $C^1_u \cap C^1_v \neq \emptyset$ then one has $v \leq u$ and thus $C^1_u \subseteq C^1_v$, or else $u \leq v$ and thus $C^1_v \subseteq C^1_u$.

(3) For any integer $k \geq 0$ one has $C^1_u = \dot{\bigcup\limits_{u_i \in u|^k}} C^1_{u_i}$.

From parts (1) and (2) of Remark 3.2 we obtain directly:

Lemma 3.3. *Given $u, u' \in F(A)$ with $|u| = |u'|$, then either $C^1_u \cap C^1_{u'} = \emptyset$, or else $u = u'$ and thus $C^1_u = C^1_{u'}$.* □

Definition 3.4. For any subset $U \subset F(A)$ we will denote by $C^1_U \subset \partial F$ the union of all cylinders C^1_u with $u \in U$:

$$C^1_U = \bigcup_{u_i \in U} C^1_{u_i}$$

From Lemma 3.3 we obtain directly:

Lemma 3.5. *Let $k \in \mathbb{N}$, $U \subset F(A)$ and $|u_i| = k$ for all $u_i \in U$. Then one obtains a disjoint union:*

$$C^1_U = \dot{\bigcup_{u_i \in U}} C^1_{u_i}$$

□

Recall that $\mathbb{U}$ denotes the set of all finite subsets of $F(A)$.

Lemma 3.6. *Let $k \in \mathbb{N}$ and $U, U' \in \mathbb{U}$, and assume for all $u \in U \cup U'$ that $|u| = k$. Then we have $C^1_U = C^1_{U'}$ if and only if $U = U'$.*

Proof. If $U = U'$ then clearly one has $C^1_U = C^1_{U'}$. Conversely, from the hypothesis $|u| = k$ for all $u \in U \cup U'$ we obtain, by Lemma 3.5, that $C^1_U = \dot{\bigcup\limits_{u_i \in U}} C^1_{u_i}$ and $C^1_{U'} = \dot{\bigcup\limits_{u'_j \in U'}} C^1_{u'_j}$. Thus, if $C^1_U = C^1_{U'}$, we obtain $\dot{\bigcup\limits_{u_i \in U}} C^1_{u_i} = \dot{\bigcup\limits_{u'_j \in U'}} C^1_{u'_j}$. From Lemma 3.3 we deduce that for any $C^1_{u_i} \subset C^1_U$ there exists a unique $C^1_{u'_j} \subset C^1_{U'}$ with $C^1_{u_i} = C^1_{u'_j}$ and thus $u_i = u'_j$ (by Remark 3.2 (1)). This shows $U \subset U'$, and from the symmetry between U and U' we obtain $U = U'$. □

We define now an "elementary" relation $\searrow$ on $\mathbb{U}$ as follows:

Definition 3.7. For any $U_1, U_2 \in \mathbb{U}$ we write $U_1 \searrow U_2$ if one of the following conditions is satisfied:

1. There are distinct elements $u_i, u_j \in U_1$ with $u_i < u_j$ such that $U_2 = U_1 \smallsetminus \{u_j\}$. In this case we sometimes specify the notation $U_1 \searrow U_2$ to $U_2 \overset{(1)}{\swarrow} U_1$.

2. There exists an element $u \in F(A) \smallsetminus U_1$ with $u|^1 \subset U_1$, and one has $U_2 = (U_1 \smallsetminus u|^1) \cup \{u\}$. In this case we write sometimes $U_2 \overset{(2)}{\swarrow} U_1$.

For example, let F be a free group with base $A = \{a, b\}$, and let $U = \{aba, abab, bba, bbb, bba^{-1}\}$. Then for $U_1 = \{aba, bba, bbb, bba^{-1}\}$ we have $U \overset{(1)}{\swarrow} U_1$, and for $U_2 = \{aba, bb\}$ we obtain $U_1 \overset{(2)}{\swarrow} U_2$.

Remark 3.8. It is clear that the relation $\searrow$ strictly decreases the cardinality of the given set U:

$$U \searrow U' \implies \#U > \#U'$$

Definition 3.9. For any $U, U' \in \mathbb{U}$ we write $U \sim U'$ if there exists a finite sequence $U_1 = U, U_2, \cdots, U_n = U'$ of elements of $\mathbb{U}$, with $U_i \searrow U_{i+1}$ or $U_{i+1} \searrow U_i$ for all $1 \leq i \leq n-1$.

In other words : The relation $\sim$ is the equivalence relation on $\mathbb{U}$ generated by the elementary relation $\searrow$.

Definition 3.10. We say that $U \in \mathbb{U}$ is reduced if and only if there is no $U' \in \mathbb{U}$ with $U \searrow U'$.

Remark 3.11. (a) For any $U \in \mathbb{U}$ there exists a reduced set $U' \in \mathbb{U}$ with $U \searrow \cdots \searrow U'$. This follows directly from the finiteness of U and from Remark 3.8.

(b) However, it is a priori not clear that the reduced set U' depends only on U and not on the particular way how one choses the reduction $U \searrow \cdots \searrow U'$. To show that in each equivalence class $[U]_\sim$ there is precisely one reduced set U' is the goal of the rest of this section.

Lemma 3.12. *(a) Let $U, U' \in \mathbb{U}$ and assume $U \searrow U'$. Then we have $C^1_U = C^1_{U'}$.*
(b) In particular, if $U \sim U'$ then one has $C^1_U = C^1_{U'}$.

Proof. (a) From the above definition of $\searrow$ we distinguish two cases:
(1) If $U \overset{(1)}{\searrow} U'$ then there exist $u_1, u_2 \in U$ with $u_1 < u_2$ and $U' = U \smallsetminus \{u_2\}$. Thus one has $U = U' \cup \{u_2\}$, and thus $C^1_U = C^1_{U'} \cup C^1_{u_2}$. But $C^1_{u_2} \subset C^1_{u_1} \subset C^1_{U'}$, so that $C^1_U = C^1_{U'}$.
(2) If $U \overset{(2)}{\searrow} U'$ then there exists $u \in F(A)$ with $u \notin U$, $u|^1 \subset U$ and $U' = (U \smallsetminus u|^1) \cup \{u\}$. Thus we have $C^1_U = C^1_{U' \smallsetminus \{u\}} \cup C^1_{u|^1}$ and $C^1_{U'} = C^1_{U \smallsetminus u|^1} \cup C^1_u$. From Remark 3.2 (3) one has $C^1_u = C^1_{u|^1}$, so that the last two equalities give $C^1_U \supset C^1_{U'}$ and $C^1_{U'} \supset C^1_U$, and thus $C^1_U = C^1_{U'}$.
(b) This is a direct consequence of (a), by the definition of $\sim$. $\square$

We now define another elementary relation $\nearrow$ which allows us to extend a set U_1 to a larger set U_2:

For any $U_1, U_2 \in \mathbb{U}$ we write $U_1 \nearrow U_2$ if $u \in U_1$ and $U_2 = U_1 \cup u|^1 \smallsetminus \{u\}$.

Remark 3.13. (a) We observe that $U_1 \nearrow U_2$ does not necessarily imply that $U_2 \overset{(2)}{\searrow} U_1$. For example, if $U_1 = \{b, ba\}$ and $U_2 = \{ba, bb, ba^{-1}\} = b|^1$ then we have $U_1 \nearrow U_2$ and $U_2 \overset{(2)}{\searrow} \{b\} \subsetneqq U_1$.
(b) If $U_1 \nearrow U_2$ then one has $U_1 \sim U_2$. To see this, we observe from $U_1 \nearrow U_2$ that there exists $u \in U_1$ such that $u \notin U_2$ and $u|^1 \subset U_2$. Now we apply $\overset{(2)}{\searrow}$ to obtain $U_2 \overset{(2)}{\searrow} U_2'$, where $U_2' = \{U_2 - u|^1\} \cup \{u\}$. Thus all elements of $U_2 - U_2'$ are contained in $u|^1$. Since $u \in U_2'$, a multiple application of $\overset{(1)}{\searrow}$ yields $U_2 \overset{(1)}{\searrow} \cdots \overset{(1)}{\searrow} U_2'$. This implies $U_1 \sim U_2'$.
(c) In particular, by Lemma 3.12 (b), if $U_1 \nearrow U_2$ then $C^1_{U_1} = C^1_{U_2}$.

Proposition 3.14. *For all $U, U' \in \mathbb{U}$ one has*:

$$C^1_U = C^1_{U'} \iff U \sim U'$$

Proof. If $U \sim U'$ then by Lemma 3.12 (b) we have $C^1_U = C^1_{U'}$. For the converse direction assume $C^1_U = C^1_{U'}$. Let $k = \max\{|u| \mid u \in U \cup U'\}$. We set $U_0 = U$ and define iteratively U_{i+1} from U_i by postulating

$$U_{i+1} = (U_i \smallsetminus \{u\}) \cup u|^1$$

for some $u \in U_i$ with $|u| < k$. Then one obtains $U = U_i \nearrow U_{i+1} \nearrow U_{i+2} \nearrow \cdots \nearrow U_n$, where for all $v \in U_n$ one can assume $|v| = k$. By part (b) of Remark 3.13 we obtain $U \sim U_n$ and thus $C^1_U = C^1_{U_n}$.

We do the same for U' to find $U' = U'_0 \nearrow U'_1 \nearrow \cdots \nearrow U'_m$, where for all $v' \in U'_m$ one has $|v'| = k$. Again we obtain $U' \sim U'_m$ and thus $C^1_{U'} = C^1_{U'_m}$. But we assumed $C^1_U = C^1_{U'}$, which gives $C^1_{U_n} = C^1_{U'_m}$ and thus, by Lemma 3.13 , $U_n = U'_m$. This gives $U \sim U_n = U'_m \sim U'$ and hence $U \sim U'$. □

Definition 3.15. For any subset $B \subset \partial F(A)$ we define

$$U^*(B) = \{u \in F(A) \mid C^1_u \subset B \text{ and } C^1_{u|_1} \not\subset B\}.$$

For $U \in \mathbb{U}$ we write $U^* := U^*(C^1_U) \in \mathbb{U}$.

Remark 3.16. From Definition 3.15 we obtain directly:
(a) If $U, V \in \mathbb{U}$, with $C^1_U = C^1_V$, then $U^* = V^*$.
(b) For all $U \in \mathbb{U}$ we have $C^1_{U^*} \subset C^1_U$.
(c) For all $U \in \mathbb{U}$ one has $(U^*)^* = U^*$.

Lemma 3.17. *For any $U \in \mathbb{U}$ one has $C^1_{U^*} = \dot{\bigcup\limits_{u \in U^*}} C^1_u$.*

Proof. If, by way of contradiction, we assume $C^1_{U^*} \neq \dot{\bigcup\limits_{u \in U^*}} C^1_u$, then there exist $u_1, u_2 \in U^*$, $u_1 \neq u_2$, with $C^1_{u_1} \cap C^1_{u_2} \neq \emptyset$. By part (2) of Remark 3.2 one has $u_1 < u_2$ or $u_2 < u_1$ and thus $u_1 \leq u_2|_1$ or $u_2 \leq u_1|_1$. This implies $C^1_{u_2|_1} \subset C^1_{u_1} \subset C^1_U$ or $C^1_{u_1|_1} \subset C^1_{u_2} \subset C^1_U$, which contradicts the assumption $u_1, u_2 \in U^*$. Hence we have proved $C^1_{U^*} = \dot{\bigcup\limits_{u \in U^*}} C^1_u$. □

Lemma 3.18. *For each $U \in \mathbb{U}$ there is no $U' \sim U$ with $U' \subsetneqq U^*$.*

Proof. From Lemma 3.17 we know $C^1_{U^*} = \dot{\bigcup}_{u \in U^*} C^1_u$, and from Remark 3.16 (b) we have $C^1_{U*} \subset C^1_U$. On the other hand, $U' \sim U$ implies by Proposition 3.14 the equality $C^1_U = C^1_{U'}$ and thus $C^1_{U^*} \subset C^1_{U'}$. As a consequence, one deduces from $U' \subset U^*$ that $\dot{\bigcup}_{u \in U^*} C^1_u = \dot{\bigcup}_{u \in U'} C^1_u$, which implies $U' = U^*$, since every C^1_u is non-empty. □

Lemma 3.19. *If $U \in \mathbb{U}$ is reduced, then one has $U = U^*$.*

Proof. By way of contraction assume $U \neq U^*$. By Lemma 3.18 this implies that $U - U^*$ is non-empty. Let $n = \max\{|u| \mid u \in U - U^*\}$, and let $u \in U - U^*$ with $|u| = n$. By definition of U^* we have that $C^1_{u|_1} \subset C^1_U$, so that one of the following three properties must hold:

(1) $u|_k \in U$ for some $k \geq 1$.

(2) $u|_1\big|^1 \subset U$.

(3) $u|_k \notin U$ for all $k \geq 1$, and there exists $v \in u|_1\big|^1$ (i.e. $|v| = n$) with $v \notin U$.

The cases (1) and (2) are impossible because U is reduced and $u \in U$. In case (3), since $C^1_v \subset C^1_{u|_1} \subset C^1_U$, there exists $v' \in v\big|^k$, with $k \geq 1$, $|v'| = n + k$, $v' \in U$ and $C^1_{v'} \subset C^1_U$. We deduce $C^1_{v'|_1} \subset C^1_{u|_1} \subset C^1_U$, and thus $v' \in U - U^*$: This contradicts the definition of n above because $|v'| > n$. □

Proposition 3.20. *(a) For every $U \in \mathbb{U}$ there is precisely one reduced set $U_{\min} \in \mathbb{U}$ with $U_{\min} \sim U$.*

(b) In particular, one has $U_{\min} = U^*$ and $C^1_U = C^1_{U_{\min}} = C^1_{U^*}$, and this is the disjoint union of all C^1_u with $u \in U_{\min}$.

Proof. Let $U' \in \mathbb{U}$ be a reduced set with $U \sim U'$. By Remark 3.11 (a) such a set U' exists. By Proposition 3.14 we have $C^1_U = C^1_{U'}$ and thus $U^* = U'^*$. As U' is reduced, by Lemma 3.19 we have $U' = U'^*$ and thus $U' = U^*$. This shows the uniqueness of the set $U' =: U_{\min}$, as well as the equalities stated in claim (b). □

We now obtain Proposition 1.3 stated in the Introduction as an immediate consequence of Remark 3.11, Proposition 3.14 and Proposition 3.20.

4 The φ-image of a cylinder C^1_w

The objective of this section is to determine the image of any cylinder C^1_w, with $w \in F(A)$, under a given automorphism φ of the free group $F(A)$. We will see that there exists a finite set $U \subset F(A)$ of words in A such that

$$\varphi(C^1_w) = \dot{\bigcup_{u \in U}} C^1_u$$

In this section we will first prove the existence of such a finite set U, and in a second step we will define an algorithm that determines U, for any given word $w \in F(A)$ and any automorphism φ of $F(A)$ (given by the finite set of words $\varphi(a_i)$ for any $a_i \in A$).

Remark 4.1. Given $w \in F(A)$, we first note that in general one has:

$$\varphi(C^1_w) \neq C^1_{\varphi(w)}$$

For example, let $F(a, b)$ be the free group with base $\{a, b\}$, and let $\varphi \in \mathrm{Aut}(F(a, b))$, given by:

$$a \mapsto aba \quad , \quad b \mapsto ba$$

We consider $w = ba$ and obtain $\varphi(w) = baaba$, as well as

$$C^1_w = \{baz_1z_2 \cdots \mid z_1 \in \{a, b, b^{-1}\}, z_i \in \{a, b, a^{-1}, b^{-1}\} \smallsetminus \{z_{i-1}^{-1}\} \ \forall i \geq 2\}$$

and

$$C^1_{\varphi(w)} = \{baabaz_1z_2 \cdots \mid z_1 \in \{a, b, b^{-1}\}, z_i \in \{a, b, b^{-1}, a^{-1}\} \smallsetminus \{z_{i-1}^{-1}\} \ \forall i \geq 2\}.$$

Then for $W = bab^{-1}a^{-1}a^{-1}a^{-1}a^{-1} \cdots \in C^1_w$ we obtain

$$\varphi(W) = bab^{-1}a^{-1}a^{-1}b^{-1}a^{-1}a^{-1}b^{-1}a^{-1}a^{-1}b^{-1}a^{-1} \cdots \in \varphi(C^1_w) \ ,$$

and we observe $\varphi(W) \notin C^1_{\varphi(w)}$, which implies $\varphi(C^1_w) \neq C^1_{\varphi(w)}$.

We'd like to thank P. Arnoux for having pointed out to us that a proof of the following statement should be possible along the lines given below in the proof.

Proposition 4.2. *For any $\varphi \in \mathrm{Aut}(F)$ and $w \in F(A)$ there is a finite set $U \subset F(A)$ such that*

$$\varphi(C_w^1) = \bigcup_{u_i \in U} C_{u_i}^1$$

Proof. With respect to its natural topology (see §2) the space ∂F is compact, and for any $u \in F(A)$ the cylinder C_u^1 is open and compact. Since every $\varphi \in \mathrm{Aut}(F)$ induces a homeomorphism on ∂F, for any $u \in F(A)$ the image set $\varphi(C_u^1)$ must also be open and compact. Thus, since the set $\{C_u^1 \mid u \in F\}$ constitutes a basis of the topology of ∂F, it follows from $\varphi(C_u^1)$ open that there is a (potentially infinite) family of $C_{u_i}^1 \subset \varphi(C_u^1)$ which covers all of $\varphi(C_u^1)$. By the compactness of the latter we can extract a finite subfamily $\{C_{u_i}^1 \mid u \in U\}$ which still covers $\varphi(C_u^1)$, while each $C_{u_i}^1$ remains a subset of $\varphi(C_u^1)$. This proves the claim. $\square$

It should be noted that the above proof of Proposition 4.2 has no algorithmic value. Indeed, it does not even allow us to find U by trial and error (unless one first derives an algorithm that verifies the equality of Proposition 4.2 for any given φ, w and U).

Lemma 4.3. *Let $\varphi \in \mathrm{Aut}(F)$ and $w \in F(A)$ with $|w| \geq S(\varphi) \cdot C(\varphi)$.*
Then one has:

$$\varphi(C_w^1) \subset C^1_{\varphi(w)|_{C(\varphi)}}$$

Proof. For all $Z \in C_w^1$ there exists $X \in \partial F(A)$ such that $Z = w \cdot X$ and hence $\varphi(Z) \in \varphi(C_w^1)$ and $\varphi(Z) = \varphi(w)\varphi(X)$. By the definition of $S(\varphi)$ (see §2) we have $|\varphi(w)| \geq \frac{|w|}{S(\varphi)}$, and by assumption we know $|w| \geq S(\varphi) \cdot C(\varphi)$, so that $|\varphi(w)| \geq C(\varphi)$. Thus we can decompose $\varphi(w) = w_1 \cdot w_2$, where $|w_2| = C(\varphi)$ and $w_1 = \varphi(w)|_{C(\varphi)}$. The cancelation between $\varphi(w)$ and $\varphi(X)$ is bounded by $C(\varphi)$ (see Proposition 2.2 and the subsequent paragraph), so that for some decomposition

$w_2 = w_2' \cdot w_2''$ we obtain $\varphi(Z) = w' \cdot X'$ with $w' = w_1 \cdot w_2'$ and $\varphi(X) = w_2'^{-1} \cdot X'$. This shows $\varphi(Z) \in C^1_{w'} \subset C^1_{w_1}$, which in turn proves $\varphi(C^1_w) \subset C^1_{\varphi(w)|_{C(\varphi)}}$. $\square$

Proposition 4.4. *Let $u, u' \in F(A)$, and assume:*

1. $u \leq u'|_k$ *for* $k = S(\varphi) \cdot C(\varphi) + C(\varphi^{-1})$
2. $|\varphi(u')| \geq S(\varphi) \cdot C(\varphi^{-1}) + C(\varphi)$

Then one has:

$$C^1_{\varphi(u')|_{C(\varphi)}} \subset \varphi(C^1_u)$$

Proof. From hypothesis 1. we obtain that $|u'| \geq S(\varphi) \cdot C(\varphi)$, and thus we deduce from Lemma 4.3 that
(I) $\varphi(C^1_{u'}) \subset C^1_{\varphi(u')|_{C(\varphi)}}$.
As a direct consequence we obtain that
(II) $C^1_{u'} = \varphi^{-1}(\varphi(C^1_{u'})) \subset \varphi^{-1}(C^1_{\varphi(u')|_{C(\varphi)}})$.
Now we apply hypothesis 2. to obtain $|\varphi(u')|_{C(\varphi)}| \geq S(\varphi) \cdot C(\varphi^{-1})$. This allows us to again apply Lemma 4.3, with $w = \varphi(u')|_{C(\varphi)}$ and with φ^{-1} instead of φ, to obtain
(III) $\varphi^{-1}(C^1_{\varphi(u')|_{C(\varphi)}}) \subset C^1_{\varphi^{-1}(\varphi(u')|_{C(\varphi)})|_{C(\varphi^{-1})}}$.
From (II) and (III) we deduce
(IV) $C^1_{u'} \subset C^1_{\varphi^{-1}(\varphi(u')|_{C(\varphi)})|_{C(\varphi^{-1})}}$,
which is equivalent to
(V) $\varphi^{-1}(\varphi(u')|_{C(\varphi)})|_{C(\varphi^{-1})} \leq u'$.
By hypothesis 2. we can write $\varphi(u') := u'' \cdot u'''$ with $|u'''| = C(\varphi)$ and $u'' = \varphi(u')|_{C(\varphi)}$. We calculate

$$\begin{aligned} |u'| &= |\varphi^{-1}(u'' \cdot u''')| \\ &\leq |\varphi^{-1}(u'')| + |\varphi^{-1}(u''')| \\ &\leq |\varphi^{-1}(u'')| + S(\varphi) \cdot C(\varphi) \end{aligned}$$

and thus obtain

$$|\varphi^{-1}(u'')| - C(\varphi^{-1}) \geq |u'| - S(\varphi) \cdot C(\varphi) - C(\varphi^{-1}) \, .$$

As $u'' = \varphi(u')|_{C(\varphi)}$, we can rewrite the last inequality as:

$$|\varphi^{-1}\left(\varphi(u')|_{C(\varphi)}\right)| - C(\varphi^{-1}) \geq |u'| - S(\varphi) \cdot C(\varphi) - C(\varphi^{-1})$$

But

$$\left|\varphi^{-1}\left(\varphi(u')|_{C(\varphi)}\right)|_{C(\varphi^{-1})}\right| = \left|\varphi^{-1}\left(\varphi(u')|_{C(\varphi)}\right)\right| - C(\varphi^{-1})$$

so that we obtain $\left|\varphi^{-1}\left(\varphi(u')|_{C(\varphi)}\right)|_{C(\varphi^{-1})}\right| \geq |u'| - k$. Hence we obtain from (V) that $u'|_k \leq \varphi^{-1}(\varphi(u')|_{C(\varphi)})|_{C(\varphi^{-1})}$, and thus from hypothesis 1. that

$$u \leq \varphi^{-1}(\varphi(u')|_{C(\varphi)})|_{C(\varphi^{-1})}.$$

This is equivalent to $C^1_{\varphi^{-1}\left(\varphi(u')|_{C(\varphi)}\right)|_{C(\varphi^{-1})}} \subset C^1_u$. From (III) we then deduce that $\varphi^{-1}(C^1_{\varphi(u')|_{C(\varphi)}}) \subset C^1_u$, which is equivalent to

$$C^1_{\varphi(u')|_{C(\varphi)}} \subset \varphi(C^1_u)$$

□

Proposition 4.5. *Let $u \in F(A)$ with $|u| \geq S^2(\varphi)C(\varphi^{-1}) - C(\varphi^{-1})$, and let $k = S(\varphi) \cdot C(\varphi) + C(\varphi^{-1})$. Then one has:*

$$\varphi(C^1_u) = \bigcup_{u' \in u|^k} C^1_{\varphi(u')|_{C(\varphi)}}$$

Proof. For all $u' \in u|^k$ one has $|u'| \geq k \geq S(\varphi) \cdot C(\varphi)$. Thus by Lemma 4.3 we obtain $\varphi(C^1_{u'}) \subset C^1_{\varphi(u')|_{C(\varphi)}}$. Recall from part (3) of Lemma 3.2 that $C^1_u = \bigcup_{u' \in u|^k} C^1_{u'}$, which gives $\varphi(C^1_u) = \varphi(\bigcup_{u' \in u|^k} C^1_{u'}) = \bigcup_{u' \in u|^k} \varphi(C^1_{u'})$, so that one obtains

1. $\varphi(C^1_u) \subset \bigcup_{u' \in u|^k} C^1_{\varphi(u')|_{C(\varphi)}}$.

On the other hand, the hypothesis $|u| \geq S^2(\varphi)C(\varphi^{-1}) - C(\varphi^{-1})$ is equivalent to

$$|u| \geq S(\varphi)\left(S(\varphi)C(\varphi^{-1}) + C(\varphi)\right) - S(\varphi)C(\varphi) - C(\varphi^{-1}),$$

which gives by $|u'| = |u| + k$ the inequality
$|u'| \geq S(\varphi)\left(S(\varphi)C(\varphi^{-1}) + C(\varphi)\right) - S(\varphi)C(\varphi) - C(\varphi^{-1}) + S(\varphi)C(\varphi) + C(\varphi^{-1})$
$= S(\varphi)\left(S(\varphi)C(\varphi^{-1}) + C(\varphi)\right)$.
Since $|\varphi(u')| \geq \frac{|u'|}{S(\varphi)}$ we obtain $|\varphi(u')| \geq S(\varphi)C(\varphi^{-1}) + C(\varphi)$.
Thus we can now apply Proposition 4.4, to obtain $C^1_{\varphi(u')|_{C(\varphi)}} \subset \varphi(C^1_u)$ for all $u' \in u|^k$, so that one has
2. $\bigcup\limits_{u' \in u|^k} C^1_{\varphi(u')|_{C(\varphi)}} \subset \varphi(C^1_u)$.
From 1. and 2. together we derive

$$\varphi(C^1_u) = \bigcup_{u' \in u|^k} C^1_{\varphi(u')|_{C(\varphi)}}$$

□

Corollary 4.6. *Let $k = k_1 + k_2$, with $k_1 = S^2(\varphi)C(\varphi^{-1}) - C(\varphi^{-1})$ and $k_2 = S(\varphi)C(\varphi) + C(\varphi)$. Then for all $u \in F(A)$ we have*

$$\varphi(C^1_u) = \bigcup_{u' \in u|^k} C^1_{\varphi(u')|_{C(\varphi)}}$$

Proof. For any $v \in u|^{k_1}$ we have $|v| \geq S^2(\varphi)C(\varphi^{-1}) - C(\varphi^{-1})$. Thus we can apply Proposition 4.5 to get

$$\varphi(C^1_v) = \bigcup_{u' \in v|^{k_2}} C^1_{\varphi(u')|_{C(\varphi)}} \tag{1}$$

Recall from part (3) of Remark 3.2 that $C^1_u = \bigcup\limits_{v \in u|^{k_1}} C^1_v$ and thus $\varphi(C^1_u) = \bigcup\limits_{v \in u|^{k_1}} \varphi(C^1_v)$, so that we can deduce from equality (1):

$$\varphi(C^1_u) = \bigcup_{v \in u|^{k_1}} \Big(\bigcup_{u' \in v|^{k_2}} C^1_{\varphi(u')|_{C(\varphi)}} \Big)$$

Since $u|^k = u|^{k_1+k_2}$ this is equivalent to

$$\varphi(C_u^1) = \bigcup_{u' \in u|^k} C^1_{\varphi(u')|_{C(\varphi)}}$$

$\square$

Remark 4.7. There are several alternative approaches to determine the image of a cylinder C_u^1 under an automorphism φ. We briefly describe here two of them:

(a) Since every automorphism φ of F is a product of elementary automorphisms, one obtains a proof by induction over the length of such a product if one shows that for every elementary automorphism the image of a cylinder is a finite union of cylinders, and that those can be computed algorithmically. For permutations or inversions of the generators this is trivial; for elementary Nielsen automorphisms one has to work a little bit, but it is still not very difficult. On the other hand, this method doesn't permit one to describe $\varphi(C_u^1)$ by a closed formula as given in Corollary 4.6.

(b) Passing from $u \in F(A)$ to $u|^k$ for large k is computationally rather an effort, so that the formula exhibited in Corollary 4.6 is perhaps sometimes not very practical. We will thus sketch now a variation of the same basic approach, which has the advantage of being computationally more efficient (and also avoids some of the lengthly computations from above, after Lemma 4.3):

1. In a first step we pass from u to some $u|^k$, but we pick the smallest possible $k \geq 0$ such that any $w \in u|^k$ satisfies the hypothesis of Lemma 4.3. This gives us a finite collection W of words w_i such that $\varphi(C_u^1) \subset \bigcup\limits_{w_i \in W} C^1_{w_i}$.

2. We now prolong again every $w_i \in W$ to some $w_i|^{k_i}$, where $k_i \geq 0$ is chosen minimally to achieve two goals:

(i) We can again apply Lemma 4.3 to any $u_j \in w_i|^{k_i}$, but this time with φ^{-1} instead of φ. This gives $\varphi^{-1}(C^1_{u_j}) \subset C^1_{\varphi^{-1}(u_j)|_{C(\varphi^{-1})}}$.

(ii) For any $u_j \in w_i|^{k_i}$ the word $\varphi^{-1}(u_j)|_{C(\varphi^{-1})}$ is not a prefix of u.

3. We now check for every $u_j \in w_i|^{k_i}$ whether u is a prefix of $\varphi^{-1}(u_j)|_{C(\varphi^{-1})}$, and if this is not the case, we eliminate u_j from the collection of words given by $w_i|^{k_i}$. We do this for any of the $w_i \in W$ and obtain thus a collection U of words u_j which all have the property that u is a prefix of $\varphi^{-1}(u_j)|_{C(\varphi^{-1})}$. This is precisely the finite set $U \subset F$ with the desired property $\varphi(C^1_u) = \bigcup\limits_{u_j \in U} C^1_{u_j}$.

(The reason for this last statement is that the length bound, imposed in step 2. on all $u_j \in w_i|^{k_i}$, ensures by condition (ii) above that every $C^1_{\varphi^{-1}(u_j)|_{C(\varphi^{-1})}}$ is either contained in C^1_u or disjoint from the latter. Since from step 1 we know that $\varphi^{-1}(C^1_{u_j}) \subset C^1_{\varphi^{-1}(u_j)|_{C(\varphi^{-1})}}$, the same statement is true for $\varphi^{-1}(C^1_{u_j})$ replacing the $C^1_{\varphi^{-1}(u_j)|_{C(\varphi^{-1})}}$. Hence, if we eliminate in step 3 those $\varphi^{-1}(C^1_{u_j})$ from the collection which are disjoint from C^1_u, to determine the set U, then one obtains $\bigcup\limits_{u_j \in U} \varphi^{-1}(C^1_{u_j}) \subset C^1_u$ and thus $\bigcup\limits_{u_j \in U} C^1_{u_j} \subset \varphi(C^1_u)$.

On the other hand, the inclusion $\varphi(C^1_u) \subset \bigcup\limits_{w_i \in W} C^1_{w_i} \subset \bigcup\limits_{w_i \in W} \bigcup\limits_{u_j \in w_i|^{k_i}} C^1_{u_j}$ remains true if one eliminates from the right hand term those $C^1_{u_j}$ which are disjoint from $\varphi(C^1_u)$ (noting here that disjointness is preserved by the homeomorphism φ !), which gives the converse inclusion $\varphi(C^1_u) \subset \bigcup\limits_{u_j \in U} C^1_{u_j}$.)

We'd like to point out that Lluís Bacardit and Ilya Kapovich have informed us that each of them observed independently the fact stated in part (1) of Remark 4.7. Furthermore, the Examples 3.9 and 3.10 in the paper [1] by Berstock-Bestvina-Clay make us feel that the authors probably also had some knowledge along the lines of part (b) of Remark 4.7. We would also like to point the reader's attention to the forthcoming paper [5], which is in many ways a continuation of the work started here. In particular, we will treat there the question of the complexity of the algorithmic determination of the image of a given cylinder.

We now use the results of §3 to define a "dual map" φ^*, for any automorphism φ of F. It is important, however, to always keep in mind that the definition of this map depends (heavily !) on the choice of the basis A of F.

Definition 4.8. Let A be a basis of F. For any $u \in F(A)$ we consider the finite set $U = \{\varphi(u')|_{C(\varphi)} \mid u' \in u|^k\}$, for k as in Corollary 4.6. Let $U_{\min}$ be the unique minimal set which satisfies $C^1_{U_{\min}} = C^1_U$ $(= \varphi(C^1_u)$, see Proposition 3.20). We define:

$$\varphi^*_A(u) = U_{\min}$$

Similarly, for any $U \in \mathbb{U}$ we define $\varphi^*_A(U)$ as the unique minimal set which defines the same cylinder as $\bigcup\limits_{u_i \in U} \varphi^*_A(u_i)$.

Remark 4.9. Note that this last definition gives directly, via Corollary 4.6 and Proposition 3.20, that $\varphi^*_A(u)$ does not depend on U but only on $C^1_U = \varphi(C^1_u)$, and that $\varphi(C^1_u) = \dot{\bigcup\limits_{u' \in \varphi^*_A(u)}} C^1_{u'}$.

5 Double cylinders $C^2_{[u,v]}$

Definition 5.1. Let A be a basis for the free group F. We say that u, v are *anti-prefix* if u is not prefix of v and v is not prefix of u. Similarly, we say that $U, V \in \mathbb{U}$ are *anti-prefix* if any two elements $u \in U$ and $v \in V$ are anti-prefix.

Remark 5.2. Recall from Remark 3.2 (2) that for any $u, v \in F(A)$ the cylinders C^1_u and C^1_v are disjoint if and only if u and v are anti-prefix.

Lemma 5.3. *If $u, v \in F(A)$ are anti-prefix, then $\varphi^*_A(u)$, $\varphi^*_A(v)$ are anti-prefix as well.*

Proof. This is a direct consequence of Remark 5.2, since φ acts as homeomorphism and hence as bijection on $\partial F(A)$, so that it preserves disjointness of subsets. $\square$

We now consider the Cayley graph (a tree !) $\Gamma := \Gamma(F, A)$ of the free group F with respect to the basis A. There is a canonical identification between the vertices of Γ and the elements of F, which in turn induces a canonical identification between the boundary ∂F and the set $\partial\Gamma$ of ends of Γ. For any two $X, Y \in \partial F$ there is a well defined biinfinite reduced path $\gamma(X, Y)$ in Γ which connects the point of $\partial\Gamma$ associated to X to that associated to Y.

Definition 5.4. For any $u, v \in F(A)$ with $u \neq v$ we define the *double cylinder* $C^2_{[u,v]}$ as follows:

$$C^2_{[u,v]} = \{(X, Y) \in \partial^2 F_N \mid \gamma(X, Y) \text{ passes through } u \text{ and } v \text{ (in that order)}\}$$

Lemma 5.5. *If $u, v \in F(A)$ are anti-prefix, then one has:*

$$C^2_{[u,v]} = C^1_u \times C^1_v$$

Proof. For $w := u \wedge v$ (see §2) it follows from the assumption "u and v are anti-prefix" that $|w| < |u|$ and $|w| < |v|$. Hence for every $(X, Y) \in C^2_{[u,v]}$ the geodesic $\gamma(X, Y)$ must pass (in the given order) through the points u, w and v. In particular, it follows that $w < u < X$ and $w < v < Y$ and hence that $X \in C^1_u$ and $Y \in C^1_v$.

Conversely, for every pair $(X, Y) \in C^1_u \times C^1_v$ it follows that $w < u < X$ and $w < v < Y$, and that for $X = w \cdot X'$ and $Y = w \cdot Y'$ the biinfinite word $X'^{-1}Y'$ is reduced. Hence the geodesic $\gamma(X, Y)$ must pass (in the given order) through the points u, w and v, which implies $(X, Y) \in C^2_{[u,v]}$. $\square$

Proposition 5.6. *Let $u, v \in F(A)$ be anti-prefix. Then one has*

$$\varphi(C^2_{[u,v]}) = \dot{\bigcup_{\substack{u_i \in \varphi^*_A(u) \\ v_j \in \varphi^*_A(v)}}} C^2_{[u_i, v_j]}$$

Proof. Since u, v are anti-prefix, by Lemma 5.5 we have $C^2_{[u,v]} = C^1_u \times C^1_v$, which gives $\varphi(C^2_{[u,v]}) = \varphi(C^1_u) \times \varphi(C^1_v)$. By Remark 4.9 we have $\varphi(C^1_u) = \dot{\bigcup}_{u_i \in \varphi^*_A(u)} C^1_{u_i}$ and $\varphi(C^1_v) = \dot{\bigcup}_{v_j \in \varphi^*_A(v)} C^1_{v_j}$ and thus:

$$\varphi(C^2_{[u,v]}) = \dot{\bigcup_{u_i \in \varphi^*_A(u)}} C^1_{u_i} \times \dot{\bigcup_{v_j \in \varphi^*_A(v)}} C^1_{v_j} = \dot{\bigcup_{\substack{u_i \in \varphi^*_A(u) \\ v_j \in \varphi^*_A(v)}}} \left(C^1_{u_i} \times C^1_{v_j}\right)$$

By Lemma 5.3 the sets $\varphi_A^*(u)$, $\varphi_A^*(v)$ are anti-prefix, so that by Lemma 5.5 we have $C^1_{u_i} \times C^1_{v_j} = C^2_{[u_i,v_j]}$ for all $u_i \in \varphi_A^*(u), v_j \in \varphi_A^*(v)$, which gives

$$\varphi(C^2_{[u,v]}) = \dot{\bigcup_{\substack{u_i \in \varphi_A^*(u) \\ v_j \in \varphi_A^*(v)}}} C^2_{[u_i,v_j]} \,.$$

$\square$

Lemma 5.7. *For all $u, v, w \in F(A)$ one has $wC^2_{[u,v]} = C^2_{[wu,wv]}$.*

Proof. This is a direct consequence of the definition of $C^2_{[u,v]}$, see Definition 5.4. $\square$

Before passing to the general case of double cylinders, we need to consider the following "small" special cases, the proof of which follows directly from the definitions:

Lemma 5.8. *For any $a_i \in A$ one has:*
$C^2_{[1,a_i]} = \dot{\bigcup}_{a_j \in A \cup A^{-1} \smallsetminus \{a_i\}} C^2_{[a_j,a_i]}$.
$C^2_{[1,1]} = \dot{\bigcup}_{a_i \in A \cup A^{-1}} C^2_{[1,a_i]} = \dot{\bigcup}_{\substack{a_j, a_i \in A \cup A^{-1} \\ a_i \neq a_j}} C^2_{[a_j,a_i]}$. $\square$

Proposition 5.9. *For any two distinct $u, v \in F(A)$ there exist finite computable sets $U, V \subset F(A)$ such that*

$$\varphi(C^2_{[u,v]}) = \dot{\bigcup_{\substack{u_i \in U \\ v_j \in V}}} C^2_{[u_i,v_j]}$$

Proof. If u and v are anti-prefix, then Proposition 5.6 gives the desired statement (and furthermore a precise description of the sets U and V).

Otherwise, one has $u \leq v$ or $v \leq u$, and if $\big||u| - |v|\big| \geq 2$ we can find some $w \in F(A)$ with $u < w < v$ or $v < w < u$. Hence Lemma 5.7 allows us to replace u by $w^{-1}u$ and v by $w^{-1}v$, which reduces this case to the one treated in the previous paragraph.

Finally, if $\big||u| - |v|\big| \leq 1$ we can first again apply Lemma 5.7 to achieve that $u = 1$ or $v = 1$. But then Lemma 5.8 brings us again back to the case treated in the first paragraph. $\square$

Remark 5.10. From the arguments given in the last proof one can derive the following improvement of Proposition 5.6:

For any two distinct $u, v \in F(A)$ (i.e. without supposing that they are anti-prefix) one has:

$$\varphi(C^2_{[u,v]}) = \dot{\bigcup_{\substack{u_i \in \varphi(v)\varphi^*_A(v^{-1}u) \\ v_j \in \varphi(u)\varphi^*_A(u^{-1}v)}}} C^2_{[u_i,v_j]}$$

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