# Group extensions with special properties 

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#### Abstract

Given a group $G$ and a $G$-module $A$, we show how to determine up to isomorphism the extensions $E$ of $A$ by $G$ so that $A$ embeds as smallest non-trivial term of the derived series or of the lower central series into $E$.


## 1 Introduction

Given a group $G$ and a $G$-module $A$ an extension of $A$ by $G$ is a group $E$ that satisfies a short exact sequence $A \hookrightarrow E \rightarrow G$. The determination of extensions facilitates the construction of new groups from given ones and is an important tool in group theory. The isomorphism problem for group extensions asks to determine extensions up to isomorphism: given $G$ and $A$, the aim is to determine a complete and irredundant set of isomorphism type representatives of extensions of $A$ by $G$.
We consider two particular types of extensions. Let $E$ be an extension of $A$ by $G$ and denote by $\bar{A}$ the image of the embedding $A \hookrightarrow E$. We call $E$ a lower central series extension if $E$ is nilpotent and $\bar{A}$ coincides with the smallest non-trivial term of the lower central series of $E$. We call $E$ a derived series extension if $\bar{A}$ coincides with the smallest non-trivial term of the derived series of $E$.
As central part of this paper we present criteria which allow us to identify those elements in the cohomology group $H^{2}(G, A)$ that correspond to the desired extensions. For this purpose we use the Schur multiplier of a pair of groups and a map that can be considered as generalisation of the projection map in the universal coefficients theorem. Based on this, we then describe practical methods to determine up to isomorphism all lower central series extensions respectively all derived series extensions of $A$ by $G$.
We apply our algorithms in the construction of groups with large derived length and small composition length. In particular, we construct two new examples of groups of derived length 10 and composition length 24 . It is conjectured that 24 is the minimal possible composition length for a group of derived length 10, see [8].

## 2 Extensions

In this section we briefly recall some notation. For a more detailed introduction into the theory of group extensions we refer to [11, Chapter 11]. Throughout this section, let $G$ be an arbitrary group and let $A$ be an abelian group equipped with a $G$-module structure.

We write $G$ as multiplicative group and $A$ as additive group. We denote the action of $G$ on $A$ by $a^{g}$ for $a \in A$ and $g \in G$ and we write $\bar{g}: A \rightarrow A: a \mapsto a^{g}$.
Let $E=\{(g, a) \mid g \in G, a \in A\}$. Then $\delta \in Z^{2}(G, A)$ defines a group structure on $E$ via

$$
(g, a)\left(g^{\prime}, a^{\prime}\right)=\left(g g^{\prime}, a^{g^{\prime}}+a^{\prime}+\delta\left(g, g^{\prime}\right)\right) .
$$

The module $A$ embeds into $E$ via $A \hookrightarrow E: a \mapsto(1, a)$ and $E$ projects onto $G$ via $E \rightarrow G:(g, a) \mapsto g$. Thus $E$ is an extension of $A$ by $G$. It is well-known that each extension of $A$ by $G$ is isomorphic to an extension obtained by some $\delta \in Z^{2}(G, A)$. We usually identify $A$ with its image $\bar{A}=\{(1, a) \mid a \in A\}$.
Let $E_{1}$ and $E_{2}$ be two extensions of $A$ by $G$. We say that $E_{1}$ is strongly isomorphic to $E_{2}$ if there exists an isomorphism $\iota: E_{1} \rightarrow E_{2}$ with $A^{\iota}=A$. Further, $E_{1}$ is equivalent to $E_{2}$ if there exists an isomorphism $\iota: E_{1} \rightarrow E_{2}$ with $A^{\iota}=A$ so that $\iota$ induces the identity on $A$ and on $E_{1} / A \cong G \cong E_{2} / A$.
The group of compatible pairs $\operatorname{Comp}(G, A)$ is defined as subgroup of the direct product $\operatorname{Aut}(G) \times \operatorname{Aut}(A)$ via

$$
\operatorname{Comp}(G, A)=\left\{(\eta, \nu) \in \operatorname{Aut}(G) \times \operatorname{Aut}(A) \mid \overline{g^{\eta}}=\nu^{-1} \bar{g} \nu \text { for all } g \in G\right\}
$$

If the action of $G$ on $A$ is trivial then $\operatorname{Comp}(G, A)$ equals $\operatorname{Aut}(G) \times \operatorname{Aut}(A)$. An action of $\operatorname{Comp}(G, A)$ on $Z^{2}(G, A)$ is given via

$$
\delta^{(\eta, \nu)}: G \times G \rightarrow A:(g, h) \mapsto\left(\delta\left(g^{\eta^{-1}}, h^{\eta^{-1}}\right)\right)^{\nu}
$$

For $\delta \in Z^{2}(G, A)$ we denote $[\delta]=\delta+B^{2}(G, A) \in H^{2}(G, A)$. The subgroup $B^{2}(G, A)$ of $Z^{2}(G, A)$ is setwise invariant under the action of $\operatorname{Comp}(G, A)$ and hence $\operatorname{Comp}(G, A)$ acts on $H^{2}(G, A)$ via $[\delta]^{(\eta, \nu)}=\left[\delta^{(\eta, \nu)}\right]$.

1 Theorem: (Robinson (10])
Let $\delta, \delta_{1}, \delta_{2} \in Z^{2}(G, A)$ and denote their corresponding extensions by $E, E_{1}$ and $E_{2}$. Write $A u t_{A}(E)=\left\{\alpha \in \operatorname{Aut}(E) \mid A^{\alpha}=A\right\}$.
(1) Then $E_{1}$ is strongly isomorphic to $E_{2}$ if and only if there exists $(\eta, \nu) \in \operatorname{Comp}(G, A)$ with $\left[\delta_{1}\right]^{(\eta, \nu)}=\left[\delta_{2}\right]$.
(2) The homomorphism $\varphi:$ Aut $_{A}(E) \rightarrow \operatorname{Aut}(G) \times \operatorname{Aut}(A): \alpha \mapsto \alpha_{E / A} \times \alpha_{A}$ satisfies $\operatorname{ker}(\varphi) \cong Z^{1}(G, A)$ and $\operatorname{im}(\varphi)=\operatorname{Stab}_{\operatorname{Comp}(G, A)}([\delta])$.

## 3 Cohomology and Schur multipliers

Let $G$ be an arbitrary group and let $A$ be a trivial $G$-module. In this section we introduce a map that links $H^{2}(G, A)$ to the Schur multiplier of a pair of groups. This map will play a central role in our later applications.

### 3.1 Twisted cocycles

For $\delta \in Z^{2}(G, A)$ we define

$$
\begin{equation*}
\hat{\delta}: G \times G \rightarrow A:(g, h) \mapsto \delta(g, h)-\delta(h, g)-\delta\left((h g)^{-1}, h g\right)+\delta\left((h g)^{-1}, g h\right) . \tag{1}
\end{equation*}
$$

For later use we note that $h \in Z(G)$ implies

$$
\begin{equation*}
\hat{\delta}(g, h)=\delta(g, h)-\delta(h, g) \tag{2}
\end{equation*}
$$

The following lemma gives an alternative description for $\hat{\delta}$. Its proof is a direct computation which we omit here. For any two group elements $g$ and $h$ let $[g, h]=g^{-1} h^{-1} g h$ the commutator of $g$ and $h$.

2 Lemma: Let $G$ be a group, $A$ a trivial $G$-module and $\delta \in Z^{2}(G, A)$. Then in the extension of $A$ by $G$ via $\delta$ the equation $[(g, a),(h, b)]=([g, h], \hat{\delta}(g, h))$ holds for all $g, h \in G$ and $a, b \in A$.

### 3.2 The Schur multiplier of a pair of groups

We briefly recall the construction of the non-abelian exterior product and the non-abelian tensor product as introduced by Brown and Loday [3], see also [2] for details.
Let $H \leq G$ and let $F$ be the free group on the symbols $\{g \wedge h \mid g \in G, h \in H\}$. For $g, h \in G$ denote ${ }^{h} g:=h g h^{-1}=g^{h^{-1}}$. Let $R$ be the normal subgroup of $F$ generated by the relations

$$
\begin{array}{rlrl}
g g^{\prime} \wedge h & =\left({ }^{g} g^{\prime} \wedge{ }^{g} h\right)(g \wedge h) & \text { for } g, g^{\prime} \in G, h \in H \\
g \wedge h h^{\prime} & =(g \wedge h)\left({ }^{h} g \wedge{ }^{h} h^{\prime}\right) & \text { for } g \in G, h, h^{\prime} \in H \\
h \wedge h & =1 \quad \text { for } h \in H . & &
\end{array}
$$

Then $G \wedge H:=F / R$ is the non-abelian exterior product of $G$ and $H$. If the relations $h \wedge h=1$ are omitted, then the resulting quotient is the non-abelian tensor product of $G$ and $H$.
By construction, there is a natural homomorphism

$$
\varphi: G \wedge H \rightarrow[G, H]: g \wedge h \mapsto g h g^{-1} h^{-1}=\left[g^{-1}, h^{-1}\right] .
$$

The kernel of $\varphi$ is denoted with $M(G, H)$ and is called the Schur multiplier of the pair of groups $(G, H)$, see [6] for background. It is known that $M(G, H)$ is an abelian group and the (ordinary) Schur multiplier $M(G)$ of the group $G$ can be obtained as $M(G)=M(G, G)$ by Hopf's formula. Further, if $H \leq Z(G)$, then $M(G, H)=G \wedge H$ holds.

### 3.3 The linking map

The following lemma provides the first step for the definition of the linking map.

3 Lemma: Let $G$ be a group, $A$ a trivial $G$-module, $\delta \in Z^{2}(G, A)$ and $H \leq Z(G)$. Then the following map is a group homomorphism

$$
\bar{\delta}: M(G, H) \rightarrow A: g \wedge h \mapsto \hat{\delta}(g, h) .
$$

Proof: Let $E$ be the extension of $A$ by $G$ defined by $\delta$ and define $\alpha: F \rightarrow E$ as the group homomorphism extending $g \wedge h \mapsto([g, h], \hat{\delta}(g, h))$. We note that $[(g, 0),(h, 0)]=$ $([g, h], \hat{\delta}(g, h))=(1, \hat{\delta}(g, h))$ for each $g \in G$ and $h \in H$ by Lemma 2 and the fact that $H$ is central in $G$.
We show that $R$ is contained in the kernel of $\alpha$. First, consider the relation $h \wedge h$ for $h \in H$. As $\hat{\delta}(g, g)=0$ for all $g \in G$ it follows that $\alpha(h \wedge h)=([h, h], \hat{\delta}(h, h))=(1,0)$ in $E$. Next, consider the relation $g \wedge h h^{\prime}=(g \wedge h)\left({ }^{h} g \wedge^{h} h^{\prime}\right)=(g \wedge h)\left(g \wedge h^{\prime}\right)$, as $H$ is central in $G$. Then

$$
\begin{aligned}
\left(1, \hat{\delta}\left(g, h h^{\prime}\right)\right) & =\left[(g, 0),\left(h h^{\prime}, 0\right)\right] \\
& =\left[(g, 0),\left(h,-\delta\left(h, h^{\prime}\right)\right)\left(h^{\prime}, 0\right)\right] \\
& =\left[(g, 0),\left(h^{\prime}, 0\right)\right]\left[(g, 0),\left(h,-\delta\left(h, h^{\prime}\right)\right)\right]\left(h^{\prime}, 0\right) \\
& =\left(\left[g, h^{\prime}\right], \hat{\delta}\left(g, h^{\prime}\right)\right)([g, h], \hat{\delta}(g, h))^{\left(h^{\prime}, 0\right)} \\
& =\left(1, \hat{\delta}\left(g, h^{\prime}\right)\right)(1, \hat{\delta}(g, h))^{\left(h^{\prime}, 0\right)} \\
& =\left(1, \hat{\delta}\left(g, h^{\prime}\right)\right)(1, \hat{\delta}(g, h)) \\
& =\left(1, \hat{\delta}(g, h)+\hat{\delta}\left(g, h^{\prime}\right)\right) .
\end{aligned}
$$

Thus $\alpha\left(g \wedge h h^{\prime}\right)=\alpha\left((g \wedge h)\left(g \wedge h^{\prime}\right)\right)$ as desired. Finally consider the relation $g g^{\prime} \wedge h=$ $\left({ }^{g} g^{\prime} \wedge{ }^{g} h\right)(g \wedge h)=\left({ }^{g} g^{\prime} \wedge h\right)(g \wedge h)$, as $H$ is central in $G$. Then using a similar computation as above we obtain

$$
\begin{aligned}
\left(1, \hat{\delta}\left(g g^{\prime}, h\right)\right) & =\left[\left(g g^{\prime}, 0\right),(h, 0)\right] \\
& =\left[\left(\left({ }^{g} g^{\prime}\right) \cdot g, 0\right),(h, 0)\right] \\
& =\left(1, \hat{\delta}\left({ }^{g} g^{\prime}, h\right)+\hat{\delta}(g, h)\right)
\end{aligned}
$$

as desired. In summary, $R \leq \operatorname{ker}(\alpha)$ and thus $\alpha$ induces a group homomorphism $G \wedge H \rightarrow$ $E$ whose image is contained in $A$. As $G \wedge H=M(G, H)$ for the central subgroup $H$ of $G$, the result now follows.

Lemma 3 leads to the following definition for the linking map between $H^{2}(G, A)$ and the Schur multiplier of a pair of groups.

4 Theorem: Let $G$ be a group, $A$ a trivial $G$-module and $H \leq Z(G)$. Then

$$
\begin{equation*}
\kappa_{H}: Z^{2}(G, A) \rightarrow \operatorname{Hom}(M(G, H), A): \delta \mapsto \bar{\delta} \tag{3}
\end{equation*}
$$

is a homomorphism of abelian groups with $B^{2}(G, A) \leq \operatorname{ker}\left(\kappa_{H}\right)$. Thus it induces

$$
\begin{equation*}
\bar{\kappa}_{H}: H^{2}(G, A) \rightarrow \operatorname{Hom}(M(G, H), A):[\delta] \mapsto \bar{\delta} \tag{4}
\end{equation*}
$$

Proof: From Lemma 3 it follows that $\kappa_{H}$ is well-defined. The definition of $\hat{\delta}$ yields that $\kappa_{H}$ is compatible with the addition and inversion in $Z^{2}(G, A)$ and hence is a group homomorphism. It remains to show that $B^{2}(G, A) \leq \operatorname{ker}\left(\kappa_{H}\right)$. Let $\delta \in B^{2}(G, A)$. Then there exists $\epsilon \in C^{1}(G, A)$ with $\delta(g, h)=\epsilon(g)+\epsilon(h)-\epsilon(g h)$ for all $g, h \in G$. Using (2) it follows for all $g \in G, h \in H$ that $\hat{\delta}(g, h)=\epsilon(h g)-\epsilon(g h)=0$. Thus $\delta \in \operatorname{ker}\left(\kappa_{H}\right)$.

We note that the universal coefficients theorem for cohomology asserts the existence of a short exact sequence

$$
\operatorname{Ext}\left(H_{1}(G, \mathbb{Z}), A\right) \hookrightarrow H^{2}(G, A) \rightarrow \operatorname{Hom}(M(G), A)
$$

If $G$ is abelian, then $\kappa_{G}$ coincides with the projection in the universal coefficients sequence.
5 Remark: Let $H \leq Z(G)$ be a characteristic subgroup of $G$. Then the action of $\operatorname{Comp}(G, A)$ on $Z^{2}(G, A)$ is compatible with $\kappa_{H}$ and thus defines an action of $\operatorname{Comp}(G, A)$ on $\operatorname{Hom}(M(G, H), A)$.

For a non-trivial nilpotent group $G$ let $\lambda(G)$ denote its smallest non-trivial subgroup of the lower central series of $G$. To shorten notation in our later section we then also denote

$$
\kappa=\kappa_{\lambda(G)} \quad \text { and } \quad \bar{\kappa}=\bar{\kappa}_{\lambda(G)}
$$

## 4 Lower central series extensions

In this section we describe a construction for a set of isomorphism type representatives of lower central series extensions of $A$ by $G$. If any such extension exists, then $G$ is a nilpotent group and $A$ is a non-trivial abelian group with a trivial $G$-module structure. We assume this throughout the section and we also assume that $G$ is non-trivial to obtain proper extensions of $A$. The following theorem provides a characterisation of the cocycles defining lower central series extensions.

6 Theorem: Let $G$ be a non-trivial nilpotent group, A a non-trivial group with trivial $G$-module structure and $\delta \in Z^{2}(G, A)$. Then the extension of $A$ by $G$ via $\delta$ is a lower central series extension if and only if $\kappa(\delta)$ is surjective.

Proof: Let $E$ denote the extension of $A$ by $G$ via $\delta$ and let $E=\lambda_{1}(E)>\lambda_{2}(E)>\ldots$ be the lower central series of $E$. Further let $c$ be the class of $G$.
Suppose that $E$ is a lower central series extension. Then $E$ is nilpotent of class $c+1$ and the image $\bar{A}$ of $A$ in $E$ satisfies that $\bar{A}=\lambda_{c+1}(E)=\left[E, \lambda_{c}(E)\right]$. This implies that $\bar{A}=\langle[(g, a),(h, b)] \mid a, b \in A, g \in G, h \in \lambda(G)\rangle$. As $[(g, a),(h, b)]=(1, \hat{\delta}(g, h))$ by Lemma 2 and $\hat{\delta}(g, h)=\bar{\delta}(g \wedge h)$ by definition of $\bar{\delta}$, it follows that $\bar{\delta}=\kappa(\delta)$ is surjective.
Now suppose that $\kappa(\delta)$ is surjective. Using the same calculation as in the first part of the proof, it follows that $\bar{A}=\lambda_{c+1}(E)$. As $A$ is a trivial $G$-module, this implies that $\lambda_{c+2}(E)$ is trivial and thus $E$ is nilpotent. Hence $E$ is a lower central series extension of $A$ by $G$. •

The following theorem exhibits an explicit construction for the isomorphism types of lower central series extensions of $A$ by $G$. Recall that $\operatorname{Comp}(G, A)=A u t(G) \times A u t(A)$, as $G$ acts trivially on $A$.

7 Theorem: Let $G$ be a non-trivial nilpotent group and $A$ a non-trivial group with trivial $G$-module structure. Define

$$
\begin{equation*}
\Delta=\left\{\gamma \in H^{2}(G, A) \mid \bar{\kappa}(\gamma) \text { is surjective }\right\} . \tag{5}
\end{equation*}
$$

(a) Then $\Delta$ is invariant under the action of $\operatorname{Comp}(G, A)$; denote by $\Omega$ a set of orbit representatives of the action of $\operatorname{Comp}(G, A)$ on $\Delta$.
(b) The extensions of $A$ by $G$ defined by the elements in $\Omega$ form a complete and irredundant set of isomorphism type representatives of lower central series extensions of $A$ by $G$.
(c) For $\delta$ with $[\delta] \in \Delta$ denote the corresponding extension by $E$. Then $\operatorname{Aut}(E)=A u t_{A}(E)$ and there exists a short exact sequence

$$
Z^{1}(G, A) \hookrightarrow \operatorname{Aut}(E) \longrightarrow \operatorname{Stab}_{\operatorname{Comp}(G, A)}([\delta])
$$

Proof: Part (a) follows from Theorem 6. Parts (b) and (c) follow from Theorem 11, as lower central series extensions are isomorphic if and only if they are strongly isomorphic.

The following remark recalls the structure of $M(G, \lambda(G))$ and thus gives further insight into the range $\operatorname{Hom}(M(G, \lambda(G)), A)$ of $\bar{\kappa}$.

8 Remark: (See Prop. 3.2 of [6]) Let $G$ be a non-trivial nilpotent group.
(a) If $G$ has class 1 , then $\lambda(G)=G$ and $M(G, \lambda(G))=M(G) \cong G \wedge G$, the abelian exterior product.
(b) If $G$ has class at least 2 , then $\lambda(G) \leq G^{\prime}$ and $M(G, \lambda(G)) \cong G / G^{\prime} \otimes \lambda(G)$, the abelian tensor product.

## 5 Derived series extensions

In this section we describe a construction for a set of isomorphism type representatives of derived series extensions of $A$ by $G$. Any such extension is solvable and if one exists, then $G$ is solvable and $A$ is a non-trivial abelian group and a $G$-module. We assume this throughout the section and further suppose that $G$ is non-trivial to obtain proper extensions of $A$. Let $\gamma(G)$ denote the smallest non-trivial subgroup of the derived series of $G$ and set $U=[A, \gamma(G)]$. We consider the sequence

$$
\begin{equation*}
Z^{2}(G, A) \xrightarrow{\sigma} Z^{2}(\gamma(G), A / U) \xrightarrow{\kappa} \operatorname{Hom}(M(\gamma(G)), A / U), \tag{6}
\end{equation*}
$$

where $\sigma(\delta): \gamma(G) \times \gamma(G) \rightarrow A / U:(g, h) \mapsto \delta(g, h)+U$. Regarding $\kappa$ note that $\lambda(\gamma(G))=$ $\gamma(G)$ and thus $M(\gamma(G), \lambda(\gamma(G)))=M(\gamma(G))$. As $\sigma$ maps $B^{2}(G, A)$ into $B^{2}(\gamma(G), A / U)$ equation (6) induces the sequence of maps

$$
\begin{equation*}
H^{2}(G, A) \xrightarrow{\bar{\sigma}} H^{2}(\gamma(G), A / U) \xrightarrow{\bar{\kappa}} \operatorname{Hom}(M(\gamma(G)), A / U), \tag{7}
\end{equation*}
$$

The following theorem characterises the derived series extensions of $G$ by $A$.

9 Theorem: Let $G$ be a non-trivial solvable group, A a non-trivial abelian group with a $G$-module structure and $\delta \in Z^{2}(G, A)$. Then the extension of $A$ by $G$ via $\delta$ is a derived series extension if and only if $\kappa(\sigma(\delta))$ is surjective.

Proof: Let $E$ denote the extension of $A$ by $G$ via $\delta$. Consider $E \rightarrow G:(g, a) \mapsto g$, the natural epimorphism from $E$ onto $G$ and denote by $M$ the full preimage of $\gamma(G)$ in $E$. By definition $E$ is a derived series extension if $M^{\prime}=A$. As $U=[A, M] \leq M^{\prime}$, this is equivalent to the condition $(M / U)^{\prime}=A / U$. The induced action of $\gamma(G)$ on $A / U$ is trivial. Thus $M / U$ has class at most 2 and $\lambda(M / U)=(M / U)^{\prime}$. Hence $E$ is a derived series extension if and only if $M / U$ is a lower central series extension of $\gamma(G)$ by $A / U$ via the cocycle $\sigma(\delta)$. The result now follows from Theorem 6 ,

This allows the following explicit construction for the isomorphism types of derived series extensions of $A$ by $G$.

10 Theorem: Let $G$ be a non-trivial solvable group and $A$ a non-trivial group with an arbitrary $G$-module structure. Define

$$
\begin{equation*}
\Delta=\left\{\gamma \in H^{2}(G, A) \mid \bar{\kappa}(\bar{\sigma}(\gamma)) \text { is surjective }\right\} . \tag{8}
\end{equation*}
$$

(a) Then $\Delta$ is invariant under the action of $\operatorname{Comp}(G, A) ;$ let $\Omega$ denote a set of orbit representatives of the action of $\operatorname{Comp}(G, A)$ on $\Delta$.
(b) The extensions of $A$ by $G$ defined by the elements in $\Omega$ form a complete and irredundant set of isomorphism type representatives of derived series extensions of $A$ by $G$.
(c) For $\delta$ with $[\delta] \in \Delta$ denote the corresponding extension by $E$. Then $\operatorname{Aut}(E)=A u t_{A}(E)$ and there exists a short exact sequence

$$
Z^{1}(G, A) \hookrightarrow \operatorname{Aut}(E) \longrightarrow \operatorname{Stab}_{\operatorname{Comp}(G, A)}([\delta]) .
$$

Proof: The proof is similar to that of Theorem 7 .

## 6 Computational methods

In the previous sections we established criteria to decide whether an extension is either a lower central series extension or a derived series extension. Here we exploit these descriptions to obtain effective algorithms to construct those extensions. We have implemented the algorithms in GAP [7] for the special case that the module is elementary abelian. The implementation is available in the GAP package SpecialExt [4].

### 6.1 Computing $\mathbf{H}^{2}(\mathbf{G}, \mathbf{A})$ and the action of $\operatorname{Comp}(\mathbf{G}, \mathbf{A})$

If $G$ is a polycyclic group defined by a consistent polycyclic presentation and $A$ is a finitely generated abelian group and a $G$-module, then $H^{2}(G, A)$ can be computed effectively. We recall the basic ideas of the underlying algorithm here briefly for the case that $G$ is finite; see also [9, Section 8.7.2].

Let $\mathfrak{g}=\left\{g_{1}, \ldots, g_{n}\right\}$ be a polycyclic generating sequence of $G$ and let $\left\{a_{1}, \ldots, a_{s}\right\}$ be a generating set of $A$. Then there exists a (unique) consistent polycyclic presentation of $G$ on the generators $\mathfrak{g}$. This has relations of the form

$$
\begin{aligned}
& g_{i}^{r_{i}}=g_{i+1}^{e_{i, i+1}} \cdots g_{n}^{e_{i, n}} \text { for } 1 \leq i \leq n, \text { and } \\
& g_{i}^{g_{j}}=g_{j+1}^{e_{i, j, j+1}} \cdots g_{n}^{e_{i, j, n}} \text { for } 1 \leq j<i \leq n
\end{aligned}
$$

for certain $r_{i} \in \mathbb{N}$ and $e_{i, j}, e_{i, j, k} \in \mathbb{Z}$.
Let $l=(n+1) n / 2$. For an extension $E$ of $A$ by $G$ via $\delta \in Z^{2}(G, A)$ we define the tuple $t_{\delta}=\left(t_{i, j} \mid 1 \leq j \leq i \leq n\right) \in A^{l}$ via

$$
\begin{gathered}
\left(1, t_{i, i}\right)=\left(g_{n}, 0\right)^{-e_{i, n}} \cdots\left(g_{i+1}, 0\right)^{-e_{i, i+1}}\left(g_{i}, 0\right)^{r_{i}} \text { for } 1 \leq i \leq n \text {, and } \\
\left(1, t_{i, j}\right)=\left(g_{n}, 0\right)^{-e_{i, j, n}} \cdots\left(g_{j+1}, 0\right)^{-e_{i, j, j+1}}\left(g_{i}, 0\right)^{\left(g_{j}, 0\right)} \text { for } 1 \leq j<i \leq n .
\end{gathered}
$$

By [9, Lemma 8.47] the map

$$
\varphi: Z^{2}(G, A) \rightarrow A^{l}: \delta \mapsto t_{\delta}
$$

is a group homomorphism with $\operatorname{ker}(\varphi) \leq B^{2}(G, A)$. Denote $Z=\operatorname{im}(\varphi) \leq A^{l}$ and $B=$ $B^{2}(G, A)^{\varphi} \leq Z$. Then we obtain an isomorphism

$$
\bar{\varphi}: H^{2}(G, A) \rightarrow Z / B
$$

For $t \in A^{l}$ we define a presentation $P(t)$ with generators $g_{1}, \ldots, g_{n}, a_{1}, \ldots, a_{s}$ and two types of relations: firstly relations defining $A$ as group and $G$-module and secondly

$$
\begin{gathered}
g_{i}^{r_{i}}=g_{i+1}^{e_{i, i+1}} \cdots g_{n}^{e_{i, n}} \cdot t_{i, i} \text { for } 1 \leq i \leq n \\
g_{i}^{g_{j}}=g_{j+1}^{e_{i, j, j+1}} \cdots g_{n}^{e_{i, j, n}} \cdot t_{i, j} \text { for } 1 \leq j<i \leq n .
\end{gathered}
$$

If $t \in Z$, then the presentation $P(t)$ defines a group that is an extension of $A$ by $G$ via a cocycle $\delta$ in the preimage of $t$ under $\varphi$.
The subgroups $Z$ and $B$ of $A^{l}$ can be computed effectively if $A$ is elementary-abelian (see [9, Section 8.7.2]) and we use $Z / B$ to represent $H^{2}(G, A)$ in our applications. The group $\operatorname{Comp}(G, A)$ can be computed via its definition The action of $\operatorname{Comp}(G, A)$ on $H^{2}(G, A)$ translates to an action of $\operatorname{Comp}(G, A)$ on $Z / B$. A pair $(\eta, \nu) \in \operatorname{Comp}(G, A) \leq \operatorname{Aut}(G) \times$ $\operatorname{Aut}(A)$ acts on $t \in Z$ via

$$
\begin{gathered}
\left(1, t_{i, i}\right)^{(\eta, \nu)}=\left(\left(g_{n}^{\eta^{-1}}, 0\right)^{-e_{i, n}} \cdots\left(g_{i+1}^{\eta^{-1}}, 0\right)^{-e_{i, i+1}}\left(g_{i}^{\eta^{-1}}, 0\right)^{r_{i}}\right)^{\nu} \text { for } 1 \leq i \leq n \text {, and } \\
\left(1, t_{i, j}\right)^{(\eta, \nu)}=\left(\left(g_{n}^{\eta^{-1}}, 0\right)^{-e_{i, j, n}} \cdots\left(g_{j+1}^{\eta^{-1}}, 0\right)^{-e_{i, j, j+1}}\left(g_{i}^{\eta^{-1}}, 0\right)^{\left(g_{j}^{\eta^{-1}}, 0\right)}\right)^{\nu} \text { for } 1 \leq j<i \leq n .
\end{gathered}
$$

### 6.2 Lower central series extensions

Let $G$ be a finite non-trivial nilpotent group and $A$ a finite non-trivial abelian group with a trivial $G$-module structure. Our aim is to compute a complete and irredundant set of isomorphism type representatives of lower central series extensions of $A$ by $G$.
We choose a polycyclic generating sequence $\mathfrak{g}=\left\{g_{1}, \ldots, g_{n}\right\}$ such that it refines the lower central series of $G$. As $G^{\prime}=[G, G]$ and $\lambda(G)$ are subgroups in the lower central series of $G$, it then follows that there exist indices $d$ and $m$ in $\{1, \ldots, n\}$ with $G^{\prime}=\left\langle g_{d+1}, \ldots, g_{n}\right\rangle$ and $\lambda(G)=\left\langle g_{m}, \ldots, g_{n}\right\rangle$. Let $J=\{(i, j) \mid 1 \leq j \leq i \leq n\}$ and $I=\{(i, j) \mid 1 \leq j \leq d, m \leq$ $i \leq n, j<i\} \subseteq J$. Then $|J|=l$ and we denote $|I|=h$. We further denote the natural projection corresponding to $I$ and $J$ by

$$
\pi: A^{l} \rightarrow A^{h}:\left(t_{i, j} \mid(i, j) \in J\right) \mapsto\left(t_{i, j} \mid(i, j) \in I\right)
$$

By Theorem 7 we have achieved our aim if we find $\varphi(\Omega)$ for a set $\Omega$ as defined in Theorem 7 , Recall that $\Omega$ is a set of orbit representatives in the set $\Delta$ from (5). The following lemma provides a criterion to check whether a given $\delta \in Z^{2}(G, A)$ satisfies $[\delta] \in \Delta$. We say that a tuple $t \in A^{k}$ for $k \in \mathbb{N}$ is full if the entries in $t$ generate $A$.

11 Lemma: Let $G$ be a non-trivial nilpotent group, A a non-trivial abelian group with trivial $G$-module structure and $\delta \in Z^{2}(G, A)$. Then $\kappa(\delta)$ is surjective if and only if $\pi\left(t_{\delta}\right)$ is full.

Proof: Recall that $\kappa(\delta): M(G, \lambda(G)) \rightarrow A$ with $\kappa(\delta)(g \wedge h)=\hat{\delta}(g, h)=\delta(g, h)-\delta(h, g)$. Thus $\kappa(\delta)$ is surjective if and only if $\{\hat{\delta}(g, h) \mid g \in G, h \in \lambda(G)\}=A$. As $\lambda(G)$ is central in $G$, the definition of $M(G, \lambda(G))$ and Lemma 3 yield that $\hat{\delta}\left(g, h h^{\prime}\right)=\hat{\delta}(g, h)+\hat{\delta}\left(g, h^{\prime}\right)$ and $\hat{\delta}\left(g g^{\prime}, h\right)=\hat{\delta}\left({ }^{g} g^{\prime}, h\right)+\hat{\delta}(g, h)$. By Remark 8 we note that the first argument of $\hat{\delta}$ depends on $G / G^{\prime}$ only. This also yields that $\hat{\delta}$ is multiplicative in both components. In summary, it follows that $\kappa(\delta)$ is surjective if and only if $\left\langle\hat{\delta}\left(g_{j}, g_{i}\right) \mid(i, j) \in I\right\rangle=A$ and thus if and only if $\pi\left(t_{\delta}\right)$ is full.

Following the approach described in Section 6.1 we first compute $Z / B$ and then its orbits under the action of $\operatorname{Comp}(G, A)$. We next choose orbit representatives $r_{1}, \ldots, r_{m} \in Z / B$ and then a representative $t_{i}$ in $A^{l}$ for each $r_{i}, 1 \leq i \leq m$. According to Theorem 7 (a) and Lemma 11 we then have $\varphi(\Omega)=\left\{t_{i} \mid \pi\left(t_{i}\right)\right.$ is full $\}$ for some $\Omega$ as desired and $\{P(t) \mid t \in$ $\varphi(\Omega)\}$ is a set of presentations for a complete and irredundant list of lower central series extensions of $A$ by $G$.
To avoid redundant computations it is useful to know a priori when there are no lower central series extensions of $A$ by $G$. The following remark collects some elementary conditions for this purpose.

12 Remark: There are no lower central series extensions of $A$ by $G$ if either $h<d(A)$, where $d(A)$ is the minimal generator number of $A$, or $\pi(\operatorname{im}(\varphi)) \leq M^{h}$ for some proper subgroup $M$ of $A$.

A computationally more involved criterium enables us to always detect when no lower central series extension exists. Denote the maximal subgroups of $A$ by $M_{1}, \ldots, M_{s}$ and let $Z_{i}:=\left\{t \in Z \mid \pi(t) \in M_{i}^{h}\right\}$. Then

$$
\varphi(\Delta)=Z \backslash \cup_{i=1}^{s} Z_{i} .
$$

We use the Inclusion-Exclusion Principle to determine the cardinality of $\varphi(\Delta)$ via

$$
\begin{aligned}
|\varphi(\Delta)| & =|Z|-\left|\cup_{i=1}^{s} Z_{i}\right| \\
& =|Z|-\left(\sum_{k=1}^{s}(-1)^{k+1}\left(\sum_{1 \leq i_{1}<\ldots<i_{k} \leq s}\left|Z_{i_{1}} \cap \ldots \cap Z_{i_{k}}\right|\right)\right)
\end{aligned}
$$

Each intersection $Z_{i_{1}} \cap \ldots \cap Z_{i_{k}}$ can be computed readily from $M_{i_{1}}, \ldots, M_{i_{k}}$ and $Z$ by solving a system of linear equations. In the special case that $A$ is elementary abelian this approach simplifies as the automorphism group acts as full symmetric group on the maximal subgroups and only one intersection of every type has to be determined.

### 6.3 Derived series extensions

Let $G$ be a finite non-trivial solvable group and $A$ a finite non-trivial abelian group with a $G$-module structure. Our aim is to compute a complete and irredundant set of isomorphism type representatives of derived series extensions of $A$ by $G$.
We choose a polycyclic generating sequence $\mathfrak{g}=\left\{g_{1}, \ldots, g_{n}\right\}$ such that it refines the derived series of $G$. As $\gamma(G)$ is a subgroup of this series, it follows that there exists $m \in\{1, \ldots, n\}$ with $\gamma(G)=\left\langle g_{m}, \ldots, g_{n}\right\rangle$. Let $J=\{(i, j) \mid 1 \leq j \leq i \leq n\}$ and $I=\{(i, j) \mid m \leq j<i \leq n\}$. Then $|J|=l$ and we denote $|I|=k$. We denote the natural projection corresponding to $I, J$ and $A /[A, \gamma(G)]$ by

$$
\mu: A^{l} \rightarrow(A /[A, \gamma(G)])^{k}:\left(t_{i, j} \mid(i, j) \in J\right) \mapsto\left(t_{i, j}+[A, \gamma(G)] \mid(i, j) \in I\right)
$$

By Theorem 10 we have achieved our aim if we find $\varphi(\Omega)$ for a set $\Omega$ as defined in Theorem [10, Recall that $\Omega$ is a set of orbit representatives in the set $\Delta$ from (8). The following lemma provides a criterion to check whether a given $\delta \in Z^{2}(G, A)$ satisfies $[\delta] \in \Delta$.
13 Lemma: Let $G$ be a non-trivial solvable group, $A$ a non-trivial abelian group with a $G$-module structure and $\delta \in Z^{2}(G, A)$. Then $\kappa(\sigma(\delta))$ is surjective if and only if $\mu\left(t_{\delta}\right)$ is full.

Proof: The induced cocycle $\sigma(\delta)$ defines an extension of $A /[A, \gamma(G)]$ by $\gamma(G)$. Applying Lemma 11 yields that $\kappa(\sigma(\delta))$ is surjective if and only if $\pi\left(t_{\sigma(\delta)}\right)$ is full. The latter equals $\mu\left(t_{\delta}\right)$ which completes the proof.

Following the approach described in Section 6.1 we first compute $Z / B$ and then its orbits under the action of $\operatorname{Comp}(G, A)$. We next choose orbit representatives $r_{1}, \ldots, r_{m} \in Z / B$ and then a representative $t_{i}$ in $A^{l}$ for each $r_{i}, 1 \leq i \leq m$. According to Theorem 10(a) and Lemma 13 we then have $\varphi(\Omega)=\left\{t_{i} \mid \mu\left(t_{i}\right)\right.$ is full $\}$ for some $\Omega$ as desired and $\{P(t) \mid$ $t \in \varphi(\Omega)\}$ is a set of presentations for a complete and irredundant list of derived series extensions of $A$ by $G$.

## 7 Application

Groups of given derived length and minimal composition length are known up to derived length 8 , see [8]. Here we use the method from Section 6.3 to obtain new information on groups of derived length 10. For our computations we utilised the GAP package SpecialExt [4] which implements the methods from Section 6 for elementary abelian groups. The sporadic simple group $F i_{23}$ has a maximal solvable subgroup $M$ of order $2^{11} 3^{13}$. This group has the form $M=\mathrm{GL}(2,3) \ltimes 3^{2+1} \ltimes 2^{6+1} \ltimes 3^{8+1}$, where $p^{r+s}$ is used to denote an $r$-generator group of order $p^{r+s}$ and class 2. The group $M$ has derived length 10 and composition length 24 . It is conjectured that 24 is the minimal possible composition length for a group of derived length 10 , see [ 8 . Previously the group $M$ has been the only group known to achieve this bound.
Let $M=M^{(1)} \geq M^{(2)} \geq \ldots \geq M^{(11)}=\{1\}$ denote the derived series of $M$. We have used the method from Section 6.3 to determine a complete and irredundant set of isomorphism type representatives of derived series extensions of $A:=M^{(i)} / M^{(i+1)}$ by $G:=M / M^{(i)}$ for $2 \leq i \leq 10$ using the $G$-module structure of $A$ inside $M$. The following four values of $i$ yield more than one isomorphism type representative.
$\mathbf{i}=4$ : Here $G \cong S_{4}$ and $A \cong C_{2}$ is a trivial $G$-module. We obtain two non-isomorphic derived series extensions: the groups with the numbers 28 and 29 in the SmallGroups Library described in [1].
$\mathbf{i}=\mathbf{6}$ : Here $G \cong \mathrm{GL}_{2}(3) \ltimes 3^{2}$ and $A \cong C_{3}^{2}$. We obtain three non-isomorphic derived series extensions: the groups with the numbers 2889, 2890 and 2891 in the SmallGroups Library.
$\mathbf{i}=\mathbf{8}$ : Here $G \cong \mathrm{GL}_{2}(3) \ltimes 3^{2+1} \ltimes 2^{6}$ and $A \cong C_{2}$ is a trivial $G$-module. We obtain two non-isomorphic derived series extensions.
$\mathbf{i}=\mathbf{1 0}$ : Here $G \cong \mathrm{GL}_{2}(3) \ltimes 3^{2+1} \ltimes 2^{6+1} \ltimes 3^{8}$ and $A \cong C_{3}$. We obtain three non-isomorphic derived series extensions.

In particular, the computation for $i$ equal to 10 yields two new examples of groups of derived length 10 and composition length 24 . We describe all three groups arising from this case via a parametrised polycyclic presentation. This has 24 generators $g_{1}, \ldots, g_{24}$ and the relations exhibited in Figures 1 and 2 where conjugation relations of the form $g_{i}^{g_{j}}=g_{i}$ are omitted in the latter. The relations with left hand sides $g_{2}^{3}$ and $g_{2}^{g_{1}}$ contain a parameter $k$ and the three different groups are obtained for $k \in\{0,1,2\}$. The presentations are also available as examples in the GAP package SpecialExt. Using GAP it is straightforward to determine the orders of conjugacy class representatives of each of these three groups, which yields an independent check for the non-isomorphism of the groups.

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We thank Stephen Glasby for pointing us at the interesting example of the group $M$ from Section 7 .

$$
\begin{aligned}
& g_{1}^{2}=1, g_{2}^{3}=g_{24}^{k}, g_{3}^{2}=g_{5}, g_{4}^{2}=g_{5}, g_{5}^{2}=1, g_{6}^{3}=1, g_{7}^{3}=1, g_{8}^{3}=1, g_{9}^{2}=g_{15}, g_{10}^{2}=g_{15}, g_{11}^{2}=g_{15} \\
& g_{12}^{2}=g_{15}, g_{13}^{2}=g_{15}, g_{14}^{2}=g_{15}, g_{15}^{2}=1, g_{16}^{3}=1, g_{17}^{3}=1, g_{18}^{3}=1, g_{19}^{3}=1, g_{20}^{3}=1, g_{21}^{3}=1 \\
& g_{22}^{3}=1, g_{23}^{3}=1, g_{24}^{3}=1
\end{aligned}
$$

Figure 1: Power relations

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$$
\begin{aligned}
& g_{2}^{g_{1}}=g_{2}^{2} g_{24}^{3-k}, g_{3}^{g_{1}}=g_{3} g_{5}, g_{3}^{g_{2}}=g_{3} g_{4}, g_{4}^{g_{1}}=g_{3} g_{4} g_{5}, g_{4}^{g_{2}}=g_{3}, g_{4}^{g_{3}}=g_{4} g_{5}, g_{6}^{g_{3}}=g_{6}^{2} g_{7} g_{8}, g_{6}^{g_{4}}=g_{6} g_{7} g_{8}^{2}, \\
& g_{6}^{g_{5}}=g_{6}^{2}, g_{7}^{g_{1}}=g_{6}^{2} g_{7}^{2} g_{8}^{2}, g_{7}^{g_{2}}=g_{6} g_{7} g_{8}^{2}, g_{7}^{g_{3}}=g_{6} g_{7} g_{8}^{2}, g_{7}^{g_{4}}=g_{6} g_{7}^{2} g_{8}, g_{7}^{g_{5}}=g_{7}^{2}, g_{7}^{g_{6}}=g_{7} g_{8}, g_{8}^{g_{1}}=g_{8}^{2} \text {, } \\
& g_{9}^{g_{1}}=g_{10} g_{11} g_{12} g_{13} g_{14} g_{15}, g_{9}^{g_{2}}=g_{9} g_{12} g_{14}, g_{9}^{g_{3}}=g_{9} g_{12} g_{14}, g_{9}^{g_{4}}=g_{9} g_{11} g_{12} g_{13} g_{14}, g_{9}^{g_{6}}=g_{13}, g_{9}^{g_{8}}=g_{10}, \\
& g_{10}^{g_{1}}=g_{9} g_{12} g_{14} g_{15}, g_{10}^{g_{2}}=g_{10} g_{11} g_{12} g_{13} g_{14}, g_{10}^{g_{3}}=g_{10} g_{11} g_{12} g_{13} g_{14}, g_{10}^{g_{4}}=g_{10} g_{11} g_{13}, g_{10}^{g_{6}}=g_{14}, \\
& g_{10}^{g_{8}}=g_{9} g_{10}, g_{10}^{g_{9}}=g_{10} g_{15}, g_{11}^{g_{1}}=g_{9} g_{10} g_{12} g_{13} g_{14} g_{15}, g_{11}^{g_{2}}=g_{10} g_{11} g_{14}, g_{11}^{g_{3}}=g_{9} g_{10} g_{12} g_{13} g_{14}, \\
& g_{11}^{g_{4}}=g_{10} g_{12} g_{13} g_{14}, g_{11}^{g_{5}}=g_{13}, g_{11}^{g_{6}}=g_{9}, g_{11}^{g_{7}}=g_{12}, g_{11}^{g_{8}}=g_{12}, g_{12}^{g_{1}}=g_{10} g_{11} g_{14} g_{15} \text {, } \\
& g_{12}^{g_{2}}=g_{9} g_{10} g_{12} g_{13} g_{14}, g_{12}^{g_{3}}=g_{9} g_{11} g_{12} g_{13}, g_{12}^{g_{4}}=g_{9} g_{10} g_{11} g_{12} g_{13}, g_{12}^{g_{5}}=g_{14}, g_{12}^{g_{6}}=g_{10}, g_{12}^{g_{7}}=g_{11} g_{12} \text {, } \\
& g_{12}^{g_{8}}=g_{11} g_{12}, g_{12}^{g_{11}}=g_{12} g_{15}, g_{13}^{g_{1}}=g_{9} g_{10} g_{11} g_{12} g_{14} g_{15}, g_{13}^{g_{2}}=g_{10} g_{12} g_{13}, g_{13}^{g_{3}}=g_{9} g_{10} g_{11} g_{12} g_{14} \text {, } \\
& g_{13}^{g_{4}}=g_{10} g_{11} g_{12} g_{14}, g_{13}^{g_{5}}=g_{11}, g_{13}^{g_{6}}=g_{11}, g_{13}^{g_{7}}=g_{13} g_{14}, g_{13}^{g_{8}}=g_{14}, g_{14}^{g_{1}}=g_{10} g_{12} g_{13} g_{15} \text {, } \\
& g_{14}^{g_{2}}=g_{9} g_{10} g_{11} g_{12} g_{14}, g_{14}^{g_{3}}=g_{9} g_{11} g_{13} g_{14}, g_{14}^{g_{4}}=g_{9} g_{10} g_{11} g_{13} g_{14}, g_{14}^{g_{5}}=g_{12}, g_{14}^{g_{6}}=g_{12}, g_{14}^{g_{7}}=g_{13}, \\
& g_{14}^{g_{8}}=g_{13} g_{14}, g_{14}^{g_{13}}=g_{14} g_{15}, g_{16}^{g_{1}}=g_{16}^{2} g_{17}^{2} g_{18}^{2} g_{20}^{2} g_{24}, g_{16}^{g_{2}}=g_{16} g_{17} g_{18} g_{20} g_{24}^{2}, g_{16}^{g_{3}}=g_{16} g_{17} g_{18} g_{20} g_{24}^{2}, \\
& g_{16}^{g_{4}}=g_{16}^{2} g_{17} g_{18} g_{20}, g_{16}^{g_{5}}=g_{16}^{2} g_{24}, g_{16}^{g_{9}}=g_{16}^{2} g_{20}, g_{16}^{g_{10}}=g_{16} g_{20} g_{24}^{2}, g_{16}^{g_{11}}=g_{16}^{2} g_{18}, g_{16}^{g_{12}}=g_{16} g_{18} g_{24}^{2}, \\
& g_{16}^{g_{13}}=g_{16}^{2} g_{17} g_{24}^{2}, g_{16}^{g_{14}}=g_{16} g_{17} g_{24}, g_{16}^{g_{15}}=g_{16}^{2} g_{24}, g_{17}^{g_{1}}=g_{16}^{2} g_{17}^{2} g_{18} g_{19}^{2} g_{20} g_{21}^{2} g_{22}^{2}, \\
& g_{17}^{g_{2}}=g_{16}^{2} g_{17}^{2} g_{18} g_{19}^{2} g_{20} g_{21}^{2} g_{22}^{2} g_{24}, g_{17}^{g_{3}}=g_{16} g_{17} g_{19}^{2} g_{21}^{2} g_{22}^{2}, g_{17}^{g_{4}}=g_{16} g_{18}^{2} g_{19}^{2} g_{21}^{2} g_{22}^{2} g_{24}, g_{17}^{g_{5}}=g_{18}^{2}, \\
& g_{17}^{g_{6}}=g_{18} g_{24}, g_{17}^{g_{7}}=g_{16} g_{17} g_{24}^{2}, g_{17}^{g_{8}}=g_{16}^{2} g_{17} g_{24}^{2}, g_{17}^{g_{9}}=g_{17}^{2} g_{21}, g_{17}^{g_{10}}=g_{17} g_{21} g_{24}^{2}, g_{17}^{g_{11}}=g_{17}^{2} g_{19}, \\
& g_{17}^{g_{12}}=g_{17} g_{19} g_{24}^{2}, g_{17}^{g_{13}}=g_{16} g_{17} g_{24}, g_{17}^{g_{14}}=g_{16} g_{17}^{2} g_{24}^{2}, g_{17}^{g_{17}}=g_{17}^{2} g_{24}, g_{18}^{g_{1}}=g_{16}^{2} g_{17} g_{18}^{2} g_{19}^{2} g_{20} g_{21}^{2} g_{22}^{2} g_{24}^{2}, \\
& g_{18}^{g_{2}}=g_{16}^{2} g_{17} g_{18}^{2} g_{19}^{2} g_{20} g_{21}^{2} g_{22}^{2} g_{24}, g_{18}^{g_{3}}=g_{16} g_{18} g_{19}^{2} g_{21}^{2} g_{22}^{2}, g_{18}^{g_{4}}=g_{16} g_{17}^{2} g_{19}^{2} g_{21}^{2} g_{22}^{2} g_{24}, g_{18}^{g_{5}}=g_{17}^{2}, \\
& g_{18}^{g_{6}}=g_{20}, g_{18}^{g_{7}}=g_{16}^{2} g_{18} g_{24}^{2}, g_{18}^{g_{8}}=g_{16}^{2} g_{18} g_{24}^{2}, g_{18}^{g_{9}}=g_{18}^{2} g_{22}, g_{18}^{g_{10}}=g_{18} g_{22} g_{24}, g_{18}^{g_{11}}=g_{16} g_{18} g_{24}, \\
& g_{18}^{g_{12}}=g_{16} g_{18}^{2}, g_{18}^{g_{13}}=g_{18}^{2} g_{19} g_{24}, g_{18}^{g_{14}}=g_{18} g_{19} g_{24}^{2}, g_{18}^{g_{15}}=g_{18}^{2} g_{24}^{2}, g_{19}^{g_{1}}=g_{17}^{2} g_{18}^{2} g_{19} g_{20}^{2} g_{21}^{2} g_{22}^{2} g_{23}^{2}, \\
& g_{19}^{g_{2}}=g_{17} g_{18} g_{19}^{2} g_{20} g_{21} g_{22} g_{23}, g_{19}^{g_{3}}=g_{17}^{2} g_{18}^{2} g_{19} g_{23} g_{24}, g_{19}^{g_{4}}=g_{17}^{2} g_{18}^{2} g_{19}^{2} g_{23} g_{24}, g_{19}^{g_{5}}=g_{19}^{2} g_{24}^{2}, \\
& g_{19}^{g_{6}}=g_{22} g_{24}^{2}, g_{19}^{g_{7}}=g_{16}^{2} g_{17}^{2} g_{18} g_{19}, g_{19}^{g_{8}}=g_{16} g_{17}^{2} g_{18}^{2} g_{19} g_{24}, g_{19}^{g_{9}}=g_{19}^{2} g_{23} g_{24}, g_{19}^{g_{10}}=g_{19} g_{23} g_{24}^{2}, \\
& g_{19}^{g_{11}}=g_{17} g_{19} g_{24}, g_{19}^{g_{12}}=g_{17} g_{19}^{2}, g_{19}^{g_{13}}=g_{18} g_{19} g_{24}^{2}, g_{19}^{g_{14}}=g_{18} g_{19}^{2} g_{24}, g_{19}^{g_{15}}=g_{19}^{2} g_{24}^{2}, \\
& g_{20}^{g_{1}}=g_{16}^{2} g_{17} g_{18} g_{19}^{2} g_{20}^{2} g_{21}^{2} g_{22}^{2}, g_{20}^{g_{2}}=g_{16}^{2} g_{17} g_{18} g_{19}^{2} g_{20}^{2} g_{21}^{2} g_{22}^{2} g_{24}^{2}, g_{20}^{g_{3}}=g_{16}^{2} g_{17} g_{18} g_{19}^{2} g_{20}^{2} g_{21}^{2} g_{22}^{2} g_{24}^{2}, \\
& g_{20}^{g_{4}}=g_{16}^{2} g_{17}^{2} g_{18}^{2} g_{19}^{2} g_{20} g_{21}^{2} g_{22}^{2}, g_{20}^{g_{5}}=g_{20}^{2} g_{24}^{2}, g_{20}^{g_{6}}=g_{17} g_{24}^{2}, g_{20}^{g_{8}}=g_{16}^{2} g_{20} g_{24}^{2}, g_{20}^{g_{9}}=g_{16} g_{20} g_{24}, \\
& g_{20}^{g_{10}}=g_{16} g_{20}^{2}, g_{20}^{g_{11}}=g_{20}^{2} g_{22}, g_{20}^{g_{12}}=g_{20} g_{22} g_{24}, g_{20}^{g_{13}}=g_{20}^{2} g_{21} g_{24}, g_{20}^{g_{14}}=g_{20} g_{21} g_{24}^{2}, g_{20}^{g_{15}}=g_{20}^{2} g_{24}^{2}, \\
& g_{20}^{g_{19}}=g_{20} g_{24}^{2}, g_{21}^{g_{1}}=g_{17}^{2} g_{18}^{2} g_{19}^{2} g_{20}^{2} g_{21} g_{22}^{2} g_{23}^{2} g_{24}, g_{21}^{g_{2}}=g_{17} g_{18} g_{19} g_{20} g_{21}^{2} g_{22} g_{23} g_{24}, \\
& g_{21}^{g_{3}}=g_{16} g_{19}^{2} g_{20}^{2} g_{22}^{2} g_{23} g_{24}, g_{21}^{g_{4}}=g_{16}^{2} g_{19} g_{20}^{2} g_{21} g_{23} g_{24}^{2}, g_{21}^{g_{5}}=g_{22}^{2}, g_{21}^{g_{6}}=g_{19}, g_{21}^{g_{7}}=g_{20} g_{21} g_{24}^{2}, \\
& g_{21}^{g_{8}}=g_{16} g_{17}^{2} g_{20}^{2} g_{21} g_{24}, g_{21}^{g_{9}}=g_{17} g_{21} g_{24}, g_{21}^{g_{10}}=g_{17} g_{21}^{2}, g_{21}^{g_{11}}=g_{21}^{2} g_{23} g_{24}, g_{21}^{g_{12}}=g_{21} g_{23} g_{24}^{2}, \\
& g_{21}^{g_{13}}=g_{20} g_{21} g_{24}^{2}, g_{21}^{g_{14}}=g_{20} g_{21}^{2} g_{24}, g_{21}^{g_{15}}=g_{21}^{2} g_{24}^{2}, g_{21}^{g_{18}}=g_{21} g_{24}, g_{22}^{g_{1}}=g_{17}^{2} g_{18}^{2} g_{19}^{2} g_{20}^{2} g_{21}^{2} g_{22} g_{23}^{2} g_{24}, \\
& g_{22}^{g_{2}}=g_{17} g_{18} g_{19} g_{20} g_{21} g_{22}^{2} g_{23} g_{24}, g_{22}^{g_{3}}=g_{16} g_{19}^{2} g_{20}^{2} g_{21}^{2} g_{23} g_{24}, g_{22}^{g_{4}}=g_{16}^{2} g_{19} g_{20}^{2} g_{22} g_{23} g_{24}^{2}, g_{22}^{g_{5}}=g_{21}^{2} \text {, } \\
& g_{22}^{g_{6}}=g_{21} g_{24}, g_{22}^{g_{7}}=g_{20}^{2} g_{22} g_{24}, g_{22}^{g_{8}}=g_{16} g_{18}^{2} g_{20}^{2} g_{22}, g_{22}^{g_{9}}=g_{18} g_{22} g_{24}^{2}, g_{22}^{g_{10}}=g_{18} g_{22}^{2}, g_{22}^{g_{11}}=g_{20} g_{22} g_{24}^{2}, \\
& g_{22}^{g_{12}}=g_{20} g_{22}^{2}, g_{22}^{g_{13}}=g_{22}^{2} g_{23}, g_{22}^{g_{14}}=g_{22} g_{23} g_{24}^{2}, g_{22}^{g_{15}}=g_{22}^{2} g_{24}, g_{22}^{g_{17}}=g_{22} g_{24}, g_{23}^{g_{1}}=g_{19}^{2} g_{21}^{2} g_{22}^{2} g_{23} g_{24}^{2}, \\
& g_{23}^{g_{2}}=g_{19}^{2} g_{21}^{2} g_{22}^{2} g_{23} g_{24}, g_{23}^{g_{3}}=g_{16}^{2} g_{17}^{2} g_{18}^{2} g_{19}^{2} g_{21} g_{22} g_{23}^{2} g_{24}^{2}, g_{23}^{g_{4}}=g_{16}^{2} g_{17} g_{18} g_{19}^{2} g_{21} g_{22} g_{23}, g_{23}^{g_{5}}=g_{23}^{2} g_{24}^{2}, \\
& g_{23}^{g_{7}}=g_{20}^{2} g_{21}^{2} g_{22} g_{23}, g_{23}^{g_{8}}=g_{16}^{2} g_{17} g_{18} g_{19}^{2} g_{20} g_{21}^{2} g_{22}^{2} g_{23} g_{24}^{2}, g_{23}^{g_{9}}=g_{19} g_{23} g_{24}^{2}, g_{23}^{g_{10}}=g_{19} g_{23}^{2} g_{24}, \\
& g_{23}^{g_{11}}=g_{21} g_{23} g_{24}^{2}, g_{23}^{g_{12}}=g_{21} g_{23}^{2} g_{24}, g_{23}^{g_{13}}=g_{22} g_{23} g_{24}, g_{23}^{g_{14}}=g_{22} g_{23}^{2}, g_{23}^{g_{15}}=g_{23}^{2} g_{24}^{2}, g_{23}^{g_{16}}=g_{23} g_{24}^{2}, \\
& g_{24}^{g_{1}}=g_{24}^{2}
\end{aligned}
$$

Figure 2: Conjugation relations

