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Research Article

Jack O. Button*

Free by cyclic groups and linear groups with restricted unipotent elements

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Abstract: We introduce the class of linear groups that do not contain unipotent elements of infinite order, which includes all linear groups in positive characteristic. We show that groups in this class have good closure properties, in addition to having properties akin to non-positive curvature, which were proved in [6]. We give examples of abstract groups lying in this class, but also show that Gersten's free by cyclic group does not. This implies that it has no faithful linear representation of any dimension over any field of positive characteristic, nor can it be embedded in any complex unitary group.

Keywords: Linear, positive characteristic, unipotent, free by cyclic

MSC 2010: Primary 20F67; secondary 20F65

1 Introduction

A common question to ask about an abstract group G is whether it is linear, meaning that it is isomorphic to a subgroup of $\mathrm{GL}(d, \mathbb{F})$ for some dimension d and some field $\mathbb{F}$. If so, then we can conclude that various further properties hold for G, for instance, if G is also finitely generated, then it will be residually finite and satisfy the Tits alternative. However, if conversely G is an abstract group known to satisfy all such properties, then how do we determine whether G is actually linear? Some results are known but still many questions remain. For instance linearity of the Braid groups was a longstanding open question which was eventually resolved in the positive by [3] and [14]. On the other hand, [9] showed that for the free group F_m of rank m, the automorphism group $\mathrm{Aut}(F_m)$ and the outer automorphism group $\mathrm{Out}(F_n)$ fail to be linear for $m \geq 3$ and $n \geq 4$, respectively (though linearity of $\mathrm{Out}(F_3)$ is still open). But linearity of the mapping class groups $\mathcal{M}_g$ for genus $g \geq 3$ is a notorious unsolved problem, and also it is not known if all free by cyclic groups $F_n \rtimes \mathbb{Z}$ or all closed 3-manifold groups $\pi_1(M^3)$ are linear.

Now although the definition of linearity only refers to some field, in practise we tend to think of linearity over $\mathbb{C}$. In fact, if a group is finitely generated, then it embeds in $\mathrm{GL}(d, \mathbb{F})$ for $\mathbb{F}$ some field of characteristic zero if and only if it embeds in $\mathbb{C}$, thus as our main focus here is indeed on finitely generated groups, we can therefore take $\mathbb{F} = \mathbb{C}$ without loss of generality when in characteristic zero. However, we will also want to examine linearity in positive characteristic and we will be arguing that such groups are much better behaved in general than those that are linear over fields of characteristic zero. This is because often the problems in characteristic zero are due to unipotent elements (those with all eigenvalues equal to 1), but in positive characteristic, such matrices always have finite order.

Consequently, in this paper and its companion [6], we introduce the class of linear groups where every unipotent matrix has finite order (which we call NIU-linear groups for "no infinite order unipotents"). This means that in characteristic zero, the only unipotent element is the identity, however, it includes all linear groups in positive characteristic. We show that this class of groups has very good properties which

***Corresponding author: Jack O. Button:** Selwyn College, University of Cambridge, Cambridge CB3 9DQ, United Kingdom, e-mail: j.o.button@dpmms.cam.ac.uk

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are certainly not shared by linear groups in general. In [6] the focus was on what we called properties of non-positive curvature and we proved the following theorem.

Theorem 1.1. *If G is a finitely generated group, which is linear in positive characteristic, or which is linear in characteristic zero and contains no unipotent elements except the identity, then the following hold:*
(i) *Every polycyclic subgroup of G is virtually abelian.*
(ii) *All finitely generated abelian subgroups of G are undistorted in G.*
(iii) *G does not contain subgroups of the form $\langle a, t \mid t^{-1}a^p t = a^q \rangle$ for non-zero p, q, with $|p| \neq |q|$.*
(iv) *If $A \cong \mathbb{Z}^n$ is central in G, then there exists a subgroup of finite index in G that contains A as a direct factor.*

This means that we can use any of these four properties in Theorem 1.1 (each of which fails to hold when considered over all linear groups) as obstructions for a group being NIU-linear. Moreover, we can use NIU-linearity as a test case for linearity in general, in that we can regard it as a stronger version of linearity which is equivalent in positive characteristic. In [6] we showed that the mapping class groups $\mathcal{M}_g$ with genus $g \geq 3$ are not NIU-linear by utilising the fact that they fail a property which is similar to Theorem 1.1 (iv). But in this paper we will be considering groups which pass all four of these properties, such as free by cyclic groups, and will determine whether or not they are NIU-linear.

The paper is organised as follows. In Section 2 we provide the basic facts we will need and introduce this class of NIU-linear groups, as well as a stricter class of groups called VUF-linear groups (for “virtually unipotent free”), though in characteristic zero, these two classes are the same. One reason for introducing this stricter class is that for these groups, we showed in [6] that Theorem 1.1 (i) can be strengthened to: any solvable subgroup is both finitely generated and virtually abelian. We also show in Section 2 that both of these two classes of groups have the good closure properties possessed by linear groups in general, namely, they are preserved by taking subgroups, commensurability classes, and by free and direct products.

In Section 3 we concentrate on giving examples of which well-known linear groups are in fact NIU-linear. Section 3.1 shows that this holds for all limit groups, whereas Section 3.2 looks at the fundamental groups of compact 3-manifolds. As graph manifolds cause problems here because linearity is still open for some of these fundamental groups, we concentrate on 3-manifolds with a geometric structure. First we assume that M^3 is closed, whereupon Theorem 3.3 shows that if M^3 has one of the eight Thurston geometries, then $\pi_1(M^3)$ is NIU-linear (and indeed VUF-linear) if and only if it is not modelled on Nil, Sol or $\widetilde{\mathrm{PSL}(2, \mathbb{R})}$. This is straightforward because we can eliminate these geometries using the various obstructions in Theorem 1.1.

For compact 3-manifolds with boundary where the interior has a geometric structure, we show in Theorem 3.4 that their fundamental groups are all NIU-linear. The interesting thing here is that whilst this fact for closed hyperbolic 3-manifolds follows immediately, we require the machinery of Agol and Wise in the finite volume case in order to alter the linear representation of the fundamental group so that it has no unipotent elements. We also look at RAAGs in Section 3.3.

In Section 4 we take a look at free by cyclic groups. As mentioned above, it is not known if they are all linear but this does hold for all groups of the form $F_2 \rtimes \mathbb{Z}$, so we first show that these are all NIU-linear and VUF-linear. However, Gersten's famous group of the form $F_3 \rtimes \mathbb{Z}$, introduced in [10], is not known to be linear and provides a good test case for linearity of free by cyclic groups in general.

We finish with Theorem 4.5 which shows that Gersten's group, despite satisfying all the conditions for NIU-linearity in Theorem 1.1, is in fact not NIU-linear. In particular, it is not linear over any field of positive characteristic, and in characteristic zero it cannot embed in any orthogonal or unitary group. This result might be seen as evidence that not all free by cyclic groups are linear, and it certainly suggests that if these groups are all linear, then the resulting faithful matrix representations will be rather nasty and will have to contain many unipotent elements, just as would be the case for the mapping class groups.

2 Preliminaries

We begin this section with the definitions and basic facts that we will need. If $\mathbb{F}$ is any field and $d \in \mathbb{N}$ is any positive integer, then we say that an element g of the general linear group $\mathrm{GL}(d, \mathbb{F})$ is *unipotent* if all its

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eigenvalues (considered over the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$) are equal to 1, or equivalently some positive power of $g - I$ is the zero matrix.

Proposition 2.1. (i) *If $\mathbb{F}$ is a field of characteristic $p > 0$ and $M \in \mathrm{GL}(d, \mathbb{F})$ is unipotent, then M has finite order equal to some power of p. Conversely, if M is any element of $\mathrm{GL}(d, \mathbb{F})$ with order n which is a multiple of p, then $M^{n/p}$ is a non-identity unipotent element.*
(ii) *If $\mathbb{F}$ has zero characteristic, then the only unipotent element M having finite order in $\mathrm{GL}(d, \mathbb{F})$ is I.*

Proof. For (i), there exists $r > 0$ with $N^l = 0$ for $l \geq r$, where $N = M - I$. If we take k to be any power of p which is at least r, then

$$M^k = (I + N)^k = I^k + \binom{k}{1} I^{k-1} N + \cdots + \binom{k}{k-1} I N^{k-1} + N^k.$$

But $N^k = 0$ because $k \geq r$ and $\binom{k}{i} \equiv 0$ modulo p for $0 < i < k$, as k is a power of p, thus M has order dividing k.

We now assume for the rest of the proof that $\mathbb{F}$ is algebraically closed. As $1 \leq n/p < n$, we know that $M^{n/p}$ has order exactly p and hence has minimum polynomial $p(x)$ dividing $x^p - 1 = (x - 1)^p$. But any eigenvalue of a matrix must be a root of its minimum polynomial, so $M^{n/p}$ is unipotent.

For (ii), if M has finite order, then the minimum polynomial of M is $x^n - 1$ for some $n \in \mathbb{N}$ and in characteristic zero this has no repeated roots, so M is diagonalisable over $\mathbb{F}$ but has all eigenvalues equal to 1, and so is the identity. □

We now come to the two key definitions of this work.

Definition 2.2. If $\mathbb{F}$ is any field and $d \in \mathbb{N}$ any dimension, then we say that a subgroup G of $\mathrm{GL}(d, \mathbb{F})$ is *NIU-linear* (standing for linear with no infinite order unipotents) if every unipotent element of G has finite order.

Note. By Proposition 2.1, if $\mathbb{F}$ has positive characteristic, then G is automatically NIU-linear. If $\mathbb{F}$ has characteristic zero, then the definition says that the only unipotent element of G is the identity.

Example. If G is any subgroup of the real orthogonal group $O(d)$ or of the complex unitary group $U(d)$ in any dimension d, then G is NIU-linear because all orthogonal or unitary matrices are diagonalisable over $\mathbb{C}$.

Definition 2.3. If $\mathbb{F}$ is any field and $d \in \mathbb{N}$ any dimension, then we say that a subgroup G of $\mathrm{GL}(d, \mathbb{F})$ is *VUF-linear* (standing for linear and virtually unipotent free) if G has a finite index subgroup H, where the only unipotent element of H is the identity.

Note. Clearly VUF-linear implies NIU-linear, and they are the same in characteristic zero. As for the case when G is linear in positive characteristic, clearly if G is also virtually torsion free, then it is VUF-linear. Although this need not be true the other way round, it does hold if G is finitely generated, say by [22, Corollary 4.8]. This states that any finitely generated linear group has a finite index subgroup whose elements of finite order are all unipotent (which might be thought of as "Selberg's theorem in arbitrary characteristic").

When we have a group G which is only given in abstract form, then to say G is NIU-linear or VUF-linear will mean that there exists some field $\mathbb{F}$ and dimension d such that G has a faithful representation in $\mathrm{GL}(d, \mathbb{F})$, and the image of this representation has the respective property.

2.1 Closure properties

Let us first examine the closure properties of arbitrary linear groups. Clearly subgroups of linear groups are also linear, as well as supergroups of finite index by induced representations, so that linearity is a commensurability invariant. Indeed, as here we do not have to vary the field, commensurability holds more specifically for linearity over a particular field, and certainly in a particular characteristic.

If we have two groups which are linear over different fields of the same characteristic, say $A \leq \mathrm{GL}(d_1, \mathbb{F}_1)$ and $B \leq \mathrm{GL}(d_2, \mathbb{F}_2)$, then not only can we take $d_1 = d_2$ without loss of generality, but we can also assume that $\mathbb{F}_1 = \mathbb{F}_2 = \mathbb{F}$ as well. (As they have the same prime subfield $\mathbb{P}$, we can adjoin enough transcendental

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elements to $\mathbb{P}$ which are all algebraically independent, resulting in a field $\mathbb{F}'$ where all elements of $\mathbb{F}_1$ and $\mathbb{F}_2$ are algebraic over $\mathbb{F}'$, so that $\mathbb{F}_1$ and $\mathbb{F}_2$ both embed in the algebraic closure $\mathbb{F}$ of $\mathbb{F}'$.) We then see that the direct product of $A \times B$ is linear over $\mathbb{F}$ in the usual way by combining the two blocks representing A and B.

As for free products, it was shown by Nisnevič [18] that the free product of linear groups is linear. In particular, if A and B are subgroups of $\mathrm{GL}(d, \mathbb{F})$ for $\mathbb{F}$ a field of characteristic p (zero or prime), then $A * B$ embeds in $\mathrm{GL}(d+1, \mathbb{E})$, where $\mathbb{E}$ is some extension field of $\mathbb{F}$ (and so has the same characteristic). Moreover, if neither A nor B contain any scalar matrices except the identity, then $A * B$ embeds in $\mathrm{GL}(d, \mathbb{E})$. Further results are in [21] and [20], with the latter giving a useful general result to establish the linearity of free products with an abelian amalgamated subgroup, which we state here as follows.

Proposition 2.4 ([20, Proposition 1.3]). *Suppose $G_1 *_H G_2$ is a free product with abelian amalgamated subgroup H and suppose we have faithful representations $\rho_i\colon G_i \hookrightarrow \mathrm{GL}(d, \mathbb{F})$ for $d \geq 2$ and $i = 1, 2$ over any field $\mathbb{F}$ such that:*
(a) *ρ_1 and ρ_2 agree on H,*
(b) *$\rho_1(h)$ is diagonal for all $h \in H$,*
(c) *for all $g \in G_1 \setminus H$, we have that the bottom left-hand entry of $\rho_1(g)$ is non-zero, and similarly for the top right-hand entry of $\rho_2(g)$ for all $g \in G_2 \setminus H$.*

Then $G_1 *_H G_2$ embeds in $\mathrm{GL}(d, \mathbb{F}(t))$, where t is a transcendental element over $\mathbb{F}$. Moreover, any $g \in G_1 *_H G_2$ either has transcendental trace in the resulting faithful representation or g is conjugate into G_1 or G_2.

Proof. In [20] the result was stated just for $\mathbb{C}$ (and for $\mathrm{SL}(d, \mathbb{C})$ rather than $\mathrm{GL}(d, \mathbb{C})$), but the proof goes through in general, so here we just give a summary.

Define the representation $\rho\colon G_1 *_H G_2 \to \mathrm{GL}(d, \mathbb{F}(t))$ as equal to ρ_2 on G_2, but on G_1 we replace ρ_1 by the conjugate representation $T\rho_1 T^{-1}$, where T is the diagonal matrix $\mathrm{diag}(t, t^2, \dots, t^n)$, and then extend to all of $G_1 *_H G_2$. Now it can be shown straightforwardly that any element not conjugate into $G_1 \cup G_2$ is conjugate in $G_1 *_H G_2$ to something with normal form

$$g = \gamma_1 \delta_1 \cdots \gamma_l \delta_l,$$

where $l \geq 1$, $\gamma_i \in G_1 \setminus H$ and $\delta_i \in G_2 \setminus H$ for $i = 1, \dots, l$. Induction on l then yields that the entries of g are Laurent polynomials in $t^{\pm 1}$ with coefficients in $\mathbb{F}$ and with the bottom right-hand entry of g equal to $\alpha t^{l(d-1)} + \cdots$, where all other terms are of strictly lower degree in t. But it can be checked that α is actually just a product of these respective bottom-left and top-right entries, thus it is a non-zero element of $\mathbb{F}$, so this bottom right-hand entry is transcendental over $\mathbb{F}$ and thus g is not the identity matrix. Furthermore, every other diagonal element of g is also a Laurent polynomial but with leading term having degree in t strictly less than $l(d-1)$, thus the trace of g is transcendental too. □

We now examine the closure properties of our two classes of linear groups defined above, which happily turn out to be the same as for arbitrary linear groups.

Proposition 2.5. *If G_1 and G_2 are groups which are both NIU-linear over fields $\mathbb{F}_1, \mathbb{F}_2$ having the same characteristic, then the following hold:*
(i) *Any subgroup S of G_1 (or G_2) is NIU-linear.*
(ii) *Any group G commensurable with G_1 (or G_2) is NIU-linear.*
(iii) *The direct product $G_1 \times G_2$ is NIU-linear.*
(iv) *The free product $G_1 * G_2$ is NIU-linear.*

The same holds with NIU-linear replaced throughout by VUF-linear.

Proof. We will proceed in the following order. First, NIU-linear groups in a given positive characteristic just mean arbitrary linear groups in this characteristic, in which case the closure properties have been mentioned above. We next argue for NIU-linear groups in characteristic zero, which here are the same as VUF-linear groups. We then finish with the necessary adjustments in our proof for VUF-linear groups in positive characteristic.

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Part (i) is immediate for NIU-linear groups and follows straight away for VUF-linear groups, because if S is a subgroup of G and H is the given finite index subgroup which is unipotent free, then $S \cap H$ has finite index in S. This now reduces (ii) to saying that if G_1 is NIU-linear and has index i in the group G, then G is NIU-linear too (and the same holds for VUF-linearity but this is immediate). We certainly know that G is linear over $\mathbb{F}$ too by taking the induced representation, whereupon for elements $g_1 \in G_1$ the induced representation is a direct sum of i blocks consisting of conjugates of the original representation of g_1, with the first block equal to the original matrix for g_1. Thus, if g_1 is not given by a unipotent matrix in the original representation of G_1, then it is still not unipotent in the induced representation of G. But if $g \in G$ is a unipotent element of infinite order in this induced representation, then so are all its positive powers g^n and some of these will lie in G_1, thus G is also NIU-linear.

For (iii) and (iv), we can again assume as above that G_1 and G_2 are both subgroups of the same linear group $\mathrm{GL}(d, \mathbb{F})$ say. On forming $G_1 \times G_2 \leq \mathrm{GL}(2d, \mathbb{F})$ in the usual way by combining the two blocks representing G_1 and G_2, we note that the eigenvalues of an element $g \in G_1 \times G_2$ are just the union of the eigenvalues in each of the two blocks. Thus, a unipotent element of $G_1 \times G_2$ is unipotent in both the G_1 and G_2 blocks, hence in characteristic zero it is the identity in $G_1 \times G_2$ if G_1 and G_2 are both NIU-linear. If instead they are VUF-linear with finite index subgroups H_1 and H_2, respectively, that are unipotent free, then so is $H_1 \times H_2$, which has finite index in $G_1 \times G_2$.

As we know free products of linear groups over the same characteristic are also linear, we have shown NIU-linearity in the case of positive characteristic. For $\mathbb{F}$ of characteristic zero, we first assume, by increasing the size of the matrices and adding ones on the diagonal if needed, that the NIU-linear groups G_1 and G_2 both embed in $\mathrm{GL}(d, \mathbb{F})$, with neither subgroup containing any scalar matrices apart from the identity. Moreover, these embeddings will still be NIU-linear. Then [20, Lemma 2.2] says that there is a conjugate of G_1 (which will henceforth be called G_1) in $\mathrm{GL}(d, \mathbb{F})$ such that no non-identity element of this conjugate has zero in its top right-hand entry. Similarly, by taking a conjugate of G_2, we can assume that no non-identity element of G_2 has zero in its bottom left-hand entry. Thus, Proposition 2.4 applies to show that $G_1 * G_2$ embeds in $\mathrm{GL}(d, \mathbb{F}(t))$ for t any element which is transcendental over $\mathbb{F}$. It further shows that any element $g \in G_1 * G_2$ which is not conjugate into G_1 or G_2 has trace which is also transcendental over $\mathbb{F}$, thus cannot equal d, and so g is not unipotent. Now being unipotent is a conjugacy invariant and as there are no unipotents in G_1 or G_2 either, the resulting faithful linear representation of $G_1 * G_2$ is NIU-linear.

Thus, we are done in characteristic zero for both our classes of linear groups. As for preservation of VUF-linearity under free products in characteristic $p > 0$, we can use a trick: by dropping down further if necessary, we can assume that both the unipotent free finite index subgroups H_1 of G_1 and H_2 of G_2 are normal. This then gives us two homomorphisms (for $i = 1, 2$) $q_i\colon G_i \to G_i/H_i$ onto finite groups and these can be both be extended from $G_1 * G_2$ to G_i/H_i with kernels which we will call K_1 and K_2. Now note that $K_1 \cap K_2$ is also normal and has finite index in $G_1 * G_2$, and that both maps from $G_1 * G_2$ to G_i are retractions, so we have $G_i \cap K_i = H_i$.

So suppose that there is a non-identity unipotent element $k \in K_1 \cap K_2$. By Proposition 2.1 (i), we can assume that k has order p. Thus, in the free product $G_1 * G_2$ we must have that k is conjugate into G_1 or G_2. This conjugate also lies in $K_1 \cap K_2$ and is unipotent, so if it is in G_1, then it is also in H_1, and the same holds for G_2 and H_2. But both H_1 and H_2 are unipotent free, so either way we are done. □

Note. To see that we cannot mix and match different characteristics in (iii) and (iv), even for finitely generated groups, the lamplighter group $C_2 \wr \mathbb{Z}$ is linear in characteristic 2 but only in this characteristic, whereas the "trilamplighter" group $C_3 \wr \mathbb{Z}$ is linear only in characteristic 3. Thus, any group containing them both (such as their direct or free product) is not NIU-linear, or even linear over any field.

In the 1-dimensional case where $\mathrm{GL}(1, \mathbb{F}) = \mathbb{F}^*$, we have $\mathbb{Z} \cong \langle t \rangle \leq \mathbb{F}_p(t)^*$ is clearly VUF-linear in any characteristic, thus so is $\mathbb{Z}^m$ and the free group F_r, as well as direct and free products formed from these groups and other groups commensurable with these ones. In the next section we will provide many more examples.

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3 A range of linear groups with restricted unipotent elements

3.1 Limit groups

Here we merely define a limit group to be a finitely generated group G which is fully residually free, that is, given any finite list of elements $g_1, \dots, g_k \in G$, there exists a homomorphism θ from G to a free group with $\theta(g_i) \neq 1$ unless $g_i = 1$. A result that essentially dates back to Malce'ev is the following.

Proposition 3.1. *Given any characteristic p (zero or prime) and any limit group G, there is a field $\mathbb{F}$ of characteristic p such that G embeds in* $\mathrm{GL}(2, \mathbb{F})$. *Consequently, any limit group is VUF-linear by taking any $p > 0$.*

Proof. The free group F_r is well known to embed in $\mathrm{GL}(2, \mathbb{C})$, and indeed it embeds in $\mathrm{GL}(2, \mathbb{E})$ for $\mathbb{E}$ some field of arbitrary characteristic $p > 0$ as well. For instance, [22, Exercise 2.2] is to show that

$$\begin{pmatrix} t & 0 \\ s & t^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t & s \\ 0 & t^{-1} \end{pmatrix}$$

generate F_2 over the field $\mathbb{E} = \mathbb{F}_p(s, t)$, with s, t algebraically independent elements which are both transcendental over $\mathbb{F}_p$. Now Malce'ev's argument, as given in [15, Window 8, Theorem 1], is that a group G embeds in $\mathrm{GL}(d, \mathbb{F})$ for some field $\mathbb{F}$ of characteristic p if and only if there exists a field $\mathbb{E}$ of characteristic p with the property that for every finite subset $T \subseteq G$, there exists a linear representation $\langle T \rangle \to \mathrm{GL}(d, \mathbb{E})$ such that $\theta(t) \neq 1$ for all $t \in T \setminus \{1\}$. Thus, we can set $\mathbb{E} = \mathbb{F}_p(s, t)$ as above and on being given a limit group G and any finite subset $T \subseteq G$, we have our homomorphism θ from G to this embedding of F_r in $\mathrm{GL}(2, \mathbb{E})$, which is injective on T. On restricting this homomorphism to $\langle T \rangle$, Malce'ev's argument applies. □

Consequently, by combining this with Theorem 1.1 (ii), we immediately obtain the following corollary.

Corollary 3.2. *Any finitely generated abelian subgroup is undistorted in any limit group.*

Whilst this is certainly already known, we emphasise that we have shown this directly from the "fully residually free" definition of limit groups. We remark that [7] and [8] show how to construct explicit faithful representations of limit groups in $\mathrm{PSL}(2, \mathbb{C})$ and in $\mathrm{SL}(2, \mathbb{C})$.

3.2 Three-manifold groups

Here a 3-manifold group will mean the fundamental group of a *compact* 3-manifold, so the group will be finitely presented. Linearity of such groups has been studied over the years but a surprising consequence obtained from applications of the recent Agol–Wise results is that most 3-manifold groups are linear even over $\mathbb{Z}$. Indeed, on taking compact orientable irreducible 3-manifolds M^3, we have that if M^3 admits a metric of non-positive curvature, then $\pi_1(M^3)$ is linear over $\mathbb{Z}$. (It should be said though that these representations over $\mathbb{Z}$ are likely to be of vast dimension and very hard to construct explicitly.) Linearity of some other special cases such as Seifert fibred spaces is also known, meaning that amongst the fundamental groups of these 3-manifolds, linearity is only open for closed graph manifolds which do not admit a metric of non-positive curvature. In [4] we gave an example of one of these closed graph manifolds where linearity of its fundamental group is unknown, and showed that this group did not embed in $\mathrm{GL}(4, \mathbb{F})$ for any field $\mathbb{F}$, thus answering a question of Thurston. We note that the resulting 3-manifold was already known to be virtually fibred, so that it is even unknown whether all semidirect products of the form $\pi_1(S_g) \rtimes \mathbb{Z}$ (where S_g is the closed orientable surface of genus g) are linear.

Thus, here we might ask which 3-manifolds M^3 have a fundamental group which is NIU-linear or VUF-linear and how this relates to the non-positive curvature of M^3. Whilst we cannot answer this in full here because of the open cases of graph manifolds, we can at least do this quickly for 3-manifolds admitting a geometric structure. Note that for very small fundamental groups, namely, those which are virtually cyclic

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and so have geometry modelled on S^3 or $S^2 \times \mathbb{R}$, we have a dichotomy because the manifolds fail to be non-positively curved, but the fundamental groups are (elementary) word hyperbolic. However, apart from these examples the correspondence works. We first consider the case of closed 3-manifolds.

Theorem 3.3. *Let M^3 be a closed orientable irreducible 3-manifold which admits one of Thurston's eight model geometries and where $\pi_1(M^3)$ is not virtually cyclic. Then $\pi_1(M^3)$ is NIU-linear if and only if it is VUF-linear if and only if M^3 is non-positively curved.*

Proof. We will work in characteristic zero, so that the difference between NIU and VUF-linearity disappears (the arguments do all work in positive characteristic except for the hyperbolic case, where it is not clear if we have faithful linear representations). As $\pi_1(M^3)$ is not virtually cyclic, we have six geometries to check. First if M^3 is hyperbolic, then it has a faithful (and indeed discrete) representation in $\mathrm{PSL}(2, \mathbb{C})$, and also in $\mathrm{SL}(2, \mathbb{C})$, where every element is loxodromic, thus without unipotents. Also if M^3 is Euclidean, then its fundamental group is virtually $\mathbb{Z}^3$ and all such groups are NIU-linear (by commensurability and taking direct products).

If M^3 has Nil or Sol geometry, then it cannot admit a metric of non-positive curvature and its fundamental group is virtually polycyclic, but not virtually abelian, so $\pi_1(M^3)$ is not NIU-linear by Theorem 1.1 (i).

We are now left with the $\widetilde{\mathrm{PSL}(2, \mathbb{R})}$ and $\mathbb{H}^2 \times \mathbb{R}$ geometries, leading us to Seifert fibred spaces, which here can be defined as closed 3-manifolds M^3, that are finitely covered by circle bundles over surfaces B^3, and where $\pi_1(M^3)$ is not virtually cyclic. By say [1, Sections C.10 and C.11], we have two possibilities for B^3. The first is that it has a finite cover which is of the form $S^1 \times F$ for F a closed orientable surface, whereupon M^3 is non-positively curved. Thus, again by commensurability and direct products, the fundamental group of this finite cover, and hence of M^3 itself, is NIU-linear.

The other case, where $\pi_1(M^3)$ will fail to be non-positively curved, is when B^3 has non-zero Euler class, in which case the infinite order central element of $\pi_1(B^3)$ has finite order in the abelianisation of $\pi_1(B^3)$. This means that Theorem 1.1 (iv) immediately applies to tell us that $\pi_1(B^3)$ and $\pi_1(M^3)$ cannot be NIU-linear. □

For compact 3-manifolds M^3 with non-empty boundary, there are fewer possibilities for a geometric structure on M^3. Indeed, Thurston's work shows us that either we are in the Seifert fibred case, whereupon M^3 is finitely covered by a manifold with fundamental group $F_r \times \mathbb{Z}$ and so $\pi_1(M^3)$ is NIU-linear and VUF-linear, or the interior of M^3 has a hyperbolic structure. We can deal with the second case here too but, unlike the closed hyperbolic case, this proof will depend on the work of Agol and Wise.

Theorem 3.4. *If M^3 is a compact orientable 3-manifold with non-empty boundary (whose components could be tori or higher genus surfaces) and where the interior of M^3 has a complete hyperbolic structure, then the fundamental group $\pi_1(M^3)$ is NIU-linear and VUF-linear.*

Proof. We will require two facts: the first is that $G = \pi_1(M^3)$ has the property that it virtually retracts onto all infinite cyclic subgroups. This can be thought of as saying that for any infinite order element $g \in G$, there exists a finite index subgroup H of G with $g \in H$, and where g has infinite order in the abelianisation of H. This does hold for our groups G, for instance, see [1, Corollary 5.31], though we emphasise that the proof is highly non-trivial.

However, we also need to know about the unipotent elements, which here will manifest themselves as parabolic elements in $\pi_1(M^3)$. We have (say by [1, Theorem 5.28]) that there exists an embedding of G in $\mathrm{SL}(2, \mathbb{C})$ such that the virtually unipotent elements of G are exactly the elements conjugate in G to the torus boundary components $\mathbb{Z}^2$. In particular, as there are finitely many boundary components, we can take a complete list of peripheral elements $a_1, b_1, \dots, a_m, b_m \in G$ under this embedding, where $\langle a_i, b_i \rangle \cong \mathbb{Z}^2$ is the fundamental group of one of these boundary components and such that any unipotent element in G is conjugate into one of these peripheral subgroups $P_i = \langle a_i, b_i \rangle$.

We now explain how to combine these two facts to obtain a faithful linear representation of G in $\mathrm{GL}(d, \mathbb{C})$ for some d which is unipotent free. First take a finite index subgroup H of G, where a_1 has infinite order in the abelianisation H/H' of H. By dropping down further if necessary, we can assume that H is normal in G. Now the subgroup $P_1 \cap H$ of H has finite index in P_1, so it is also isomorphic to $\mathbb{Z}^2$. If $P_1 \cap H$ injects under the abelianisation of H, then set $N_1 = H$ and move on to the next peripheral subgroup P_2 in G. Otherwise,

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there exists some infinite cyclic subgroup $\langle p_1 \rangle$ of $P_1 \cap H$ which does not inject when abelianising H. In this case, we now drop down further to obtain a subgroup $N_1 \leq H$, which is also normal and of finite index in G, but where $\langle p_1 \rangle$ injects under the abelianisation of N_1. As the $\mathbb{Z}$-rank of N_1/N_1' is more than that of H/H', we see that the rank 2 free abelian group $P_1 \cap N_1$ injects in N_1/N_1'. Now we run through this process for each peripheral subgroup P_i to obtain subgroups N_i, all of finite index and normal in G, and where $P_i \cap N_i$ injects in the abelianisation of N_i.

Let us first describe what to do next when there is only one peripheral subgroup P_1. Here we form a faithful representation of N_1 in $\mathrm{GL}(3, \mathbb{C})$ by taking the direct sum of the original representation in $\mathrm{SL}(2, \mathbb{C})$, but restricted to N_1, along with a 1-dimensional representation of N_1 to its free abelianisation $\mathbb{Z}^r$ for some $r \geq 2$ (namely, take the abelianisation N_1/N_1' and then quotient out by the torsion), where this will be thought of as a subgroup of $\mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^*$ which contains a copy of $\mathbb{Z}^r$ for any r. The idea now is that in this new representation, any element of $P_1 \cap N_1$, which was a unipotent element under the original representation of G, will now have a non-unit entry in the bottom right-hand corner, since $P_1 \cap N_1$ injects in the free abelianisation of N_1. In particular, this element (or any power of this element) is no longer unipotent in our new representation of N_1. However, a typical unipotent element of G need not lie in P_1 but will be conjugate in G to an element of P_1, so we will need to finish the argument as below for these more general unipotent elements. Now suppose we have m peripheral subgroups P_i, whereupon we obtain N_i as above for each i and form the intersection $N = \cap N_i$, which is still of finite index and normal in G. We then create a faithful representation of N in $\mathrm{GL}(2 + m, \mathbb{C})$ by taking the original representation of N in $\mathrm{SL}(2, \mathbb{C})$ along with the direct sum of each of the 1-dimensional abelianisations of N_i, but also restricted to N. This will have the property, as above, that a non-identity element x of $P_i \cap N$ will no longer be unipotent, because x has infinite order when evaluated at the restriction of the abelianisation map of N_i to N, and this is recorded as the diagonal entry coming from the ith 1-dimensional representation, thus it is an eigenvalue. Finally, we induce up to a faithful representation of G in $\mathrm{GL}(i(2 + m), \mathbb{C})$, where i is the index of N in G.

To show that G is VUF-linear, we can show this for the finite index subgroup N. Thus, suppose we have $n \in N$ which is unipotent under our final representation of N. When we induce from N to G, for any element $n \in N$, the first $2 + m$ by $2 + m$ block in the representation of G is just the matrix of n in the representation of N, and thus the first 2 by 2 subblock is the original representation of N into $\mathrm{SL}(2, \mathbb{C})$. This implies that n was unipotent in that representation too, thus there exists $g \in G$ with gng^{-1} lying in some peripheral subgroup P_i. Since N is normal in G, we have $gng^{-1} \in P_i \cap N$. Now as n and gng^{-1} are conjugate in G, we see that these elements will have the same eigenvalues in the final linear representation of G, thus if gng^{-1} is not unipotent in this representation, then nor is n. But as gng^{-1} lies in P_i, as well as in N, one of the 1-dimensional representations above will be non-trivial and this is seen on the diagonal in the first block of the representation for G. Thus, gng^{-1} and hence n are not unipotent in this representation, and so N and G are VUF-linear. □

3.3 Right angled Artin groups

Right angled Artin groups (RAAGs) have been much studied, especially in recent years. Here we will assume throughout that the underlying graph Γ is finite with n vertices, in which case we can say something strong about the linearity of the corresponding RAAG $R(\Gamma)$. As $R(\Gamma)$ embeds in a right angled Coxeter group with $2n$ vertices by [12], we see that $R(\Gamma)$ is a subgroup of $\mathrm{GL}(2n, \mathbb{C})$, using the standard faithful linear representation of Coxeter groups. In fact, because these Coxeter groups are right angled, all entries in the resulting reflection matrix obtained from each Coxeter generator are equal to $0, \pm 1$ or 2. As these matrices have order 2 and thus determinant ± 1, we further see that the RAAG $R(\Gamma)$ actually embeds in $\mathrm{GL}(2n, \mathbb{Z})$. This immediately implies that any virtually special group G is also linear over $\mathbb{Z}$, where our definition of virtually special is that G has a finite index subgroup H which embeds in a RAAG. Thus, the work of Agol and Wise has had profound consequences for linearity (as well as for other group theoretic properties), though we emphasise that generally the hard step is in showing that a given group is virtually special, rather than establishing its linearity directly. This raises the obvious question.

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Question 3.5. Is every RAAG NIU-linear?

(This is equivalent to being VUF-linear as these groups are torsion free). We think the answer is very likely to be yes. If so, then we would be able to use NIU-linearity as a first step in showing a group is virtually special, or alternatively the failure of NIU-linearity would be an obstruction to being virtually special. For instance, limit groups are known to be virtually special, but whereas the short argument in Proposition 3.1 suffices to show NIU-linearity, the full weight of Sela's splitting hierarchy and Wise's results (both for hyperbolic groups and groups hyperbolic relative to abelian subgroups) are required to show that they are virtually special groups. We note that some NIU-linear groups are not virtually special, for instance, higher rank lattices are not as they have property (T), but in positive characteristic, these will be NIU-linear groups.

We do have one partial positive result which encompasses a range of examples. For the RAAG $R(\Gamma)$ given by the graph $\Gamma = \bullet - \bullet - \bullet - \bullet$, we showed in [5], by a direct application of Proposition 2.4, that $R(\Gamma)$ embeds in $\mathrm{GL}(3, \mathbb{F})$ for some field $\mathbb{F}$ which can be taken to have arbitrary characteristic. Thus, this RAAG $R(\Gamma)$ is NIU-linear and VUF-linear. However, there has been particular interest recently in trying to determine when one RAAG is a subgroup of another RAAG. In particular, [13, Theorem 1.8] states that any RAAG defined by a finite forest is a subgroup of our RAAG above, and so this immediately gives us the following proposition.

Proposition 3.6. *If Γ is a finite forest, namely, a graph with finitely many connected components, each of which is a finite tree, then the resulting RAAG $R(\Gamma)$ is NIU-linear and (as it is torsion free) VUF-linear.*

4 NIU-linearity of free by cyclic groups

In this section a free by cyclic group will mean the semidirect product $F_n \rtimes_\alpha \mathbb{Z}$ formed by taking an automorphism α of the free group F_n of rank $n \geq 2$. (Thus, "free" here means non-abelian free of finite rank and "cyclic" means infinite cyclic.) Given $\alpha \in \mathrm{Aut}(F_n)$ (or rather in $\mathrm{Out}(F_n)$ as automorphisms that are equal in $\mathrm{Out}(F_n)$ give rise to isomorphic groups), we are interested in when $F_n \rtimes_\alpha \mathbb{Z}$ is a linear group, and in particular when is it NIU-linear (equivalently, VUF-linear as again these groups are all torsion free). Free by cyclic groups are good test cases for these sorts of questions, as they have a restricted subgroup structure and good group theoretic properties (for instance, all the properties in Theorem 1.1 hold for free by cyclic groups), but there can be considerable variation in their behaviour.

As for the linearity question for free by cyclic groups, this has been looked at by various authors. A big step forward recently was the result of [11], which proved that any word hyperbolic free by cyclic group acts properly and cocompactly on a CAT(0) cube complex. By the Agol–Wise results, this implies that such groups will be virtually special and so have faithful linear representations over $\mathbb{Z}$. However, for non-word hyperbolic free by cyclic groups (which here is equivalent to containing $\mathbb{Z} \times \mathbb{Z}$), linearity is still open in general. We first look at the case where the free part has rank 2, where linearity does hold. One way of seeing this is to embed such a group in $\mathrm{Aut}(F_2)$, where linearity of the Braid groups implies that this is linear too. However, we will now show quickly that NIU-linearity also holds for these groups.

Proposition 4.1. *Any free by cyclic group $F_2 \rtimes_\alpha \mathbb{Z}$ with free part having rank 2 is NIU-linear and VUF-linear.*

Here we will work over $\mathbb{C}$. In order to describe the element α when regarded as an outer automorphism, we have that $\mathrm{Out}(F_2)$ is $\mathrm{GL}(2, \mathbb{Z})$, and we first consider the case when α is orientation preserving, which means that it is represented by a matrix $M \in SL(2, \mathbb{Z})$. Then we have that M is hyperbolic, elliptic or parabolic. In the first case the automorphism α is a pseudo-Anosov homeomorphism of the once punctured torus, and Thurston showed that the corresponding mapping torus has a hyperbolic structure of finite volume on its interior, so we are covered by Theorem 3.4. For the orientation reversing elements α whose square is hyperbolic, we know that these groups have an index 2 subgroup which is the fundamental group of a finite volume hyperbolic 3-manifold, and so we again obtain NIU-linearity in characteristic zero.

If α has finite order in $\mathrm{Out}(F_2)$, then (whether or not it is orientation preserving) $F_2 \rtimes_\alpha \mathbb{Z}$ contains $F_2 \times \mathbb{Z}$ with finite index, and thus we again have NIU-linearity.

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However, this still leaves the parabolic case (which will be orientation preserving), where $M = \pm\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right)$ for $n \in \mathbb{Z} \setminus \{0\}$. These groups are all commensurable and in fact the question of their linearity has received a lot of attention. Indeed, whether $F_2 \rtimes_\alpha \mathbb{Z}$ is linear for $M = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$, which is $\alpha(x) = x$ and $\alpha(y) = yx$, was Question 18.86 in the Kourovka notebook [16]. However, [2] and [19] pointed out that linearity of the Braid group B_4 provides a positive answer for this group (and thus for the others too). Here we can use a result of Niblo and Wise to provide a quick proof that does not require linearity of B_4, and which gives us NIU-linearity as well.

Proposition 4.2. *For the group $G = F_2 \rtimes_\alpha \mathbb{Z}$, where $\alpha(x) = x$ and $\alpha(y) = yx$, we have that G embeds in* $\mathrm{GL}(6, \mathbb{F})$ *for $\mathbb{F}$ some field of arbitrary characteristic. Consequently, G is NIU-linear and VUF-linear.*

Proof. Let the stable letter of G be t, so that $G = \langle x, y, t\rangle = \langle y, t\rangle$ because $y^{-1}tyt^{-1} = x$. It was shown in [17] that the index 2 subgroup $H = \langle y^2, t, y^{-1}ty\rangle$ of G embeds into the RAAG $R(\Gamma)$ immediately before Proposition 3.6. There we mentioned that $R(\Gamma)$ embeds in $\mathrm{GL}(3, \mathbb{F})$, so taking the induced representation shows that G embeds in $\mathrm{GL}(6, \mathbb{F})$. □

Combining the results above, we have the following corollary.

Corollary 4.3. *If $G = F_2 \rtimes_\alpha \mathbb{Z}$ is any free by cyclic group where the free part has rank* 2, *then G is NIU-linear and VUF-linear.*

We will now see that things are very different when the rank of the free group is greater than 2. We have already mentioned that the following question is open.

Question 4.4. If $F_n \rtimes_\alpha \mathbb{Z}$ is a free by cyclic group for $n \geq 3$ which contains $\mathbb{Z} \times \mathbb{Z}$, then is it linear over some field?

In Gersten's paper [10] the free by cyclic group $F_3 \rtimes_\alpha \mathbb{Z} = \langle a, b, c, t\rangle$ with $tat^{-1} = a$, $tbt^{-1} = ba$, $tct^{-1} = ca^2$ is introduced and shown to have very strange properties. In particular, an argument using translation length proves that it cannot act properly and cocompactly by isometries on a CAT(0) space. Now for this linearity question, Gersten's group G would seem like an important test case. In this section we prove that G is not NIU-linear, and hence not linear over any field of positive characteristic. This will be a consequence of showing that the most tractable linear representations of this group, namely, the ones where the elements t, a are both diagonalisable, and thus simultaneously diagonalisable as $ta = at$ are never faithful over any field.

Theorem 4.5. *Suppose we have commuting elements $T, A \in$* $\mathrm{GL}(d, \mathbb{F})$, *for d any dimension and $\mathbb{F}$ any field, such that the matrix T is conjugate to TA and also to TA^2. Then all eigenvalues of A must be roots of unity.*

Proof. We replace $\mathbb{F}$ by its algebraic closure, which we will also call $\mathbb{F}$, and first suppose that both A and T are diagonalisable, so that we can choose a basis $e_1, \ldots, e_d$ in which both

$$T = \mathrm{diag}(t_1, \ldots, t_d) \quad \text{and} \quad A = \mathrm{diag}(a_1, \ldots, a_d)$$

are simultaneously diagonal. As TA and TA^2 are also then diagonal, each has entries which are a permutation of the diagonal entries of T. Although these permutations, which we will name π and σ, respectively, will not in general be well defined because of repeated eigenvalues, we choose appropriate π and σ defined in some suitable way. We now permute our basis so that π is a product of disjoint consecutive cycles, that is,

$$\pi = (12\cdots k_1)(k_1 + 1\, k_1 + 2 \cdots k_2)\cdots(k_{r-1} + 1 \cdots k_r)$$

for $k_r = d$.

First suppose that the number of cycles r in π is 1. Then we have

$$T = \mathrm{diag}(t_1, \ldots, t_d) \quad \text{and} \quad TA = \mathrm{diag}(t_2, \ldots, t_d, t_1),$$

so that

$$A = \mathrm{diag}(t_2/t_1, t_3/t_2, \ldots, t_1/t_d),$$

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and thus

$$TA^2 = \mathrm{diag}(t_2^2/t_1, t_3^2/t_2, \ldots, t_1^2/t_d) = \mathrm{diag}(t_{\sigma(1)}, t_{\sigma(2)}, \ldots, t_{\sigma(d)})$$

for the permutation σ above.

Now all t_i are in the abelian group $\mathbb{F}^*$ written multiplicatively, but on changing to additive notation, we can regard the two expressions for TA^2 as providing a system of linear equations. Thus, let us work in an arbitrary abelian group $\mathbb{A}$ written additively, so that we are replacing $(\mathbb{F}^*, \times)$ above with $(\mathbb{A}, +)$. We are hence trying to solve the homogeneous system of equations

$$2x_2 = x_1 + x_{\sigma(1)}, \quad 2x_3 = x_2 + x_{\sigma(2)}, \ldots, 2x_1 = x_d + x_{\sigma(d)}$$

in the unknown variables $x_1, \ldots, x_d \in \mathbb{A}$. We note that there exist solutions where $x_1 = \cdots = x_d$ (which would result in A being the identity) and we are trying to rule out other solutions.

Let us first set $\mathbb{A} = \mathbb{R}$, so that we can use the usual order on $\mathbb{R}$ as well as linear algebra. Given a non-zero solution $(r_1, \ldots, r_d) \in \mathbb{R}^d$, let M be $\max_{1\le i\le d}|r_i|$ and let $|r_k|$ attain M, so that $r_k = M > 0$ without loss of generality by multiplying the solution by -1 if necessary. Now one equation is $2r_k = r_{k-1} + r_{\sigma(k-1)}$ (where all subscripts are taken modulo d), so $2M = r_{k-1} + r_{\sigma(k-1)} \le |r_{k-1}| + |r_{\sigma(k-1)}| \le 2M$, thus for equality we need r_{k-1} and $r_{\sigma(k-1)}$ both to have modulus M and be positive. Thus, now we can replace the equation with left-hand side $2r_k$ by the equation having left-hand side $2r_{k-1}$ and continue until we have the constant solution.

However, this assumed that the initial permutation π was just the cycle $(12\cdots d)$. Let us consider the general case

$$\pi = (12\cdots k_1)(k_1+1\ k_1+2\cdots k_2)\cdots(k_{r-1}+1\cdots k_r),$$

so that the two expressions for TA^2 now read

$$\begin{aligned}&\mathrm{diag}(t_2^2/t_1, \ldots, t_1^2/t_{k_1}, t_{k_1+2}^2/t_{k_1+1}, \ldots, t_{k_1+1}^2/t_{k_2}, \ldots, t_{k_{r-1}+1}^2/t_d)\\ &\quad = \mathrm{diag}(t_{\sigma(1)}, \ldots, t_{\sigma(k_1)}, t_{\sigma(k_1+1)}, \ldots, t_{\sigma(k_2)}, \ldots, t_{\sigma(d)}).\end{aligned}$$

Then if S is the subgroup of $\mathrm{Sym}(d)$ generated by π, σ, we have that the orbits of S are unions of the disjoint cycles for π as above. However, if S is not transitive, then it should be clear that we have solutions $x_1, \ldots, x_d \in \mathbb{A}$ which are constant on orbits of S but which are not constant overall, because the equations in separate orbits involve disjoint sets of variables (however, these solutions still give rise to the matrix A being the identity). Therefore, let us consider the orbit O under S of some point $x \in \{1, \ldots, d\}$ which will be a union of the cycles for π. Let us assume without loss of generality that $j \in O$ is such that $x_j = M > 0$ maximises $|x_i|$ over all $i \in O$. Then j sits in some cycle $(k_{l-1}+1\cdots k_l)$ and, as before, on considering the equation $2r_j = r_{j-1} + r_{\sigma(j-1)}$, then $2r_{j-1} = r_{j-2} + r_{\sigma(j-2)}$ and so on, where our subscripts are taken from numbers in this cycle, and where by subtracting 1 we mean shifting backwards round the cycle. This implies not only that

$$r_{k_{l-1}+1} = r_{k_{l-1}+2} = \cdots = r_{k_l} = M$$

but also that any subscript s which is an image under σ of a point in this cycle will satisfy $r_s = M$ too. Thus, we now move to another cycle until we see that our solution is constant on the whole of O.

We now deduce the same conclusion for solutions of these equations over any torsion free abelian group $\mathbb{A}$. If we have a solution $(x_1, \ldots, x_d) \in \mathbb{A}^d$, then we replace $\mathbb{A}$ with the finitely generated subgroup $\langle x_1, \ldots, x_d\rangle = \mathbb{A}_0$ and work in $\mathbb{A}_0$ which, being a finitely generated torsion free abelian group, is just a copy of $\mathbb{Z}^m$ for some m. If we now take a particular $\mathbb{Z}$-basis for $\mathbb{A}_0$, we can express $x_1, \ldots, x_d$ as elements of $\mathbb{Z}^m$, and then our d equations become m lots of d equations, with one set of d equations for each coefficient of $\mathbb{Z}^m$. But as each system of equations over $\mathbb{Z}$ can be thought of as also over $\mathbb{Q}$ and indeed over $\mathbb{R}$, our above argument tells us that our solution must be constant over each orbit coordinate-wise, so indeed our elements $x_1, \ldots, x_d$ are equal amongst subscripts in the same orbit.

However, this assumed that $\mathbb{A}$ is torsion free, whereas in the multiplicative group $\mathbb{F}^*$ of a field we will have roots of unity. To deal with this case, note that if the element $X \in G$ is such that $XTX^{-1} = TA$, then $XT^nX^{-1} = T^nA^n$ for any $n \in \mathbb{Z}$ and similarly we have $YT^nY^{-1} = T^nA^{2n}$ if $YTY^{-1} = TA^2$. Thus, on initially being given our diagonal elements $t_1, \ldots, t_d \in \mathbb{F}^*$ of T, where now we finally return to multiplicative notation,

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we have that $\langle t_1, \ldots, t_d\rangle$, considered as an abstract finitely generated abelian group, must be isomorphic to $\mathbb{Z}^r \oplus R$ for some $r \leq d$ and R a finite subgroup consisting of the torsion elements. Hence, there exists an exponent $e > 0$ such that $t_1^e, \ldots, t_d^e$ all lie in the $\mathbb{Z}^r$ part and so these elements generate a torsion free abelian subgroup. Now we can run through the whole proof above with T and A replaced by T^e and A^e (but the conjugating elements X and Y remain the same), whereupon we conclude that A^e must be the identity and so the eigenvalues of A are all roots of unity.

If our elements are not diagonalisable, then, as they commute, we can still find a basis in which both T and A are upper triangular. We can then work through the above proof using the diagonal elements of T and of A, which will multiply in the same way to give the diagonal entries of TA and of TA^2, whereupon we will conclude that some power A^e of A is upper triangular with all ones down the diagonal, thus again the eigenvalues of A are all roots of unity. □

Corollary 4.6. *If a group G has commuting elements T, A, with A of infinite order, such that T, TA, TA^2 are all conjugate in G, then G is not NIU-linear.*

Proof. Theorem 4.5 tells us that a power A^e of A is unipotent, but if G were NIU-linear, then A^e would have finite order. □

Corollary 4.7. *Gersten's group G is not NIU-linear or VUF-linear, and in particular is not linear in any positive characteristic nor embeds in any complex unitary group of any finite dimension.*

Proof. We have $b^{-1}tb = at = ta$ and $c^{-1}tc = a^2t = ta^2$, with a of infinite order. □

Gersten constructed this example specifically because it embeds in $\mathrm{Aut}(F_3)$, and thus in $\mathrm{Aut}(F_m)$ for $m \geq 3$ and $\mathrm{Out}(F_n)$ for $n \geq 4$. Consequently, we have a proof that none of these groups are linear over a field of positive characteristic without using the theory of algebraic groups in [9]. However, as long as the question of linearity of Gersten's group over $\mathbb{C}$ is still open, we will not know if this approach can be made to work in characteristic zero. We remark here that it can be shown that there is no faithful linear representation of Gersten's group in $\mathrm{GL}(d, \mathbb{C})$ for $d \leq 4$ (as was done in [5]). We understand that I. Soroko has extended this to $d = 5$. As this involves working through and eliminating the various possible Jordan normal forms (or other suitable canonical forms), this approach becomes unwieldy as the dimension increases.

References

[1] M. Aschenbrenner, S. Friedl and H. Wilton, *3-Manifold Groups*, EMS Ser. Lect. Math., European Mathematical Society, Zürich, 2015.
[2] V. G. Bardakov and O. V. Bryukhanov, On linear representations of some extensions (in Russian), *Vestnik Novosib. Univ. Ser. Mat. Mekh. Inf.* **7** (2007), 45–58.
[3] S. J. Bigelow, Braid groups are linear, *J. Amer. Math. Soc.* **14** (2001), no. 2, 471–486.
[4] J. O. Button, A 3-manifold group which is not four dimensional linear, *J. Pure Appl. Algebra* **218** (2014), no. 9, 1604–1619.
[5] J. O. Button, Minimal dimension faithful linear representations of common finitely presented groups, preprint (2016), https://arxiv.org/abs/1610.03712.
[6] J. O. Button, Properties of non positive curvature for linear groups with restricted unipotent elements, submitted.
[7] B. Fine and G. Rosenberger, A note on faithful representations of limit groups, *Groups Complex. Cryptol.* **3** (2011), no. 2, 349–355.
[8] B. Fine and G. Rosenberger, Faithful representations of limit groups II, *Groups Complex. Cryptol.* **5** (2013), no. 1, 91–96.
[9] E. Formanek and C. Procesi, The automorphism group of a free group is not linear, *J. Algebra* **149** (1992), no. 2, 494–499.
[10] S. M. Gersten, The automorphism group of a free group is not a CAT(0) group, *Proc. Amer. Math. Soc.* **121** (1994), no. 4, 999–1002.
[11] M. F. Hagen and D. T. Wise, Cubulating hyperbolic free-by-cyclic groups: The general case, *Geom. Funct. Anal.* **25** (2015), no. 1, 134–179.
[12] T. Hsu and D. T. Wise, On linear and residual properties of graph products, *Michigan Math. J.* **46** (1999), no. 2, 251–259.
[13] S.-H. Kim and T. Koberda, Embedability between right-angled Artin groups, *Geom. Topol.* **17** (2013), no. 1, 493–530.
[14] D. Krammer, Braid groups are linear, *Ann. of Math. (2)* **155** (2002), no. 1, 131–156.
[15] A. Lubotzky and D. Segal, *Subgroup Growth*, Progr. Math. 212, Birkhäuser, Basel, 2003.

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[16] V. D. Mazurov and E. I. Khukhro, *The Kourovka Notebook. Unsolved Problems in Group Theory*, 18th ed., Russian Academy of Sciences Siberian Division, Novosibirsk, 2014.
[17] G. A. Niblo and D. T. Wise, Subgroup separability, knot groups and graph manifolds, *Proc. Amer. Math. Soc.* **129** (2001), no. 3, 685–693.
[18] V. L. Nisnewitsch, Über Gruppen, die durch Matrizen über einem kommutativen Feld isomorph darstellbar sind, *Rec. Math. N.S.* **8 (50)** (1940), 395–403.
[19] V. A. Roman'kov, The linearity problem for the unitriangular automorphism groups of free groups, *J. Siberian Federal Univ. Math. Phys.* **6** (2013), 516–520.
[20] P. B. Shalen, Linear representations of certain amalgamated products, *J. Pure Appl. Algebra* **15** (1979), no. 2, 187–197.
[21] B. A. F. Wehrfritz, Generalized free products of linear groups, *Proc. Lond. Math. Soc. (3)* **27** (1973), 402–424.
[22] B. A. F. Wehrfritz, *Infinite Linear Groups*, Ergeb. Math. Grenzgeb. (3) 76, Springer, New York, 1973.

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