# CONJUGACY SEARCH PROBLEM AND THE ANDREWS-CURTIS CONJECTURE 

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#### Abstract

We develop new computational methods for studying potential counterexamples to the Andrews-Curtis conjecture, in particular, Akbulut-Kurby examples AK(n). We devise a number of algorithms in an attempt to disprove the most interesting counterexample AK(3). To improve metric properties of the search space (which is a set of balanced presentations of $\mathbb{1}$ ) we introduce a new transformation (called an ACMmove here) that generalizes the original Andrews-Curtis transformations and discuss details of a practical implementation. To reduce growth of the search space we introduce a strong equivalence relation on balanced presentations and study the space modulo automorphisms of the underlying free group. Finally, we prove that automorphism-moves can be applied to AK(n)-presentations. Unfortunately, despite a lot of effort we were unable to trivialize any of $\mathrm{AK}(\mathrm{n})$-presentations, for $n>2$. Keywords. Andrews-Curtis conjecture, Akbulut-Kurby presentations, trivial group, conjugacy search problem, computations.


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## 1. Introduction

The Andrews-Curtis conjecture (AC-conjecture, or ACC) is a long-standing open problem in low-dimensional topology and combinatorial group theory. It was proposed by Andrews and Curtis in [1 while categorizing possible counterexamples to the Poincaré conjecture. Later, Wright in [29] formulated an equivalent conjecture about 3 -deformations of 2 -CW-complexes associated with all finitely presented groups, thus showing that Zeeman conjecture 30 implies AC-conjecture. It is known that Zemman conjecture also implies the Poincaré conjecture and is implied by Poincaré in some cases, e.g. [12]. Despite recent progress and solution to Poincaré conjecture, validity of the AC-conjecture remains open.

Although most of motivating examples come from topology, the conjecture is usually formulated in the language of combinatorial group theory, as a question of equivalence of presentations of the trivial group. In this paper we use the language of combinatorial group theory, omitting any topological aspects of the problem.

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1.1. Balanced presentations of the trivial group. Let $X=\left\{x_{1}, \ldots, x_{n}\right\}$, $F=F(X)$ be the free group on $X$, and $R$ a finite subset of $F$. The normal closure of $R$, denoted by $\operatorname{ncl}(R)$, is the smallest normal subgroup in $F(X)$ containing $R$. A pair $(X ; R)$ defines a quotient $\operatorname{group} F / \mathbf{n c l}(R)$, denoted by $\langle X \mid R\rangle$, and is called a presentation of $\langle X \mid R\rangle$. The sum $\sum_{r \in R}|r|$ is called the total length of the presentation $(X ; R)$ and is denoted by $L(R)$. We say that $R \subseteq F(X)$ is symmetrized if $R$ contains only cyclically reduced words and is closed under taking inverses and cyclic permutations. Denote by $R^{\star}$ the minimal symmetrized set containing $R$ (with all the words cyclically reduced). A presentation $(X ; R)$ is symmetrized if $R=R^{\star}$. A finite presentation can be efficiently symmetrized and symmetrization does not change the computational properties of the fundamental problems.

We say that a group presentation $(X ; R)$ is balanced if $|X|=|R|$. Some group presentations define the trivial group $\mathbb{1}$. The "most trivial" presentation of $\mathbb{1}$ on generators $\left\{x_{1}, \ldots, x_{n}\right\}$ is, of course, $\left(x_{1}, \ldots, x_{n} ; x_{1}, \ldots, x_{n}\right)$ called the canonical presentation of $\mathbb{1}$ on $\left\{x_{1}, \ldots, x_{n}\right\}$. Define a set $\mathcal{B}_{n} \subseteq F_{n}^{n}$ of balanced relator-tuples of the trivial group:

$$
\mathcal{B}_{n}=\left\{\left(r_{1}, \ldots, r_{n}\right) \mid \boldsymbol{\operatorname { n c l }}\left(r_{1}, \ldots, r_{n}\right)=F_{n}\right\}
$$

We use vector notation for tuples in $F_{n}^{n}$. The problem of deciding if $(X ; R)$ defines the trivial group is undecidable (see [24, 4]). It is an open problem if the same is true for balanced presentations (see Magnus' problem [18, Problem 1.12]).
1.2. Transformations of group presentations. There are several types of transformations that for a general group presentation $\left(x_{1}, \ldots, x_{n} ; r_{1}, \ldots, r_{n}\right)$ produce a new presentation $\left(x_{1}, \ldots, x_{n} ; r_{1}^{\prime}, \ldots, r_{n}^{\prime}\right)$ on the same set of generators of the same group. The Andrews-Curtis transformations $A C_{1}, A C_{2}, A C_{3}$ (or simply AC-moves) are of that type:

$$
\begin{aligned}
& \left(A C_{1}\right) r_{i} \rightarrow r_{i} r_{j} \text { for } i \neq j \\
& \left(A C_{2}\right) r_{i} \rightarrow r_{i}^{-1} \\
& \left(A C_{3}\right) r_{i} \rightarrow w^{-1} r_{i} w \text { for some } w \in F_{n}
\end{aligned}
$$

The transformations $A C_{1}, A C_{2}$ can be recognized as Nielsen transformations of the tuple $\left(r_{1}, \ldots, r_{n}\right)$ and the transformation $A C_{3}$ is a conjugation of any element in a tuple. Since the AC-moves are invertible, we can say that $\bar{u}$ and $\bar{v}$ are AC-equivalent (and write $\bar{u} \sim_{A C} \bar{v}$ ) if there exists a sequence of AC-moves transforming $\bar{u}$ into $\bar{v}$.

More generally, a transformation (named here an ACM-move) that replaces a single element $u_{i}$ in $\bar{u}$ with an element $u_{i}^{\prime}$ satisfying:

$$
u_{i}^{ \pm 1} \sim u_{i}^{\prime} \text { in }\left\langle x_{1}, \ldots, x_{n} \mid u_{1}, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{n}\right\rangle
$$

produces an isomorphic presentation. It is easy to see that AC-moves are particular types of the ACM-move. Also, the ACM-move can be recognized as a slightly generalized $M$-transformation of [9]. It is easy to see that $\bar{u}$
can be transformed to $\bar{v}$ by AC-moves if and only if the same can done by ACM-moves. Therefore, to check AC-equivalence one can use ACM-moves.

Yet another transformation of a group presentation that does not change the group is an automorphism move, which is an application of $\varphi \in \operatorname{Aut}\left(F_{n}\right)$ to every component of $\bar{u}$. It is not known if the system of AC-moves with automorphism-moves is equivalent to the system of AC-moves. More on automorphism-moves in Section 3 .
1.3. The conjecture. Denote by $\mathcal{C}$ the set of all tuples that can be obtained from the canonical tuple by a sequence of AC-moves. More generally, for $\bar{u} \in \mathcal{B}_{n}$ denote by $\mathcal{C}_{\bar{u}}$ the set of tuples in $\mathcal{B}_{n}$ AC-equivalent to $\bar{u}$.

The Andrews-Curtis conjecture [2] states that $\mathcal{C}=\mathcal{B}_{n}$, i.e., every balanced presentation of the trivial group can be converted to the canonical presentation by a sequence of AC-moves.

Despite nearly 50 years of research the conjecture is still open. It is widely believed that the AndrewsCurtis conjecture is false with most theoretic works attempting to disprove it. A common approach is to fix some group $G$, a homomorphism $\varphi: F_{n} \rightarrow G$, and investigate if for any $\bar{u} \in \mathcal{B}_{n}$ there exists a sequence of AC-moves taking $\varphi(\bar{u})$ into $\left(\varphi\left(x_{1}\right), \ldots, \varphi\left(x_{n}\right)\right)$. Clearly, if the answer is negative for some choice of $G$ and $\varphi$, then the original conjecture does not hold. Several classes of groups were investigated that way, e.g., solvable groups [21], finite groups [5], the Grigorchuk group [23], but the (negative) answer is not found.
1.4. Potential counterexamples. A big obstacle towards the solution of the problem is that there is no algorithm to test if a particular balanced presentation of the trivial group satisfies the conjecture, or not. There is a number of particular balanced presentations that are not known to satisfy the conjecture.

- Akbulut-Kurby examples: $\operatorname{AK}(n)=\left\langle x, y \mid x y x=y x y, x^{n+1}=y^{n}\right\rangle$ for $n \geq 3$.
- Miller-Schupp examples: $\left\langle x, y \mid x^{-1} y^{2} x=y^{3}, x=w\right\rangle$, where $w$ has exponent sum 0 on $x$.
- B. H. Neumann example $\left\langle x, y, z \mid z^{-1} y z=y^{2}, x^{-1} z x=z^{2}, y^{-1} x y=x^{2}\right\rangle$.

These examples are referred to as potential counterexamples to ACC. More examples of balanced presentations of $\mathbb{1}$ (known to be AC-equivalent to the canonical presentation) can be found in [15, 16, 8]. It was shown in [19] by means of a computer experiment that there are no counterexamples of total length 12 or less. Later it was shown in [13] that every balanced presentation of total length 13 is either AC-equivalent to the canonical presentation or to

$$
\mathrm{AK}(3)=\left\langle x, y \mid x^{3}=y^{4}, x y x=y x y\right\rangle,
$$

which makes $\mathrm{AK}(3)$ the shortest potential counterexample.
1.5. Computational approach to disproving a counterexample. To check if a given tuple $\bar{u}$ is AC-equivalent to the canonical presentation, one can enumerate equivalent presentations (by applying AC-moves) until the canonical presentation is found (see [19, 13, 6]). There are several general computational problems associated with that approach that we would like to mention here:

- $\mathcal{C}_{\bar{u}}$ is infinite and there is no terminating condition which allows an enumeration procedure to stop with a negative answer. The enumeration procedure can only terminate with a positive answer when it finds the canonical presentation.
- Lengths of tuples are unbounded.
- $\mathcal{C}_{\bar{u}}$ has exponential growth.

To alleviate some of the problems one can bound the lengths of the words in tuples by some constant $L$ and do not process a tuple $\bar{v}$ which is ACequivalent to the given $\bar{u}$ if $\bar{v}$ contains a (cyclic) word of length greater than $L$. This approach (used in [13, 6]) allows to use fixed memory slots for words and makes the search space finite. Also, it is a good heuristic to process shorter tuples first.

In this paper we consider the case $n=2$ only. We use compact memory representation for balanced pairs $(u, v) \in \mathcal{B}_{2}$. We represent each letter by a 2 -bit number, thus packing 32 letters into a 64 -bit machine-word. This approach saves memory and allows to implement operations such as a cyclic shift in just a few processor instructions, compared to a usual approach which includes several memory writes.
1.6. Our work. In this paper we develop new efficient techniques to enhance algorithmic search in $\mathcal{C}$ ( or $\mathcal{C}_{\bar{u}}$ ). Our work is similar to previous computational investigations of ACC , but goes much further. The presentation $\mathrm{AK}(3)$ is the main object of study and most of the algorithms are tested on $\mathrm{AK}(3)$. Our big goal was to prove that $\mathrm{AK}(3)$ is not a counterexample, i.e., it satisfies ACC. Unfortunately, we were not able to achieve our goal. Below we list the key features of our work.

- In Section 2 we show that ACM-moves can be used in practice. Notice that for $n=2$ that requires enumerating (short!) conjugates in a one relator group for a given element. The later problem does not have an efficient solution as of now. It is not even known if the conjugacy problem is decidable or not in one relator groups. Based on techniques described in [22] we design a heuristic procedure enumerating short conjugates and discuss the details of implementation.
- We prove in Section 3 that automorphism-moves can be used with the AC-moves for Akbulut-Kurby presentations $\mathrm{AK}(n)$, regardless of whether the conjecture holds for $\operatorname{AK}(n)$.
- In Section 4 we introduce an equivalence relation $\sim$ on pairs in $\mathcal{B}_{2}$ and define normal forms for the equivalence classes. We show that in practice equivalence of two pairs can be checked and normal forms
computed. That allows us to work with the quotient space $\mathcal{B}_{2} / \sim$ which elements are (infinite) equivalence classes. Working in $\mathcal{B}_{2} / \sim$ we work with large blocks of elements from $\mathcal{B}_{2}$. Thus, we can say that the space $\mathcal{B}_{2} / \sim$ is much smaller than $\mathcal{B}_{2}$, even though both sets are infinite countable.
- In Section 5 we use heuristics to investigate if certain properties of one-relator groups described in [8] and [16] could be the reason of our unsuccessful search of trivialization for $\operatorname{AK}(3)$.
- In Section 6 we present results of our experiments.


## 2. ACM-move

In this section we describe our implementation of the ACM-move, i.e., an algorithm which for a given pair $u, v \in F=F(x, y)$ constructs a subset of the set:

$$
U=U(u, v)=\left\{u^{\prime} \in F(x, y) \mid u^{\prime} \sim u \text { in }\langle x, y \mid v\rangle \text { and }\left|u^{\prime}\right| \leq L\right\}
$$

where $L \in \mathbb{N}$ is a fixed parameter value. Ideally, the algorithm should construct the whole set $U(u, v)$. The algorithm is based on weighted $X$ digraphs.
2.1. Weighted $X$-digraphs. Formally, a weighted $X$-digraph is a tuple $(V, E, \mu, \gamma)$ where $(V, E)$ defines a directed graph, $\mu: E \rightarrow X^{ \pm}$is the labeling function, and $\gamma: E \rightarrow \mathbb{Z}$ is the weight function.

We often use the following notation $a \xrightarrow{x, k} b$ for the edge with origin $a$, terminus $b$, label $x$, and weight $k$. We say that an edge $b \xrightarrow{x^{-1},-k} a$ is the inverse to $e=a \xrightarrow{x, k} b$ and denote it by $e^{-1}$. We say that a weighted $X$ digraph $\Gamma$ is:

- folded, if for every $a \in V$ and $x \in X^{ \pm}$there exists at most one edge with the origin $a$ labeled with $x$;
- inverse, if with every edge $e$ the graph $\Gamma$ contains $e^{-1}$;
- rooted, if $\Gamma$ comes with a designed vertex called the root.

A path $p$ in $\Gamma$ is a sequence of adjacent edges $e_{1} \ldots e_{n}$, its label is $\mu(p)=$ $\mu\left(e_{1}\right) \ldots \mu\left(e_{k}\right)$ and the weight $\gamma(p)=\gamma\left(e_{1}\right)+\ldots+\gamma\left(e_{k}\right)$. A circuit is a path with the same origin and terminus.

An inverse weighted labeled digraph $\Gamma$ with a root $v_{0}$ and a number $N \in$ $\mathbb{N} \cup\{\infty\}$ is called a pseudo conjugacy graph for $u$ in $G=\langle x, y \mid v\rangle$ if the following conditions are satisfied:
(CG1) $\mu(l) \sim_{v} u^{\gamma(l)}$ for any circuit $l$.
(CG2) $N=\infty$ or $u^{N}=1$ in $G$.
The simplest nontrivial example of a pseudo conjugacy graph $\operatorname{Loop}(u)$ for $u$ in $\langle x, y \mid v\rangle$ is shown in Fig. 1.

To implement ACM-move we generate a sufficiently large pseudo-conjugacy graph $\Gamma$ for $u$ and then "harvest" circuits of weight 1. A large pseudoconjugacy graph $\Gamma$ can be generated starting with Loop $(u)$ and applying


Figure 1. A pseudo conjugacy graph Loop $(u)$.
$R$-completion procedure $D$ times. $R$-completion is a variation of coset enumeration first described in [28, 22] and reviewed for more precise complexity bounds below. Harvest is shortly discussed in Section 2.4
2.2. Operations on weighted $X$-digraphs. Here we shortly describe several operations with graphs used later in the sequel.
2.2.1. Vertex-identification. Given a folded $X$-digraph $\Gamma$ and distinct $v_{1}, v_{2} \in$ $V(\Gamma)$ define a graph $\operatorname{Id}\left(\Gamma, v_{1}, v_{2}\right)$ obtained from $\Gamma$ as follows:

- add a new vertex $v$ to $V(\Gamma)$;
- for each edge $v_{i} \xrightarrow{x, a} u$ add an edge $v \xrightarrow{x, a} u$;
- for each edge $u \xrightarrow{x, a} v_{i}$ add an edge $u \xrightarrow{x, a} v$;
- remove $v_{1}$ and $v_{2}$.

In general, the graph $\operatorname{Id}\left(\Gamma, v_{1}, v_{2}\right)$ is not folded.
2.2.2. Weight-shift. For $v \in V(\Gamma)$ and $\delta \in \mathbb{Z}$ define a $\operatorname{graph} \operatorname{Shift}(v, \delta)$ obtained from $\Gamma$ by changing weights of edges incident to $v$ as follows:

- the weight $a$ of $e=v \xrightarrow{x, a} u$ (where $v \neq u$ ) is increased by $\delta$;
- the weight $a$ of $e=u \xrightarrow{x, a} v$ (where $v \neq u$ ) is decreased by $\delta$.

It is easy to see that a weight-shift preserves the weights of circuits in $\Gamma$ and, hence, preserves the property to be a pseudo-conjugacy graph. The number of arithmetic operations required for this operation is clearly bounded by the number of edges $E_{v}$ incident on $v$.
2.2.3. Folding. If $\Gamma$ is not folded, then it contains two edges $e_{1}=u \xrightarrow{x, a} v_{1}$ and $e_{2}=u \xrightarrow{x, b} v_{2}$. Consider several cases.

- If $v_{1}=v_{2}$ and $a \equiv b \bmod N$, then we remove one of the edges.
- If $v_{1}=v_{2}$ and $a \not \equiv b \bmod N$, then we replace the modulus $N$ with the number $\operatorname{gcd}(N, b-a)$ and remove one of the edges.
- If $v_{1} \neq v_{2}$, then:
- apply $\operatorname{Shift}\left(v_{1}, a-b\right)\left(\right.$ or $\left.\operatorname{Shift}\left(v_{2}, b-a\right)\right)$ to achieve $\gamma\left(e_{1}\right)=\gamma\left(e_{2}\right)$;
- identify $v_{1}$ and $v_{2}$;
- remove $e_{2}$.

It is straightforward to check that the described operation produces a pseudoconjugacy graph from a pseudo-conjugacy graph $\Gamma$ (see [28] for more detail). Since folding $e_{1}$ with $e_{2}$ decreases the number of edges, a sequence of folds eventually stops with a folded graph. The final result of folding is unique up to shifts of weights. Denote it by Fold $(\Gamma)$.
2.2.4. $R$-completion. Recall that a (finite) set $R \subset F(X)$ is called symmetrized if $R$ with every $r \in R$ contains all cyclic permutations of $r$. To complete a given weighted $X$-digraph $\Gamma$ with (symmetrized) relators $R \subset F(X)$ means to add a circuit at $v$ labeled with $r$ to $\Gamma$ of weight 0 for every $v \in V$ and $r \in R$. It is easy to check that if $\Gamma$ is a pseudo-conjugacy graph then the result is a pseudo-conjugacy graph as well. It clearly requires linear time (in $|\Gamma|$ ) to $R$-complete a given graph $\Gamma$. In general, the result is not folded.
2.3. Complexity of weighted $X$-digraph folding. Let $\Gamma=(V, E, \mu, \gamma)$ be a weighted $X$-digraph. It follows from the description of folding that $\operatorname{Fold}(\Gamma)$ as an $X$-digraph (i.e., if we forget about the weight function) is the same as the result of folding of the $X$-digraph $(V, E, \mu)$. The only difference between weighted $X$-digraph folding and $X$-digraph folding is weightprocessing (applications of $\operatorname{Shift}\left(v_{1}, a-b\right)$ or $\operatorname{Shift}\left(v_{2}, b-a\right)$ in Section 2.2.3). The idea is to modify the procedure and take weights into account.
2.3.1. $X$-digraph folding. Recall that $X$-digraph folding can be done in nearly linear time $O\left(|V| \log ^{*}(|V|)\right)$, where $\log ^{*}$ is inverse-Ackermann function (see [27]). In some sense, folding of an $X$-digraph $\Gamma$ describes the following equivalence relation $\sim$ on $V(\Gamma)$ :

- $v_{1} \sim v_{2} \Leftrightarrow$ there exists a path from $v_{1}$ to $v_{2}$ in $\Gamma$ with $\mu(p)=\varepsilon$;
and results into the graph $\Gamma / \sim$. To effectively represent the sets of identified vertices (equivalence classes) one can use compressed tree representation (as in [26]). Each vertex $v \in V$ contains a pointer $p(v)$ to its parent, the root points to itself. Vertex and its parent always belong to the same equivalence class, thus each tree represents an equivalence class. This presentation allows to compare and merge two classes very efficiently, which results in $O\left(|V| \log ^{*}(|V|)\right)$ complexity bound.
2.3.2. Weighted $X$-digraph folding. To achieve a similar complexity bound for weighted $X$-digraph folding we need to take into account shifts of weight. Since in the middle of the folding process we work with equivalence classes (as in Section 2.3.1 above), weight-shift requires shifting weights for a whole class. To avoid shifting weights many times with each vertex $v$ we keep a number $\delta(v)$ called shift value. That defines the total shift $\Delta(v)$ of a vertex $v$ as:

$$
\Delta(v)= \begin{cases}0, & \text { if } p(v)=v \\ \delta(v)+\Delta\left(v^{\prime}\right), & \text { if } p(v)=v^{\prime} \neq v\end{cases}
$$

Hence, to perform weight-shift of $v_{2}$ while identifying $v_{1}$ and $v_{2}$ (case $v_{1} \neq v_{2}$ in the Folding procedure, section 2.2.3 we set $p\left(v_{2}\right)=v_{1}$ and instead of
doing $\operatorname{Shift}(v, c)$ immediately, we just set $\delta(v)=c$. Comparison and merge of two vertex equivalence classes can be easily extended to tree presentations with shifts $\delta$. Therefore, the following proposition holds.

Proposition 2.1. The number of additions performed by the folding procedure is $O\left(|V| \log ^{*}(|V|)\right)$.

We would like to point out that the values of the weight function can grow exponentially fast (linearly in binary). Nevertheless, in all our experiments we never encountered values greater than $2^{64}$.
Corollary 2.2. Let $(X ; R)$ be a symmetrized presentation. The total number of additions required to apply $R$-completion to a weighted graph $\Gamma$ is $O\left(|\Gamma| L(R) \log ^{*}(|\Gamma| L(R))\right)$.
2.4. Harvest. Here we describe a procedure that for a weighted folded $X$ digraph $\Gamma$ and $L \in \mathbb{N}$ finds all circuits in $\Gamma$ of weight 1 and length up to $L$. Since the number of such circuits is expected to grow exponentially with $L$ one can not expect a very efficient implementation. Nevertheless, certain heuristics allow to speed up enumeration significantly.

For every vertex $v \in \Gamma$ we find all reduced paths $P$ in $\Gamma$ from $v$ of length up to $\lceil L / 2\rceil$ and distribute them into bins $P_{u, x, \nu, l}$ :

$$
P_{u, x, \gamma, l}=\{p \in P|t(p)=u, \gamma(p)=\gamma,|p|=l, \mu(p) \text { ends with } x\} .
$$

Then we consider pairs of "compatible" bins:
$T=\left\{\left(P_{u_{1}, x_{1}, \gamma_{1}, l_{1}}, P_{u_{2}, x_{2}, \gamma_{2}, l_{2}}\right) \mid u_{1}=u_{2}, x_{1} \neq x_{2}, \gamma_{1}-\gamma_{2} \equiv_{N} 1,0 \leq l_{1}-l_{2} \leq 1\right\}$.
The set of circuits at $v$ is constructed as:

$$
\left\{p_{1} p_{2}^{-1} \mid p_{1} \in P_{1}, p_{2} \in P_{2},\left(P_{1}, P_{2}\right) \in T\right\}
$$

Finally, the vertex $v$ is removed from $\Gamma$ and the same process is applied to another vertex in $\Gamma$. Note that we may skip vertices which has no adjacent edges of a non-trivial weight. The number of operations is bounded by $(2 \cdot|X|)^{\lceil L / 2\rceil}$, but it was much smaller in practice.
2.5. ACM-move implementation efficiency. The implementation of the ACM-move described above constructs a subset of the set of conjugates for a given $u$ in $\langle x, y \mid v\rangle$ of bounded length. In general, it can be a proper subset of $U(u, v)$. The result depends on the value of $D$ - the number of completion steps used to construct pseudo-conjugacy graphs. We denote it by $U_{D}(u, v)$. In this section we shortly prove that:

$$
\bigcup_{D=1}^{\infty} U_{D}(u, v)=U(u, v)
$$

and define the parameter $\delta$ (called depth) of a conjugate $u^{\prime} \in U(u, v)$ responsible for "complexity" of $u$.

For a set $S \subset \mathbb{R}^{2}$ let $\partial S$ be its boundary and $\bar{S}$ the closure of $S$ in $\mathbb{R}^{2}$. Let $D$ be a finite connected planar $X$-digraph with set of vertices $V(D)$ and set of
edges $E(D)$. Let $C(D)$ be a set of cells of $D$ which are connected and simply connected bounded components of $\mathbb{R}^{2} \backslash D$. The unbounded component of $\mathbb{R}^{2} \backslash D$ is called the outer cell of $D$ denoted by $c_{\text {out }}$. An edge $e \in E(D)$ is free if it does not belong to $\partial c$ for any $c \in C(D)$. For any $e \in E(D)$ we denote its label by $\mu(e) \in X^{ \pm}$. The boundary of a cell $c \in C(D)$ traversed in a counterclockwise direction starting from some vertex of $c$ makes a closed path $e_{1} \ldots e_{n}$ giving the word $\mu(c)=\mu\left(e_{1}\right) \ldots \mu\left(e_{n}\right) \in\left(X^{ \pm}\right)^{*}$ called a boundary label of $c$. Depending on a starting vertex we get a cyclic permutation of the same word.

For the rest of this subsection let $D$ be a finite connected planar $X$-digraph with a base vertex $v_{0} \in V(D) \cap \partial c_{\text {out }}$. The graph $D$ is a van Kampen diagram over $\langle X \mid R\rangle$ if $\mu(c) \in R^{\star}$ for every $c \in C(D)$. The boundary label $\mu(D)$ of $D$ is the boundary label of $\partial c_{\text {out }}$ read starting from $v_{0}$ in a counterclockwise direction. Note that we need also to specify the first edge to read from $v_{0}$, that is, the starting boundary position, but it is not important for our considerations so we omit this issue.

Now let us exclude one of the cells from $C(D)$ and call it the inner cell $c_{\text {in }}$ of $D$. Denote $v_{0}$ by $v_{\text {out }}$ and pick any vertex $v_{\text {in }} \in V(D) \cap \partial c_{\text {in }}$. We call $D$ an annular (Schupp) diagram (see [25]) over $\langle X \mid R\rangle$ if $\mu(c) \in R^{\star}$ for any $c \in C(D)$. Its two boundary labels $\mu_{\text {in }}(D)=\mu\left(c_{\text {in }}\right)$ and $\mu_{\text {out }}(D)=\mu\left(c_{\text {out }}\right)$ read in a counterclockwise direction from $v_{\text {in }}$ and $v_{\text {out }}$ correspondingly, are called the inner and outer labels of $D$. For any $w_{1}, w_{2} \in\left(X^{ \pm}\right)^{*}$ we have that $w_{1} \sim_{G} w_{2}$ if and only if there exists an annular diagram $D$ over $\langle X \mid R\rangle$ with $\mu_{\mathrm{in}}(D)=w_{1}$ and $\mu_{\text {out }}(D)=w_{2}$.

We measure diagram complexity using a notion of depth (introduced in [28]). For a van Kampen or annular diagram $D$ define the dual graph $D^{*}=\left(V^{*}, E^{*}\right)$ as an undirected graph with $V^{*}=C(D) \cup c_{\text {out }}$ (for annular diagrams we add $c_{\text {in }}$ ) and $E^{*}=\left\{\left(c_{1}, c_{2}\right) \mid \partial c_{1} \cap \partial c_{2} \neq \emptyset\right\}$. We denote the graph distance in $D^{*}$ by $d^{*}$.

The depth of a (generalized) van Kampen diagram $D$ is defined by:

$$
\delta(D)=\max _{c \in C(D)} d^{*}\left(c, c_{\mathrm{out}}\right)
$$

The depth of an annular diagram $D$ is:

$$
\delta_{\sim}(D)=\max _{c \in C(D)}\left[\min \left(d^{*}\left(c, c_{\mathrm{out}}\right), d^{*}\left(c, c_{\mathrm{in}}\right)\right)\right]
$$

(There is a similar notion of a diagram radii (see [10, 7]).) Define the conjugate depth of two words $w_{1}, w_{2} \in F(X)$ as:

$$
\delta_{\sim}\left(w_{1}, w_{2}\right)=\min _{\substack{D \text { is } \\ \text { an annular } \\ \text { diagram }}}\left\{\delta(D) \mid \mu_{\text {in }}(D)=w_{1}, \mu_{\text {out }}(D)=w_{2}\right\}
$$

if $w_{1} \sim_{G} w_{2}$ and $\infty$ otherwise. The next theorem shows a relation between complexity of the conjugacy search problem and the conjugacy depth.

Theorem (Theorem 3.5 in [20]). There exists an algorithm which for a given finite symmetrized presentation $\langle X \mid R\rangle$ and words $w_{1}, w_{2} \in F(X)$
terminates with the affirmative answer if and only if $w_{1} \sim_{G} w_{2}$. Furthermore, its complexity can be bounded above by:

$$
\tilde{O}\left(\left|w_{1}\right|\left|w_{2}\right| L(R)^{2 \delta \sim\left(w_{1}, w_{2}\right)}\right)
$$

For our purposes it will be useful to define another characteristic of annular diagrams, the inner conjugacy depth:

$$
\delta_{\sim}^{i n}(D)=\max _{c \in C(D)} d^{*}\left(c, c_{\text {in }}\right)
$$

and the conjugacy depth from $w_{1}$ to $w_{2}$ as:

$$
\delta_{\sim}^{w_{1}}\left(w_{2}\right)=\min _{\substack{D \text { is } \\ \text { an annular } \\ \text { diagram }}}\left\{\delta_{\sim}^{w_{1}}(D) \mid \mu_{\text {in }}(D)=w_{1}, \mu_{\text {out }}(D)=w_{2}\right\}
$$

if $w_{1} \sim_{G} w_{2}$ and $\infty$ otherwise.
Theorem 2.3. Assume that $u$ and $u^{\prime}$ are conjugate in $\langle x, y \mid v\rangle$ and $\delta=$ $\delta_{\sim}^{u}\left(u^{\prime}\right)$. If $\delta \leq D$ (where $D$ is the number of $R$-completions applied to Loop $(u)$ ), then our implementation of the ACM-move applied for $(u, v)$ produces the pair $\left(u^{\prime}, v\right)$.

Proof. Same proof as that of [22, Theorem 17.6.12].
It easily follows from Corollary 2.2 that, in general, $D$ iterations of $R$ completion procedure require exponential time. Fortunately, in our experiments with $\mathrm{AK}(3)$, we observed that the value $D=2$ is sufficient. Application of more than two $R$-completions did not produce any additional conjugates and did not change highlighted figures in Table 1, section 6.1.

## 3. Nielsen automorphisms and AC-EQuivalence

In this section we discuss automorphism-moves, namely applications of an automorphism $\varphi \in \boldsymbol{\operatorname { A u t }}\left(F_{2}\right)$ :

$$
(u, v) \rightarrow(\varphi(u), \varphi(v))
$$

It is not known if adding these transformations to AC-moves results in an equivalent system of transformations or not (even for presentations of $\mathbb{1}$ ). Nevertheless, the following is true.

Lemma 3.1 ( 9 , Proposition 1(iii)] or [19, Lemma 1]). If (u,v) can be transformed into $(x, y)$ using $A C$-moves and automorphisms, then $(u, v)$ can be transformed into $(x, y)$ using $A C$-moves only.

With any $(u, v)$ we can associate $\varphi_{(u, v)} \in \mathbf{E n d}\left(F_{2}\right)$ defined by $\varphi_{(u, v)}(x)=$ $u$ and $\varphi_{(u, v)}(y)=v$. That way we can treat $\mathcal{B}_{2}$ as a monoid. By Lemma $3.2, \mathcal{B}_{2}$ naturally acts on AC-components.

Lemma 3.2. Assume that $(u, v) \sim_{\mathrm{AC}}\left(u^{\prime}, v^{\prime}\right)$ and $\varphi \in \mathcal{B}_{2}$. Then:

$$
(\varphi(u), \varphi(v)) \sim_{\mathrm{AC}}\left(\varphi\left(u^{\prime}\right), \varphi\left(v^{\prime}\right)\right)
$$

Proof. Clearly, it is sufficient to prove the result for the case when $\left(u^{\prime}, v^{\prime}\right)$ is obtained from $(u, v)$ by a single ACM-move, i.e., we may assume that $u \sim u^{\prime}$ in $\langle x, y \mid v\rangle$ and $v=v^{\prime}$. Hence, $\varphi(u) \sim \varphi\left(u^{\prime}\right)$ in $\langle x, y \mid \varphi(v)\rangle$ and $(\varphi(u), \varphi(v)) \sim_{\mathrm{AC}}\left(\varphi\left(u^{\prime}\right), \varphi\left(v^{\prime}\right)\right)$.

Lemma 3.3 immediately implies Lemma 3.1 since $(x, y) \sim_{A C}(\varphi(x), \varphi(y))$ for every $\varphi \in \boldsymbol{\operatorname { A u t }}(F(X))$.

Lemma 3.3. Let $u, v \in F(X), \varphi \in \operatorname{End}(F(X))$, and $(u, v) \sim_{\mathrm{AC}}(\varphi(u), \varphi(v))$. Then for any $u^{\prime}, v^{\prime} \in F(X)$ the following holds:

$$
(u, v) \sim_{\mathrm{AC}}\left(u^{\prime}, v^{\prime}\right) \quad \Rightarrow \quad\left(u^{\prime}, v^{\prime}\right) \sim_{\mathrm{AC}}\left(\varphi\left(u^{\prime}\right), \varphi\left(v^{\prime}\right)\right)
$$

Proof. As above, we may assume that $u \sim u^{\prime}$ in $\langle x, y \mid v\rangle$ and $v=v^{\prime}$. Hence $\left(\varphi\left(u^{\prime}\right), \varphi\left(v^{\prime}\right)\right) \sim_{A C}(\varphi(u), \varphi(v)) \sim_{A C}(u, v) \sim_{A C}\left(u^{\prime}, v^{\prime}\right)$.

With $(u, v) \in \mathcal{B}_{2}$ we can associate a monoid:

$$
\operatorname{End}_{\mathrm{AC}}(u, v)=\left\{\varphi \in \operatorname{End}\left(F_{2}\right) \mid(u, v) \sim_{\mathrm{AC}}(\varphi(u), \varphi(v))\right\},
$$

under the usual composition. Lemma 3.3 implies that:

- $\operatorname{End}_{\mathrm{AC}}(u, v)=\operatorname{End}_{\mathrm{AC}}\left(u^{\prime}, v^{\prime}\right)$ whenever $(u, v) \sim_{A C}\left(u^{\prime}, v^{\prime}\right)$;
- $\boldsymbol{\operatorname { A u t }}\left(F_{2}\right) \leq \operatorname{End}_{\mathrm{AC}}(x, y)$.
- $\operatorname{End}_{\mathrm{AC}}(x, y)=\left\{\varphi \mid \varphi(x)=u, \varphi(y)=v,(u, v) \in \mathcal{B}_{2}\right\}$ if and only if ACC holds.
Below we prove that $\boldsymbol{\operatorname { A u t }}\left(F_{2}\right) \leq \operatorname{End}_{\mathrm{AC}}(\operatorname{AK}(n))$ for every $n \in \mathbb{N}$.
3.1. Akbulut-Kurby examples. Lemmas 3.4, 3.5, and 3.6 show that:

$$
\varphi(\operatorname{AK}(n)) \sim_{A C} \operatorname{AK}(n)
$$

for some $\varphi \in \boldsymbol{\operatorname { A u t }}\left(F_{2}\right)$. Proofs were obtained using a computer program.
Below for brevity we use $X$ and $Y$ as $x^{-1}$ and $y^{-1}$.
Lemma 3.4. $\varphi(\operatorname{AK}(n)) \sim_{A C} \operatorname{AK}(n)$ for $\varphi \in \boldsymbol{\operatorname { A u t }}\left(F_{2}\right)$ defined by $\varphi(x)=y$, $\varphi(y)=x$.

Proof. The pair $\left(\varphi(x y x Y X Y), \varphi\left(x^{k} Y^{k+1}\right)\right)=\left(y x y X Y X, y^{k} X^{k+1}\right)$ can be modified as follows:

$$
\begin{aligned}
\sim_{A C_{2}} & \left(x y x Y X Y, y^{k} X^{k+1}\right) \\
\sim_{A C M} & \left(x y x Y X Y, c^{-1} x^{k} Y^{k+1} c\right) \text { where } c=x y x \\
\sim_{A C_{3}} & \left(x y x Y X Y, x^{k} Y^{k+1}\right) .
\end{aligned}
$$

Appendix A provides more detail for each ACM-move used.
Lemma 3.5. $\varphi(\operatorname{AK}(n)) \sim_{A C} \operatorname{AK}(n)$ for $\varphi \in \boldsymbol{\operatorname { A u t }}\left(F_{2}\right)$ defined by $\varphi(x)=x$, $\varphi(y)=Y$.

Proof. The pair $\left(\varphi(x y x Y X Y), \varphi\left(x^{k} Y^{k+1}\right)\right)=\left(x Y x y X y, x^{k} y^{k+1}\right)$ can be modified as follows:

$$
\begin{aligned}
\sim_{A C_{2}} & \left(x Y x y X y, Y^{k+1} X^{k}\right) \\
\sim_{A C 3} & \left(x y X y x Y, Y^{k+1} X^{k}\right) \\
\sim_{A C M} & \left(x y X y x Y, c^{-1} x y X y^{k+1} x Y^{k+2} c\right), c=y^{k+1} \\
\sim_{A C 3} & \left(x y X y x Y, x y X y^{k+1} x Y^{k+2}\right) \\
\sim_{A C M} & \left(c^{-1} x y X y x Y Y X y c, x y X y^{k+1} x Y^{k+2}\right), c=y^{k+2} X y x Y \\
\sim_{A C 3} & \left(x y X y x Y Y X y, x y X y^{k+1} x Y^{k+2}\right) \\
\sim_{A C 2} & \left(x y X y x Y Y X y, y^{k+2} X Y^{k+1} x Y X\right) \\
\sim_{A C M} & \left(x y X y x Y Y X y, c^{-1} x y X y^{k} X Y^{k} c\right), c=x y X y \\
\sim_{A C 3} & \left(x y X y x Y Y X y, x y X y^{k} X Y^{k}\right) \\
\sim_{A C 2} & \left(Y x y y X Y x Y X, x y X y^{k} X Y^{k}\right) \\
\sim_{A C M} & \left(c^{-1} Y x y y X Y x Y X c, x y X y^{k} X Y^{k}\right), c=Y^{k-1} x Y^{k-1} x Y Y X y \\
\sim_{A C 3} & \left(Y x y y X Y x Y X, x y X y^{k} X Y^{k}\right) \\
\sim_{A C M} & \left(c^{-1} x^{k} Y^{k+1} c, x y X y^{k} X Y^{k}\right), c=Y^{k-1} x Y^{k-1} x Y Y X y \\
\sim_{A C 3} & \left(x^{k} Y^{k+1}, x y X y X Y^{k}\right) \\
\sim_{A C M} & \left(x^{k} Y^{k+1}, c^{-1} x y x Y X Y c\right), c=Y^{k} \\
\sim_{A C 3} & \left(x^{k} Y^{k+1}, x y x Y X Y\right)
\end{aligned}
$$

Lemma 3.6. $\varphi(\operatorname{AK}(n)) \sim_{A C} \operatorname{AK}(n)$ for $\varphi \in \boldsymbol{\operatorname { A u t }}\left(F_{2}\right)$ defined by $\varphi(x)=x$, $\varphi(y)=y x$.
Proof. The pair $\left(\varphi(x y x Y X Y), \varphi\left(x^{k} Y^{k+1}\right)\right)=\left(x y x Y X X Y, x^{k-1}(Y X)^{k} Y\right)$ can be modified as follows:

$$
\begin{aligned}
= & \left(x y x Y X X Y, x^{k-1}(Y X)^{k} Y\right) \\
\sim_{A C_{3}} & \left(Y x y x Y X X, x^{k-1}(Y X)^{k} Y\right) \\
\sim_{A C_{3}} & \left(x x y X Y X y, x^{k-1}(Y X)^{k} Y\right) \\
\sim_{A C M} & \left(x x y X Y X y, c^{-1} x^{k-1} y X^{k-1} y X Y c\right), c=x^{k-3}(Y X)^{k} Y \\
\sim_{A C_{3}} & \left(x x y X Y X y, x^{k-1} y X^{k-1} y X Y\right) \\
\sim_{A C_{2}} & \left(Y x y x Y X X, x^{k-1} y X^{k-1} y X Y\right) \\
\sim_{A C M} & \left(c^{-1} x^{k} Y^{k+1} c, x^{k-1} y X^{k-1} y X Y\right), c=x y X^{k-1} y X Y X y \\
\sim_{A C_{3}} & \left(x^{k} Y^{k+1}, x^{k-1} y X^{k-1} y X Y\right) \\
\sim_{A C_{2}} & \left(x^{k} Y^{k+1}, y x Y x^{k-1} Y X^{k-1}\right) \\
\sim_{A C M} & \left(x^{k} Y^{k+1}, c^{-1} x y x Y X Y c\right), c=y X^{k-1} y X Y \\
\sim_{A C 3} & \left(x^{k} Y^{k+1}, x y x Y X Y\right)
\end{aligned}
$$

Proposition 3.7. $\varphi(\operatorname{AK}(n)) \sim_{A C} \operatorname{AK}(n)$ for any $n>2$ and $\varphi \in \operatorname{Aut}\left(F_{2}\right)$.
Proof. Automorphisms considered in Lemmas 3.4, 3.5 and 3.6 generate Aut $\left(F_{2}\right)$. Hence the proposition holds for any $\varphi \in \boldsymbol{\operatorname { A u t }}\left(F_{2}\right)$.

The next corollary implies that adding automorphism-moves to AC-moves does not increase orbits for $\operatorname{AK}(n)$ presentations.

Corollary 3.8. If $(u, v) \sim_{A C} \operatorname{AK}(3)$, then $(\varphi(u), \varphi(v)) \sim_{A C} \operatorname{AK}(3)$ for every $\varphi \in \boldsymbol{\operatorname { A u t }}\left(F_{2}\right)$.

It is natural to raise a question if a similar result holds for all balanced presentations of $\mathbb{1}$. We performed experiments with several randomly generated Miller-Schupp presentations. The results were not always positive, i.e., for some presentations we were unable to prove AC-equivalence to their automorphic images.

Conjecture. It is not true that for every $(u, v) \in \mathcal{B}_{2}$ and $\varphi \in \boldsymbol{\operatorname { A u t }}\left(F_{2}\right)$ $(u, v) \sim_{\mathrm{AC}}(\varphi(u), \varphi(v))$.

Note that the conjecture above immediately implies negative answer to ACC. In Table 2 the reader can find particular balanced presentations suspected to satisfy the conjecture above.

## 4. Canonical forms of presentations

For a given relation tuple $\bar{u} \in \mathcal{B}_{n}$ the search space $\mathcal{C}_{\bar{u}}$ is infinite and no computer procedure can exhaust all of its elements. To reduce the search space one can introduce an equivalence relation $\sim$ on $\mathcal{C}_{\bar{u}}$ (or on $F_{n}^{n}$ ), define efficiently computable representatives $N F(\bar{u})$ for equivalence classes, and study the quotient space:

$$
\mathcal{B}_{n} / \sim=\left\{\operatorname{NF}(\bar{u}) \mid \bar{u} \in \mathcal{B}_{n}\right\} .
$$

That way one can achieve compression of the search space as a single element $\mathrm{NF}(\bar{u})$ representing its (infinite) equivalence class. Clearly, coarser relations on $F_{n}^{n}$ give better compression.

Below we consider two equivalence relations on $\mathcal{B}_{2}$. Same results hold for $\mathcal{B}_{n}$ with $n>2$. The first one (referred to as a cyclic relation here) was used by Casson in a series of unpublished work (according to Bowman-McCaul) and by Bowman-McCaul (who "followed" Casson). The second relation is new and is significantly stronger.
4.1. Cyclic relation. Let $\sim$ be the transitive closure of the following pairs in $F_{2}^{2}$ :

- $(u, v) \sim(v, u)$,
- $(u, v) \sim\left(u^{-1}, v\right)$,
- $(u, v) \sim\left(u, c^{-1} v c\right)$,
where $u, v, c$ are arbitrary words in $F_{2}$. We call $\sim$ a cyclic relation on $F_{2}^{2}$.
To define canonical representatives for the cyclic relation we do the following. Fix any order on generators, say $x_{1}<x_{1}^{-1}<x_{2}<x_{2}^{-1}$ and denote by $<$ the corresponding shortlex order on $F_{2}$ and, further, the corresponding lexicographic order on $F_{2}^{2}$. Let $(u, v) \in F_{2}^{2}$. It is easy to see that taking the least cyclic permutation of $u^{ \pm 1}$, the least cyclic permutation of $v^{ \pm 1}$, and "sorting" the obtained words, produces the least representative of the equivalence class of $(u, v)$, denoted by $\mathrm{NF}(u, v)$. Clearly, so-defined normal form is an efficiently computable.

It easily follows from the definition that $\mathrm{NF}(u, v) \in \mathcal{B}_{2}$ for any $(u, v) \in \mathcal{B}_{2}$. Hence, AC-moves can be naturally defined on $\mathcal{B}_{2} / \sim$ :

$$
\mathrm{NF}(u, v)=(u, v) \quad \stackrel{\nu}{\mapsto} \quad \mathrm{NF}(\nu(u, v)),
$$

where $\nu$ is an AC-move. The problem with this approach is that computing the cyclic normal form negates applications of the $A C_{3}$-move. That can result in the component $\mathcal{C}_{\bar{u}}$ being broken into subcomponents (i.e., $\mathcal{C}_{\bar{u}}$ can become disconnected). In particular, Bowman-McCaul implementation (found here http://www.math.utexas.edu/users/sbowman/ac-bfs.tar. $\mathrm{gz})$ does not take the normal form of a pair obtained by an $\mathrm{AC}_{3}$-move, which completely negates the advantage of using normal forms. As we explain below, ACM-moves can solve this problem.
4.2. Cyclic relation with automorphisms. Define an equivalence relation $\sim$ on $F_{2}^{2}$ by taking a closure of the following pairs:

- $(u, v) \sim(v, u)$,
- $(u, v) \sim\left(u^{-1}, v\right)$,
- $(u, v) \sim\left(u, c^{-1} v c\right)$,
- $(u, v) \sim(\varphi(u), \varphi(v))$,
where $u, v, c$ are arbitrary words in $F_{2}$ and $\varphi$ is an arbitrary automorphism in $\boldsymbol{A u t}\left(F_{2}\right)$. Note that the defined relation makes $(u, v)$ equivalent to $(\varphi(u), \varphi(v))$ which, in general, is not known to be AC-equivalent to $(u, v)$. Hence, it is possible that an equivalence class of $(u, v)$ contains an element which is not AC-equivalent to $(u, v)$. Nevertheless, the following is true.

Proposition 4.1. For every $u, v, u^{\prime}, v^{\prime} \in F_{2}$ if

- $\operatorname{AK}(n) \sim_{A C}(u, v)$, and
- $(u, v) \sim\left(u^{\prime}, v^{\prime}\right)$,
then $\operatorname{AK}(n) \sim_{A C}\left(u^{\prime}, v^{\prime}\right)$.
Proof. Follows from Corollary 3.8 .
Proposition 4.1 allows us to replace the original component $\mathcal{C}=\mathcal{C}_{\mathrm{AK}(3)}$ with its factor $\mathcal{C} \sim$ which is much smaller. The problem that taking a normal form of a pair negates $A C_{3}$-moves is still relevant if we use the original AC-moves. That is where ACM-moves really help. It follows from Theorem 2.3 that choosing sufficiently large value of the parameter $D$ we can produce any conjugate of $u$ in $\langle x, y \mid v\rangle$.

As in Section 4.1, the normal form of the pair $(u, v)$ is defined as the minimal pair in its equivalence class. Below we show that normal forms can be computed efficiently. Our main tool is the following classic result.

Theorem (Whitehead theorem, see [17, Proposition 4.20]). Let $w_{1}, \ldots, w_{t}$, $w_{1}^{\prime}, \ldots, w_{t}^{\prime}$ be cyclic words in a free group $F$ such that:

$$
w_{1}^{\prime}=\alpha\left(w_{1}\right), \ldots, w_{t}^{\prime}=\alpha\left(w_{t}\right)
$$

for some $\alpha \in \operatorname{Aut}(F)$. If $\sum\left|w_{k}^{\prime}\right|$ is minimal among all $\sum\left|\alpha^{\prime}\left(w_{k}\right)\right|$ for $\alpha^{\prime} \in \boldsymbol{\operatorname { A u t }}(F)$, then $\alpha=\tau_{1} \ldots \tau_{n}, n \geq 0$, where $\tau_{1}, \ldots, \tau_{n}$ are Whitehead automorphisms and for each $i$ the length $\sum_{k}\left|\tau_{1} \cdots \tau_{i}\left(w_{k}\right)\right| \leq \sum_{k}\left|w_{k}\right|$ with strict inequality unless $\sum\left|w_{k}\right|=\sum\left|w_{k}^{\prime}\right|$.

Recall [17, Section 1.4] that Whitehead automorphisms are automorphisms of two types:

- (length-preserving) automorphisms that permute the letters $X^{ \pm}$;
- automorphisms that for some fixed "multiplier" $a \in X^{ \pm}$carry each of the elements $x \in X$ into one of $x, x a, a^{-1} x$, or $a^{-1} x a$.
There are exactly 20 Whitehead automorphisms for a free group of rank 2. According to the Whitehead theorem if the total length of a given tuple of cyclic words can be decreased by an application of an automorphism, then it can be decreased by an application of a single Whitehead automorphism. Hence, to compute the normal form of a pair $(u, v)$ we do the following.
- First, minimize the total length $|u|+|v|$ of $(u, v)$ by applying 12 non-length-preserving Whitehead automorphisms while the total length decreases.
- Then, construct a set of all equivalent pairs of the least total length by applying all automorphisms.
- Finally, choose the least cyclic normal form among the pairs of the least length.
The procedure described above is efficient except, maybe, the second step, where we construct the set of all pairs of the least total length. Currently there are no theoretical polynomial bounds on the size of that set. Nevertheless, in our computations the maximal size observed was 112 for AK (3) equivalent presentations with $|r| \leq 20$ bound. The average size of the set of all pairs of the least total length was 9 .


## 5. Groups with high Dehn function

One potential challenge for computer enumeration techniques is described by Bridson in [8] and Lishak in [16]. Both papers use a similar idea based on properties of the following one-relator group:

$$
\begin{equation*}
\left\langle x, y \mid y^{-1} x^{-1} y x y^{-1} x y=x^{2}\right\rangle, \tag{1}
\end{equation*}
$$

introduced by Baumslag in [3], satisfying the inequality:

$$
\begin{equation*}
\operatorname{Dehn}(n) \geq \operatorname{Tower}_{2}\left(\log _{2}(n)\right), \tag{2}
\end{equation*}
$$

first observed in [11. Lishak constructs a particular sequence of balanced presentations parametrized by $n \in \mathbb{N}$ :

$$
\bar{u}_{n}=\left(r, w_{n} y^{-1}\right) \in \mathcal{B}_{2},
$$

where $w_{n} \in F(X)$, satisfying the following conditions.

- $\bar{u}_{n}$ is AC-equivalent to the canonical presentation.
- The number of steps required to obtain the canonical presentation is super-exponential in $n$.
The later property comes as a consequence of the inequality (2).
Being very curious about possibility that that is the reason why our program fails to find AC-trivialization of $\operatorname{AK}(3)$ we tested all words obtained in our experiments. For each word $r$ we attempted to bound the Dehn function of the group $\langle x, y \mid r\rangle$. For that purpose we used D. Holt's package [14] to identify automatic groups (automatic groups have at most quadratic Dehn functions) that left us with 5356 "perhaps non-automatic" one-relator groups. Among those 1205 we classified as Baumslag-Solitar type presentations, i.e., the presentations with the relation $\left(u^{n}\right)^{v}=u^{m}$ for some $u, v \in F_{2}$. None of Baumslag-Solitar type presentations satisfied the condition $u \sim_{F} v$, i.e., no presentations were identified as Baumslag-type presentations (1). Clearly, this is a heuristic approach and we can not guarantee that our list of presentations does not contain Baumslag groups, as isomorphism problem for one-relator groups is not known to be decidable/undecidable. Also, we were unable to classify 4151 remaining presentations. In case someone would like to further investigate this, we have published the obtained lists here: https://github.com/stevens-crag/ak3_types.

In light of these heuristic results it seems to be a very interesting computational problem to classify short one-relator groups $\langle x, y \mid r\rangle$ with $|r| \leq 20$. Find precise upper bounds on their Dehn functions.

Conjecture. Baumslag's group $\left\langle x, y \mid y^{-1} x^{-1} y x y^{-1} x y=x^{2}\right\rangle$ has the highest Dehn function among all one-relator groups.

## 6. Results

The described algorithms were tested on several known potential counterexamples. Our attention was mainly focused on $\operatorname{AK}(3)$ and MillerSchupp presentations. To test performance and compare with other experimental results we also ran our programs on $\operatorname{AK}(2)$ and other presentations that are known to be AC-equivalent to the canonical presentation.

As we already mentioned in Section 2.4, we set a bound $L$ on the length on the conjugates obtained during harvest phase. We also set a limit on the total length of pairs to be $2 L+2$. Notice that we need to do that as taking a normal form described in Section 4.2 can increase length of one of the words beyond $L$ (which is allowed in our implementation). Experiments were run on a machine with two 8-core 3.1 Ghz Intel Xeon CPU E5-2687W and 64 GB RAM.
6.1. Enumeration of $\operatorname{AK}(3)$-equivalent presentations. As shown in Section 3, automorphism-moves can be used together with ACM-moves when applied to $\mathrm{AK}(3)$-equivalent presentations. In particular, one can use normal forms from Section 4.2 to compress the array of stored presentations. Table 1 shows dynamics of growth of a component of $\mathcal{C}_{\mathrm{AK}(3)}$ constructed by
our program for different values of $L$. Each cell in Table 1 corresponds to a value $L$ and a value $T$ and presents the number of pairs of the total length equal to $T$ constructed by the program with the single-word-bound $L$.

| $\mathrm{T} \backslash \mathrm{L}$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 13 | $\mathbf{4}$ | $\mathbf{4}$ | $\mathbf{4}$ | $\mathbf{4}$ | $\mathbf{4}$ | $\mathbf{4}$ | $\mathbf{4}$ | $\mathbf{4}$ | $\mathbf{4}$ | $\mathbf{4}$ | $\mathbf{4}$ |
| 14 | $\mathbf{1 0}$ | $\mathbf{1 0}$ | $\mathbf{1 0}$ | $\mathbf{1 0}$ | $\mathbf{1 0}$ | $\mathbf{1 0}$ | $\mathbf{1 0}$ | $\mathbf{1 0}$ | $\mathbf{1 0}$ | $\mathbf{1 0}$ | $\mathbf{1 0}$ |
| 15 | $\mathbf{7 0}$ | $\mathbf{7 0}$ | $\mathbf{7 0}$ | $\mathbf{7 0}$ | $\mathbf{7 0}$ | $\mathbf{7 0}$ | $\mathbf{7 0}$ | $\mathbf{7 0}$ | $\mathbf{7 0}$ | $\mathbf{7 0}$ | $\mathbf{7 0}$ |
| 16 | 64 | $\mathbf{8 6}$ | $\mathbf{8 6}$ | $\mathbf{8 6}$ | $\mathbf{8 6}$ | $\mathbf{8 6}$ | $\mathbf{8 6}$ | $\mathbf{8 6}$ | $\mathbf{8 6}$ | $\mathbf{8 6}$ | $\mathbf{8 6}$ |
| 17 | 220 | 416 | 454 | $\mathbf{4 5 8}$ | $\mathbf{4 5 8}$ | $\mathbf{4 5 8}$ | $\mathbf{4 5 8}$ | $\mathbf{4 5 8}$ | $\mathbf{4 5 8}$ | $\mathbf{4 5 8}$ | $\mathbf{4 5 8}$ |
| 18 | 98 | 392 | 398 | $\mathbf{5 9 0}$ | $\mathbf{5 9 0}$ | $\mathbf{5 9 0}$ | $\mathbf{5 9 0}$ | $\mathbf{5 9 0}$ | $\mathbf{5 9 0}$ | $\mathbf{5 9 0}$ | $\mathbf{5 9 0}$ |
| 19 | 240 | 764 | 1382 | 2854 | $\mathbf{3 2 2 6}$ | $\mathbf{3 2 2 6}$ | $\mathbf{3 2 2 6}$ | $\mathbf{3 2 2 6}$ | $\mathbf{3 2 2 6}$ | $\mathbf{3 2 2 6}$ | $\mathbf{3 2 2 6}$ |
| 20 | 10 | 442 | 522 | 2004 | 2082 | 3352 | 3352 | $\mathbf{3 3 5 6}$ | $\mathbf{3 3 5 6}$ | $\mathbf{3 3 5 6}$ | $\mathbf{3 3 5 6}$ |
| 21 | 20 | 746 | 1624 | 3870 | 8334 | 16948 | 19666 | 19690 | 19690 | $\mathbf{1 9 6 9 2}$ | $\mathbf{1 9 6 9 2}$ |
| 22 | 0 | 438 | 570 | 2812 | 3714 | 12288 | 12584 | 23174 | 23174 | 23188 | 23192 |
| 23 | 0 | 112 | 1462 | 4474 | 9194 | 21678 | 41492 | 101544 | 128356 | 128380 | 128388 |
| 24 | 0 | 6 | 42 | 3400 | 3858 | 12978 | 15458 | 61100 | 64686 | 150264 | 150276 |
| 25 | 0 | 0 | 110 | 4350 | 11246 | 22422 | 42550 | 102262 | 236860 | 631000 | 843778 |
| 26 | 0 | 0 | 0 | 4306 | 5384 | 17930 | 19668 | 62874 | 83902 | 375818 | 394172 |
| 27 | 0 | 0 | 0 | 710 | 13548 | 28176 | 51590 | 96714 | 196098 | 538380 | 1269016 |
| 28 | 0 | 0 | 0 | 52 | 494 | 26008 | 27874 | 76930 | 83864 | 289920 | 364040 |
| 29 | 0 | 0 | 0 | 0 | 1652 | 30934 | 77162 | 123178 | 230774 | 445036 | 953378 |
| 30 | 0 | 0 | 0 | 0 | 2 | 20430 | 24146 | 128556 | 138478 | 355754 | 405746 |
| 31 | 0 | 0 | 0 | 0 | 0 | 5854 | 62178 | 159086 | 368336 | 546680 | 1041462 |
| 32 | 0 | 0 | 0 | 0 | 0 | 326 | 3338 | 122164 | 130302 | 597064 | 639362 |
| 33 | 0 | 0 | 0 | 0 | 0 | 0 | 6314 | 151550 | 353810 | 730650 | 1758270 |
| 34 | 0 | 0 | 0 | 0 | 0 | 0 | 62 | 128556 | 150518 | 538278 | 585132 |
| 35 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 22772 | 374246 | 872784 | 1519374 |
| 36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1848 | 19030 | 762768 | 813708 |
| 37 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 51496 | 1016332 | 2112918 |
| 38 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 522 | 848998 | 946260 |
| 39 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 209668 | 2414958 |
| 40 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 19332 | 120852 |
| 41 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 270942 |
| 42 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 12062 |

Table 1. Each cell shows the number of pairs AC-equivalent to AK (3) of total length $T$ obtained by the program when run with the length bound $L$. Highlighted cells do not increase when $L$ is increased.

It took our program 10 days to finish enumeration with the bound $L=20$, consuming 207 days of CPU time. The running time with the bound $L=21$ is expected to be 60 days. We decided not to proceed beyond the value $L=20$. Memory usage during the experiments was moderate and never exceeded 8 Gb . CPU time is the main obstacle. However, we can notice that the numbers in rows of Table 1 stabilize, at least for values $T=13, \ldots, 20$. For instance, we can conjecture that the number of normal forms of AK(3)equivalent presentations of total length 20 or less is 3356 and there is no canonical one among them.
6.2. Old non-counterexamples. We also tested our program on some balanced presentations that were eliminated from the list of potential counterexamples before us. Our program trivializes any of them almost immediately (in less than 10 seconds on a single computational core) in less than 5 ACM-moves.

- $\operatorname{AK}(2):\left(x^{2} y^{-3}, x y x(y x y)^{-1}\right) \sim_{A C}\left(x, x y x^{-1} y x y^{-3}\right) \sim_{A C}(x, y)$
- Gordon presentation $\left(x^{-1} y x^{2} y^{-1}: x y^{3} x^{-1} y^{-4}\right) \sim_{A C}\left(x y x^{-1} y^{-2}, x\right) \sim_{A C}$ $(x, y)$, also considered in [6].
6.3. Miller-Schupp type presentations. We analyzed several randomly generated Miller-Schupp presentations:

$$
\left\langle x, y \mid x^{-1} y^{2} x=y^{3}, x=w\right\rangle
$$

where $w$ has exponent sum 0 on $x$. We attempted to trivialize them or show AC-equivalence with their automorphic images. Both tasks were dealt with different success. Table 2 contains pairs $(u, v)$ for which the program failed to prove equivalence $(u, v) \sim_{\mathrm{AC}}(\varphi(u), \varphi(v))$ for $\varphi \in \boldsymbol{\operatorname { A u t }}\left(F_{2}\right)$ defined by $y \rightarrow y^{-1}, y \rightarrow y x$ and $x \rightarrow y, y \rightarrow x$. (In particular, we could not trivialize the corresponding presentations.) Table 3 contains Miller-Schupp type presentations for which the program proved automorphic equivalence $(u, v) \sim_{\mathrm{AC}}(\varphi(u), \varphi(v))$ for any $\varphi \in \boldsymbol{\operatorname { A u t }}\left(F_{2}\right)$, but was not able to trivialize them. Table 4 contains trivializable Miller-Schupp presentations. The purpose of Tables 2, 3, and 4 is to provide reference for future experiments.

| (xyyyXYY,xxxyXYXY) | (xyyyXYY,xxxYXyXy) | (xyyyXYY,xxxYxyXXXy) |
| :---: | :---: | :---: |
| (xyyyXYY,xxxYXXyXXY) | (xyyyXYY,xxxxYXyXXy) | (xyyyXYY,xxxxyXXYXy) |
| (xyyyXYY,xxxyxyXXXy) | (xyyyXYY,xxxxYXYXXY) | (xyyyXYY,xxxxyXYXXY) |
| (xyyyXYY,xxxxyXyXXY) | (xyyyXYY,xxxxyXXyXy) | (xyyyXYY, xxxyXyXXXY) |
| (xyyyXYY,xxxyXyXXXy) | (xyyyXYY,xxxxYXXYXY) | (xyyyXYY,xxxxyXXYXY) |
| (xyyyXYY,xxxYxyXXXY) | (xyyyXYY,xxxxYXXyXy) | (xyyyXYY,xxxxYXXYXy) |
| (xyyyXYY,xxxxYXyXXY) | (xyyyXYY,xxxyxYXXXy) | (xyyyXYY,xxxxYXXyXY) |
| (xyyyXYY,xxxxyXXyXY) | (xyyyXYY,xxxxYXYXXy) | (xyyyXYY,xxxyxyXXXY) |
| (xyyyXYY,xxxyxYXXXY) | (xyyyXYY,xxxxyXXyXy) | (xyyyXYY,xxxxyXYXXy) |
| (xyyyXYY,xxxyXXYXXy) | (xyyyXYY,xxxxyXyXXy) |  |

Table 2. Miller-Schupp pairs ( $u, v$ ) with unknown equivalence $(u, v) \sim_{\mathrm{AC}}(\varphi(u), \varphi(v))$ for $\varphi \in \boldsymbol{\operatorname { A u t }}\left(F_{2}\right)$.

| (xxxyXXY,xyyyyXYYY) | (xxxyXXY,xyyyXYYYY) | (xyyyXYY,xxxyyXXY) |
| :--- | :--- | :--- |
| (xxxyXXY,xxyyyXYY) | (xxxYXXy,xxYYYXyy) | (xyyyXYY,xxxyXyXY) |
| (xxxYXXy,xyyyxYYYY) | (xyyyXYY,xxxxYXXXY) | (xxxyXXY,xyyyyxYYYY) |
| (xxxYXXy,xxYYxyyy) | (xyyyXYY,xxxyyXXy) | (xyyyXYY,xxxxyXXXy) |
| (xyyyXYY,xxxyXyXy) | (xxxyXXY,xxyyxYYY) | (xyyyXYY,xxxYXyXXY) |
| (xyyyXYY,xxxyXYXy) |  |  |

Table 3. Miller-Schupp pairs $(u, v)$ with equivalence $(u, v) \sim_{\mathrm{AC}}(\varphi(u), \varphi(v))$ for $\varphi \in \boldsymbol{\operatorname { A u t }}\left(F_{2}\right)$, but not known if trivializable.

| (XyyxYYY,xxYYYXYxYXYY) | (XyyxYYY,xxyyyXYYXyxY) | (XyyxYYY,xxYXyxyyyXY) |
| :--- | :--- | :--- |
| (XyyxYYY,xxYXyXyyxyy) | (XyyxYYY,xxyyyXYxYXYY) | (XyyxYYY,xyxYYYYYYXy) |
| (XyyxYYY,xyxYXyy) | (XyyxYYY,xxYYXYXYXyy) | (XyyxYYY,xyyxYYYXYYY) |
| (XyyxYYY,xyxYYXyyyyyy) | (XyyxYYY,xyXyyxYXyyxY) | (XyyxYYY,xyyyXyyXyy) |

Table 4. Trivializable Miller-Schupp presentations.

## 7. Conclusion

Despite a lot of effort, we were unable to disprove any new AkbulutKurby type presentations. In fact, the numbers in Table 1 rows stabilize as the value of the parameter $L$ increases, suggesting that the AC-equivalence class of $\mathrm{AK}(3)$ does not contain the canonical presentation, thus supporting a common opinion that ACC does not hold.

## Appendix A. Used ACM-moves justification

In this section we prove the one-relator groups identities used in lemmas $3.4,3.5$ and 3.6 for the ACM moves. Every proof demonstrates that $c^{-1} u c\left(u^{\prime}\right)^{-1}=1$ in $\langle x, y \mid v\rangle$.

## A.1. Used in lemma 3.4.

$c=x y x, u=x^{k} Y^{k+1}, v=x y x Y X Y, u^{\prime}=y^{k} X^{K+1}$ :

$$
\begin{aligned}
& X Y X \cdot x^{k}\left(Y^{k+1} \cdot x y x \cdot x^{k+1}\right) Y^{k} \\
= & X Y X\left(x^{k} \cdot x y x \cdot Y^{K}\right) \quad(Y x y x=x y) \\
= & X Y X x y x \quad(x y x Y=y x) \\
= & 1
\end{aligned}
$$

## A.2. Used in lemma 3.5.

$$
\begin{aligned}
& u= x y X y^{k+1} x Y^{k+2}, v=x y X y x Y, c=y^{k+1}, u^{\prime}=Y^{k+1} X^{k}: \\
&\left(Y^{k+1}\right) \cdot x y X y^{k+1} x\left(Y^{k+2} \cdot y^{k+1}\right) \cdot x^{k}\left(y^{k+1}\right)=\left(x^{k+1} y X y^{k+1}\right) x \cdot Y \\
&=\left(x^{k} y X y^{k}\right) \cdot x Y=\ldots=y X x Y=1 \\
& u= x y X y x Y Y X y, v=x y X y^{k+1} x Y^{k+2}, c=y^{k+2} X y x Y, u^{\prime}=x y X y x Y: \\
& y X Y x Y^{k+2} \cdot x y X y x Y Y X\left(y \cdot y^{k+2} X y x Y \cdot y X Y x Y\right) X \\
&= y X Y\left(x Y^{k+2} x y X y\right) x Y Y X \cdot y^{k+2} X=y X Y \cdot Y^{k} \cdot x Y\left(Y X y^{k+2} X\right) \\
&=(y X) Y^{k+1}\left(x Y \cdot X y^{k+1}\right)=Y^{k+1} \cdot y^{k+1}=1 \\
& u=x y X y^{k} X Y^{k}, v=x y X y x Y Y X y, c=x y X y, u^{\prime}=y^{k+2} X Y^{k+1} x Y X: \\
&(Y x Y X) \cdot\left(x y X y^{k}\right) X Y^{k} \cdot x y X y \cdot x y X y^{k+1} x\left(Y^{k+2}\right) \\
&= X Y^{k} x y\left(X y x y X y^{k+1}\right) x \cdot Y^{3}=X\left(Y^{k} x y \cdot y\right) y X y^{k} \cdot x Y^{3} \\
&= X \cdot Y^{k-1} x y\left(X y x \cdot y X y^{k}\right) x Y^{3}=X\left(Y^{k-1} x y \cdot y\right) y X y^{k-1} \cdot x Y^{3} \\
&= X x y \cdot y y X \cdot x Y^{3}=1
\end{aligned}
$$

$$
\begin{aligned}
& u=x^{k} Y^{k+1}, v=x y X y^{k} X Y^{k}, c=Y^{k-1} x Y^{k-1} x Y Y X y \\
& u^{\prime}=Y x y y X Y x Y X:
\end{aligned}
$$

$$
\begin{aligned}
& (Y x y y X) y^{k-1} X y^{k-1} \cdot x^{k} Y^{k+1} \cdot Y^{k-1} x Y^{k-1} x Y Y X y \\
& \quad \cdot x y X y(x Y Y X y)=\left(y^{k-1} X\right) y^{k-1} x^{k} Y^{2 k} x Y^{k-1} x Y Y X y(x y X y) \\
& =\left(y^{k} \cdot y^{k-1}\right) x^{k} Y^{2 k} x Y^{k-1} x Y Y X(y) \quad\left(x y X y^{k} X=y^{k}\right) \\
& =y^{k} x\left(X y^{k} \cdot x^{k}\right) Y^{2 k} x Y^{k-1} x Y Y X \quad(\text { shift }) \\
& =y^{k} x \cdot y\left(X y^{k} \cdot x^{k-1}\right) Y^{2 k} x Y^{k-1} x Y Y X \quad\left(X y^{k} x=y X y^{k}\right) \\
& \quad \cdots \\
& =y^{k} x \cdot y^{k} X\left(y^{k} \cdot Y^{2 k}\right) x Y^{k-1} x Y Y X \\
& =y^{k} x\left(y^{k} X \cdot Y^{k} \cdot x\right) Y^{k-1} x Y Y X=\left(y^{k} x\right) \cdot x Y \cdot Y^{k-1} x Y(Y X) \\
& =\left(X y^{k} \cdot x\right) Y^{k} x Y=\left(X y^{k} \cdot x\right) Y^{k}(x Y)=y^{k} \cdot Y^{k}=1
\end{aligned}
$$

$$
u=x y x Y X Y, v=x^{k} Y^{k+1}, c=Y^{k}, u^{\prime}=y^{k} x Y^{k} x Y X
$$

$$
\begin{aligned}
& \left(y^{k} \cdot x y\right) x Y X Y \cdot Y^{k} \cdot x y X\left(y^{k} X Y^{k}\right)=x Y X\left(Y Y^{k}\right) x y X \cdot y^{k+1}= \\
& x Y\left(X \cdot X^{k} x\right) y X y^{k+1}=x\left(Y \cdot Y^{k+1} \cdot y\right) X y^{k+1}=\ldots=Y^{k+1} y^{k+1}
\end{aligned}
$$

## A.3. Used in lemma 3.6,

$u=x^{k-1} y X^{k-1} y X Y, v=x x y X Y X y, c=x^{k-3}(Y X)^{k} Y$, $u^{\prime}=x^{k-1}(Y X)^{k} Y$ :

$$
\begin{aligned}
& y(x y)^{k}\left(X^{k-3} \cdot x^{k-1}\right) y X^{k-1} y X Y \cdot\left(x^{k-3}(Y X)^{k} Y \cdot y(x y)^{k} X^{k-1}\right) \\
= & y(x y)^{k} \cdot x^{2} y X^{k-1}(y X Y \cdot X) X=(y x)^{k} y x^{2} y X^{k-1} \cdot X X(Y \cdot X) \\
= & \ldots=y x^{2} y X^{k-1} X X \cdot x^{k} Y X=y x^{2} y X Y X=1
\end{aligned}
$$

$$
u=x^{k} Y^{k+1}, v=x^{k-1} y X^{k-1} y X Y, c=x y X^{k-1} y X Y X y
$$

$$
u^{\prime}=Y x y x Y X X:
$$

$$
\begin{aligned}
&(Y x y x Y) x^{k-1} Y\left(X \cdot x^{k}\right) Y^{k+1} \cdot x y X^{k-1} y X Y X y \cdot x x(y X Y X y) \\
&=\left(x^{k-1} Y x^{k-1} Y^{k+1}\right) \cdot x y X^{k-1} y X Y X y(x x)=\ldots \\
&= Y\left(x^{k-1} \cdot x y X^{k-1} y X Y\right) X y=Y \cdot x \cdot X y=1 \\
& u=x y x Y X Y, v=x^{k} Y^{k+1}, c=y X^{k-1} y X Y \\
& u^{\prime}=y x Y x^{k-1} Y X^{k-1}: \\
&\left(y x Y x^{k-1} Y\right) \cdot x y x Y X(Y \cdot y) X^{k-1} y X Y \cdot x^{k-1}\left(y X^{k-1} y X Y\right) \\
&=(x) y x Y X^{k} y X\left(Y x^{k-1}\right)=(y) x Y X^{k} y X \cdot\left(y^{k}\right) \\
&=(x Y) X^{k} y\left(X \cdot x^{k}\right)=X^{k} y \cdot y^{k}=1
\end{aligned}
$$

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