# Polynomial Time Conjugacy in Wreath Products and Free Solvable Groups 

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#### Abstract

We prove that the complexity of the Conjugacy Problems for wreath products and for free solvable groups is decidable in polynomial time. For the wreath product $A \mathrm{wr} B$, we must assume the decidability in polynomial time of the Conjugacy Problems for $A$ and $B$ and of the power problem in $B$. We obtain the result by making the algorithm for the Conjugacy Problem in Matthews [11] run in polynomial time. Using this result and properties of the Magnus embedding, we show that the Conjugacy and Conjugacy Search Problems in free solvable groups are computable in polynomial time.


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## 1 Introduction

In this paper we attack the computational complexity of the Conjugacy and Conjugacy Search Problems in free solvable groups. We show that they are both solvable in polynomial time and that the degree of the polynomial is uniform for the class of free solvable groups. Further, we show that the Conjugacy

Problem and the Conjugacy Search Problem in wreath products are solvable in polynomial time modulo some natural conditions.

Algorithmic problems in group theory were considered as early as 1910, when Dehn introduced the now famous Word and Conjugacy Problems. Briefly, for a finitely generated group $G$, given two words as a product of generators, the Word Problem asks whether they are equal as elements of $G$ and the Conjugacy Problem asks whether they are conjugate to each other in $G$. Both of these decision problems quickly became an active area of research. Novikov ([15], [16]) gave the first example of a finitely presented group with undecidable Word (and hence Conjugacy) Problem. A beautiful result of Miller exhibits a group which has decidable Word Problem and undecidable Conjugacy Problem [12]. At present, there are many interesting classes of groups where these problems are decidable. Here we mention only a few positive results about nonsolvable groups and discuss solvable groups in more detail below. The Word and Conjugacy Problems are decidable in braid groups (Artin, [1]), hyperbolic groups (Gromov, [6]), wreath products of groups under some natural additional conditions (Matthews [11]), the Grigorchuk group (Grigorchuk [5], Leonov [8]), bi-automatic groups (Gersten and Short, [4]), toral relatively hyperbolic groups, free solvable groups (Remeslennikov, Sokolov [18]).

Nowadays, while decidability is still an open area of research, the emphasis has shifted to complexity of decidable problems. It is worth mentioning the work of Lysenok, Miasnikov, and Ushakov who showed in 10 that the Conjugacy Problem in the Grigorchuk group is decidable in polynomial time, the work the work of Lipton and Zalenstein on the polynomial time decidability of the Word Problem in linear groups [9, the work of Marshall, Bridson and Haefliger, Epstein and Holt which, through successively improving time bounds, culminates in showing that the Conjugacy Problem in word-hyperbolic groups is decidable in linear time [3 and the work of Cannon, Goodman and Shapiro, and Holt and Rees [7] in giving a linear time algorithm for deciding the Word Problem in nilpotent groups.

Solvable groups offer a whole new world on their own. An example of Kharlampovich of a solvable group with undecidable Word, and hence Conjugacy, Problem shows that one cannot derive any positive results about the entire class of solvable groups. However, there are many interesting subclasses in which the Conjugacy Problem is decidable, for instance finitely generated metabelian groups (Noskov [14]), nilpotent groups (Blackburn [2]), polycyclic groups (Remeslennikov [17]) and free solvable groups (Remeslennikov - Sokolov 18. In all of the above cases, however, the results are about decidability without mention of the time complexity. The complexity of algorithmic problems in solvable groups has recently become an active area of research with a paper by Miasnikov, Roman'kov, Ushakov and Vershik 13 which presents a cubic time algorithm to decide the Word Problem in free solvable groups.

Most complexity results concern a fixed group. To the knowledge of the author, there is no other studied class of infinite solvable groups for which the Word and Conjugacy Problems can be decided uniformly in polynomial time. Even in the cases where one can solve the given problem using a general
description of the group, the algorithm involves heavy pre-computations specific to this group which cannot be generalized to produce a uniformly polynomialtime algorithm.

In this paper we use this result in [13] to show that the Conjugacy Problem in free solvable groups is decidable in quintic time. The proof follows the ideas of Remeslennikov and Sokolov ([18). First, we embed the free solvable group of degree $(d+1)$ and rank $r$ in a wreath product of an abelian group and a free solvable group of degree $d$. The image of a word of length $n$ can be found in time $O\left(r d n^{3}\right)$. Since the images of two words under the Magnus embedding are conjugate if and only if these words are conjugate, we can apply our general result, namely that the Conjugacy Problem in this wreath product is decidable in polynomial time, provided the Conjugacy Problems in each factor (and the Power Problem in the second factor) are decidable in polynomial time. The second factor is a free solvable group of lesser degree, so we proceed by induction. Similarly, we solve the Conjugacy Search Problem.

## 2 Preliminaries

### 2.1 Wreath products and the Magnus embedding

We start by defining the objects essential to this paper - wreath products and the Magnus embedding.

Let $G$ be a group generated by a fixed finite set of generators $Y$. We represent elements in $G$ by words $w$ over $Y^{ \pm}$and denote by $|w|$ the length of the word $w$.

Let $A$ and $B$ be groups. The restricted wreath product $A \mathrm{wr} B$ is the group formed by the set

$$
A \mathrm{wr} B=\left\{b f \mid b \in B, f \in A^{(B)}\right\}
$$

with multiplication defined by $b f c g=b c f^{c} g$, where $f^{c}(x)=f\left(x c^{-1}\right)$ for $x \in B$, where $A^{(B)}$ denotes the set of functions from $B$ to $A$ with finite support (i.e., functions from $B$ to $A$ which take non-zero values only for finitely many elements of $B$ ). Note that $A^{(B)}$ is a group under pointwise multiplication of functions with identity $1: B \rightarrow 1$, so we can view $A \mathrm{wr} B$ as the semi-direct product $B \ltimes A^{(B)}$.

Let $X=\left\{x_{1}, \ldots, x_{n}\right\}$ and $Y=\left\{y_{1}, \ldots, y_{m}\right\}$ be the generating sets for $A$ and $B$, respectively. $A \mathrm{wr} B$ is generated by $X, Y$ in the following sense: every function, $f \in A^{(B)}$ can be written as a product $f=\prod_{i} a_{i}^{b_{i}}$. Indeed consider the functions of the form

$$
f_{a_{i}, b_{i}}(x)=\left\{\begin{aligned}
a_{i} & \text { if } x=b_{i} \\
1 & \text { otherwise }
\end{aligned}\right.
$$

For simplicity, we denote $f_{a_{i}, 1}$ by $f_{a_{i}}$. Then for any $f \in A^{(B)}$, one can write $f=\prod_{i} f_{a_{i}, b_{i}}=\prod_{i} f_{a_{i}}^{b_{i}}$. There is clearly an identification between $f_{a_{i}}$ and $a_{i}$.

Remark 2.1. One can rewrite a word $w=b_{1} a_{1} \ldots b_{k} a_{k}$ in generators $X$ and $Y$ as $w=b f$ in polynomial time. Observe that

$$
w=b_{1} \ldots b_{k} a_{1}^{b_{2} \ldots b_{k}} \ldots a_{2}^{b_{3} \ldots b_{k}} a_{k-1}^{b_{k}} a_{k}
$$

Here $b=b_{1} \ldots b_{k} \in B$ and $a_{1}^{b_{2} \ldots b_{k}} a_{2}^{b_{3} \ldots b_{k}} \ldots a_{k-1}^{b_{k}} a_{k}$ corresponds to a function in $A^{(B)}$ as follows. Denote $B_{i}=b_{i} \ldots b_{k}$. For each $1<i<j \leq k$, check whether $B_{i}=B_{j}$ in $B$. This amounts to solving $\binom{k-1}{2}$ Word Problems in $B$. For each $B_{i_{1}}=B_{i_{2}}=\ldots=B_{i_{j}}$, write $f\left(B_{i_{1}}\right)=a_{1} \ldots a_{j}$. This determines $f$ completely and we can change presentations in time $O\left(|w|^{2} T_{W B}(|w|)\right)$, where $T_{W B}$ is the time function for the Word Problem in $B$. Note that if a word is given as a product of generators, converting it to standard (or pair) form gives an ordering for $\operatorname{supp}(f)=\left\{B_{i}\right\}_{i}$ determined by the indices $i$. More precisely, $B_{i}<B_{j}$ whenever $i<j$.

Fix a free group $F$ of rank $r$ with basis $X$. The derived subgroup $F^{(d)}$ is defined by induction as follows: $F^{\prime}=[F, F]$ and $F^{(d+1)}=\left[F^{(d)}, F^{(d)}\right]$. Define the free solvable group, $S_{d, r}=F / F^{(d+1)}$.

Let $N$ be a normal subgroup of $F$. Denote by $\mu: F \rightarrow F / N$ the canonical epimorphism. Let $U$ be a free $\mathbb{Z}(F / N)$-module with basis $\left\{u_{1}, \ldots, u_{r}\right\}$, so $U \simeq$ $\mathbb{Z}(F / N) \oplus \ldots \oplus \mathbb{Z}(F / N)$. Then the set of matrices

$$
M(F / N)=\left(\begin{array}{cc}
F / N & U \\
0 & 1
\end{array}\right)=\left\{\left.\left(\begin{array}{ll}
g & u \\
0 & 1
\end{array}\right) \right\rvert\, g \in F / N, u \in U\right\}
$$

forms a group with respect to matrix multiplication. One can see that (see for example, [18]) $M(F / N) \simeq F / F^{\prime} \mathrm{wr} F / N$.

The map $\varphi: F(X) \rightarrow M(F / N)$ defined by

$$
x_{i} \mapsto\left(\begin{array}{cc}
\mu\left(x_{i}\right) & u_{i} \\
0 & 1
\end{array}\right), i=1, \ldots, r
$$

extends to an injective homomorphism $\varphi: F / N^{\prime} \rightarrow M(F / N)$, called the Magnus embedding.

In the sequel, for $x \in F$ put

$$
\varphi(x)=\left(\begin{array}{cc}
\mu(x) & u_{x} \\
0 & 1
\end{array}\right)
$$

### 2.2 Algorithmic Results for the Magnus Embedding

Here we present and prove a few preliminary results on the Magnus embedding that we will need in Section 5
Theorem 2.2 ([18]). Let $\bar{f}, \bar{g} \in F / N^{\prime}$, where $N$ is normal in $F$ and $N^{\prime}$ is torsion-free. Then $\bar{f}$ and $\bar{g}$ are conjugate in $F / N^{\prime}$ if and only if their images in $M(F / N)$ are conjugate.

In particular, the theorem above holds for the free solvable group $F / F^{(d+1)}$, which is $F / N^{\prime}$ for $N=F^{(d)}$.

Theorem 2.3 ([13]). The following hold:

1) For a given $w \in S_{d, r}$, one can compute $\varphi(w)$ in time $O\left(d r|w|^{3}\right)$;
2) The Word Problem in $S_{d, r}$ is solvable in time $O\left(d r|w|^{3}\right)$, where $w$ is the input word.

Corollary 2.4. The Conjugacy Problem in $S_{d, r}$ reduces to the Conjugacy Problem in $F / F^{\prime}$ wr $S_{d-1, r}$ it time $O\left(r d L^{3}\right)$, where $L$ is the length of the input words.

The Power Problem for a group $G$ for given elements $x, y \in G$ consists of determining whether there exists an integer $n \in \mathbb{Z}$ such that $x=y^{n}$ and if so, to find it.

Theorem 2.5. The power problem in $F / F^{(d)}$ is decidable in time $O\left(r d L^{6}\right)$, where $r$ is the rank of $F$ and $L=|x|+|y|$ is the length of the input.

Proof. Let $x$ and $y$ be elements in $F / F^{(d)}$ given as products of generators. Consider first the two trivial cases. If $y=1$, which can be checked in time $O\left(r d|y|^{3}\right)$, the problem reduces to a Word Problem, which is decidable in $O\left(r d|x|^{3}\right)$. If $x=1$, then $n=0$ is always a solution. Hence, after some preliminary computation which can be done in $O\left(r d L^{3}\right)$, we can assume without loss of generality that both $x$ and $y$ are non-trivial elements in $F / F^{(d)}$. Observe the following.
Fact 2.6. 1. If there exists $n \in \mathbb{Z}$ such that $x=y^{n}$ in $F / F^{(d)}$, then $x=y^{n}$ in $F / F^{\prime}$.
2. If there exists $n \in \mathbb{Z}$ such that $x=y^{n}$ in $F / F^{(d)}$, then $n$ is unique with this property.

The first claim follows easily since $F / F^{\prime}$ is a quotient of $F / F^{(d)}$ and the second one follows from the fact that free solvable groups are torsion-free. We proceed to solve the general case of the Power Problem in a free solvable group $F / F^{(d)}$.

Step 1: Solve the Power Problem in $F / F^{\prime}$. It is a free abelian group, so the elements $x$ and $y$ can be uniquely presented in the form $x=x_{1}^{a_{1}} \ldots x_{r}^{a_{r}}$ and $y=x_{1}^{b_{1}} \ldots x_{r}^{b_{r}}$, where $X=\left\{x_{1}, \ldots, x_{r}\right\}$ is the basis for $F$. Obviously, this decomposition can be found in log-linear time, which is certainly in $O\left(r L^{6}\right)$. Then for each $1 \leq i \leq r$ set $n_{i}=a_{i} / b_{i}$. If all $n_{i}$ are equal and integer, then $x=y^{n_{1}}$, as required. Otherwise, $x \notin\langle y\rangle$ and we are done. Clearly, this can be done in time $O(r(|x|+|y|))$.
Note that the exponent $n$ satisfies $n \leq|x|+|y|=L$.
Step 2: Using $n$ from Step 1, check whether the equation

$$
\begin{equation*}
x=y^{n} \tag{1}
\end{equation*}
$$

holds in $F / F^{(d)}$. By Theorem [2.3] this can be done in time $O(r d(|x|+$ $\left.n|y|)^{3}\right) \subseteq O\left(r d L^{6}\right)$. If (11) does not hold, then $x \neq y^{m}$ for all integers $m$.

Indeed, if there were some $m \in \mathbb{Z}$ for which $x=y^{m}$ in $F / F^{(d)}$, then by Fact 1 the same equation would hold in $F / F^{\prime}$. But by the uniqueness of $n$ (Fact2), this is impossible.

## 3 Complexity of the Conjugacy Problem in Wreath Products

We establish a bound on the complexity of the Conjugacy Problem in wreath products $A \mathrm{wr} B$ by giving a bound for a variant of the algorithm developed by Matthews [11.

Let $x=b f, y=c g \in A \mathrm{wr} B$, where $b, c \in B$ and $f, g \in A$. Denote $\operatorname{supp}(f)=$ $\left\{b_{1}, \ldots, b_{n}\right\}$ and $\operatorname{supp}(g)=\left\{\beta_{1}, \ldots, \beta_{m}\right\}$ where the $b_{i}$ and $\beta_{j}$ are ordered as in Remark 2.1 Recall that all elements are given as words in generators. Let $\bar{b}$ and $\bar{\beta}$ be the longest elements in $\operatorname{supp}(f)$ and in $\operatorname{supp}(g)$, and $\bar{a}$ and $\bar{\alpha}$ be the longest element in the image of $f$ and of $g$, respectively.

For each left $\langle b\rangle$-coset in $B$ that intersects $\operatorname{supp}(f) \cup \operatorname{supp}(g)$, choose a coset representative from $\operatorname{supp}(f) \cup \operatorname{supp}(g)$ and let $T_{b}=\left\{t_{i}\right\}_{i \in I_{1} \cup I_{2}}$, where $I_{1}$ indexes the coset representatives we just chose and $I_{2}$ indexes the remaining ones. Deciding whether $b_{i}, b_{j} \in \operatorname{supp}(f) \cup \operatorname{supp}(g)$ are in the same coset is a Power Problem, since $b_{i}, b_{j}$ are in the same coset if and only if $b_{i} b_{j}^{-1}=b^{k}$ for some $k$. To find $T_{b}$ one needs to solve the Power Problem $\binom{(n+m)}{2}$ times (for all pairs $\left.\left(b_{i}, b_{j}\right)\right)$. Hence it takes time $\binom{(n+m)}{2} T_{P B}(2|\bar{b}|+2|\bar{\beta}|+|b|)$, where $T_{P B}$ is the time function for the power problem in $B$. For each $\gamma \in B$ and $i \in I_{1} \cup I_{2}$, associate with $T_{b}$ the following map $\pi_{t_{i}}^{(\gamma)}: A^{(B)} \rightarrow A$ :

$$
\pi_{t_{i}}^{(\gamma)}(f)= \begin{cases}\prod_{j=0}^{N-1} f\left(t_{i} b^{j} \gamma^{-1}\right) & \text { if } b \text { is of finite order } N \\ \prod_{j=-\infty}^{\infty} f\left(t_{i} b^{j} \gamma^{-1}\right) & \text { if } b \text { is of infinite order. }\end{cases}
$$

Note that in the above all the products are finite, since $f$ has finite support. Denote $\pi_{t_{i}}^{(1)}(f)$ by $\pi_{t_{i}}(f)$. Matthews gives a condition to check conjugacy, which will be used here.

Theorem 3.1 (11). Let $A, B$ be finitely generated groups. Two elements $x=b f, y=c g \in A \mathrm{wr} B$ are conjugate if and only if there exists $d \in B$ such that for all $t_{i} \in T_{b}$ the following hold:
(1) $d b=c d$,
(2) when the order of $b$ is finite, $\pi_{t_{i}}^{(d)}(g)$ is conjugate to $\pi_{t_{i}}(f)$ in $A$,
(3) when the order of $b$ is infinite, $\pi_{t_{i}}^{(d)}(g)=\pi_{t_{i}}(f)$ in $A$.

In order to use this criterion computationally, we need to circumvent the use of the conjugator $d$.

Lemma 3.2. Let $\left\{\bar{s}_{i}\right\}_{i \in I}$ and $\left\{\tilde{s}_{i}\right\}_{i \in I}$ be two sets of left $\langle c\rangle$-coset representatives such that $\overline{s_{i}}\langle c\rangle=\tilde{s}_{i}\langle c\rangle$. Then $\pi_{\overline{s_{i}}}(g)$ and $\pi_{\tilde{s}_{i}}(g)$ are conjugate for any $i \in I$.

Proof. Since $\bar{s}_{i}\langle c\rangle=\tilde{s}_{i}\langle c\rangle$, there is some integer $k_{i}$ for which $\bar{s}_{i}=\tilde{s}_{i} c^{k_{i}}$ and hence,

$$
\pi_{\bar{s}_{i}}(g)=\prod_{j} g\left(\overline{s_{i}} c^{j}\right)=\prod_{j} g\left(\tilde{s_{i}} c^{k_{i}} c^{j}\right)=\prod_{j} g\left(\tilde{s_{i}} c^{k_{i}+j}\right)
$$

This last product is a cyclic permutation of the factors in $\prod_{j} g\left(\tilde{s_{i}} c^{j}\right)=\pi_{\tilde{s_{i}}}(g)$ and so is conjugate to $\pi_{\tilde{s_{i}}}(g)$.

Using the Theorem 3.1 and Lemma 3.2 we show that the time complexity of the Conjugacy Problem in wreath products is polynomial.

Theorem 3.3. Let $A$ and $B$ be finitely generated groups such that the following hold:

1) there are decision algorithms for the Conjugacy Problem in $A$ and in $B$ with polynomial time functions, $T_{C A}, T_{C B}$, respectively;
2) there is an algorithm with polynomial time function $T_{P B}$ for the Power Problem in B.

Then the Conjugacy Problem in $A \mathrm{wr} B$ is decidable with complexity

$$
\begin{equation*}
O\left(L^{2} T_{C A}\left(L^{2}\right)+L T_{C B}(L)+L^{2} T_{P B}(L)\right) \tag{2}
\end{equation*}
$$

where $L=|x|+|y|$ is the length of the input pair $x, y \in A \mathrm{wr} B$.
Remark 3.4. Note that every Word Problem "s $x=1$ ?" is precisely the Conjugacy Problem "Is $x$ conjugate to 1 "? To simplify the presentation, the complexities of all Word Problems considered in this section will be bounded by the complexities of the corresponding Conjugacy Problems.

Proof. Let $x=b f, y=c g \in A \mathrm{wr} B$. The notation from the beginning of this section will be used throughout. In order to simplify the subsequent treatment of complexity in this section, we will implicitly use the bounds

$$
|x|,|y|, n, m,|c|,|b|,|\bar{b}|,\left|t_{i}\right|,|\bar{a}| \leq L
$$

Claim 3.5. There is a polynomial time algorithm which computes $\pi_{t_{i}}^{(\gamma)}(f)$. More precisely,

- $\pi_{t_{i}}^{(\gamma)}(f)$ can be computed in time $L T_{P B}(L)$.
- $\left|\pi_{t_{i}}^{(\gamma)}(f)\right| \leq L^{2}$.

Proof. The algorithm is as follows:

Step 1: For each $b_{k} \in \operatorname{supp}(f)$ check whether there is some $j$ such that $t_{i} b^{j} \gamma^{-1}=b_{k}$, i.e., $t_{i}^{-1} b_{k} \gamma=b^{j}$. This is an instance of the Power Problem in $B$ and so can be done in time $T_{P B}(2|\bar{b}|+|b|+|\gamma|)$. If such $j$ exists, look up the corresponding value $a_{j}=f\left(b_{k}\right)$. Otherwise, $a_{j}$ does not occur in the product.

Step 2: There are $n$ elements in $\operatorname{supp}(f)$ to perform computations on, so computing $\pi_{t_{i}}^{(\gamma)}(f)$ takes time $n T_{P B}(2|\bar{b}|+|b|+|\gamma|)$.

Step 3: Set $\pi_{t_{i}}^{(\gamma)}=\prod_{j} a_{j}$. Note that the order in which the factors are multiplied is a priori determined by the solution $j$ to the Power Problem. However, if the order of $b$ is finite, by the definition of $\pi$ we take $j \bmod N$, and if the order of $b$ is infinite, then the solution to the Power Problem is unique because in this case $b$ has no torsion. Thus, a fortiori, $\pi_{t_{i}}^{(\gamma)}$ is indeed equal to $\prod_{j} a_{j}$, where the $a_{j}$ are computed as above.

Note that $\left|\pi_{t_{i}}^{(\gamma)}(f)\right| \leq n|\bar{a}|$, since each factor in the product $\pi_{t_{i}}^{(\gamma)}(f)$ is in the image of $f$.

We modify the algorithm from [11] so that it runs in polynomial time as follows:

Step 1. Determine whether $b$ and $c$ are conjugate in $B$. This takes time $T_{C B}(|x|+|y|) \in O\left(T_{C B}(L)\right)$. If not, $x$ and $y$ are not conjugate. If $b$ and $c$ are conjugate in $B$, let $d \in B$ be such that $d b=c d$ (it is not required to find this $d$ ).

Step 2. Consider the following three cases.
Case 1: $g=1$. Then $\pi_{t_{i}}^{(d)}(g)=1$, so $x$ and $y$ are conjugate if and only if $\pi_{t_{i}}(f)=1$. To check this compute $\pi_{t_{i}}(f)$ as in Claim 3.5 and solve the Word Problem in $A$. This takes time

$$
\begin{equation*}
O\left(L T_{P B}(L)+T_{C A}\left(L^{2}\right)\right) \tag{3}
\end{equation*}
$$

Case 2: $g \neq 1$, and $\pi_{t_{i}}(f)=1$ for all $i \in I_{1}$. In order to check the latter, simply compute $\pi_{t_{i}}(f)$ for all $i \in I_{1}$. This will take time $O\left(L^{2} T_{P B}(L)\right)$. Then, by Theorem 3.1 $x$ is conjugate to $y$ if and only if $\pi_{t_{i}}^{(d)}(g)=1$ for all $i \in I_{1}$ (since the $\pi_{t_{i}}^{(d)}(g)=1$ for $i \in I_{2}$ ). Note that we need not know what $d$ actually is - its existence is enough. Indeed, since $d b=c d$, $g\left(t_{i} b^{j} d^{-1}\right)=g\left(t_{i} d^{-1} c^{j}\right)$ and hence

$$
\pi_{t_{i}}^{(d)}(g)=\prod_{j} g\left(t_{i} b^{j} d^{-1}\right)=\prod_{j} g\left(t_{i} d^{-1} c^{j}\right)=\pi_{t_{i} d^{-1}}(g)
$$

where $\left\{t_{i} d^{-1}\right\}_{i \in I_{1} \cup I_{2}}$ is a set of left $\langle c\rangle$-coset representatives. Moreover, by Lemma 3.2 $\pi_{t_{i} d^{-1}}(g)$ is conjugate to $\pi_{s_{i}}(g)$ for any other set of left $\langle c\rangle$-coset representatives $\left\{s_{i}\right\}_{i \in I_{1} \cup I_{2}}$ for which $t_{i} d^{-1}\langle c\rangle=s_{i}\langle c\rangle$. It follows that $\pi_{t_{i}}^{(d)}(g)=1$ for all $i \in I_{1} \cup I_{2}$ if and only if $\pi_{s_{i}}(g)=1$ for all $i \in I_{1} \cup I_{2}$. Since $\pi_{s_{i}}(g)=1$ for all $i \in I_{2}$, to check whether $x$ and $y$ are conjugate, it is enough to check whether for some set of left $\langle c\rangle$-coset representatives $T_{c}=\left\{s_{i}\right\}_{i \in I_{1}}, \pi_{s_{i}}(g)=1$ for all $i \in I_{1}$. Choosing $T_{c}$ can be done in time $O\left(L^{2} T_{P B}(L)\right)$ and by Claim 3.5, checking whether $\pi_{s_{i}}(g)=1$ for all $i \in I_{1}$ can be done in time $L^{2} T_{C A}\left(L^{2}\right)$. Thus checking whether $x$ and $y$ are conjugate takes time

$$
\begin{equation*}
O\left(L^{2} T_{P B}(L)+L^{2} T_{C A}\left(L^{2}\right)\right) \tag{4}
\end{equation*}
$$

Case 3: $g \neq 1$ and some $\pi_{t_{i}}(f) \neq 1$. There are two subcases:

1) The order of $b$ is finite. By Theorem 3.1, $x$ and $y$ are conjugate if and only if $\pi_{t_{i}}(f)$ and $\pi_{t_{i}}^{(d)}(g)$ are conjugate. As in Case $2, \pi_{t_{i}}^{(d)}(g)=\pi_{t_{i} d^{-1}}(g)$, which is conjugate to $\pi_{s_{i}}(g)$ if $t_{i} d^{-1}\langle c\rangle=s_{i}\langle c\rangle$. This does not have to be the case for the set $T_{c}=\left\{s_{i}\right\}_{i \in I_{1}}$ computed in Case 2, but we know that for each $i \in I_{1} \cup I_{2}$ there is a unique $k \in I_{1} \cup I_{2}$ such that $t_{i} d^{-1}\langle c\rangle=s_{k}\langle c\rangle$. Hence, for each $i \in I_{1}$, it is enough to check for all $k \in I_{1}$ whether

$$
\pi_{t_{i}}(f) \text { and } \pi_{s_{k}}(g) \text { are conjugate. }
$$

If for each $i \in I_{1}$ there is some $k \in I_{1}$ for which this is true, then $x$ and $y$ are conjugate. Otherwise, they are not. Note that the above computations amount to solving $L^{2}$ instances of the Conjugacy Problem in $A$ and so determining whether $x$ and $y$ are conjugate can be done in time

$$
\begin{equation*}
O\left(L^{2} T_{P B}(L)+L^{2} T_{C A}\left(L^{2}\right)\right) \tag{5}
\end{equation*}
$$

2) The order of $b$ is infinite. Let $k$ be a fixed integer such that $\pi_{t_{k}}(f) \neq 1$ (such a $k$ must be found already in the beginning of Case 3 ). We proceed to check that $\pi_{t_{k}}(f)=\pi_{t_{k}}^{(d)}(g)$ without finding $d$. Assume that $\pi_{t_{k}}^{(d)}(g)=1$ as otherwise, by Theorem [3.1, we can conclude that $x$ and $y$ are not conjugate. Since $\pi_{t_{k}}^{(d)}(g)=\prod_{j} g\left(t_{k} b^{j} d^{-1}\right) \neq 1$, there is some integer $l$ for which $g\left(t_{k} b^{l} d^{-1}\right) \neq 1$. Then $t_{k} b^{l} d^{-1}=\beta_{p}$ for some $\beta_{p} \in \operatorname{supp}(g)$ and so $d=\beta_{p}^{-1} t_{k} b^{l}$. It would suffice to check for all $d$ of the form $d=\beta_{p}^{-1} t_{k} b^{l}$ such that $d b=c d$ whether $\pi_{t_{i}}(f)=\pi_{t_{i}}^{(d)}(g)$.
In order to check the former, we need to check for all $\beta_{p} \in \operatorname{supp}(g)$ whether $\beta_{p}^{-1} t_{k} b^{l} b=c \beta_{p}^{-1} t_{k} b^{l}$, i.e., it is enough to check whether $\beta_{p}^{-1} t_{k} b=c \beta_{p}^{-1} t_{k}$. These are $m$ instances of the Word Problem in $B$ which do not involve $l$, so they can be decided in time $m T_{C B}(6 L)$. Thus checking whether $d$ satisfies $d b=c d$ can be done in time $O\left(L T_{C B}(L)\right)$.

It remains to check whether $\pi_{t_{i}}(f)=\pi_{t_{i}}^{(d)}(g)$. Notice that

$$
\begin{aligned}
\pi_{t_{i}}^{(d)}(g) & =\prod_{j=-\infty}^{\infty} g\left(t_{i} b^{j} d^{-1}\right)=\prod_{j=-\infty}^{\infty} g\left(t_{i} b^{j} b^{-l} t_{k}^{-1} \beta_{p}\right) \\
& =\prod_{j=-\infty}^{\infty} g\left(t_{i} b^{j-l} t_{k}^{-1} \beta_{p}\right)=\prod_{j=-\infty}^{\infty} g\left(t_{i} b^{j} t_{k}^{-1} \beta_{p}\right)=\pi_{t_{i}}^{\left(\beta_{p}^{-1} t_{k}\right)}(g)
\end{aligned}
$$

So we need to check whether $\pi_{t_{i}}^{\left(\beta_{p}^{-1} t_{k}\right)}(g)=\pi_{t_{i}}(f)$. Using 3.5 this can be done in time

$$
\begin{equation*}
O\left(L T_{C B}(L)+T_{C A}\left(L^{2}\right)+L T_{P B}(L)\right) \tag{6}
\end{equation*}
$$

The complexity of the conjugacy problem in $A \mathrm{wr} B$ is

$$
O\left(L^{2} T_{C A}\left(L^{2}\right)+L T_{C B}(L)+L^{2} T_{P B}(L)\right),
$$

which is clearly polynomial since $T_{C A}, T_{C B}$ and $T_{P B}$ are polynomial.

Remark 3.6. The algorithm described above differs from the algorithm described in [11] in item 2) of Case 3. The original algorithm is not polynomial in this part.

## 4 Complexity of the Conjugacy Search Problem in Wreath Products

We use the same notation as in the previous section. The following result is a corollary of several propositions in [11], together with their proofs.
Lemma 4.1. Let $A$ and $B$ be finitely generated groups and let $x=b f, y=c g$ be conjugate in $A \mathrm{wr} B$. Then $z=d h \in A \mathrm{wr} B$ conjugates $x$ to $y$ if and only if $z$ satisfies

1. $d b=c d$ in $B$;
2. when the order of $b$ is finite, $h$ satisfies

$$
\begin{equation*}
h\left(t_{i} b^{k}\right)=\left(\prod_{j=0}^{k} g\left(t_{i} b^{j} d^{-1}\right)\right)^{-1} \alpha_{i} \prod_{j=0}^{k} f\left(t_{i} b^{j}\right) \tag{7}
\end{equation*}
$$

where $\alpha_{i}$ is such that $\pi_{t_{i}}^{(d)}(g)=\alpha_{i} \pi_{t_{i}}(f) \alpha_{i}^{-1} ;$
3. when the order of $b$ is infinite, $h$ satisfies

$$
\begin{equation*}
h\left(t_{i} b^{k}\right)=\left(\prod_{j=0}^{k} g\left(t_{i} b^{j} d^{-1}\right)\right)^{-1} \prod_{j=0}^{k} f\left(t_{i} b^{j}\right) \tag{8}
\end{equation*}
$$

Note that it follows from [11] that the formulas (7) and (8) define $h(\beta)$ for all $\beta \in B$ and do not depend on the choice of coset representatives. With this, we can now prove the following theorem.

Theorem 4.2. Let $A$ and $B$ be finitely generated groups such that the following hold:

1) there are algorithms which solve the Conjugacy Search Problem in $A$ and in $B$ with polynomial time functions, $T_{C S A}, T_{C S B}$, respectively;
2) there is an algorithm with polynomial time function $T_{P B}$ for the Power Problem in B.

Then the Conjugacy Search Problem in AwrB is solvable with complexity

$$
O\left(T_{C S B}(L)+T_{C S A}(L)+L^{2} T_{P B}(L)\right)
$$

where $L=|x|+|y|$ is the length of the input pair $x, y \in A \mathrm{wr} B$.
Proof. Let $x=b f, y=c g$ be conjugate in $A \mathrm{wr} B$ (this can be checked in polynomial time using Theorem 3.3). Using the algorithm to solve the Conjugacy Search Problem in $B$, one can find $d \in B$ such that $d b=c d$ in time $T_{C S B}(L)$. It remains to show that the function $h$ as described in Lemma ?? can be described by a finite set of pairs $\left\{\left(b_{i}, h\left(b_{i}\right)\right)\right\}$.

First, assume that the order of $b$ in $B$ is infinite. Let

$$
M=\max \left\{M_{i} \mid t_{i} b^{M_{i}} \in \operatorname{supp}(f) \cup \operatorname{supp}(g), \text { and } i \in I_{1}\right\} .
$$

We show that $M$ can be found in polynomial time. For each $b_{j} \in \operatorname{supp}(f) \cup$ $\operatorname{supp}(g)$ and for each $t_{i} \in T_{b}$, compute $M_{i j}$ such that $t_{i} b^{M_{i j}}=b_{j}$. This can be done in time $O\left(L^{2} T_{P B}(L)\right)$. Let $M=\max \left\{M_{i} \mid b_{j} \in \operatorname{supp}(f) \cup \operatorname{supp}(g)\right\}$. Then $M=\max \left\{M_{i} \mid i \in I_{1}\right\}$ can be computed in $O\left(L^{2} T_{P B}(L)\right)$ steps. Consider the following cases.

1. $k \geq M$. Then $h\left(t_{i} b^{k}\right)=\left(\pi_{t_{i}}^{(d)}(g)\right)^{-1} \pi_{t_{i}}(f)=1$, by Theorem 3.1. Hence $h\left(t_{i} b^{k}\right)=1$.
2. $k<M$.
(a) If $t_{i} \notin \operatorname{supp}(f) \cup \operatorname{supp}(g)$ and $t_{i} d^{-1} \notin \operatorname{supp}(f) \cup \operatorname{supp}(g)$, then $f\left(t_{i} b^{j}\right)=1$ and $g\left(t_{i} d^{-1} b^{j}\right)=1$ for all $j$ and hence $h\left(\tilde{t_{i}} b^{k}\right)=1$.
(b) If $t_{i} \in \operatorname{supp}(f) \cup \operatorname{supp}(g)$, but $t_{i} d^{-1} \notin \operatorname{supp}(f) \cup \operatorname{supp}(g)$, then

$$
h\left(\tilde{t}_{i} b^{k}\right)=\left(\prod_{j \leq k} g\left(\tilde{t}_{i} d^{-1} c^{j}\right)\right)^{-1} \prod_{j \leq k} f\left(t_{i} b^{j}\right)=\prod_{j \leq k} f\left(t_{i} b^{j}\right)
$$

which can be computed in time $\left.O\left(M L T_{P B}(L)\right)\right)$.
(c) If $t_{i} \notin \operatorname{supp}(f) \cup \operatorname{supp}(g)$, but $t_{i} d^{-1} \in \operatorname{supp}(f) \cup \operatorname{supp}(g)$, $h\left(\tilde{t_{i}} b^{k}\right)=\prod_{j \leq k} g\left(\tilde{t_{i}} b^{j} d^{-1}\right)$ which can be similarly computed in time $\left.O\left(M L T_{P B}(L)\right)\right)$.
(d) If $t_{i}, t_{i} d^{-1} \in \operatorname{supp}(f) \cup \operatorname{supp}(g)$, then

$$
h\left(\tilde{t}_{i} b^{k}\right)=\left(\prod_{j \leq k} g\left(t_{i} b^{j} d^{-1}\right)\right)^{-1} \prod_{j \leq k} f\left(t_{i} b^{j}\right)
$$

can be computed in time $O\left(M L T_{P B}(L)\right)$.
Thus, if $k<M, h\left(t_{i} b^{k}\right)$ can be computed in time $O\left(M L T_{P B}(L)\right)$. It is clear from the definition of $M$ that $M<L$, so one can compute $h\left(t_{i} b^{k}\right)$ in time $O\left(L^{2} T_{P B}(L)\right)$.

Assume that the order of $b$ is finite, say $N$. Using the algorithm to solve the Conjugacy Search Problem in $A$, one can find in time $T_{C S A}\left(L^{2}\right)$, for each $i \in I_{1}$, an $\alpha_{i} \in A$ such that $\pi_{t_{i}}^{(d)}(g)=\alpha_{i} \pi_{t_{i}}(f) \alpha_{i}^{-1}$. Then $h\left(t_{i} b^{k}\right)=$ $\left(\prod_{j=0}^{k} g\left(t_{i} b^{j} d^{-1}\right)\right)^{-1} \alpha_{i} \prod_{j=0}^{k} f\left(t_{i} b^{j}\right)$ can be found in time $O\left(T_{C S A}(L)+L^{2} T_{P B}(L)\right)$ by arguing as in the infinite order case (here instead of $M$, we use the order $N$ of $b$ ).

Thus the conjugacy search problem in $A \mathrm{wr} B$ is solvable in time

$$
O\left(T_{C S B}(L)+T_{C S A}(L)+L^{2} T_{P B}(L)\right)
$$

## 5 Complexity of the Conjugacy and Conjugacy Search Problems in Free Solvable Groups

By Corollary 2.4 the Conjugacy Problem in free solvable groups can be reduced in polynomial time to the Conjugacy Problem in a wreath product. Then the result from Section 3 can be applied to deduce that the Conjugacy Problem in free solvable groups is solvable in polynomial time. Though the bound for the complexity will be polynomial, the degree of the polynomial will depend on the degree of solvability (this is because of the factor of $L$ in front of $T_{C B}(L)$ in (21)). However, by making a modification to the algorithm, the complexity of the Conjugacy Problem in free solvable groups is shown to be a polynomial of degree eight.

Theorem 5.1. The Conjugacy Problem in a wreath product $A \mathrm{wr} B$, in which $A$ is abelian is in

$$
O\left(T_{C A}\left(L^{2}\right)+T_{C B}(L)+L^{2} T_{P B}(L)\right),
$$

where $L$ is the length of the input pair $(x, y)$.

Proof. The algorithm is similar to the one in Theorem 3.3. The only alteration to be made is in Case 3, where the order of $b$ is infinite. Let $\left\{s_{i}\right\}_{i \in I_{1}}$ be the set of coset representatives computed in Case 2. Then $\pi_{s_{i}}(g)$ is conjugate to $\pi_{t_{i}}^{(d)}(g)$. Since $A$ is now abelian, $\pi_{s_{i}}(g)=\pi_{t_{i}}^{(d)}(g)$. Thus $\pi_{t_{i}}(f)=\pi_{t_{i}}^{(d)}(g)$ if and only if $\pi_{t_{i}}(f)=\pi_{s_{i}}(g)$. Checking this requires

$$
\begin{equation*}
O\left(|x| T_{P B}(|x|)+|y| T_{P B}(|y|)+T_{C A}\left(|x|^{2}+|y|^{2}\right)\right) . \tag{9}
\end{equation*}
$$

As a result the overall complexity of the modified algorithm is

$$
O\left(T_{C A}\left(L^{2}\right)+T_{C B}(L)+L^{2} T_{P B}(L)\right)
$$

Theorem 5.2. The Conjugacy Problem in $S_{d, r}$ is in $O\left(r d L^{8}\right)$, where $L=$ $|x|+|y|$ is the input length.

Proof. We proceed by induction on the degree of solvability, $d$. The base case is the abelian group $F / F^{\prime}$, where the Conjugacy Problem is in $O(r L)$. Now suppose there is an algorithm, which solves the Conjugacy Problem in $F / F^{(d)}$ in $O\left(r d L^{8}\right)$. By Corollary 2.4, one can reduce the Conjugacy Problem in $F / F^{(d+1)}$ to the Conjugacy Problem in $F / F^{\prime}$ wr $F / F^{(d)}$ in time $O\left(r d L^{3}\right)$. Since $F / F^{\prime}$ is abelian, we apply Theorem 5.1. In order to do this we need polynomial bounds for the Conjugacy Problems of $F / F^{\prime}, F / F^{(d)}$ and the Power Problem in $F / F^{(d)}$.

The Conjugacy Problem in $F / F^{\prime}$ is in $O(r L)$. By the induction hypothesis, there is an algorithm which solves the Conjugacy Problem in $F / F^{(d)}$ in $O\left(r d L^{8}\right)$. By Theorem 2.5there is an algorithm which solves the Power Problem in $F / F^{(d)}$ in $O\left(r d L^{6}\right)$. Then from Theorem 5.1] the complexity of the Conjugacy Problem in $F / F^{(d+1)}$ is

$$
O\left(r L^{2}+r d L^{8}+L^{2} r d L^{6}\right)
$$

It is easily seen now that the complexity of the Conjugacy Problem in free solvable groups is

$$
O\left(r d L^{8}\right)
$$

Since all the proofs of the decidability results are constructive one can also deduce the following theorem.

Theorem 5.3. The Conjugacy Search Problem in $S_{d, r}$ is solvable in time $O\left(r d L^{8}\right)$, where $L=|x|+|y|$ is the input length.

Proof. Again we proceed by induction on the degree of solvability $d$, this time making sure that at each step we are effectively finding the required object. When $d=1$ the group is abelian and so deciding the Conjugacy Search Problem there is trivial - two words are conjugate if and only if the identity is a
conjugator. Now suppose that there is an algorithm running in time $O\left(r d L^{8}\right)$, which, if two words $\bar{x}, \bar{y} \in F / F^{(d)}$ are conjugate, exhibits a conjugator. We proceed to describe an algorithm which does the same for two conjugate elements $x, y \in F / F^{(d+1)}$ given as products of generators of $F$. As before, by Corollary 2.4, we reduce the Conjugacy Problem in $F / F^{(d+1)}$ to the Conjugacy Problem in $F / F^{\prime}$ wr $F / F^{(d)}$. Hence by Theorem 4.2 there is an algorithm running in time $O\left(r d L^{8}\right)$, which finds a conjugator for $\varphi(x)$ and $\varphi(y)$. The proof of Theorem 2 in [18] gives a pre-image $s \in F / F^{(d+1)}$ for this conjugator. One can see easily that computing $s$ can be done in time $O\left(r(d+1) L^{3}\right)$. Thus, the overall complexity of this algorithm is

$$
O\left(r(d+1) L^{8}\right)
$$

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