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PAPER

Nested Circular Array and Its Concentric Extension for Underdetermined Direction of Arrival Estimation

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SUMMARY In this paper, a new array geometry is proposed which is capable of performing underdetermined Direction-Of-Arrival (DOA) estimation for the circular array configuration. DOA estimation is a classical problem and one of the most important techniques in array signal processing as it has applications in wireless and mobile communications, acoustics, and seismic sensing. We consider the problem of estimating DOAs in the case when we have more sources than the number of physical sensors where the resolution must be maintained. The proposed array geometry called Nested Sparse Circular Array (NSCA) is an extension of the two level nested linear array obtained by nesting two sub-circular arrays and one element is placed at the origin. In order to extend the array aperture, a Khatri-Rao (KR) approach is applied to the proposed NSCA which yields the virtual array structure. To utilize the increase in the degrees of freedom (DOFs) that this new array provides, a subspace based approach (MUSIC) for DOA estimation and ℓ_1-based optimization approach is extended to estimate DOAs using NSCA. Simulations show that better performance for underdetermined DOA estimation is achieved using the proposed array geometry.

key words: *array antenna, degrees of freedom (DOF), direction of arrival estimation, Khatri-Rao product, nested array.*

1. Introduction

Direction of arrival (DOA) estimation, which is also called spatial spectra estimation has been an active research area, playing an important role in many applications, such as electromagnetic, acoustic, and seismic sensing [1]–[3]. The development in the array signal processing discipline has led to high resolution DOA estimation techniques for narrowband signals and wideband [3], [4].

DOA estimation in antenna arrays however has been mostly confined to uniform linear arrays (ULA) and uniform circular arrays (UCA) [5]. Subspace based methods like MUltiple SIgnal Classification (MUSIC) [4] can resolve up to $(M-1)$ sources for an M element ULA and UCA [5], [6]. In order to estimate more sources than the number of physical sensors, [7] proposed nested linear arrays. This work was further extended to arrays with higher geometries in [8], [9] as well as co-prime arrays [10]. In this paper, we consider nonuniform circular arrays and propose a novel array structure which has the ability to provide an increase in DOFs and hence is capable of resolving more sources than physical sensors available called Nested Sparse Circular Array (NSCA). This proposed array is obtained by combining two or more sub-circular arrays. We demonstrate that by using NSCA, we can achieve underdetermined DOA estimation.

In recent years, underdetermined DOA estimation has received considerable interest [11]–[13]. One of the most effective approach to underdetermined DOA estimation is to construct a new array that has an extended aperture and obtains higher DOFs as compared to DOFs obtained from the physical array. Sparse spatial sampling in this case provides a remarkable improvement in DOFs, and typical array structures employed include nested linear arrays [7] and co-prime arrays [10]. In recent years, a different kind but effective DOA estimation technique called ℓ_1-SVD based on sparse signal reconstruction emerged [14]. In single measurement case, ℓ_1 optimization is considered attractive to sparse signal recovery due to its guaranteed recovery accuracy [15]. However, for an array with M sensors, the ℓ_1-based approach in [14] can resolve up to $M-1$ signals impinging on the array.

Earlier works fail to fully consider underdetermined DOA estimation for UCA. In [16], to resolve more than $(M-1)$ sources, the Khatri-Rao (KR) [11] subspace approach was considered for quasi-stationary signals applied to UCA. Quasi-stationary signals are a class of nonstationary signals in which the signal statics are locally static over a short period of time [11]. Speech and audio signals are some of the examples of quasi-stationary signals. The problem with using quasi-stationary signals is that this method can not be applied to stationary sources [7].

Reference [17] proposed a Nested Circular Array (NCA) to perform wideband estimation in which microphone pairs are used to eliminate spatial aliasing for counting and DOA estimation of multiple simultaneous speakers. Although this is the case, NCA is basically a Uniform Circular Array in structure and not extension of nested arrays as proposed in [7]. In [18], optimal array structures were surveyed in which optimal and nearly optimal schemes operating both in a periodic and non-periodic fashion were designed by considering linear compression schemes classified as dense or sparse. Although in linear case, the length-10 minimal sparse ruler (SR) is an optimal sparse array, its counterpart, length-20 circular SR designed with a length-10 linear SR has sensors positioned on one side of the array which results in angular dependency for DOA estimation hence reduced performance. We carry out a performance comparison of the length-20 circular SR to the proposed array configuration NSCA in underdetermined DOA estimation.

In this paper, we propose a new array structure called "*Nested Sparse Circular Array*" that has the ability to esti-

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mate more sources than the number of physical sensors. The proposed array is further synthesized into a non-uniform concentric array through the KR approach. The virtual elements synthesized by the KR approach are inside the original circle which works as a concentric circular array and is effective for wideband DOA estimation. The synthesized non-uniform concentric array increases the DOFs and helps to perform underdetermined DOA estimation. This makes the proposed NSCA a good candidate for both narrowband and wideband underdetermined DOA estimation. We extend the subspace based approach MUSIC in [4] and used in [11] for quasi-stationary signals to the proposed NSCA. Therefore, the proposed NSCA is capable of performing underdetermined DOA estimation for quasi-stationary signals as reported in [19] and not stationary signals. Furthermore, an ℓ_1 optimization method based on compressive sensing or sparse signal recovery is used which takes advantage of the KR product of covariance matrix.

Notations

Notations used in this paper are given as follows. We represent matrices by capital boldface letters (e.g., $\mathbf{A}$) whilst vectors are represented by lowercase boldface letters (e.g., $\mathbf{a}$). The superscript ^{T} represents transpose, and superscript ^{H} represents conjugate transpose, whereas superscript * represents conjugation without transpose. $\|\bullet\|_p$ represents the ℓ_p norm. The symbol $\odot$ represents the Khatri-Rao product between two matrices of appropriate size and the symbol $\otimes$ is used to represent the left Kronecker product.

2. Preliminaries

2.1 The Signal Representation

We consider an M element omnidirectional, non-uniform circular antenna array. We assume that D narrowband sources with wavenumber $k = 2\pi/\lambda$ are impinging on this array from the directions $\Theta_d = (\theta_d, \phi_d)$, where $d = 1, 2, \cdots, D$ with receiver noise powers σ^2. θ is the elevation angle while ϕ is the azimuth angle and λ is wavelength. The received signal vector is therefore given by

$$\boldsymbol{x}(t) = \mathbf{A}(\Theta)\mathbf{s}(t) + \mathbf{n}(t) \tag{1}$$

where $\boldsymbol{x}$(t) is an $M \times 1$ noise-corrupted array snapshot vector, $\mathbf{s}$(t) is a $D \times 1$ signal vector, and $\mathbf{n}$(t) is an $M \times 1$ noise vector. The noise is assumed to be additive white gaussian noise. The array manifold matrix $\mathbf{A}$ is an $M \times D$ matrix, the columns of which are steering vectors $\mathbf{a}(\theta_d, \phi_d)$, $d = 1, 2, ..., D$. Thus, we have the array manifold matrix given by:

$$\mathbf{A} = [\mathbf{a}(\theta_1, \phi_1), \mathbf{a}(\theta_2, \phi_2), ..., \mathbf{a}(\theta_D, \phi_D)] \tag{2}$$

The source number D is a *priori* known or accurately estimated [6]. We further assume that the sources are uncorrelated such that the source autocorrelation matrix of $\boldsymbol{s}$(t) is diagonal. Thus,

$$\mathbf{R}_{xx} = E[\boldsymbol{x}\boldsymbol{x}^H] = \mathbf{A}\mathbf{R}_{ss}\mathbf{A}^H + \sigma^2\mathbf{I} \tag{3}$$

where $\mathbf{R}_{ss}$ is the signal covariance matrix given by the diagonal of signal powers and $\mathbf{I}$ is an identity matrix.

2.2 Nested Array

The class of arrays called nested array was first proposed in [7]. The "two level" nested array as defined in [7] is in fact similar to the array structure originally proposed in [20]. However in [7] the concept of nested arrays was generalized to more than two levels so that there is a considerable increase in the DOFs. A two level nested linear array is a series of two interconnected uniform linear array (inner and outer) with the inner ULA having M_1 and the outer ULA having M_2 elements. A two level nested linear array can therefore achieve $2M_2(M_1 + 1) - 1$ freedoms in the co-array using $M_1 + M_2$ elements only [7].

3. The Concept of Nested Sparse Circular Array and its Application to Underdetermined DOA Estimation

3.1 Nested Sparse Circular Array

In case of nested linear array, its ability to resolve more sources than physical sensors is that its difference co-array has significantly more DOFs than the original array. In other works, types of Minimum Redundancy arrays (MRAs) [21], were utilized to achieve increased DOFs. The problem with MRAs is that they require an extensive computer search to construct the array [7]. In [22], a joint sparsity approach was used by adopting convex relaxation idea with co-prime arrays for off-grid targets in sparse DOA estimation. A generalization of the co-prime array concept was proposed in [23]. Until now, so much work related to underdetermined DOA estimation has been performed mainly considering nested linear arrays, co-prime arrays, and MRAs.

In this subsection, we consider circular arrays and attempt to provide a solution for underdetermined DOA estimation using NSCA. Figure 1 shows a nested sparse circular array with two sub-circular arrays concatenated and one element at the origin which is used in KR formulation since we consider the center of the NSCA as the origin. For an M-element NSCA, the first and second sub-circular arrays have $(M - 1)$ physical elements in total. The first sub-circular array has M_1 physical elements with inter-element spacing of d_1 whilst the second sub-circular array has M_2 physical elements with inter-element spacing of d_2.

For NSCA, the spacing $d_1 = 2\pi r/((M - 1)^2/2)$ and $d_2 = (M_1 + 1)d_1$. Thus, d_1 is the distance between elements in the dense part of the array and d_2 is the distance between elements in the sparse part of the array. Using the union of the first and second sub-circular array, we obtain the NSCA. The element positions are therefore given by $C_{first} = m_1 d_1$, where $m_1 = 1, 2, \cdots, M_1$ and $C_{second} = m_2(M_1 + 1)d_1$, where $m_2 = 1, 2, \cdots, M_2$. Thus the steering vector of the NSCA will be given by;

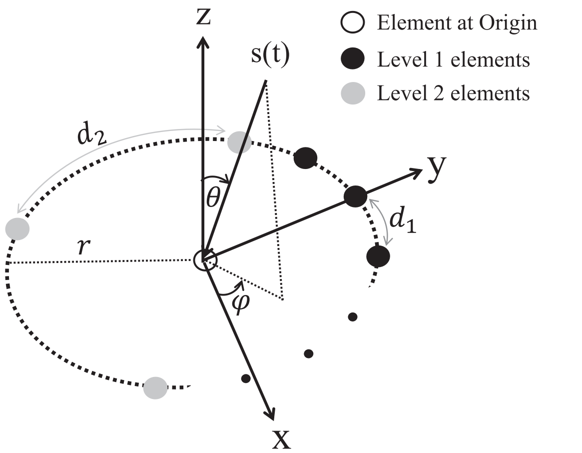

Fig. 1 A 7 element Nested Sparse Circular Array with one element at the origin.

$$\mathbf{a}(\theta,\phi) = [1, e^{jkr\sin\theta\cos(\phi-\gamma_1)}, e^{jkr\sin\theta\cos(\phi-\gamma_2)}, \ldots, e^{jkr\sin\theta\cos(\phi-\gamma_{M-1})}]^T \tag{4}$$

where γ_m is the angular position of the m-th element. We assume that the elevation angle θ is fixed at 90° [6], therefore the steering vector of the NSCA given in (4) as $\mathbf{a}(\theta,\phi)$ will be reduced to $\mathbf{a}(\phi)$.

To find the virtual elements and extend the array aperture, the KR subspace approach [11], [24] is applied. By using this approach, we can extend the DOFs of the NSCA and be able to perform underdetermined DOA estimation. This approach therefore is described in the following subsection.

3.2 The Khatri-Rao Subspace Approach

In many works, increased DOFs has been exploited using various techniques for example augmented matrix approach [21], fourth-order-cumulant based methods [25] and quasi-stationary signal based methods [11]. However, these methods are used in linear arrays and there is little in terms of circular array DOA estimation for more sources than physical sensors. Hence, in this subsection and the following, we will exploit the increased DOFs by using the KR subspace approach [11] and extend the KR-MUSIC and ℓ_1-based optimization approach to underdetermined DOA estimation using NSCA.

We apply the KR subspace approach proposed in [11] to DOA estimation. Consider $\mathbf{A}$ which is an $L \times D$ matrix and $\mathbf{B}$ an $M \times D$ matrix having an identical number of columns, their KR product is given by

$$\mathbf{A} \odot \mathbf{B} = [\mathbf{a}_1 \otimes \mathbf{b}_1, \mathbf{a}_2 \otimes \mathbf{b}_2, \ldots, \mathbf{a}_D \otimes \mathbf{b}_D] \tag{5}$$

where $\mathbf{A} \odot \mathbf{B}$ results in an $LM \times D$ matrix. For two vectors $\mathbf{a}$ and $\mathbf{b}$, their Kronecker product is given by

$$\mathbf{a} \otimes \mathbf{b} = \begin{bmatrix} a_1\mathbf{b} \\ a_2\mathbf{b} \\ \vdots \\ a_L\mathbf{b} \end{bmatrix} = \mathrm{vec}(\mathbf{b}\mathbf{a}^T) \tag{6}$$

From the KR subspace approach, we find a new array model for our proposed NSCA. For the DOA estimation problem that has been formulated in Sect. 2.1, we apply vectorization to (3) to obtain

$$\begin{aligned} \mathbf{y} &= \mathrm{vec}(\mathbf{R}_{xx}) \\ &= \mathrm{vec}\left(\mathbf{A}\mathbf{R}_{ss}\mathbf{A}^H\right) + \mathrm{vec}\left(\sigma^2\mathbf{I}\right) \\ &= (\mathbf{A}^* \odot \mathbf{A})\mathbf{p} + \sigma^2\mathbf{1}_M^T \end{aligned} \tag{7}$$

where $\mathbf{p} = [\sigma_1^2, \sigma_2^2, \ldots, \sigma_D^2]^T$ and $\mathbf{1}_m = [\mathbf{e}_1^T, \mathbf{e}_2^T, \ldots, \mathbf{e}_M^T]$ and $\mathbf{e}_i$ is a column vector having all zeros except a 1 at the i-th position. $\mathbf{p}$ therefore is equivalent to source signal vector and noise becomes a deterministic vector that is given by $\sigma^2\mathbf{1}_M$ which can be eliminated easily. In this case, $\boldsymbol{y}$ behaves like the array's received signal whose manifold is given by $(\mathbf{A}^* \odot \mathbf{A})$. Thus $(\mathbf{A}^* \odot \mathbf{A})$ is a manifold of a longer array i.e., array with virtual elements with a larger array aperture than the one when it is not vectorized. Let $\mathbf{B} = (\mathbf{A}^* \odot \mathbf{A})$, The steering matrix of array with virtual elements will be given by $\mathbf{B} = [\mathbf{b}(\phi_1), \mathbf{b}(\phi_2), \ldots, \mathbf{b}(\phi_D)]^T$ which is an $M^2 \times D$ matrix. Using the Kronecker product, the $M^2 \times 1$ steering vector is

$$\begin{aligned} \mathbf{b}(\phi) &= \mathrm{vec}(\mathbf{a}(\phi)\mathbf{a}^H(\phi)) = \mathbf{a}^*(\phi) \otimes \mathbf{a}(\phi) \\ &= \begin{bmatrix} 1 \\ e^{jkr\cos(\phi-\gamma_1)} \\ e^{jkr\cos(\phi-\gamma_2)} \\ \vdots \\ e^{jkr\cos(\phi-\gamma_{M-1})} \end{bmatrix}^* \otimes \begin{bmatrix} 1 \\ e^{jkr\cos(\phi-\gamma_1)} \\ e^{jkr\cos(\phi-\gamma_2)} \\ \vdots \\ e^{jkr\cos(\phi-\gamma_{M-1})} \end{bmatrix} \\ &= \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_{M^2} \end{bmatrix} \end{aligned} \tag{8}$$

for $i = 1, 2, \ldots M^2$. Therefore instead of using (1), we can apply the problem of DOA estimation to the data obtained in (7).

3.3 Concentric Extension of Nested Sparse Circular Array

In this subsection, we discuss the concept of a non-uniform concentric array obtained as a result of a co-array of the NSCA. This concentric extension of the NSCA enables the increase in the DOFs provided by the co-array, such that we can perform underdetermined DOA estimation.

For an array with M sensors, with the position of the ith sensor denoted by $\vec{x}_i$, the difference co-array is defined as

$$\mathbf{C}_d = \vec{\boldsymbol{x}}_i - \vec{\boldsymbol{x}}_j, \; \forall i, j = 1, 2, \ldots, M \tag{9}$$

For difference co-arrays, $\mathbf{C}_d(i,i)$ refers to the co-array origin since $\vec{\boldsymbol{x}}_i - \vec{\boldsymbol{x}}_i = 0$. In our case, although we have M^2 distinct pairs of array elements in the co-array using the KR product, our co-array contains some redundant co-array points, thus we end up having less than (M^2) distinct points [26]. The other co-array points therefore are influenced by two distinct pairs $(\vec{\boldsymbol{x}}_i, \vec{\boldsymbol{x}}_j)$ and $(\vec{\boldsymbol{x}}_j, \vec{\boldsymbol{x}}_i)$ with $\vec{\boldsymbol{x}}_i \neq \vec{\boldsymbol{x}}_j$. As illustrated in [26], the co-array of a circular array with odd number of elements is $M(M-1)+1$ while for an array with even number of elements is $(M^2/2+1)$ but in our case, we synthesized more co-array points than in [26].

To synthesize the virtual elements formed from the NSCA using the KR product, we consider $\mathbf{b}(\phi)$. From (8), let $m, l = 1, 2, \cdots, M$, therefore the first $2M$ points will be determined by $1 \times e^{jkr\cos(\phi-\gamma_l)}$ and $e^{jkr\cos(\phi-\gamma_m)} \times 1$ which are redundant points and the remaining points will be given by;

$$\begin{aligned} b_{ml}(\phi) &= e^{jkr\cos(\phi-\gamma_m)} \times e^{jkr\cos(\phi-\gamma_l)} \\ &= e^{jkr\{\cos(\phi-\gamma_m)-\cos(\phi-\gamma_l)\}} \end{aligned} \tag{10}$$

from trigonometric addition, $\mathbf{b}(\phi)$ becomes;

$$\begin{aligned} b_{ml}(\phi) &= e^{-j2kr\{\sin((2\phi-\gamma_m-\gamma_l)/2)\sin((\gamma_m-\gamma_l)/2)\}} \\ &= e^{-j2kr\{\sin(\phi-(\gamma_m+\gamma_l)/2)\sin((\gamma_m-\gamma_l)/2)\}} \end{aligned} \tag{11}$$

From this equation, we observe that we end up with virtual elements located on positions having different radius from the origin which implies that using the KR approach with the NSCA, we synthesize the virtual elements onto a non-uniform concentric circular array. Figure 1 shows NSCA while in Fig. 2, we show the synthesized version of the array with virtual elements which is basically a concentric extension of the NSCA.

The number of elements in the concentric extension of the NSCA in Fig. 2 (given by $\mathbf{C}_d$) decides the values of the cross correlation values in the covariance matrix of the received signal by the NSCA. By carefully using cross correlation terms, we substantially increase the DOFs, thus, be able to detect more number of sources than the number of physical elements using the NSCA. From the synthesized non-uniform concentric circular array, we can easily perform underdetermined DOA estimation.

3.4 DOA Estimation Based on KR-MUSIC

We use an ideal MUSIC [4] based approach for exploiting the DOFs of the NSCA with virtual elements. Firstly, the unknown noise covariance is eliminated and then dimension reduction [11]. In order to eliminate the noise covariance, let $\mathbf{q}^{\perp}_{\mathbf{1}_M}$ denote an orthogonal complement projector, where $\mathbf{q}^{\perp}_{\mathbf{1}_M} = \mathbf{I}_M - (1/M)\mathbf{1}_M\mathbf{1}_M^T$. By performing a projection on (7), we obtain

$$\mathbf{y}\mathbf{q}^{\perp}_{\mathbf{1}_M} = (\mathbf{A}^* \odot \mathbf{A})(\mathbf{q}^{\perp}_{\mathbf{1}_M}\mathbf{p}) \tag{12}$$

The singular value decomposition (SVD) of $\mathbf{y}\mathbf{q}^{\perp}_{\mathbf{1}_M}$ is therefore

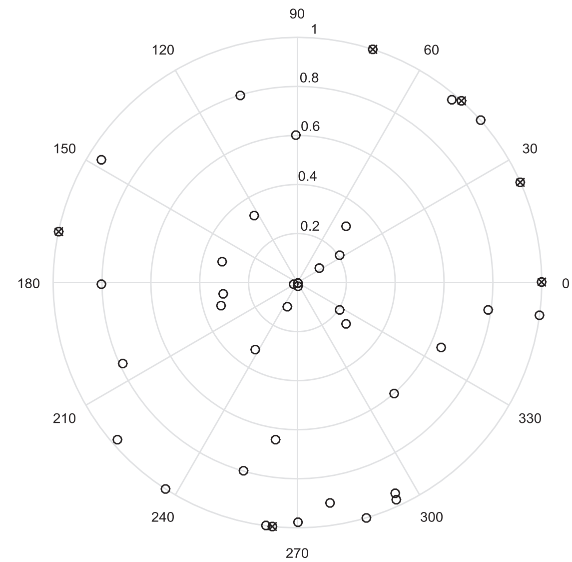

Fig. 2 Synthesized Concentric extension of a 7 element Nested Sparse Circular Array. The physical element positions are given by ⊗ and virtual element positions are given by ∘.

$$\mathbf{y}\mathbf{q}^{\perp}_{\mathbf{1}_M} = \begin{bmatrix}\mathbf{U}_s\\ \mathbf{U}_n\end{bmatrix}\begin{bmatrix}\sum_s \mathbf{0}\\ \mathbf{0}\ \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{V}_s^H\\ \mathbf{V}_n^H\end{bmatrix}^* \tag{13}$$

where $\mathbf{U}_s$ and $\mathbf{V}_s$ are the left and right singular matrices associated with nonzero singular values respectively, $\mathbf{U}_n$ and $\mathbf{V}_n$ are the left and right singular matrices associated with zero singular values respectively, and $\sum_s$ is a diagonal matrix whose diagonals contain the nonzero singular values.

In [11] it has been proved that for $(\mathbf{A}^* \odot \mathbf{A})$ to yield a full column rank i.e rank $= D$ the sufficient and necessary condition is when $D \leq 2M-1$. Thus, prior to apply subspace approach to DOA estimation problem such as MUSIC, we need to reduce the problem of dimension. Let the virtual array response matrix $(\mathbf{A}^* \odot \mathbf{A}) = \mathbf{G}\mathbf{B}$ where $\mathbf{B}$ is a dimensionally reduced virtual array response matrix which is $(2M) \times D$ given by $\mathbf{B} = [\mathbf{b}(\phi_1), \mathbf{b}(\phi_2), \ldots, \mathbf{b}(\phi_D)]^T$ as compared to one given in (8) which is an $M^2 \times D$ matrix and $\mathbf{G}$ is an $(M^2+M) \times (2M)$ matrix [11]. Let $\mathbf{F} = \mathbf{G}^T\mathbf{G}$, this implies that

$$\mathbf{F} = \mathrm{diag}(1, 2, \ldots M-1, M, M-1, \ldots, 2, 1) \tag{14}$$

from (12), this problem can therefore result in

$$\begin{aligned} \hat{\mathbf{y}} &= \mathbf{F}^{-\frac{1}{2}}\mathbf{G}^T\left[\mathbf{y}\mathbf{q}^{\perp}_{\mathbf{1}_M}\right] \\ &= \mathbf{F}^{-\frac{1}{2}}\mathbf{G}^T(\mathbf{A}^* \odot \mathbf{A})(\mathbf{q}^{\perp}_{\mathbf{1}_M}\mathbf{p}) \\ &= \mathbf{F}^{\frac{1}{2}}\mathbf{B}(\mathbf{q}^{\perp}_{\mathbf{1}_M}\mathbf{p}) \end{aligned} \tag{15}$$

the dimension reducing transformation $\mathbf{F}^{-\frac{1}{2}}\mathbf{G}^T$ has orthonormal rows. We can therefore apply subspace based approach to $\hat{\mathbf{y}}$ which is $(2M-1) \times 1$. We therefore apply MUSIC to the dimensionally reduced problem, thus the MUSIC spectrum is given by

$$P(\phi) = \frac{1}{\left\| \mathbf{U}_n^H \mathbf{F}^{\frac{1}{2}} \mathbf{b}(\phi) \right\|^2} \tag{16}$$

over $\phi \in [0, \pi]$, and then we pick D largest peaks of $P(\phi)$ as the DOA estimates.

3.5 DOA Estimation with ℓ_1-Based Optimization

Sparse signal representation employs the ideas of enforcing sparsity by ℓ_1 penalization and restricting error by ℓ_2-norm which enables reconstruction of sparse signal [14]. Under the sparsity framework multiple measurements vectors are employed to the problem of estimating a sparse unknown parameter. Thus, we can extend the DOA estimation problem as a problem of finding the sparsest solution to underdetermined linear system [27]. In this paper we extend ℓ_1-based optimization proposed in [14] to underdetermined DOA estimation such that we increase the DOFs to $2M$. We consider (7) as a sparse signal representation problem which is given by

$$\mathbf{y} = \mathbf{B}\mathbf{p} + \sigma^2 \mathbf{1}_m \tag{17}$$

where $\mathbf{B}$ is as defined in Sect. 3.2. To extend ℓ_1 penalization to (17), we need to appropriately choose the optimization criteria which is $\min \|\mathbf{p}\|_1$ subject to $\|\mathbf{y} - \mathbf{B}\mathbf{p}\|_2^2 \leq \beta^2$, where β is a parameter specifying how much noise we wish to allow. Therefore, an unconstrained form of this objective function is

$$\min \|\mathbf{y} - \mathbf{B}\mathbf{p}\|_2^2 + \lambda \|\mathbf{p}\|_1 \tag{18}$$

The ℓ_2 term in (18) forces the residual $\mathbf{y} - \mathbf{B}\mathbf{p}$ to be small and λ controls the tradeoff between the sparsity of the spectrum and residual norm.

In a practical setting, $\boldsymbol{y}$ in (17) can be estimated from N snapshots such that $\Delta\mathbf{y} = \hat{\mathbf{y}} - \mathbf{y}$. The estimate error from is asymptotically normal distribution (AsN), thus

$$\Delta\mathbf{y} = \text{vec}(\Delta\mathbf{R}_{xx}) \sim AsN\left(0_{M^2,1}, \frac{1}{N}\mathbf{R}_{xx}^T \otimes \mathbf{R}_{xx}\right) \tag{19}$$

which leads to

$$\mathbf{W}^{-\frac{1}{2}}\Delta\mathbf{y} \sim AsN\left(0_{M^2,1}, \mathbf{I}_{M^2}\right) \tag{20}$$

where the weighting matrix $\mathbf{W}^{-\frac{1}{2}} = \sqrt{N}\left[\mathbf{R}_{xx}^T\right]^{-\frac{1}{2}} \otimes \mathbf{R}_{xx}^{-\frac{1}{2}}$ with $\mathbf{W} = \frac{1}{N}\mathbf{R}_{xx}^T \otimes \mathbf{R}_{xx}$. Let $\hat{\mathbf{p}}$ be the estimate of $\mathbf{p}$, the DOA estimation problem can then be given by the following ℓ_1-norm minimization

$$\min_{\hat{\mathbf{p}}} \|\hat{\mathbf{y}} - \mathbf{B}\hat{\mathbf{p}}\|_2^2 + \lambda \|\hat{\mathbf{p}}\|_1 \tag{21}$$

from (20) and (21) we further deduce that

$$\mathbf{W}^{-\frac{1}{2}}\left[\hat{\mathbf{y}} - \mathbf{B}\hat{\mathbf{p}}\right] \sim AsN\left(0_{M^2,1}, \mathbf{I}_{M^2}\right) \tag{22}$$

which then results in

$$\mathbf{W}^{-\frac{1}{2}}\|\hat{\mathbf{y}} - \mathbf{B}\hat{\mathbf{p}}\|_2^2 \sim As\chi^2(M^2) \tag{23}$$

where $As\chi^2(M^2)$ denotes the asymptotic chi-square distribution with M^2 degrees of freedom. Thus a parameter β is introduced such that $\left\|\mathbf{W}^{-\frac{1}{2}}\left[\hat{\mathbf{y}} - \mathbf{B}\hat{\mathbf{p}}\right]\right\|_2^2 \leq \beta^2$. Therefore, DOA estimation can be reduced to

$$\min_{\hat{\mathbf{p}}} \|\hat{\mathbf{p}}\|_1 \text{ subject to } \left\|\mathbf{W}^{-\frac{1}{2}}\left[\hat{\mathbf{y}} - \mathbf{B}\hat{\mathbf{p}}\right]\right\|_2^2 \leq \beta^2 \tag{24}$$

where $\beta = \sqrt{\chi^2(M^2)}$. The problem (24) is a second-order cone program problem. For numerical solution of our SOC problem, we can efficiently solve the SOC problem by using off the shelf optimization software CVX [28].

4. Simulation Results

4.1 Specifications of Simulation

In this section, we carry out simulation experiments to assess the capability of NSCA in estimating more sources than the number of physical sensors. Numerical examples in this section shows superior performance of the proposed array geometry in terms of DOFs for underdetermined DOA estimation and RMSE. In the examples, we examine a 7 element NSCA antenna system ($M = 7$) as shown in Fig. 1 with two subcircular arrays concatenated and one element at the origin. 8 narrowband sources ($D = 8$) are impinging on the array from the directions $\phi = [15°, 36°, 46°, 70°, 90°, 112°, 130°, 145°]$, all with the same amount of power. In case of 10 narrowband sources ($D = 10$) impinging on the array, $\phi = 236°$ and $284°$ are added. The radius of the nested circular array is $r = \lambda$. The noise is assumed to be spatially and temporally white.

4.2 Spectra of Underdetermined DOA Estimation

Figure 3(a) shows the spectra after applying the subspace based approach MUSIC for underdetermined DOA estimation. In this case, we observe that we are able to resolve all DOAs correctly and the peaks are sharp but we have low dynamic range. In Fig. 3(b) we observe the spectra of ℓ_1-based optimization for sparse signal recovery in an underdetermined case. In the case of ℓ_1-based optimization, we are able to resolve all DOAs and estimate them accurately. The peaks in this method are very sharp and we have very high dynamic range. Both methods in Fig. 3 requires more snapshots to resolve DOAs correctly but the ℓ_1-based technique have higher dynamic range as compared to MUSIC based method. Both methods use a total number of snapshots of 2000, and an SNR of 0 dB.

4.3 SNR Dependency for Underdetermined DOA Estimation

In this subsection, we examine the performance of MUSIC, the ℓ_1-based optimization technique with the cramer-rao lower bound (CRLB) [29] by examining RMSE of the

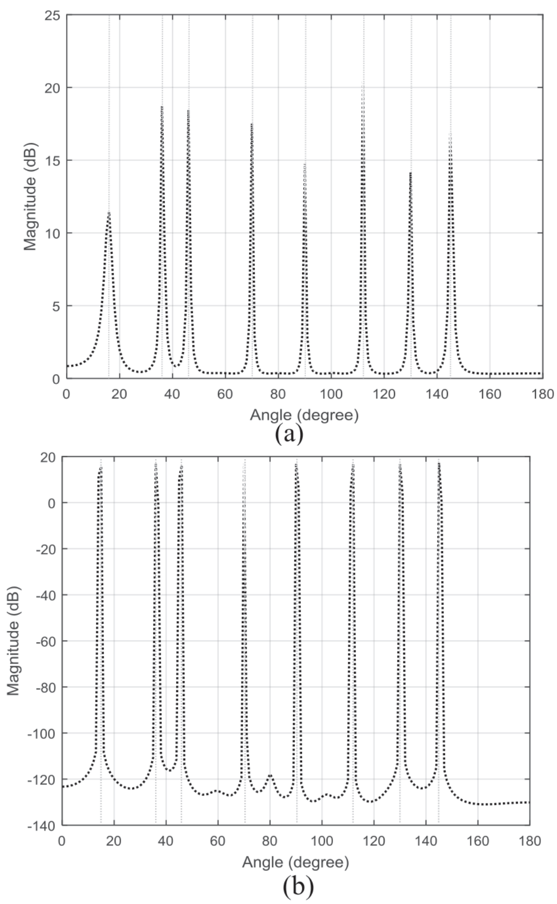

Fig. 3 Spectra for underdetermined DOA estimation for NSCA with $M = 7$, $D = 8$, snapshots = 2000, and SNR = 0 dB for (a) MUSIC and (b) ℓ_1-based optimization.

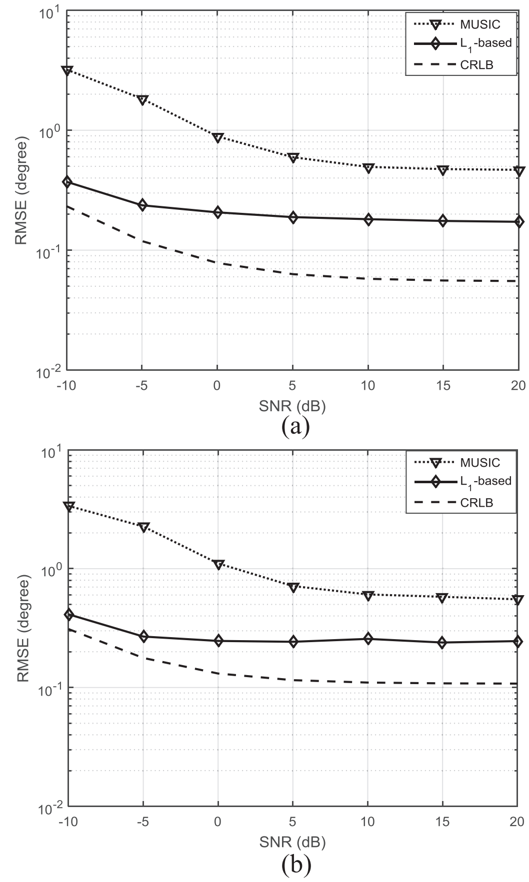

Fig. 4 RMSE performance versus SNR of MUSIC and ℓ_1-based optimization for underdetermined DOA estimation using NSCA with $M = 7$, Snapshots = 10000 for (a) $D = 8$, and (b) $D = 10$.

angle estimates as a function of SNR. Thus, CRLB is used as the benchmark at high SNR conditions. We show the plots for ($D = 8$) and ($D = 10$). The performance is not angular dependent because we observe similar result for different angles. The number of trials used in this example is 100.

Figure 4(a) shows the RMSE as a function of SNR for 8 DOAs impinging on the NSCA for 10000 snapshots averaged over 100 monte carlo simulations. The performance of both methods improves as the SNR is increasing. The subspace based technique MUSIC has lower performance compared to ℓ_1-based optimization method but becomes closer to ℓ_1-based method at 20 dB. In Fig. 4(b) we observe the RMSE of MUSIC and ℓ_1-based optimization as a function of SNR for 10 DOAs impinging on the NSCA. In this case, we observe that the ℓ_1-based optimization method outperforms MUSIC, and has an RMSE of about 0.15 degrees. From the results in Fig. 4(a) and (b) ℓ_1-based optimization method is clearly better than MUSIC, and hereafter we concentrate more on the results of ℓ_1-based optimization method for underdetermined DOA estimation.

4.4 Optimal Array

The third example in this section shows optimal array configuration in underdetermined DOA estimation. In this example, we examine 3 circular array configurations; (a). Nested Sparse Circular Array (NSCA) proposed in this paper, (b). the length-20 circular sparse ruler proposed in [18], and (c). Uniform Circular Array (UCA) conventional circular array type which is also similar to the array type proposed in [17]. Figure 5, shows the RMSE performance of ℓ_1-based optimization technique for the three arrays when ($D = 8$). We observe that the proposed NSCA has superior performance as compared to the other circular array types and its performance is close to the CRLB as shown in Fig. 5.

4.5 Angular Dependency

In this example, we show the performance of ℓ_1-based optimization approach in terms of angular dependency of RMSE for the 3 circular array configurations examined in the previous subsection. With ($D = 8$), we observed the RMSE behavior of the 3 arrays when the DOA is changed from initial positions by $\Delta\phi$, where $\Delta\phi$ ranges from 0° to 180°. In Fig. 6, we observe that there is very little change in the RMSE behavior for NSCA and UCA while sparse ruler has higher angular dependency. The NSCA also has better RMSE comparing to the two configurations over all angles. UCA has no

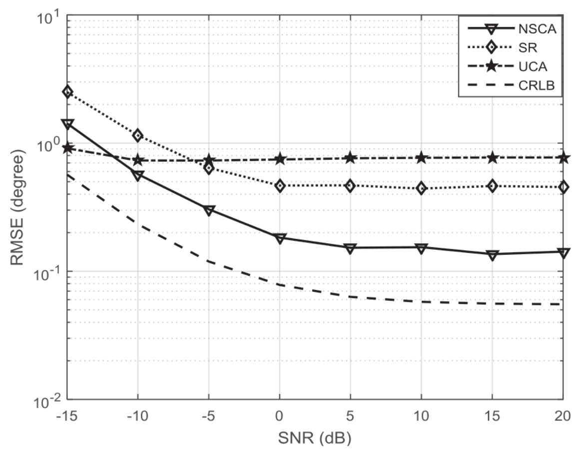

Fig. 5 Optimal array configuration comparison by ℓ_1-based optimization for underdetermined DOA estimation $D = 8$, and Snapshots = 10000.

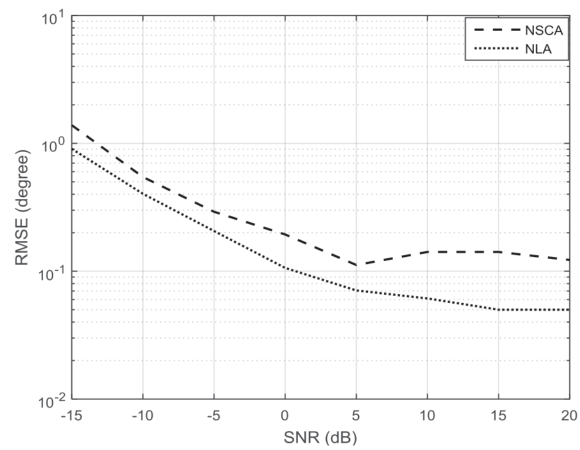

Fig. 7 Comparison of NSCA and Nested Linear array RMSE performance using ℓ_1-based optimization.

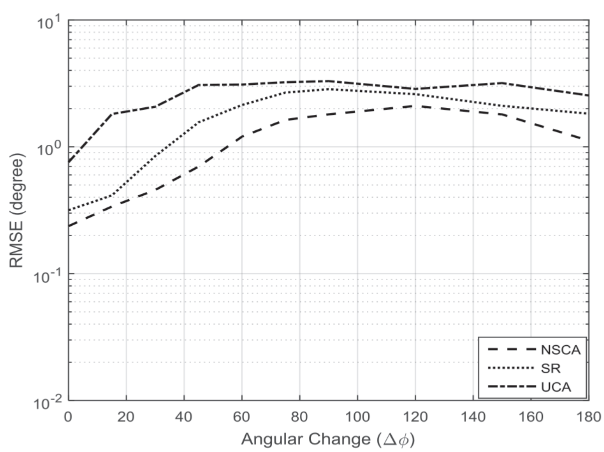

Fig. 6 Angular dependency of Nested Sparse Circular Array (NSCA), length-20 circular Sparse Ruler (SR) and Uniform Circular Array (UCA) using ℓ_1-based optimization technique for underdetermined DOA estimation. $D = 8$, SNR = 20 dB, and Snapshots = 10000.

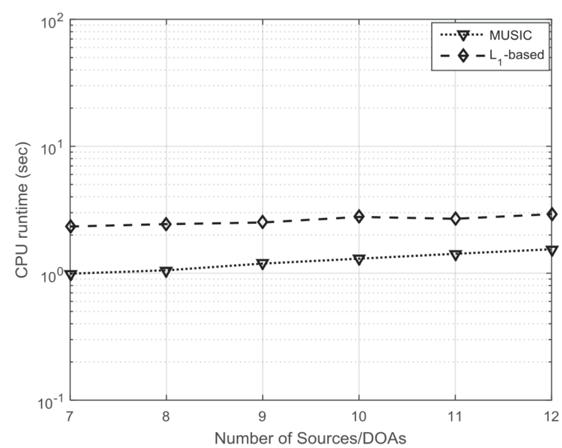

Fig. 8 Comparison of CPU runtime versus the number of impinging signals for MUSIC and ℓ_1-based optimization using NSCA. $M = 7$, $D = 8$, SNR = 20 dB, and Snapshots = 10000.

RMSE change as DOAs change in terms of angular dependency between 45° and 180°. Therefore we can conclude that NSCA has little angular dependency in the case of $\Delta\phi$ between 0° and 180° and a better RMSE performance as compared to the other two array configurations.

4.6 Performance Comparison of NSCA and Nested Linear Array

In the next example we consider the performance comparison of the proposed NSCA and nested linear array with 7 elements. Figure 7 shows the performance comparison. We observe that nested linear array obtains a good RMSE performance but it is not much different from the RMSE performance of NSCA especially in lower SNR levels. On the other hand, the performance of the proposed array (NSCA) is also good and close to that of nested linear. On top of that, NSCA has advantages over nested linear because it does not suffer from angular dependency and as a circular array type, it is capable of 2D DOA estimation.

4.7 Average Runtime

The last example in this section shows the CPU runtime versus the number of impinging signals for subspace based method MUSIC and ℓ_1-based optimization using NSCA for different number of signals impinging on the array. Figure 8 shows this comparison, where subspace based technique MUSIC requires very little amount of time to run whilst the ℓ_1-based optimization technique requires almost two times more than subspace based technique. Although ℓ_1-based optimization method requires more run time, it has superior performance as compared to subspace based method MUSIC.

5. Conclusion

In this paper, we proposed a nested sparse circular array geometry that realizes Underdetermined DOA estimation. The concentric extension of NSCA provides the virtual sensors which are synthesized on a non-uniform concentric circular array. By utilizing the virtual sensors in the concentric extension, the NSCA achieves increased degrees of freedom. We explored two strategies that can be used with the NSCA; the subspace based technique called MUSIC and the ℓ_1-based optimization method. In both methods, we confirmed that NSCA have the power to estimate more number of sources in comparison to the number of physical sensors. We also investigated the RMSE performance as related to SNR and number of snapshots in which the number of snapshots plays a crucial role in the underdetermined DOA estimation. Using the nested sparse circular array, an increase in the degrees of freedom is guaranteed, especially with the ℓ_1-based optimization method which attains $2M - 1$ degrees of freedom and has no angular dependency.

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